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Cyclic parallel CR-submanifolds of maximal CR-dimension in a complex space form

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Abstract We first classify $(2n-1)$-dimensional cyclic parallel CR-submanifold M with CR-dimension $n-1$ in a non-flat complex space form of constant holomorphic sectional curvature $4c$. Then, we prove that $||\nabla h||^2 \geq 4(n-1)c^2$, where h is the second fundamental form on M. We also completely classify $(2n-1)$-dimensional CR-submanifolds with CR-dimension $n-1$ in a non-flat complex space form which satisfy the equality case of this inequality. This generalizes an inequality for real hypersurfaces in a non-flat complex space form obtained by Maeda (J Math Soc Jpn 28:529–540; 1976) and Chen et al. (Algebras Groups Geom 1:176–212; 1984) for complex projective and hyperbolic spaces, respectively.

Keywords CR-submanifolds · Cyclic parallel submanifolds · Complex space forms

Mathematics Subject Classification (2000) 53C40 · 53C15

1 Introduction

A complex n-dimensional complex space form $\hat{M}_n(c)$ is a complete and simply connected Kaehler manifold with constant holomorphic sectional curvature $4c$, that is, it is either a complex projective space $\mathbb{C}P_n$, a complex Euclidean space $\mathbb{C}_n$, or a complex hyperbolic space $\mathbb{C}H_n$ (according to as the holomorphic sectional curvature $4c$ is positive, zero, or negative).

The study of real hypersurfaces in a Kaehler manifold has been an active field in the past few decades, especially when the ambient space is a complex space form. One of the first results in this topic is the non-existence of real hypersurfaces M with parallel shape operator

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A in a non-flat complex space form, that is, $\nabla A = 0$, where ∇ is the Levi-Civita connection on M. This fact is an immediate consequence of the Codazzi equation of such a submanifold. Several weaker notions such as η-parallelism and recurrence of the shape operator were hence studied by the researchers.

The shape operator A is said to be *recurrent* if there is a 1-form τ on M such that $\nabla A = A \otimes \tau$. It is known that there does not exist any real hypersurface in $\hat{M}_n(c)$, $c \neq 0$, with recurrent shape operator (cf. [14,21]). A real hypersurface M in $\hat{M}_n(c)$ is said to be η-*recurrent* if $\langle(\nabla_X A)Y, Z\rangle = \tau(X)\langle AY, Z\rangle$, for any tangent vector fields X, Y, $and Z$ in the maximal holomorphic distribution $\mathscr{D}$, where τ is a 1-form on M (cf. [13]). In particular, M is said to be η-parallel when $\tau = 0$ (cf. [17]).

In [18,19], the author and Kon classified η-parallel real hypersurfaces in $\hat{M}_n(c)$, $c \neq 0$, $n \geq 3$. It was also proved in [20] that a real hypersurface in $\hat{M}_n(c)$, $c \neq 0$, $n \geq 3$ is η-recurrent if and only if it is η-parallel.

A submanifold M in a Riemannian manifold $\hat{M}$ is said to be *cyclic parallel* if its second fundamental form h satisfies

$$(\nabla_X h)(Y, Z) + (\nabla_Y)h(Z, X) + (\nabla_Z h)(X, Y) = 0$$

for any vector fields X, Y, and Z tangent to M. When M is a real hypersurface in $\hat{M}_n(c)$, the cyclic parallelism is equivalent to the condition

$$(\nabla_X A)Y = -c\{\eta(Y)\phi X + \langle\phi X, Y\rangle\xi\}$$

for any vector fields X and Y tangent to M, where $(\phi, \xi, \eta, \langle, \rangle)$ is the almost contact structure on M induced by the complex structure J of the ambient space. Maeda (cf. [22]) and Chen, Ludden and Montiel (cf. [5]) classified real hypersurfaces in $\hat{M}_n(c)$, $c \neq 0$, under this condition (cf. Theorem 4). With this result, it can be proved that

$$||\nabla A||^2 \geq 4(n-1)c^2 \tag{1}$$

and equality holds if and only if the real hypersurface M is an open part of a tube over $\mathbb{C}P_k$, $1 \leq k \leq n-1$, for $c > 0$, and M is an open part of a horosphere, a geodesic hypersphere in $\mathbb{C}H_n$, or a tube over $\mathbb{C}H_k$, $1 \leq k \leq n-1$, for $c < 0$.

Note that a real hypersurface in $\hat{M}_n(c)$ is a CR-submanifold (see Definition 2 for precise definition) of maximal CR-dimension (or of hypersurface type). Hence, one of the main lines deals with generalizing these known results in real hypersurfaces in $M_n(c)$ to CR-submanifolds of maximal CR-dimension in $\hat{M}_n(c)$. A number of results were obtained by Djorić and Okumura (cf. [7]–[11]). In particular, they attempted to generalize certain results concerning relationship between A and ϕ for real hypersurfaces in a complex space form into the setting of CR-submanifolds of maximal CR-dimension.

This paper is also a contribution in this line. The main objective of this paper is to extend the inequality (1) for real hypersurfaces in a non-flat complex space form to the setting of CR-submanifolds of maximal CR-dimension. We shall first prove the following theorem.

Theorem 1 *Let M be a $(2n-1)$-dimensional CR-submanifold of maximal CR-dimension in $\hat{M}_{n+p}(c)$, $c \neq 0$, $n \geq 2$. Then, M is cyclic parallel if and only if M is an open part of one of the following spaces.*

(a) For $c < 0$:

(i) a horosphere in $\mathbb{C}H_n$,
(ii) a geodesic hypersphere or a tube over a hyperplane $\mathbb{C}H_{n-1}$ in $\mathbb{C}H_n$,
(iii) a tube over a totally geodesic $\mathbb{C}H_k$ in $\mathbb{C}H_n$, where $1 \leq k \leq n-2$.

(b) For $c > 0$*:*

(i) a geodesic hypersphere in $\mathbb{C}P_n$*,*
(ii) a tube over a totally geodesic $\mathbb{C}P_k$ *in* $\mathbb{C}P_n$*, where* $1 \leq k \leq n-2$*,*
(iii) a standard CR-product $\mathbb{C}P_{n-1} \times \mathbb{R}P^1$ *in* $\mathbb{C}P_{2n-1}$*.*

With this result, we can prove the following.

Theorem 2 *Let* M *be a* $(2n-1)$*-dimensional CR-submanifold of maximal CR-dimension in* $\hat{M}_{n+p}(c)$*,* $c \neq 0$*,* $n \geq 2$*. Then,* M *satisfies*

$$||\nabla h||^2 \geq 4(n-1)c^2$$

and equality holds if and only M *is an open part of one of the following spaces.*

(a) For $c < 0$*:*

(i) a horosphere in $\mathbb{C}H_n$*,*
(ii) a geodesic hypersphere or a tube over a hyperplane $\mathbb{C}H_{n-1}$ *in* $\mathbb{C}H_n$*,*
(iii) a tube over a totally geodesic $\mathbb{C}H_k$ *in* $\mathbb{C}H_n$*, where* $1 \leq k \leq n-2$*.*

(b) For $c > 0$*:*

(i) a geodesic hypersphere in $\mathbb{C}P_n$*,*
(ii) a tube over a totally geodesic $\mathbb{C}P_k$ *in* $\mathbb{C}P_n$*, where* $1 \leq k \leq n-2$*,*
(iii) a standard CR-product $\mathbb{C}P_{n-1} \times \mathbb{R}P^1$ *in* $\mathbb{C}P_{2n-1}$*.*

Remark 1 It is worthwhile to remark that there is an additional class of submanifolds, that is, $\mathbb{C}P_{n-1} \times \mathbb{R}P^1$ in Case (b)(iii), appeared in the list of Theorem 1 compared to the classification of real hypersurfaces under the same condition (cf. Theorem 4). Chen and Maeda (cf. [6]) proved that there do not exist real hypersurfaces which are Riemannian product of Riemannian manifolds. Hence, we can see that $\mathbb{C}P_{n-1} \times \mathbb{R}P^1$ can never be immersed in $\mathbb{C}P_n$ as a real hypersurface.

This paper is organized as follows. In the next two sections, we shall fix some notations and discuss some fundamental properties of CR-submanifolds in a Kaehler manifold. We describe the standard examples of cyclic parallel CR-submanifolds of maximal CR-dimension in a non-flat complex space form in Sect. 4. In Sect. 5, we prepare some lemmas. We prove Theorem 1 and Theorem 2 in the last two sections.

2 CR-submanifolds in a Kaehler manifold

In this section, we shall recall some structural equations in the theory of CR-submanifolds in a Kaehler manifold and fix some notations. Some fundamental properties of CR-submanifolds in a Kaehler manifold are also derived here.

Let $\hat{M}$ be a Kaehler manifold with complex structure J, and let M be a connected Riemannian manifold isometrically immersed in $\hat{M}$. The maximal J-invariant subspace $\mathscr{D}_x$ of the tangent space T_xM, $x \in M$ is given by

$$\mathscr{D}_x = T_xM \cap JT_xM.$$

Definition 1 ([4]) A submanifold M in a Kaehler manifold $\hat{M}$ is said to be a *generic submanifold* if the dimension of $\mathscr{D}_x$ is constant along M. The distribution $\mathscr{D} : x \to \mathscr{D}_x$, $x \in M$ is called the *holomorphic distribution (or Levi distribution)* on M and the complex dimension of $\mathscr{D}$ is called the CR-dimension of M.

Definition 2 ([1]) A generic submanifold M in a Kaehler manifold $\hat{M}$ is said to be a *CR-submanifold* if the orthogonal complementary distribution $\mathscr{D}^\perp$ of $\mathscr{D}$ in TM is totally real, that is, $J\mathscr{D}_x^\perp \subset T_xM^\perp$, $x \in M$.

If $\mathscr{D}^\perp = \{0\}$ (resp. $\mathscr{D} = \{0\}$), the CR-submanifold M is said to be *holomorphic* (resp. *totally real*). A CR-submanifold M is said to be *proper* if it is neither holomorphic nor totally real. Let ν be the orthogonal complementary distribution of $J\mathscr{D}^\perp$ in $TM^\perp$. Then, an *anti-holomorphic* submanifold M is a CR-submanifold with $\nu = \{0\}$, that is, $J\mathscr{D}^\perp = TM^\perp$.

Remark 2 The study of CR-submanifolds in the sense of Definition 2 was initiated by Bejancu in [1]. Generic submanifolds have been studied by some researchers under the term of "CR-submanifolds" from the CR geometric view point (cf. [12,24, pp. 345]). We will not follow this term here in order to avoid the confusion. We remark that when a generic submanifold M is of maximal CR-dimension, that is, $\dim_{\mathbb{R}} \mathscr{D} = \dim M - 1$, M will be a CR-submanifold in the sense of Definition 2.

Suppose M is a CR-submanifold in a Kaehler manifold $\hat{M}$. Denote by $\langle , \rangle$ the Riemannian metric of $\hat{M}$ as well as that induced on M. Also, we let ∇ be the Levi-Civita connection on the tangent bundle TM of M, $\nabla^\perp$ the normal connection on the normal bundle $TM^\perp$ of M, h the second fundamental form, and A_σ the shape operator of M with respect to a vector σ normal to M.

For a vector bundle $\mathscr{V}$ over M, we denote by $\Gamma(\mathscr{V})$ the $\Omega^0(M)$-module of cross sections on $\mathscr{V}$, where $\Omega^k(M)$ is the space of k-forms on M. For any $X \in \Gamma(TM)$ and $\sigma \in \Gamma(TM^\perp)$, we put $\phi X = \tan(JX)$, $\omega X = \mathrm{nor}(JX)$, $B\sigma = \tan(J\sigma)$ and $C\sigma = \mathrm{nor}(J\sigma)$. From the parallelism of J, we have (cf. [27, pp. 77])

$$(\nabla_X\phi)Y = A_{\omega Y}X + Bh(X, Y) \tag{2}$$
$$(\nabla_X\omega)Y = -h(X, \phi Y) + Ch(X, Y) \tag{3}$$
$$(\nabla_X B)\sigma = -\phi A_\sigma X + A_{C\sigma}X \tag{4}$$
$$(\nabla_X C)\sigma = -\omega A_\sigma X - h(X, B\sigma) \tag{5}$$

for any X, $Y \in \Gamma(TM)$ and $\sigma \in \Gamma(TM^\perp)$.

We denote by $H := \mathrm{Trace}(h)$. For a local frame of orthonormal vectors $e_1, e_2, \ldots, e_{2m}$ in $\Gamma(\mathscr{D})$, where $m = \dim_{\mathbb{C}} \mathscr{D}$, we define

$$H_{\mathscr{D}} := \sum_{j=1}^{2m} h(e_j, e_j).$$

Lemma 1 *Let M be a CR-submanifold in a Kaehler manifold $\hat{M}$. Then, $\langle(\phi A_\sigma + A_\sigma\phi)X, Y\rangle = 0$, for any $X, Y \in \Gamma(\mathscr{D})$ and $\sigma \in \Gamma(\nu)$. Moreover, we have $CH_{\mathscr{D}} = 0$.*

Proof By putting $X, Y \in \Gamma(\mathscr{D})$ in (3), we have

$$-\omega\nabla_X Y = -h(X, \phi Y) + Ch(X, Y).$$

Taking inner product of both sides of this equation with $\sigma \in \Gamma(\nu)$, we obtain

$$0 = \langle\phi A_\sigma X, Y\rangle - \langle A_{C\sigma}X, Y\rangle.$$

Since $A_{C\sigma}$ is self-adjoint, we obtain $\langle(\phi A_\sigma + A_\sigma\phi)X, Y\rangle = 0$, for any $X, Y \in \Gamma(\mathscr{D})$. Furthermore, for any unit vector field $X \in \Gamma(\mathscr{D})$ and $\sigma \in \Gamma(\nu)$, we have

$$0 = \langle(\phi A_\sigma + A_\sigma\phi)X, \phi X\rangle = \langle h(X, X) + h(\phi X, \phi X), \sigma\rangle.$$

This equation implies that $\langle H_{\mathscr{D}}, \sigma\rangle = 0$ and hence $CH_{\mathscr{D}} = 0$. □

A CR-submanifold M is said to be *mixed totally geodesic* if $h(X, Y) = 0$, for any $X \in \Gamma(\mathscr{D})$ and $Y \in \Gamma(\mathscr{D}^\perp)$. A CR-submanifold M is called a *CR-product* if it is locally a Riemannian product of a holomorphic submanifold and a totally real submanifold.

The following lemma characterizes CR-products in a Kaehler manifold.

Lemma 2 ([3]) *A CR-submanifold M in a Kaehler manifold is a CR-product if and only if $Bh(X, Y) = 0$, for any $X \in \Gamma(\mathscr{D})$ and $Y \in \Gamma(TM)$.*

Now suppose $\hat{M}_q(c)$ is a q-dimensional complex space form with constant holomorphic sectional curvature $4c$, and let M be a CR-submanifold in $\hat{M}_q(c)$.

Let R and $R^\perp$ be the curvature tensors associated with ∇ and $\nabla^\perp$, respectively. The equations of Gauss, Codazzi, and Ricci are then given, respectively, by

$$\begin{aligned} R(X, Y)Z &= c\{\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle \phi Y, Z\rangle \phi X - \langle \phi X, Z\rangle \phi Y \\ &\quad -2\langle \phi X, Y\rangle \phi Z\} + A_{h(Y,Z)}X - A_{h(X,Z)}Y \qquad (6)\\ (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) &= c\{\langle \phi Y, Z\rangle \omega X - \langle \phi X, Z\rangle \omega Y - 2\langle \phi X, Y\rangle \omega Z\} \\ R^\perp(X, Y)\sigma &= c\{\langle \omega Y, \sigma\rangle \omega X - \langle \omega X, \sigma\rangle \omega Y - 2\langle \phi X, Y\rangle C\sigma\} + h(X, A_\sigma Y) - h(Y, A_\sigma X) \end{aligned}$$

for any $X,\ Y,\ Z \in \Gamma(TM)$ and $\sigma \in \Gamma(TM^\perp)$.

A submanifold M in a Riemannian manifold $\hat{M}$ is said to be *cyclic parallel* if its second fundamental form h satisfies

$$(\nabla_X h)(Y, Z) + (\nabla_Y)h(Z, X) + (\nabla_Z h)(X, Y) = 0$$

for any X, Y, and $Z \in \Gamma(TM)$. When M is CR-submanifold in $\hat{M}_q(c)$, by the Codazzi equation, the cyclic parallelism of M is equivalent to the condition

$$(\nabla_X h)(Y, Z) = -c\{\langle \phi X, Z\rangle \omega Y + \langle \phi X, Y\rangle \omega Z\} \qquad (7)$$

for any X, Y, and $Z \in \Gamma(TM)$.

The second-order covariant derivative $\nabla^2 h$ on the second fundamental form h is defined by

$$\begin{aligned} (\nabla^2_{XY} h)(Z, W) = \nabla^\perp_X\{(\nabla_Y h)(Z, W)\} &- (\nabla_{\nabla_X Y} h)(Z, W) - (\nabla_Y h)(\nabla_X Z, W) \\ &-(\nabla_Y h)(Z, \nabla_X W). \end{aligned}$$

The Ricci identity gives

$$R(X, Y)h = \nabla^2_{XY} h - \nabla^2_{YX} h \qquad (8)$$

where

$$(R(X, Y)h)(Z, W) = R^\perp(X, Y)h(Z, W) - h(R(X, Y)Z, W) - h(Z, R(X, Y)W)$$

for any X, Y, Z, and $W \in \Gamma(TM)$.

Finally, we state without proof a codimension reduction theorem for real submanifolds in a non-flat complex space form.

Theorem 3 ([15,25]) *Let M be a connected real n-dimensional submanifold in $\hat{M}_{(n+p)/2}(c)$, $c \neq 0$ and let $N_0(x)$ be the orthogonal complement of the first normal space in $T_x M^\perp$. We put $H_0(x) = JN_0(x) \cap N_0(x)$ and let $H(x)$ be a J-invariant subspace of $H_0(x)$. If the orthogonal complement $H_2(x)$ of $H(x)$ in $T_x M^\perp$ is invariant under parallel translation with respect to the normal connection and if q is the constant dimension of $H_2(x)$, for each $x \in M$, then there exists a $(n+q)$-dimensional totally geodesic holomorphic submanifold $\hat{M}_{(n+q)/2}(c)$ in $\hat{M}_{(n+p)/2}(c)$ such that $M \subset \hat{M}_{(n+q)/2}(c)$.*

3 CR-submanifolds of maximal CR-dimension in a complex space form

Suppose $\hat{M}_{n+p}(c)$ is a complex $(n+p)$-dimensional complex space form of constant holomorphic sectional curvature $4c$, and M is a real $(2n-1)$-dimensional CR-submanifold of maximal CR-dimension in $\hat{M}_{n+p}(c)$. Then, $\dim_{\mathbb{C}} \mathscr{D} = n-1$ and $\dim \mathscr{D}^\perp = 1$. Let $N \in \Gamma(J\mathscr{D}^\perp)$ be a local unit vector field normal to M, $\xi = -JN$ and η the 1-form dual to ξ. Then, we have

$$\phi^2 X = -X + \eta(X)\xi$$
$$\omega X = \eta(X)N; \quad B\sigma = -\langle \sigma, N\rangle \xi$$

for any $X \in \Gamma(TM)$ and $\sigma \in \Gamma(TM^\perp)$. It follows from (2)–(5) that

$$(\nabla_X \phi)Y = \eta(Y)A_N X - \langle A_N X, Y\rangle \xi \tag{9}$$
$$\nabla_X \xi = \phi A_N X; \quad \nabla^\perp_X N = Ch(X, \xi) \tag{10}$$
$$h(X, \phi Y) = -\langle \phi A_N X, Y\rangle N - \eta(Y)Ch(X, \xi) + Ch(X, Y) \tag{11}$$
$$(\nabla_X C)\sigma = -\langle h(X, \xi), \sigma\rangle N + \langle \sigma, N\rangle h(X, \xi) \tag{12}$$

for any $X, Y \in \Gamma(TM)$ and $\sigma \in \Gamma(TM^\perp)$.

The equations of Codazzi and Ricci can also be reduced to

$$(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = c\{\eta(X)\langle \phi Y, Z\rangle - \eta(Y)\langle \phi X, Z\rangle - 2\eta(Z)\langle \phi X, Y\rangle\}N \tag{13}$$
$$R^\perp(X, Y)\sigma = -2c\langle \phi X, Y\rangle C\sigma + h(X, A_\sigma Y) - h(Y, A_\sigma X) \tag{14}$$

for any $X, Y, Z \in \Gamma(TM)$ and $\sigma \in \Gamma(TM^\perp)$. We define the covariant derivative of the shape operator as

$$(\nabla_X A)_\sigma Y = \nabla_X\{A_\sigma Y\} - A_\sigma \nabla_X Y - A_{\nabla^\perp_X \sigma} Y. \tag{15}$$

Then, we have

$$\langle (\nabla_X A)_\sigma Y, Z\rangle = \langle (\nabla_X h)(Y, Z), \sigma\rangle$$

and the Codazzi equation (13) can be rephrased as

$$(\nabla_X A)_\sigma Y - (\nabla_Y A)_\sigma X = c\langle \sigma, N\rangle\{\eta(X)\phi Y - \eta(Y)\phi X - 2\langle \phi X, Y\rangle \xi\} \tag{16}$$

for any $X, Y, Z \in \Gamma(TM)$ and $\sigma \in \Gamma(TM^\perp)$.

The following lemma can be obtained immediately from Lemma 1.

Lemma 3 *Let M be a CR-submanifold of maximal CR-dimension in a Kaehler manifold $\hat{M}$. Then, $Ch(\xi, \xi) = CH$.*

4 Examples

In this section, we discuss certain examples of cyclic parallel CR-submanifolds of maximal CR-dimension in a non-flat complex space form. From (7), it is equivalent to said that M satisfies the following condition.

$$(\nabla_X h)(Y, Z) = -c\{\eta(Y)\langle \phi X, Z\rangle + \eta(Z)\langle \phi X, Y\rangle\}N \tag{17}$$

for any X, Y, and $Z \in \Gamma(TM)$.

Let $\mathbb{C}^1_{n+1}$ be the complex Lorentzian space with Hermitian inner product

$$G(z,w) = -z_0\bar{w}_0 + \sum_{j=1}^{n} z_j\bar{w}_j$$

where $z = (z_0, z_1, \ldots, z_n)$, $w = (w_0, w_1, \ldots, w_n) \in \mathbb{C}^1_{n+1}$. Then, the anti-De Sitter space of radius 1 is given by

$$H_1^{2n+1} := H_1^{2n+1}(-1) = \left\{z \in \mathbb{C}^1_{n+1} : \langle z, z\rangle = -1\right\}$$

where $\langle z, w\rangle := \Re G(z,w)$. We denote by $\psi : H_1^{2n+1} \to \mathbb{C}H_n$ the principal S^1-bundle over $\mathbb{C}H_n$. Here, $\mathbb{C}H_n$ denotes the complex hyperbolic space with constant holomorphic sectional curvature -4.

Example 1 (Horospheres in $\mathbb{C}H_n$) Let M' be a Lorentzian hypersurface in H_1^{2n+1} given by

$$|z_0 - z_1| = 1; \qquad -|z_0| + \sum_{j=1}^{n} |z_j|^2 = -1.$$

Then, $M^* = \psi(M')$ is a real hypersurface in $\mathbb{C}H_n$, so-called a horosphere (a self-tube).

Example 2 (Tubes over $\mathbb{C}H_k$ in $\mathbb{C}H_n$, $0 \leq k \leq n-1$) Let $k, l \geq 0$ be integers with $k + l = n - 1$, $r > 0$. We consider a Lorentzian hypersurface $M'_k(r)$ in H_1^{2n+1} defined by

$$-|z_0|^2 + \sum_{j=1}^{k} |z_j|^2 = -\cosh^2 r, \qquad -|z_0| + \sum_{j=1}^{n} |z_j|^2 = -1.$$

Then, $M'_k(r)$ is the standard product $H_1^{2k+1}(-\cosh r) \times S^{2l+1}(\sinh r)$. $M_k(r) = \psi(M'_k(r))$ is a real hypersurface in $\mathbb{C}H_n$, which is a tube of radius r over a totally geodesic holomorphic submanifold $\mathbb{C}H_k$ in $\mathbb{C}H_n$. In particular, $M_k(r)$ is a geodesic hypersphere in $\mathbb{C}H_n$ when $k = 0$.

Now, we consider the complex Euclidean space $\mathbb{C}_{n+1}$ with Hermitian inner product

$$G(z,w) = \sum_{j=0}^{n} z_j\bar{w}_j$$

where $z = (z_0, z_1, \ldots, z_n)$, $w = (w_0, w_1, \ldots, w_n) \in \mathbb{C}_{n+1}$. Then, the sphere of radius 1 centered at the origin is given by

$$S^{2n+1} := S^{2n+1}(1) = \{z \in \mathbb{C}_{n+1} : \langle z, z\rangle = 1\}$$

where $\langle z, w\rangle := \Re G(z,w)$. We denote by $\psi : S^{2n+1} \to \mathbb{C}P_n$ the principal S^1-bundle over $\mathbb{C}P_n$. Here, $\mathbb{C}P_n$ denotes the complex projective space with constant holomorphic sectional curvature 4.

Example 3 (Tubes over $\mathbb{C}P_k$ in $\mathbb{C}P_n$, $0 \leq k \leq n-1$) Let $k, l \geq 0$ be integers with $k + l = n - 1$, $r \in\,]\,0, \pi/2[$. We consider a hypersurface $M'_k(r)$ in S^{2n+1} defined by

$$\sum_{j=0}^{k} |z_j|^2 = \cos^2 r, \qquad \sum_{j=0}^{n} |z_j|^2 = 1.$$

Then, $M'_k(r)$ is the standard product $S^{2k+1}(\cos r) \times S^{2l+1}(\sin r)$. $M_k(r) = \psi(M'_k(r))$ is a real hypersurface in $\mathbb{C}P_n$, which is a tube of radius r over a totally geodesic holomorphic submanifold $\mathbb{C}P_k$ in $\mathbb{C}P_n$. In particular, when $k = 0$, $M_k(r)$ is a geodesic hypersphere in $\mathbb{C}P_n$.

Theorem 4 [5,22] *Let M be a real hypersurface in $\hat{M}_n(c)$, $c \neq 0$, $n \geq 2$. Then, M satisfies*

$$(\nabla_X A)Y = -c\{\eta(Y)\phi X + \langle \phi X, Y \rangle \xi\}$$

for any $X, Y \in \Gamma(TM)$, if and only if M is an open part of one of the following spaces.

(a) For $c < 0$

(i) a horosphere,
(ii) a geodesic hypersphere or a tube over $\mathbb{C}H_{n-1}$,
(iii) a tube over $\mathbb{C}H_k$, where $1 \leq k \leq n-2$.

(b) For $c > 0$

(i) a geodesic hypersphere,
(ii) a tube over $\mathbb{C}P_k$, where $1 \leq k \leq n-2$.

Remark 3 A real hypersurface in a Kaehler manifold is said to be *Hopf* if it is mixed totally geodesic. The real hypersurfaces stated in Theorem 4 are categorized as Hopf hypersurfaces of type A in the Takagi's list (for $c > 0$) and Montiel's list (for $c < 0$) of Hopf hypersurfaces of constant principal curvatures in $\hat{M}_n(c)$, $c \neq 0$ (cf. [23,26]). These real hypersurfaces in the Takagi's list and Montiel's list are in fact the only Hopf hypersurfaces with constant principal curvatures in $\hat{M}_n(c)$, $c \neq 0$ (cf. [2,16]).

The spaces M stated in Theorem 4 can be naturally immersed into $\hat{M}_{n+p}(c)$ with higher codimension via the standard holomorphic immersion of $\hat{M}_n(c)$ into $\hat{M}_{n+p}(c)$ as follows

$$M \longrightarrow \hat{M}_n(c) \longrightarrow \hat{M}_{n+p}(c).$$

Clearly, such an immersion is not full. Next, we shall discuss an example of CR-submanifolds with maximal CR-dimension in $\mathbb{C}P_q$, which are irreducible to real hypersurfaces in a totally geodesic holomorphic submanifold of $\mathbb{C}P_q$.

We denote by $(z_0 : z_1 : \cdots : z_n)$ the homogeneous coordinates of $\mathbb{C}P_n$. Then, the Segre embedding $S_{m,l} : \mathbb{C}P_m \times \mathbb{C}P_l \to \mathbb{C}P_{m+l+ml}$ is given by

$$S_{m,l}(z, w) = (z_0 w_0 : \cdots : z_0 w_l : z_1 w_0 : \cdots : z_1 w_l : \cdots : z_m w_0 : \cdots z_m w_l)$$

where $(z_0 : z_1 : \cdots : z_m) \in \mathbb{C}P_m$ and $(w_0 : w_1 : \cdots : w_l) \in \mathbb{C}P_l$.

Example 4 (The standard CR-products $\mathbb{C}P_{n-1} \times \mathbb{R}P^1$) We consider $\mathbb{R}P^l$ as a totally geodesic, totally real submanifold in $\mathbb{C}P_l$, and $\mathbb{C}P_{m+l+ml}$ as a totally geodesic, holomorphic submanifold in $\mathbb{C}P_q$, $m + l + ml \leq q$. The *standard CR-product* $\mathbb{C}P_m \times \mathbb{R}P^l$ can be immersed into $\mathbb{C}P_q$ via $S_{m,l}$ as follows: (cf. [3])

$$\mathbb{C}P_m \times \mathbb{R}P^l \longrightarrow \mathbb{C}P_m \times \mathbb{C}P_l \xrightarrow{S_{m,l}} \mathbb{C}P_{m+l+ml} \longrightarrow \mathbb{C}P_q. \tag{18}$$

In particular, $\mathbb{C}P_{n-1} \times \mathbb{R}P^1$ is a CR-submanifold of maximal CR-dimension in $\mathbb{C}P_q$, $2n - 1 \leq q$.

Theorem 5 ([3]) *Let M be a CR-product in $\mathbb{C}P_q$, $\dim_{\mathbb{C}} \mathscr{D} = m$ and $\dim_{\mathbb{R}} \mathscr{D}^{\perp} = l$. Then, we have*

$$||h||^2 \geq 4ml$$

and equality holds if and only if M is given by the immersion (18).

Theorem 6 *Let $M = \mathbb{C}P_{n-1} \times \mathbb{R}P^1$. Then, M is a cyclic parallel CR-submanifold of maximal CR-dimension in $\mathbb{C}P_q$.*

Proof Since $\mathbb{C}P_{n-1}$ and $\mathbb{R}P^1$ are leaves of $\mathscr{D}$ and $\mathscr{D}^\perp$, respectively, and they are totally geodesic in $\mathbb{C}P_q$, by the Gauss formula, we see that

$$h(\mathscr{D}, \mathscr{D}) = 0; \qquad h(\xi, \xi) = 0.$$

Further, as M is a Riemannian product $\mathbb{C}P_{n-1} \times \mathbb{R}P^1$, both distributions $\mathscr{D}$ and $\mathscr{D}^\perp$ are auto-parallel, that is,

$$\nabla : \Gamma(\mathscr{D}) \to \Omega^1(M)\,_{\Omega^0(M)} \otimes \Gamma(\mathscr{D}); \qquad \nabla : \Gamma(\mathscr{D}^\perp) \to \Omega^1(M)\,_{\Omega^0(M)} \otimes \Gamma(\mathscr{D}^\perp).$$

Therefore, we have

$$(\nabla_X h)(Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = 0$$
$$(\nabla_X h)(\xi, \xi) = \nabla_X^\perp h(\xi, \xi) - 2h(\nabla_X \xi, \xi) = 0$$

for any $X \in \Gamma(TM)$ and $Y, Z \in \Gamma(\mathscr{D})$. By using the above two equations and the Codazzi equation, we have

$$(\nabla_\xi h)(Y, \xi) = (\nabla_Y h)(\xi, \xi) = 0$$
$$(\nabla_X h)(Y, \xi) = (\nabla_\xi h)(X, Y) - c\langle \phi X, Y\rangle N = -c\langle \phi X, Y\rangle N$$

for any $X, Y \in \Gamma(\mathscr{D})$. Hence, M satisfies (17) and so it is cyclic parallel. □

Remark 4 By using a similar manner as in the above proof, we may verify that such standard CR-products with higher CR-codimension are also cyclic parallel.

5 Lemmas

Throughout this section, suppose M is a $(2n-1)$-dimensional CR-submanifold of maximal CR-dimension in $\hat{M}_{n+p}(c)$, $c \neq 0$, $n \geq 2$ and M is cyclic parallel or equivalent, it satisfies (17), that is,

$$(\nabla_X h)(Y, Z) = -c\{\eta(Y)\langle \phi X, Z\rangle + \eta(Z)\langle \phi X, Y\rangle\}N$$

for any $X, Y, and Z \in \Gamma(TM)$. By (15), we can see that the condition (17) is equivalent to

$$(\nabla_X A)_\sigma Y = -c\langle \sigma, N\rangle\{\eta(Y)\phi X + \langle \phi X, Y\rangle \xi\} \tag{19}$$

for any $X, Y \in \Gamma(TM)$ and $\sigma \in \Gamma(\nu)$.

Lemma 4 *(a)* $CH = 0$*;*
(b) $\langle H, N\rangle Ch = 0$.

Proof Note that the Eq. (17) implies that $\nabla^\perp H = 0$, and by the Ricci equation (14), we have

$$-2c\langle \phi Y, Z\rangle CH + h(Y, A_H Z) - h(Z, A_H Y) = 0$$

for any $Y, Z \in \Gamma(TM)$. By differentiating this equation covariantly in the direction of $X \in \Gamma(TM)$, we have

$$-2c\langle(\nabla_X \phi)Y, Z\rangle CH - 2c\langle \phi Y, Z\rangle(\nabla_X C)H + (\nabla_X h)(Y, A_H Z) + h(Y, (\nabla_X A)_H Z)$$
$$-(\nabla_X h)(Z, A_H Y) - h(Z, (\nabla_X A)_H Y) = 0.$$

By using (9)–(12) and (17), we have

$$2\{-\eta(Y)\langle A_N X, Z\rangle + \eta(Z)\langle A_N X, Y\rangle\}CH + 2\langle\phi Y, Z\rangle\{\eta(A_H X)N$$
$$-\langle H, N\rangle h(X, \xi)\} + \{\eta(Z)\langle\phi X, A_H Y\rangle + \eta(A_H Y)\langle\phi X, Z\rangle$$
$$-\eta(Y)\langle\phi X, A_H Z\rangle - \eta(A_H Z)\langle\phi X, Y\rangle\}N + \langle H, N\rangle\{\eta(Y)h(Z, \phi X)$$
$$+\langle\phi X, Y\rangle h(Z, \xi) - \eta(Z)h(Y, \phi X) - \langle\phi X, Z\rangle h(Y, \xi)\} = 0. \tag{20}$$

If we substitute $X = \xi$, $Y \in \Gamma(\mathscr{D})$ and $Z = \phi Y$ in the above equation, then $\langle h(\xi, \xi), H\rangle N - \langle H, N\rangle h(\xi, \xi) = 0$ and hence $CH = 0$.

Furthermore, after putting $Y = X \in \Gamma(\mathscr{D})$ and $Z = \phi X$ in (20), we get

$$\langle H, N\rangle Ch(X, \xi) = 0$$

for any $X \in \Gamma(\mathscr{D})$. Next, by putting $Z = \xi$ in (20) and making use of the above equation, we obtain

$$\langle H, N\rangle Ch(Y, \phi X) = 0$$

for any $X, Y \in \Gamma(TM)$. By these two equations and the fact that $Ch(\xi, \xi) = 0 (= CH)$, we obtain Statement (b). □

Lemma 5 *For any $X \in \Gamma(TM)$,*

(a) $\langle\phi A_N \xi, X\rangle N = -h(\xi, \phi X) + Ch(\xi, X)$;
(b) $d\alpha(X) = 2\eta(A_N \phi A_N X)$;
(c) $2Ch(A_N X, \xi) = \alpha Ch(X, \xi)$.

Proof Statement (a) can be obtained easily from $X = \xi$ in (11) and Lemma 4.

Taking into account that $CH = 0$ again, we see that $h(\xi, \xi) = \alpha N$. It follows from (17), (10), and Statement (a) that

$$0 = (\nabla_X h)(\xi, \xi) = d\alpha(X)N + \alpha Ch(X, \xi) + 2\langle\phi A_N \xi, A_N X\rangle N - 2Ch(\xi, A_N X)$$

for any $X \in \Gamma(TM)$. Statements (b) and (c) are the $J\mathscr{D}^{\perp}$- and ν-component of this equation, respectively. □

Lemma 6 *For any X, Y, Z, and $W \in \Gamma(TM)$,*

$$c\{-\langle\phi Y, \phi Z\rangle\langle A_N X, W\rangle + \langle\phi X, \phi Z\rangle\langle A_N Y, W\rangle$$
$$-\langle\phi Y, \phi W\rangle\langle A_N X, Z\rangle + \langle\phi X, \phi W\rangle\langle A_N Y, Z\rangle$$
$$+\langle\phi Y, Z\rangle\langle(\phi A_N - A_N\phi)X, W\rangle - \langle\phi X, Z\rangle\langle(\phi A_N - A_N\phi)Y, W\rangle$$
$$+\langle\phi Y, W\rangle\langle(\phi A_N - A_N\phi)X, Z\rangle - \langle\phi X, W\rangle\langle(\phi A_N - A_N\phi)Y, Z\rangle$$
$$-2\langle\phi X, Y\rangle\langle(\phi A_N - A_N\phi)Z, W\rangle\}$$
$$-\langle h(Y, Z), h(X, A_N W)\rangle + \langle h(X, Z), h(Y, A_N W)\rangle$$
$$-\langle h(Y, W), h(X, A_N Z)\rangle + \langle h(X, W), h(Y, A_N Z)\rangle$$
$$-\langle h(Z, W), h(X, A_N Y)\rangle + \langle h(Z, W), h(Y, A_N X)\rangle = 0, \tag{21}$$
$$c\Big\{- \langle Y, Z\rangle\langle A_\sigma X, W\rangle + \langle X, Z\rangle\langle A_\sigma Y, W\rangle$$
$$-\langle Y, W\rangle\langle A_\sigma X, Z\rangle + \langle X, W\rangle\langle A_\sigma Y, Z\rangle$$
$$-\langle\phi Y, Z\rangle\langle\phi A_\sigma \phi X, \phi W\rangle + \langle\phi X, Z\rangle\langle\phi A_\sigma \phi Y, \phi W\rangle$$
$$-\langle\phi Y, W\rangle\langle\phi A_\sigma \phi X, \phi Z\rangle + \langle\phi X, W\rangle\langle\phi A_\sigma \phi Y, \phi Z\rangle$$
$$-2\langle\phi X, Y\rangle\{\eta(Z)\langle A_\sigma \xi, \phi W\rangle + \eta(W)\langle A_\sigma \xi, \phi Z\rangle + \langle A_{C\sigma} Z, W\rangle\}\Big\}$$

$$
\begin{aligned}
&-\langle h(Y,Z), h(X, A_\sigma W)\rangle + \langle h(X,Z), h(Y, A_\sigma W)\rangle \\
&-\langle h(Y,W), h(X, A_\sigma Z)\rangle + \langle h(X,W), h(Y, A_\sigma Z)\rangle \\
&-\langle h(Z,W), h(X, A_\sigma Y)\rangle + \langle h(Z,W), h(Y, A_\sigma X)\rangle = 0.
\end{aligned} \tag{22}
$$

Proof Differentiating the following equation

$$(\nabla_Y h)(Z, W) = -c\{\eta(Z)\langle\phi Y, W\rangle + \eta(W)\langle\phi Y, Z\rangle\}N$$

covariantly in the direction of $X \in \Gamma(TM)$, with the help of (9) and (10), we have

$$
\begin{aligned}
(\nabla^2_{XY} h)(Z, W) = -c\{&\langle\phi A_N X, Z\rangle\langle\phi Y, W\rangle + \eta(Z)\eta(Y)\langle A_N X, W\rangle \\
&+\langle\phi A_N X, W\rangle\langle\phi Y, Z\rangle + \eta(W)\eta(Y)\langle A_N X, Z\rangle \\
&-2\eta(Z)\eta(W)\langle A_N X, Y\rangle\}N \\
&-c\{\eta(Z)\langle\phi Y, W\rangle + \eta(W)\langle\phi Y, Z\rangle\}Ch(X, \xi).
\end{aligned}
$$

It follows from (8), (6), (14), and this equation that

$$
\begin{aligned}
&c\Big\{\{-\langle\phi A_N X, Z\rangle\langle\phi Y, W\rangle - \eta(Z)\eta(Y)\langle A_N X, W\rangle \\
&\quad -\langle\phi A_N X, W\rangle\langle\phi Y, Z\rangle - \eta(W)\eta(Y)\langle A_N X, Z\rangle \\
&\quad +\langle\phi A_N Y, Z\rangle\langle\phi X, W\rangle + \eta(Z)\eta(X)\langle A_N Y, W\rangle \\
&\quad +\langle\phi A_N Y, W\rangle\langle\phi X, Z\rangle + \eta(W)\eta(X)\langle A_N Y, Z\rangle\}N \\
&\quad -\{\eta(Z)\langle\phi Y, W\rangle + \eta(W)\langle\phi Y, Z\rangle\}Ch(X, \xi) \\
&\quad +\{\eta(Z)\langle\phi X, W\rangle + \eta(W)\langle\phi X, Z\rangle\}Ch(Y, \xi)\Big\} \\
&\quad = c\Big\{-\langle Y, Z\rangle h(X, W) + \langle X, Z\rangle h(Y, W) \\
&\quad -\langle\phi Y, Z\rangle h(\phi X, W) + \langle\phi X, Z\rangle h(\phi Y, W) \\
&\quad -\langle Y, W\rangle h(X, Z) + \langle X, W\rangle h(Y, Z) \\
&\quad -\langle\phi Y, W\rangle h(\phi X, Z) + \langle\phi X, W\rangle h(\phi Y, Z) \\
&\quad +2\langle\phi X, Y\rangle\{h(\phi Z, W) + h(\phi W, Z) - Ch(Z, W)\}\Big\} \\
&\quad -h(A_{h(Y,Z)}X, W) + h(A_{h(X,Z)}Y, W) \\
&\quad -h(A_{h(Y,W)}X, Z) + h(A_{h(X,W)}Y, Z) \\
&\quad -h(A_{h(Z,W)}X, Y) + h(A_{h(Z,W)}Y, X).
\end{aligned} \tag{23}
$$

The Eq. (21) is the $J\mathscr{D}^\perp$-component of this equation. Next, it follows from Lemma 5(a) that

$$\langle h(Z, \phi Y) - \eta(Z)Ch(\xi, Y), \sigma\rangle = -\langle h(\phi^2 Z, \phi Y), \sigma\rangle = \langle\phi A_\sigma\phi Y, \phi Z\rangle$$

for any $Y, Z \in \Gamma(TM)$ and $\sigma \in \Gamma(\nu)$. With the help of this equation, after taking inner product of both sides of (23) with $\sigma \in \Gamma(\nu)$, we obtain (22). □

Lemma 7 $A_N\xi = \alpha\xi$.

Proof Suppose that $\beta = ||\phi A_N\xi|| > 0$ at some point $x \in M$. Then, we can write

$$A_N\xi = \alpha\xi + \beta U \tag{24}$$

where $U = -\beta^{-1}\phi^2 A_N\xi$ and hence from Lemma 5(c), we have

$$Ch(\xi, U) = 0. \tag{25}$$

Next, by substituting $Z = W = \xi$ in (21), we obtain

$$\langle h(X, U), h(Y, \xi)\rangle - \langle h(Y, U), h(X, \xi)\rangle = 0 \tag{26}$$

for any $X, Y \in T_xM$. By putting $Y = \xi$ in this equation, with the help of (25), we obtain $\alpha A_N U - \beta A_N \xi = 0$ and so

$$A_N U = \beta\xi + \gamma U, \quad (\alpha\gamma = \beta^2). \tag{27}$$

Hence, from (24) and (27), we have

$$A_N^2 \xi = (\alpha^2 + \beta^2)\xi + \beta(\alpha + \gamma)U \tag{28}$$
$$A_N^2 U = \beta(\alpha + \gamma)\xi + (\beta^2 + \gamma^2)U. \tag{29}$$

On the other hand, by putting $X = U$ in (26) and using (25) and (27), we have $A_{h(U,U)}\xi - \beta A_N U = 0$ or

$$A_{C^2 h(U,U)}\xi = 0. \tag{30}$$

Finally, with the help of (24), (25), (27)–(30), Lemma 5(c) and the fact that $h(\xi, \xi) = \alpha N$, after substituting $X = W = U$, $Y = Z = \xi$ in (21), gives

$$0 = c\alpha - \alpha\langle A_N U, A_N U\rangle + \gamma\langle A_N\xi, A_N\xi\rangle = c\alpha.$$

But from (27), $\alpha\gamma = \beta^2 > 0$. This is a contradiction. Accordingly, $A_N\xi = \alpha\xi$ at each point of M. □

Lemma 8 *(a)* α *is a constant;*
(b) $(A_N\phi A_N - \alpha\phi A_N - c\phi)X + A_{h(\phi X,\xi)}\xi = 0$, *for any* $X \in \Gamma(TM)$*;*
(c) $\alpha(\phi A_N - A_N\phi) = 0$.

Proof Statement (a) is directly from Lemma 5(b) and Lemma 7. Next, from Lemma 7, we have

$$\langle h(Y, \xi), N\rangle = \alpha\eta(Y)$$

for any $Y \in \Gamma(TM)$. It follows from this equation that

$$\langle(\nabla_X h)(Y, \xi) + h(Y, \nabla_X\xi), N\rangle + \langle h(Y, \xi), \nabla_X^\perp N\rangle = d\alpha(X)\eta(Y) + \alpha\langle\nabla_X\xi, Y\rangle.$$

By applying (10), (17), Lemma 5(a), and Lemma 8(a), this equation becomes

$$\langle(A_N\phi A_N - \alpha\phi A_N - c\phi)X, Y\rangle + \langle h(\phi X, \xi), h(Y, \xi)\rangle = 0 \tag{31}$$

for any $X, Y \in \Gamma(TM)$ and so we obtain Statement (b). Finally, by letting $X = Y$ in (31), we have $\alpha\langle\phi A_N X, X\rangle = 0$, for any $X \in \Gamma(TM)$, this deduces Statement (c). □

Lemma 9 *For any* $X, Y \in \Gamma(\mathscr{D})$ *and* $\sigma \in \Gamma(\nu)$*,*

$$2\langle h(\phi X, \xi), h(Y, A_\sigma\xi)\rangle + \langle 2A_N Y - \alpha Y, A_\sigma\phi A_N X\rangle = 0.$$

Proof By using Lemma 5(c) and Lemma 7, we have

$$2h(A_N Y, \xi) = 2\langle h(A_N Y, \xi), N\rangle N - 2C^2 h(A_N Y, \xi) = \alpha^2\eta(Y)N + \alpha h(Y, \xi)$$

for any $Y \in \Gamma(TM)$. By differentiating this equation covariantly in the direction of $X \in \Gamma(TM)$, we have

$$\begin{aligned}&2\{(\nabla_X h)(A_N Y, \xi) + h((\nabla_X A)_N Y + A_{\nabla_X^\perp N}Y, \xi) + h(A_N Y, \nabla_X\xi)\}\\ &= \alpha^2\{\langle\nabla_X\xi, Y\rangle N + \eta(Y)\nabla_X^\perp N\} + \alpha\{(\nabla_X h)(Y, \xi) + h(Y, \nabla_X\xi)\}.\end{aligned}$$

By using (10), (17), Lemma 5(a), and Lemma 8(a), this equation becomes

$$-2c\{\langle\phi X, A_N Y\rangle N+\eta(Y)h(\phi X,\xi)+\alpha\langle\phi X, Y\rangle N\}+2\{h(A_{h(\phi X,\xi)}Y,\xi)+h(A_N Y,\phi A_N X)\}$$
$$=\alpha^2\{\langle\phi A_N X, Y\rangle N+\eta(Y)h(\phi X,\xi)\}+\alpha\{-c\langle\phi X, Y\rangle N+h(Y,\phi A_N X)\}.$$

By first, putting $X, Y \in \Gamma(\mathscr{D})$ and then taking inner product of both sides of this equation with $\sigma \in \Gamma(\nu)$, we obtain the lemma. □

6 Proof of Theorem 1

We shall consider two cases: (I) M is mixed totally geodesic and (II) M is non-mixed totally geodesic.

Case (I) M is mixed totally geodesic.

By Lemma 4(a) and Lemma 7, we have

$$h(Y,\xi) = \eta(Y)h(\xi,\xi) = \alpha\eta(Y)N \tag{32}$$

for any $Y \in \Gamma(TM)$. It follows from (10) that $\nabla^{\perp}N = 0$. Moreover, by applying (10), (17), and (32), we obtain

$$\begin{aligned}0 &= \langle(\nabla_X h)(Y,\xi),\sigma\rangle = \langle\nabla^{\perp}_X h(Y,\xi),\sigma\rangle - \langle h(Y,\nabla_X\xi),\sigma\rangle\\ &= \langle h(Y,\phi A_N X),\sigma\rangle\end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $\sigma \in \Gamma(\nu)$. This means that

$$A_\sigma\phi A_N = 0 \tag{33}$$

for any $\sigma \in \Gamma(\nu)$. On the other hand, by Lemma 8(b), we have

$$A_N\phi A_N - \alpha\phi A_N - c\phi = 0.$$

As $c \neq 0$, we can observe from the above equation that $A_N|_{\mathscr{D}}$ is a vector bundle automorphism on $\mathscr{D}$. Hence, for any $\sigma \in \Gamma(\nu)$, we have $A_\sigma|_{\mathscr{D}} = 0$ by (33). Also, we have $A_\sigma\xi = 0$ by using Lemma 4(a). We conclude that $A_\sigma = 0$ for any $\sigma \in \Gamma(\nu)$. Further, since $A_N \neq 0$, ν_x is the J-invariant orthogonal complementary subspace of the first normal space in $T_x M^{\perp}$, at each $x \in M$. Also, since $\nabla^{\perp}N = 0$, ν is a parallel normal subbundle of $TM^{\perp}$. By applying Theorem 3, M is contained in a totally geodesic holomorphic submanifold $\hat{M}_n(c)$ of $\hat{M}_{n+p}(c)$ as a real hypersurface.

We denote by N' a unit normal vector field, ∇', the Levi-Civita connection, A' the shape operator of M, immersed in $\hat{M}_n(c)$. Further, let (ϕ', ξ', η') denote the almost contact structure on M induced by complex structure of $\hat{M}_n(c)$.

Since $\hat{M}_n(c)$ is totally geodesic in $\hat{M}_{n+p}(c)$ and $Ch = 0$, we can see that $\nabla'_X Y = \nabla_X Y$, $A' = A_N$, $\phi' = \phi$, $\eta' = \eta$, $\xi' = \xi$, and $N' = N$. Then, by (19), we have

$$\begin{aligned}(\nabla'_X A')Y &= (\nabla_X A)_N Y = -c\{\eta(Y))\phi X + \langle\phi X, Y\rangle\xi\}\\ &= -c\{\eta'(Y))\phi' X + \langle\phi' X, Y\rangle\xi'\}\end{aligned}$$

for any vectors X, Y tangent M. By using Theorem 4, we obtain Case (a) and Case (b)(i) and (ii) in Theorem 1.

Case (II) M is non-mixed totally geodesic.

Let $x \in M$, and $X \in \mathscr{D}_x$ be a unit vector with $A_N X = \lambda X$. If $h(X, \xi) = 0$, then we also have $h(\phi X, \xi) = Ch(X, \xi) = 0$ and

$$\lambda A_N \phi X - (\alpha\lambda + c)\phi X = 0, \qquad \text{(by Lemma 8(b))}.$$

If $\alpha = 0$, then $\lambda \neq 0$ and $A_N \phi X = c\lambda^{-1}\phi X$. On the other hand, if $\alpha \neq 0$, then by Lemma 8(c), $A_N \phi X = \lambda X$. From these observations, there is an integer $m \geq 1$ and we may choose an orthonormal basis of $\mathscr{D}_x$ formed by eigenvectors $E_1, E_2 = \phi E_1, \ldots, E_{2n-1}, E_{2n-2} = \phi E_{2n-1}$ of A_N such that

$$h(E_i, \xi) \neq 0, \quad (1 \leq i \leq 2m) \tag{34}$$

$$h(E_a, \xi) = 0, \quad (2m + 1 \leq a \leq 2n - 2). \tag{35}$$

In the rest of this section, we use the following convention of indices:

$$i, j, \ldots \quad \in \{1, 2, \ldots, 2m\};$$
$$a, b, \ldots \in \{2m + 1, \ldots, 2n - 2\}.$$

For simplicity, we write $\sigma_i = h(E_i, \xi)$ and $A_i = A_{\sigma_i}$.
It follows from Lemma 5(c) and Lemma 8(b) that

$$A_N E_i = \frac{\alpha}{2} E_i \tag{36}$$

$$A_i \xi = \frac{\alpha^2 + 4c}{4} E_i \tag{37}$$

$$\langle \sigma_i, h(X, \xi) \rangle = \frac{\alpha^2 + 4c}{4} \langle E_i, X \rangle \tag{38}$$

for any $X \in T_x M$. We can further observe from (38) that

$$||\sigma_i||^2 = \frac{\alpha^2 + 4c}{4} > 0. \tag{39}$$

By using (36)–(39), after putting $X = \phi E_i$, $Y = E_j$, and $\sigma = \sigma_k$ in Lemma 9, we obtain $(\alpha^2 + 4c)\langle \sigma_i, h(E_j, E_k) \rangle = 0$, and so

$$\langle A_i E_j, E_k \rangle = 0. \tag{40}$$

Now, we wish to prove that

$$A_i E_j = \frac{\alpha^2 + 4c}{4} \delta_{ij} \xi. \tag{41}$$

If $m = n - 1$, then (37) and (40) imply (41). Next, suppose $m < n - 1$. Then, by letting $Y = Z = \xi$, $X = E_j$, $W = E_a$, and $\sigma = \sigma_i$ in (22), with the help of (35)–(37), we have $c\langle A_i E_j, E_a \rangle = 0$, that is,

$$\langle A_i E_j, E_a \rangle = 0.$$

From the above equation, (37) and (40), we also obtain (41).
By putting $X = \xi$, $Y = E_i$, $Z = E_j$, $W = E_k$, and $\sigma = \sigma_l$ in (22), we have

$$\frac{\alpha^2}{4}\{\delta_{jk}\delta_{il} + \delta_{ji}\delta_{kl} + \delta_{ki}\delta_{jl}\} = \langle Ch(E_j, E_k), Ch(E_i, E_l) \rangle. \tag{42}$$

If we first put $E_i = E_j = E_k = E_l$, and next follow by $E_j = E_i,\ E_k = E_l = \phi E_i$ in the above equation, then

$$\frac{3\alpha^2}{4} = \langle Ch(E_i, E_i), Ch(E_i, E_i)\rangle$$
$$\frac{\alpha^2}{4} = \langle Ch(E_i, \phi E_i), Ch(E_i, \phi E_i)\rangle = \langle Ch(E_i, E_i), Ch(E_i, E_i)\rangle.$$

These three equations, together with (36) and (39), give

$$\alpha = 0 \tag{43}$$
$$c > 0; \quad \text{(without loss of generality, we assume } c = 1) \tag{44}$$
$$h(E_i, E_j) = 0. \tag{45}$$

Lemma 10 *Suppose $m < n-1$ and let $A_N E_a = \lambda_a E_a$. Then,*

(a) $Ch(E_a, E_b) = 0$,
(b) $\lambda_a \in \{1, -1\}$,
(c) $\phi A_N - A_N \phi = 0$.

Proof From (31), (35), (43), and (44), we have $\lambda_a \neq 0$ and $A_N \phi E_a = \lambda_a^{-1} \phi E_a$. Hence, after putting $X = \phi E_a$ and $Y = E_b$ in Lemma 9, we obtain Statement (a). Furthermore, by putting $X = W = E_i$ and $Y = Z = E_a$, and $X = E_i,\ Y = E_a,\ Z = \phi E_a$, and $W = \phi E_i$, respectively, in (21), we have

$$0 = -\lambda_a + 2\lambda_a \langle h(E_i, E_a), h(E_i, E_a)\rangle$$
$$0 = \lambda_a - \lambda_a^{-1} + \lambda_a^{-1}\langle h(E_i, \phi E_a), h(\phi E_i, E_a)\rangle + \lambda_a \langle h(E_i, E_a), h(\phi E_i, \phi E_a)\rangle$$
$$= \{\lambda_a - \lambda_a^{-1}\}\{1 - \langle h(E_i, E_a), h(E_i, E_a)\}.$$

These two equations imply that $\lambda_a = \lambda_a^{-1}$. Hence, we obtain Statement (b) and (c) as $AE_i = A\phi E_i = 0$. □

Now, we consider two subcases: $||A_N|| = 0$ and $||A_N|| \neq 0$.

Subcase (II-a) $||A_N|| = 0$.

In this case, we have $m = n - 1$ at each $x \in M$ by (36), (43), and Lemma 10(b). From Lemma 7 and (45), we see that $\langle h(X, Y), N\rangle = 0$, for any $X \in \Gamma(\mathscr{D})$ and $Y \in \Gamma(TM)$. Hence, M is a CR-product by Lemma 2. Furthermore, it follows from (38), (45), and $h(\xi, \xi) = 0$ that $||h||^2 = 2(2n - 2)$. According to Theorem 5, M is an open part of the standard CR-product $\mathbb{C}P_{n-1} \times \mathbb{R}P^1$, and we obtain Case (b)(iii) in Theorem 1.

Subcase (II-b) $||A_N|| \neq 0$.

From Lemma 4(b), we have Trace$(A_N|_{\mathscr{D}_x}) = \langle H, N\rangle = 0$. By using (36), (43), Lemma 10(b), and the continuity of the eigenvalue functions, we can see that $m < n - 1$ and A_N has three distinct constant eigenvalues $0,\ 1$, and -1 with multiplicities $2m,\ n - m - 1$, and $n - m - 1$, respectively, at each $x \in M$.

For $\lambda \in \{0, 1, -1\}$, we denote by $\mathscr{T}_\lambda$ the subbundle of $\mathscr{D}$ foliated by eigenspace of $A_N|_{\mathscr{D}}$ corresponding to λ. From Lemma 10(c), we see that each $\mathscr{T}_\lambda$ is ϕ-invariant. We shall show that $\mathscr{T}_0$ is auto-parallel, that is,

$$\Gamma(\mathscr{T}_0) \xrightarrow{\nabla} \Omega^1(M) \otimes \Gamma(\mathscr{T}_0).$$

For any $X \in \Gamma(TM)$ and $Y \in \Gamma(\mathscr{T}_0)$, we have

$$\langle \nabla_X Y, \xi\rangle = -\langle Y, \phi A_N X\rangle = 0.$$

Next, from (17), we have

$$-\langle \phi X, Y\rangle \xi = (\nabla_X A)_N Y = -A_N \nabla_X Y - A_{Ch(X,\xi)} Y = -A_N \nabla_X Y - A_{h(\phi X,\xi)} Y.$$

If $X \in \Gamma(\mathscr{T}_1 \oplus \mathscr{T}_{-1} \oplus \text{Span}\{\xi\})$, it clearly that $A_N \nabla_X Y = 0$; if $X \in \Gamma(\mathscr{T}_0)$, then by (37) and the above equation, we have $A_N \nabla_X Y = 0$ too. From these observations, we have $\nabla_X Y \in \Gamma(\mathscr{T}_0)$, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(\mathscr{T}_0)$.

For any $X \in \Gamma(\mathscr{T}_0)$ and $Y, Z \in \Gamma(\mathscr{T}_1 \oplus \mathscr{T}_{-1} \oplus \text{Span}\{\xi\})$, from Lemma 10(a), we see that $h(Y, Z) = \langle A_N Y, Z\rangle N$. It follows that

$$\begin{aligned}(\nabla_X h)(Y, Z) &= \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)\\ &= \{X\langle A_N Y, Z\rangle - \langle A_N \nabla_X Y, Z\rangle - \langle A_N Y, \nabla_X Z\rangle\}N\\ &\quad -\langle A_N Y, Z\rangle Ch(X, \xi).\end{aligned}$$

In particular, if we choose $Y = Z \in \Gamma(\mathscr{T}_1)$ with $||Y|| = 1$, then

$$C(\nabla_X h)(Y, Z) = h(X, \xi) \neq 0.$$

This is a contradiction, so this case cannot occur.

Conversely, all these submanifolds satisfy the condition (17) as we have discussed in Sect. 4. This completes the proof.

7 Proof Theorem 2

Suppose M is a $(2n-1)$-dimensional CR-submanifold of maximal CR-dimension in $\hat{M}_{n+p}(c)$, $c \neq 0$, $n \geq 2$. We define a tensor field T on M by

$$T(X, Y, Z) = (\nabla_X h)(Y, Z) + c\{\eta(Y)\langle \phi X, Z\rangle + \eta(Z)\langle \phi X, Y\rangle\}N$$

for any $X, Y, and Z \in \Gamma(TM)$. Let $e_1, e_2, \ldots, e_{2n-1}$ be a local field of orthonormal vectors in $\Gamma(TM)$. Then,

$$||T||^2 = ||\nabla h||^2 + 4(n-1)c^2 + 4c \sum_{j=1}^{2n-1} \langle (\nabla_{e_j} h)(\xi, \phi e_j), N\rangle.$$

On the other hand, by the Codazzi equation, we have

$$\sum_{j=1}^{2n-1} \langle (\nabla_{e_j} h)(\xi, \phi e_j), N\rangle = \sum_{j=1}^{2n-1} \langle (\nabla_{\xi} h)(e_j, \phi e_j), N\rangle - 2(n-1)c = -2(n-1)c.$$

Combining these two equations, we have

$$0 \leq ||T||^2 = ||\nabla h||^2 - 4(n-1)c^2$$

and equality holds if and only if M satisfies (17). By Theorem 1, we obtain the theorem.

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