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On conditional permutability and factorized groups

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Abstract Two subgroups A and B of a group G are said to be totally completely conditionally permutable (tcc-permutable) if X permutes with Y^g for some $g \in \langle X, Y\rangle$, for all $X \leq A$ and all $Y \leq B$. In this paper, we study finite products of tcc-permutable subgroups, focussing mainly on structural properties of such products. As an application, new achievements in the context of formation theory are obtained.

Keywords Finite groups · Products of subgroups · Conditional permutability · Formations

Mathematics Subject Classification (2000) 20D40 · 20D10

1 Introduction and preliminaries

In this paper, only finite groups are considered. Over the last years, the study of groups which can be factorized as the product of two subgroups has been the target of increasing interest within the theory of groups. One of the important questions dealing with factorized groups is how the structure of the factors affects the structure of the whole group (and vice versa).

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In this setting, the fact that a product of two supersoluble groups is not necessarily supersoluble has led to consider these questions under additional assumptions looking for positive results. A natural approach, which has been revealed very useful, is to consider factorized groups in which certain subgroups of the corresponding factors permute. Its starting point can be located at Asaad and Shaalan's paper [4], where groups $G = AB$ such that every subgroup of A permutes with every subgroup of B were considered, and in particular, it was proved that such groups are supersoluble provided that the factors A and B are supersoluble. These factorized groups are said to be the *product of the totally permutable subgroups* A and B by Maier [33]. Later on, a deep understanding of the structure of such groups has been reached, and this study has been extended both in the frameworks of formation theory (see [7–10,12,15,16,33]) as well as in the theory of Fitting classes [22–24]. A detailed account on this topic can be found in the book [5].

More recently, this research program has been taken further, initially by Guo et al. [21], by considering a weaker condition of subgroup permutability, namely conditional permutability, which imposes permutability just with some conjugate of the subgroups involved. More concretely, we consider the following concepts:

Two subgroups X and Y of a group G are called *conditionally permutable* (*c-permutable*, for brevity) in G if X permutes with Y^g for some element $g \in G$.

The subgroups X and Y are called *completely c-permutable* (*cc-permutable*) in G if X permutes with Y^g for some element $g \in \langle X, Y \rangle$, the subgroup generated by X and Y.

Two subgroups A and B of G are said to be *totally completely conditionally permutable* (*tcc-permutable*) in G if X and Y are cc-permutable in G for all $X \leq A$ and all $Y \leq B$.

Using these permutability properties, new criteria for a product of finite supersoluble subgroups to be supersoluble are obtained in [21,31,32] and by the authors in [1], extending known results. Also in [1], the behavior of the supersoluble residual in products of finite groups is studied, by considering conditional permutability (not necessarily complete) as mentioned below in this Introduction (Theorem 2). Then, inspired by the previous research on totally permutable products of subgroups, an initial study on conditional permutability in the framework of formation theory has been developed in [3]. A compilation of recent results can be found in [2]. Easy examples in the previous references ([1, Examples 2, 3]; also [3, Examples 3.5, 3.6]) show that strong structural properties of products of totally permutable subgroups are missed when permutability is weakened to conditional permutability, even complete, and make evident the interest of the recent progress.

This article is a contribution to a better understanding of products of tcc-permutable subgroups, focussing mainly on structural properties of such products. This information will lead us to some new achievements in the context of formation theory.

More precisely, a celebrated result by Beidleman and Heineken [12, Theorem 1] states that a group $G = AB$ which is the product of totally permutable subgroups A and B is close to be a central product in the sense that the nilpotent residual of each factor centralizes the other factor. Example 3 in [1] (also [3, Example 3.6]) shows that this is not true if every subgroup of A is completely c-permutable with every subgroup of B, also it is not true for the supersoluble residuals. However, under this weaker hypothesis, we prove in this paper that the nilpotent residuals of the factors are normal subgroups in the product (Theorem 3). Also, the derived subgroups of the factors are proved to be subnormal subgroups in such products of subgroups (Proposition 1, Corollary 3). (A corresponding result for mutually permutable products was obtained by Beidleman and Heineken [11, Theorem 1]; a product $G = AB$ of subgroups A and B is called a *mutually permutable product* if A permutes with all subgroups of B and vice versa.) The same authors, also in [12, Corollary 2], obtained that $[A, B]$ is a nilpotent normal subgroup in a group $G = AB$ which is the product of totally permutable subgroups A and B (a weaker version of this result appears in [10, Lemma 3]). A main result in the present paper (Theorem 4) states that the result is still true if permutability is replaced by complete c-permutability. As a consequence, extensions of structural properties of totally permutable products involving non-abelian chief factors (Corollary 5) and supersoluble residuals (Corollary 6) are derived. More exactly, the supersoluble residuals of the factors centralize each other in a product of tcc-permutable subgroups. Moreover, Corollary 5 allows us to avoid restrictions to soluble groups in [3] and to extend the research in this reference to the universe of all finite groups (see Sect. 3).

To be more specific, we recall first that a *formation* is a class $\mathcal{F}$ of groups closed under homomorphic images, such that $G/M \cap N \in \mathcal{F}$ whenever G is a group and M, N are normal subgroups of G with $G/M \in \mathcal{F}$ and $G/N \in \mathcal{F}$. In this case, the $\mathcal{F}$-residual $G^{\mathcal{F}}$ of G is the smallest normal subgroup of G such that $G/G^{\mathcal{F}} \in \mathcal{F}$. The formation $\mathcal{F}$ is *saturated* if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$, where $\Phi(G)$ denotes the Frattini subgroup of G. $\mathcal{U}$ denotes the class of all finite supersoluble groups. Now, we can state the main result in [3], which is the following:

Theorem 1 [3, Theorem 1.4] *Let $\mathcal{F}$ be a saturated formation of soluble groups containing $\mathcal{U}$. Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then,*

1. *If $G_i \in \mathcal{F}$ for all $i = 1, \ldots, r$, then $G \in \mathcal{F}$.*
2. *If $G \in \mathcal{F}$, then $G_i \in \mathcal{F}$ for all $i = 1, \ldots, r$.*

As a consequence, the following stronger version was also obtained:

Corollary 1 [3, Corollary 1.5] *Let $\mathcal{F}$ be a saturated formation of soluble groups containing $\mathcal{U}$. Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then,*

1. *$G_i^{\mathcal{F}} \trianglelefteq G$ for all $i = 1, \ldots, r$.*
2. *$G^{\mathcal{F}} = G_1^{\mathcal{F}} \cdots G_r^{\mathcal{F}}$.*

As an application of the results in Sect. 2, we show in Sect. 3 that the hypothesis of solubility in Theorem 1 and Corollary 1 can be removed. However, we also provide examples showing that none of the statements in Theorem 1 remains true for non-saturated formations (even of soluble groups). This is significant since if we consider pairwise totally permutable subgroups $G_1, \ldots, G_r$ in Theorem 1, Part 1 is true for any formation $\mathcal{F}$ containing $\mathcal{U}$, and Part 2 holds if in addition $\mathcal{F}$ is either saturated or a formation of soluble groups [7,8,10].

On the other hand, we mention finally that for the particular case when $\mathcal{F} = \mathcal{U}$, the class of all finite supersoluble groups, the following result involving a weaker permutability condition was proved in [1, Theorem 2].

Theorem 2 *Let the group $G = AB$ the product of subgroups A and B such that every subgroup of A is c-permutable with every subgroup of B. Then, $G^{\mathcal{U}} = A^{\mathcal{U}} B^{\mathcal{U}}$.*

For general notation and results on classes of groups, we refer to [19]. In particular, $\mathrm{Syl}_p(G)$ denotes the set of Sylow p-subgroups of the group G, for a prime number p.

2 Main results

Lemma 1 [3, Lemma 2.1] *Let the group $G = AB$ be the product of tcc-permutable subgroups A and B. Then,*

1. *X and Y^g are tcc-permutable subgroups of G for any $X \leq A$, $Y \leq B$ and $g \in G$.*
2. *For each $Y \leq B$, A permutes with Y^b for some $b \in B$.*

As a first consequence, more detailed information about products of tcc-permutable subgroups where, at least, one factor is a nilpotent group can be given. The results in Lemma 1 and Corollary 2 will be used often without any further reference.

Corollary 2 *Let the group $G = AB$ be the product of tcc-permutable subgroups A and B. Then,*

(i) If A is a nilpotent group, then every subgroup of A permutes with B.
(ii) If A is a nilpotent normal subgroup of G, then A and B are totally permutable. If, in addition, $A \cap B = 1$, then B normalizes each subgroup of A, that is, B acts as a group of power automorphisms on A by conjugacy (it may happen eventually that B centralizes A).
(iii) If A is a nilpotent minimal normal subgroup of G, then A is a cyclic group of prime order.

Proof Part (i) follows by a straightforward inductive argument by using Lemma 1(2), and the fact that subgroups of a nilpotent group are subnormal in the group. For Part (ii), taking into account that A is normal, we note that A permutes with all subgroups of B. We deduce now that A and B are totally permutable by using Part (i). Moreover, if $A \cap B = 1$ and $L \leq A$, then LB is a subgroup of G and $L = L(A \cap B) = A \cap LB \trianglelefteq LB$, which proves Part (ii). Finally, Part (iii) follows by [10, Lemma 2] since A and B are totally permutable in this case. □

As detailed below, Corollary 2(iii) and Corollaries 5 and 6 extend previous results on the structure of totally permutable products of subgroups in [10] under the weaker permutability condition that we are considering.

Also for mutually permutable products of subgroups, Beidleman and Heineken proved in [11] that the derived subgroups of the factors are subnormal subgroups. A corresponding result for products of tcc-permutable subgroups is derived next.

Proposition 1 *Let the group $G = AB$ be the product of tcc-permutable subgroups A and B. Then, A' and B' are subnormal subgroups of G.*

Proof W.l.o.g. we prove that A' is subnormal in G. For each prime p dividing $|G|$, we may consider by Lemma 1 a Sylow p-subgroup P of B such that A permutes with P and AP is the product of tcc-permutable subgroups A and P (eventually, $P = 1$); we note that $\mathrm{Syl}_p(AP) \subseteq \mathrm{Syl}_p(G)$. We claim that A' is subnormal in AP. By Corollary 1 (also Theorem 2), we have that $A^{\mathfrak{U}} = (AP)^{\mathfrak{U}} \trianglelefteq AP$ and then $A'/A^{\mathfrak{U}} \leq (AP/A^{\mathfrak{U}})' \leq F(AP/A^{\mathfrak{U}})$, since $AP/A^{\mathfrak{U}}$ is supersoluble. Therefore, A' is subnormal in AP, since $A'/A^{\mathfrak{U}}$ is subnormal in $F(AP/A^{\mathfrak{U}})$. Consequently, there exists a Sylow p-subgroup $\bar{P}$ of G such that $\bar{P} \leq AP$ and then A' is subnormal in $\langle A', \bar{P} \rangle$. It follows from [17, Main Theorem] that A' is subnormal in G. □

Corollary 3 *Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then, G_i' is a subnormal subgroup of G, for all $i \in \{1, \ldots, r\}$.*

Proof We apply Proposition 1 for each pair (G_i, G_j) with $i, j \in \{1, \ldots, r\}$, $i \neq j$. Now, the result follows from [29, 7.7.1]. □

Remark We point out that Corollary 2(i) states in particular that if a group $G = AB$ is the product of tcc-permutable nilpotent subgroups A and B, then A and B are mutually permutable.

We note that if A and B are subgroups of order 2 of S_3, the symmetric group of degree 3, $A \neq B$, then A and B are tcc-permutable nilpotent subgroups, but they are not permutable and so also not mutually permutable.

As mentioned in the Introduction in a product of totally permutable subgroups, the nilpotent residual of each factor centralizes the other factor [12, Theorem 1]. Example 3 in [1] (also [3, Example 3.6]) shows that this property fails when considering complete c-permutability instead of permutability. Nevertheless, we prove next that in a product of tcc-permutable subgroups, each factor normalizes the nilpotent residual of the other factor. Before proving it, we gather the following lemmas, which are key facts in the research carried out.

Lemma 2 [3, Lemma 2.3] *Let the group $1 \neq G = AB$ be the product of tcc-permutable subgroups A and B. Let p be the largest prime divisor of $|G|$. W.l.o.g. let $a \in A$ be a p-element of maximal order in $A \cup B$, and let $X_0 \leq \langle a \rangle$ with $|X_0| = p$. Then,*

1. *B^g normalizes X_0 for some $g \in G$.*
2. *$1 \neq \langle X_0^A \rangle \trianglelefteq G$; in particular, $1 \neq X_0 \leq Core_G(A)$.*

Also, we point out the following consequence.

Lemma 3 [3, Lemma 2.5] *Let the group $1 \neq G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots G_r$, for $r \geq 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then, there exists $1 \neq N \trianglelefteq G$ such that $N \leq G_i$ for some $i \in \{1, \ldots, r\}$.*

Theorem 3 *Let the group $G = AB$ be the product of tcc-permutable subgroups A and B. Then, B normalizes $A^{\mathcal{N}}$ and vice versa.*

Proof Assume that this result is false, and let the group $G = AB$ be a counterexample with $|G| + |A| + |B|$ minimal. Without loss of generality, we may assume that B does not normalize $A^{\mathcal{N}}$. Then, A is not nilpotent, that is, $A^{\mathcal{N}} \neq 1$, and $B \neq 1$. We split the proof into the following steps:

1. *B is a q-group for some prime q.*
   Let q be a prime divisor of $|B|$. By Lemma 1(2), A permutes with some Sylow q-subgroup of B, say Q. If Q were a proper subgroup of B, then it would follow by the choice of (G, A, B) that Q would normalize $A^{\mathcal{N}}$. Therefore, it is easily deduced that B is a q-group.
2. *$A^{\mathcal{N}}N \trianglelefteq G$ for all normal subgroups $N \neq 1$ of G, $\mathrm{Core}_G(A^{\mathcal{N}}) = 1$, $G \in \mathcal{U}$, and $A^{\mathcal{N}}$ is a p-group where p is the largest prime dividing $|G|$.*
   Let $1 \neq N \trianglelefteq G$. It is clear that $G/N = (AN/N)(BN/N)$ is the product of the tcc-permutable subgroups AN/N and BN/N. The choice of G implies that $A^{\mathcal{N}}N$ is a normal subgroup of G. In particular, it follows that $\mathrm{Core}_G(A^{\mathcal{N}}) = 1$. Moreover, G is supersoluble because $G^{\mathcal{U}} = A^{\mathcal{U}} \leq A^{\mathcal{N}}$ by Corollary 1 (also Theorem 2), and so $G^{\mathcal{U}} = 1$. We claim next that all minimal normal subgroups of G have order p for some prime p. Let N_1, N_2 be two minimal normal subgroups of G. Note that both N_1 and N_2 have prime order. Since $A^{\mathcal{N}}N_1 \cap A^{\mathcal{N}}N_2 = A^{\mathcal{N}}(N_1 \cap A^{\mathcal{N}}N_2)$ is a normal subgroup of G and $\mathrm{Core}_G(A^{\mathcal{N}}) = 1$, we can deduce that $A^{\mathcal{N}}N_1 = A^{\mathcal{N}}N_2$ and $|N_1| = |N_2| = p$ for some prime p, which proves the claim. In particular, $F(G)$ is a p-group, and since $G \in \mathcal{U}$, we deduce that p is the largest prime dividing $|G|$. Moreover, $A^{\mathcal{N}} \leq G^{\mathcal{N}} \leq G' \leq F(G)$, and we are done.

3. *$A/A^{\mathcal{N}}$ is an r-group for some prime $r \neq p$.*
   We claim that A has a unique maximal normal subgroup. Assume that M_1 and M_2 are maximal normal subgroups of A and $M_1 \neq M_2$. By Lemma 1, we have that B permutes with both M_1 and M_2, and BM_i is the tcc-permutable product of the subgroups B and M_i, for $i = 1, 2$. The choice of (G, A, B) implies that B normalizes $M_i^{\mathcal{N}}$, $i = 1, 2$. But $A^{\mathcal{N}} = M_1^{\mathcal{N}} M_2^{\mathcal{N}}$ by [19, II.2.12] since $\mathcal{N}$ is a Fitting formation. Therefore, B normalizes $A^{\mathcal{N}}$, a contradiction which proves the claim. Now, Step 3 follows clearly.
4. *B is a p-group and* $\mathrm{Core}_G(B) \neq 1$.
   Let $a \in G$ be a p-element of maximal order in $A \cup B$, and let $X_0 \leq \langle a \rangle$ with $|X_0| = p$. If $a \in A$, then by Lemma 2, we have that $1 \neq X_0 \leq \mathrm{Core}_G(A)$, and so, $1 \neq X_0 \leq \mathrm{Core}_G(A^{\mathcal{N}})$ by Step 3, but this contradicts Step 2. Consequently, $a \in B$, B is a p-group and $1 \neq X_0 \leq \mathrm{Core}_G(B)$.
5. *$A^{\mathcal{N}}$ has exponent p.*
   By Step 4 and the fact that $G \in \mathcal{U}$, there exists a minimal normal subgroup X of G such that $X \leq B$ and $|X| = p$. Now, $A^{\mathcal{N}} \leq G' \leq F(G)$ implies that $A^{\mathcal{N}}$ is subnormal in G, and then it is known that it is normalized by any minimal normal subgroup of G. Then, $[A^{\mathcal{N}}, X] \leq A^{\mathcal{N}} \cap X = 1$, that is, X centralizes $A^{\mathcal{N}}$. Let $X = \langle x \rangle$ and set $C := \langle (ax^i)^p \mid a \in A^{\mathcal{N}},\ 0 \leq i \leq p-1 \rangle = \langle a^p \mid a \in A^{\mathcal{N}} \rangle$. Then, C is normal in G because C is characteristic in $A^{\mathcal{N}}X$, which is normal in G by Step 2. But $C \leq A^{\mathcal{N}}$, which implies $C = 1$ by Step 2 again. Consequently, $A^{\mathcal{N}}$ has exponent p, and we are done.
6. *B is normal in G.*
   Let $a \in A^{\mathcal{N}}$. From Steps 2, 4 and 5 and Corollary 2(i), we deduce that $\langle a \rangle B$ is a subgroup of G and $B \trianglelefteq \langle a \rangle B$. Consequently, $A^{\mathcal{N}}$ normalizes B.
   On the other hand, by Lemma 1(2), B permutes with some Sylow r-subgroup R of A. Since G is supersoluble by Step 2, then BR is also supersoluble and $B \trianglelefteq BR$ since p is the largest prime divisor of $|G|$. Consequently, B is normalized by $A = RA^{\mathcal{N}}$, and we are done.
7. *The final contradiction.*
   Steps 1, 6 and Corollary 2(ii) imply that $G = AB$ is the totally permutable product of the subgroups A and B. Then, B normalizes $A^{\mathcal{N}}$ by [12, Theorem 1] (also [5, Lemma 4.2.6]), a contradiction which concludes the proof. □

Corollary 4 *Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then, $G_i^{\mathcal{N}}$ is a normal subgroup of G, for all $i \in \{1, \ldots, r\}$.*

Proof We apply Theorem 3 on each pair (G_i, G_j) with $i, j \in \{1, \ldots, r\}$, $i \neq j$, and the result is clear. □

Our next goal is to prove that $[A, B]$ is a nilpotent normal subgroup of a group $G = AB$, which is the product of tcc-permutable subgroups A and B (Theorem 4 below), extending a previous result for products of totally permutable subgroups by Beidleman and Heineken [12, Corollary 2]. In order to prove this, the following lemma shows us that the hypothesis of solubility in [3, Lemma 2.8] can be removed.

Lemma 4 *Let the group $1 \neq G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Assume in addition that G is a primitive group of type 1. Let N be the unique minimal normal subgroup of G and p be a prime divisor of $|N|$. Then, either G is supersoluble or the following conditions are satisfied:*

(i) *w.l.o.g.* $N \leq G_1$;
(ii) $G_2 \cdots G_r$ *is a cyclic group whose order divides* $p-1$;
(iii) *there exists a maximal subgroup* M *of* G *with* $\mathrm{Core}_G(M)=1$ *such that* $M=(M\cap G_1)(G_2\cdots G_r)$ *and* $M\cap G_1$ *centralizes* $G_2\cdots G_r$.

Proof Assume that G is not supersoluble. Let M be a maximal subgroup of G with $\mathrm{Core}_G(M)=1$. In particular, $G=NM$, $M\cap N=1$, and N is an elementary abelian p-group. Steps 1–5 are obtained similarly as in the proof of [3, Lemma 2.8].

Step 1: $N \not\leq \cap_{i=1}^{r} G_i$.
By Lemma 3 and w.l.o.g. assume that $N \leq \cap_{i=1}^{s} G_i$ with $1 \leq s < r$ and $N \not\leq G_j$ for all $j=s+1,\ldots,r$.
Step 2: $N\cap G_j = 1$ for all $j=s+1,\ldots,r$.
Step 3: G_j normalizes every subgroup of N, and therefore, G_j is a cyclic group whose order divides $p-1$, for all $j=s+1,\ldots,r$
Step 4: $s=1$.
Step 5: $G_2\cdots G_r$ is a cyclic group whose order divides $p-1$ and $[G_2\cdots G_r, G] \leq N$.
Step 6: W.l.o.g. we may assume that $G_2\cdots G_r \leq M$ and $M=(M\cap G_1)(G_2\cdots G_r)$. Moreover, $M\cap G_1$ centralizes $G_2\cdots G_r$.

Let q be a prime divisor of $G_2\cdots G_r$ and Q be a Sylow q-subgroup of $G_2\ldots G_r$. Since $G=NM$, we deduce that $G_1=N(M\cap G_1)$ and $\mathrm{Syl}_q(M)\subseteq \mathrm{Syl}_q(G)$. In particular, w.l.o.g. we may assume that $Q\leq M$, and consequently, $[Q, M\cap G_1]\leq M\cap N=1$. Therefore, $M\cap G_1$ and $G_2\cdots G_r$ are both contained in $C:=C_G(Q)$. Since $G=G_1(G_2\cdots G_r)=N(M\cap G_1)(G_2\ldots G_r)$, we deduce that

$$C=(G_1\cap C)G_2\cdots G_r=(N\cap C)(M\cap G_1)G_2\cdots G_r.$$

We note that $N\cap C \trianglelefteq N(M\cap G_1)(G_2\cdots G_r)=G$ and then either $N\leq C$ or $N\cap C=1$. In the first case, $Q\trianglelefteq G$ which is not possible. Otherwise, $C=M=(M\cap G_1)G_2\cdots G_r=(C\cap G_1)G_2\cdots G_r$ is a maximal subgroup of G with $\mathrm{Core}_G(C)=1$. Moreover, $[C\cap G_1, G_2\cdots G_r]\leq C\cap N=1$, which concludes the proof. □

Remark We will see in Corollary 5 below that if the group $1\neq G=G_1\cdots G_r$ is the product of pairwise tcc-permutable subgroups $G_1,\ldots,G_r$, for $r\geq 2$, with $G_i\neq 1$ for all $i=1,\ldots,r$, and G is a monolithic primitive group, then G is of type 1 and Lemma 4 holds.

Before proving Theorem 4, we need still some previous results. We gather first some well known facts on power automorphisms. They can be found, for example, in [5, 1.3].

Lemma 5 *Let* α *be a power automorphism of a* p*-group* P*,* p *a prime number. Assume that* α *has prime order, say* r.

(i) *If* P *is abelian, then* α *is universal, that is, there exists a fixed integer* n *such that* $g^{\alpha}=g^{n}$*, for all* $g\in P$*. Moreover, if* $p\neq r$*, then* $C_P(\alpha)=1$.
(ii) *If* P *is non-abelian, then* $p=r$.

Lemma 6 *The symmetric group of degree* 4*,* S_4*, does not contain any subgroup of prime order which is tcc-permutable with* A_4*, the alternating group of degree* 4.

Proof Let b be an element of S_4 of prime order. Assume that $\langle b\rangle$ is tcc-permutable with A_4. It is clear that $b\notin A_4$, since A_4 does not have any subgroup of order 6. It follows that b is a transposition because b has prime order. Let V be the Klein 4-subgroup of S_4. Then, the subgroup $V\langle b\rangle$ is the product of the tcc-permutable subgroups V and $\langle b\rangle$. By Corollary 2(ii), we have that V and $\langle b\rangle$ are totally permutable, a contradiction. □

Proposition 2 (Kondratiev) *If G is a finite simple exceptional group of Lie type in which all maximal subgroups are local, then $G \cong Sz(2^r)$, where r is an odd prime.*

Proof Let $G = L(q)$ be a finite simple exceptional group of Lie type L over $GF(q)$, the finite field of q elements, where $q = p^n$ and p is a prime number, in which all maximal subgroups are local. If H is a maximal subgroup of G, then we will write $H < \cdot G$.

Suppose first that $n > 1$. Let ϕ be a field automorphism of prime order r of G (r divides n). By [13], the subgroup $C = C_G(\phi)$ is maximal in G. By [20, (9-1)], it follows that $F^*(C)$ is isomorphic with $L(q^{n/r})$. Since $F(C) \neq 1$, then G is isomorphic with $Sz(2^r)$, where r is an odd prime.

Hence, we can assume that $q = p$ and $G \in \{G_2(q);\ {}^3D_4(q);\ F_4(q);\ {}^2E_6(q);\ E_6(q);\ E_7(q);\ E_8(q)\}$. By [27], we know that $L_3(3) < \cdot\, G_2(3)$ and $U_3(3) < \cdot\, G_2(q)$ for $q \geq 5$. By [26], $G_2(q) < \cdot\, {}^3D_4(q)$. By [30, Table 5.1], it holds that ${}^3D_4(q) : 3 < \cdot\, F_4(q)$, $L_3(q^3) : 3 < \cdot\, E_6(q)$, $U_3(q^3) : 3 < \cdot\, {}^2E_6(q)$, $L_2(q^7) : 7 < \cdot\, E_7(q)$, ${}^3D_4(q^2) : 6 < \cdot\, E_8(q)$. Therefore, in all these cases, there exists a maximal subgroup H in G such that $F(H) = 1$, a contradiction. □

Theorem 4 *Let the group $G = AB$ be the product of tcc-permutable subgroups A and B. Then, $[A, B] \leq F(G)$.*

Proof Assume that the result is not true, and let the group $G = AB$ be a counterexample with $|G| + |A| + |B|$ minimal. Let $N \neq 1$ be a normal subgroup of G. We note that the factor group $G/N = (AN/N)(BN/N)$ satisfies the hypotheses of the theorem. Therefore, the choice of G implies that $[AN/N, BN/N] = [A, B]N/N \leq F(G/N)$. If N_1, N_2 were minimal normal subgroups of G such that $N_1 \neq N_2$, then $[A, B]^{\mathcal{N}} \leq N_1 \cap N_2 = 1$. Hence, $[A, B]$ would be nilpotent and $[A, B] \leq F(G)$, a contradiction. Consequently, G has a unique minimal normal subgroup, say N. If $N \leq \Phi(G)$, then $F(G/N) = F(G)/N$, which would imply $[A, B] \leq F(G)$. Hence, G is a monolithic primitive group. By Lemma 2 and w.l.o.g., we may assume that $N \leq A$. We distinguish two cases:

Case 1: N is abelian.
If $G \in \mathcal{U}$, then $[A, B] \leq G' \leq F(G)$, a contradiction. Hence, we may assume that G is not supersoluble. By Lemma 4, there exists a maximal subgroup M of G such that $\mathrm{Core}_G(M) = 1$, $M = (M \cap A)B$ and $M \cap A$ centralizes B. Hence, $[A, B] = [N(M \cap A), B] = [N, B][M \cap A, B] = [N, B] \leq N = F(G)$, a contradiction.
Case 2: N is non-abelian.
In this case, we split the proof into the following steps:

1. $A = N = \mathrm{Soc}(G)$ *is a non-abelian simple group, and* $B = \langle b \rangle$ *is a cyclic group of prime order, say* r.

In case $B \not\leq A$, if L is a proper subgroup of A and L permutes with B, then

(i) B normalizes L.
(ii) $[B, L] \leq F(L)$. If $F(L) = 1$, then B centralizes L.

In case $B \leq A$, if M is a maximal subgroup of A and M permutes with B, then $B \leq M$. In particular, r divides $|M|$ and $[B, M] \leq F(M)$.

(*We recall that for each $L \leq A$, B permutes with L^x for some $x \in A$ by Lemma* 1(2).)

Assume that there exists an element $b \in B$ of prime order such that $T := N\langle b \rangle < G$. Since T is the product of the tcc-permutable subgroups N and $\langle b \rangle$, the choice of G implies that $[N, \langle b \rangle] \leq F(T) = 1$. So, $\langle b \rangle \leq C_G(N) = 1$, a contradiction. Hence, $G = T = N\langle b \rangle$, for some element $b \in B$ of prime order. From the choice of (G, A, B), we deduce that $A = N$ and $B = \langle b \rangle$.

If $B \not\leq A$ and $L \leq A$ permutes with B, then $A \cap B = 1$ and so $L = A \cap (LB) \trianglelefteq LB$, that is, B normalizes L.

If $L \trianglelefteq N$, then L permutes with B by Lemma 1(2) and then B normalizes L, if $B \leq A$ as much as if $B \not\leq A$. Whence $A = N$ is a non-abelian simple group.

If $B \not\leq A$ and L is a proper subgroup of A which permutes with B, then the choice of (G, A, B) implies that $[B, L] \leq F(BL) \cap L = F(L)$.

Assume now that $B \leq A$ and M is a maximal subgroup of A which permutes with B. If $G = A = MB$, by Theorem 2, we deduce that $G = G^{\mathcal{U}} = M^{\mathcal{U}} \leq M$, a contradiction. Hence, $MB = M$, that is, $B \leq M$.

The remainder of Step 1 is clear.

2. *Assume that $B \not\leq A$ and A has a maximal subgroup M such that $F(M) = 1$. Then, if L is any proper subgroup of A with $F(L) = 1$, there exists $z \in A$ such that $L^z \leq M$. In particular, A has at most one conjugacy class of maximal subgroups M such that $F(M) = 1$.*

By Step 1, there exist $x, y \in A$ such that B centralizes M^x and L^y. Since M is maximal, if $L^y \not\leq M^x$, then $A = \langle M^x, L^y \rangle$ is centralized by B, a contradiction.

3. *Assume that $B \leq A$. If M is any maximal subgroup of $G = A$, then $F(M) \neq 1$.*

Let M be a maximal subgroup of G. By Step 1, we deduce that $B \leq M^x$ for some $x \in G = A$. Moreover, if $F(M) = 1$, then B centralizes M^x, and so $B \leq F(M^x)$=1, a contradiction.

4. *A is neither a sporadic group nor an alternating group A_n of degree n, with $n \geq 6$.*

Assume first that $A \cong A_n$, $n \geq 6$, is an alternating group. Then, A has at least two non-conjugate maximal subgroups M_1 and M_2 such that $F(M_1) = F(M_2) = 1$, which contradicts Steps 2 and 3. (It is known by [18] that A_6 has two non-conjugate maximal subgroups isomorphic to A_5. For $n > 6$, A_n has maximal subgroups $M_1 \cong A_{n-1}$ and $M_2 \cong S_{n-2}$.)

Suppose now that A is a sporadic group. If $A \not\cong He$, the Held group, one can check in [18] that either $Out(A) = 1$ and A has a maximal subgroup M with $F(M) = 1$, or $Out(A) \neq 1$ and A has at least two non-conjugate maximal subgroups with trivial Fitting subgroup, a contradiction with Steps 3 and 2, respectively. Finally, assume that $A \cong He$. In this case, A has a maximal subgroup $M \cong S_4(4) : 2$ and a subgroup $L \cong L_3(2)$, which contradicts again Steps 2 and 3.

5. *If A is a simple group of Lie type of characteristic p, then B normalizes a maximal parabolic subgroup M of A. Moreover, $[M, B] \leq F(M) = O_p(M) \leq U$, where U is a Sylow p-subgroup of A such that $K = N_A(U)$ is a Borel subgroup contained in M; in particular, B normalizes both K and U.*

This follows by Step 1.

6. *If A is a simple group of Lie type of characteristic p and $A \not\cong L_2(q)$, then $r = p$.*

In the considered case for A, if $r \neq p$ and with the notation of Step 5, we have that $B \cap U = 1$ and B acts as a group of power automorphisms on U, by Corollary 2(ii). Since the Sylow p-subgroups of A are non-abelian and taking into account that B does not centralize U because $C_{\mathrm{Aut}(A)}(U) \leq U$, we get a contradiction to Lemma 5(ii).

7. *A is not a simple group of Lie type of Lie rank 1.*

Assume that $A \cong L(q)$ is a group of Lie type of Lie rank 1 over a field of characteristic p, with $q = p^n$. With the notation of Step 5, we have that $M = K = N_A(U)$. Recall also that when $A \not\cong L_2(q)$, we have $p = r$ by Step 6.

- $A \not\cong L_2(q)$, $q = p^n$.
  Assume that $A \cong L_2(q)$, $q = p^n$. (Note that this includes the case $A_5 \cong L_2(4) \cong L_2(5)$.) Suppose first that $B = \langle b \rangle \not\leq A$; in particular, b acts as a power automorphism on U by Corollary 2(ii). If b were a field automorphism, then it is known that $C_U(b) \neq 1$. Since

U is abelian, b centralizes U by Lemma 5(i). This is a contradiction, since $|C_A(b)|_p < |U|$ (see, for example, [20, 9.1]). Hence, we may assume that b is either a diagonal automorphism or a diagonal-field automorphism, and so $o(b) = 2$ and $p \neq 2$. But in this case, A always has a subgroup X isomorphic to A_4, the alternating group of degree 4, which is normalized but not centralized by b. This means that $XB \cong S_4$ and B is tcc-permutable with X, a contradiction by Lemma 6.

Assume now $B \leq A$. Then, by Step 1, it follows that B is contained in all maximal subgroups of $L_2(q)$ (up to conjugacy), so by order arguments we have that $|B| = r = 2$. In particular, $B \leq K = N_A(U)$. Suppose first $p = 2$. Let H be a Cartan subgroup in K, which has order $2^n - 1$, such that $BH = HB$. We note that $H \trianglelefteq HB$ because $|HB : H| = 2$. Then, by Step 5, $[B, H] \leq U \cap H = 1$, which is a contradiction since the centralizer of any 2-element in $L_2(q)$ is a 2-element. Hence, we may consider the case when p is odd. In this case, $A \cong L_2(q)$ contains a subgroup T isomorphic either to A_4 or to S_4, by Dickson's theorem [25, II.8.27], and we may assume that B permutes with T. Hence, BT is a subgroup of $L_2(q)$, which implies, again by Dickson's theorem, that $B \leq BT \cong S_4$. But this is a contradiction by Lemma 6.

– *$A \not\cong U_3(q)$, $q = p^n$.*
Suppose that $A \cong U_3(q)$, $q = p^n$. Recall that from Step 6, $r = p$.
Assume first that $B \leq A$. If $A \not\cong U_3(3)$ and $A \not\cong U_3(5)$, then A has a maximal subgroup which is a p'-subgroup (see [28] or [34]), a contradiction by Step 1. Now, if $A \cong U_3(3)$ or $A \cong U_3(5)$, then A has a maximal subgroup M with $F(M) = 1$. In both cases, we get a contradiction to Step 3.
Hence, we may consider $B \not\leq A$. Since $B = \langle b \rangle$, then b is an outer automorphism of A of order $r = p$. Let $K = N_A(U)$ be the Borel subgroup normalized by B. We may consider a Cartan subgroup $H \leq K$ such that H permutes with B. Then, B normalizes H by Step 1, and consequently, $[H, B] \leq H \cap [K, B] \leq H \cap U = 1$. This is only possible when b induces a diagonal automorphism, which is not the case since $o(b) = p$.

– *$A \not\cong Sz(q)$, $q = 2^n > 2$, n odd.*
Assume that the claim is not true. Here, we may consider that $B \leq A$, since $p = r = 2$ and $|Out(A)| = n$ is odd. This case can be treated with similar arguments as in the case $L_2(2^n)$, because there are no $2'$-elements in the maximal subgroup $K = N_A(U)$ centralized by any 2-element.

– *$A \not\cong {}^2G_2(q)$, $q = 3^n > 3$, n odd.*
Assume the assertion is false. In this case, as in the proof of Proposition 2, if s is a prime number dividing n and ϕ is a field automorphism of order s, then $M = C_A(\phi) \cong {}^2G_2(q^{1/s})$ is a maximal subgroup of A and $F(M) = 1$. Hence, if n is divisible by two prime numbers r_1, r_2, then A has at least two conjugacy classes of maximal non-local subgroups, and we get a contradiction by Steps 2 and 3. If n is divisible just by a single prime, we may assume that $B \not\leq A$ and this prime is $r = 3$, so $A \cong {}^2G_2(3^3)$ (recall that $r = p = 3$). In this case, A has a maximal non-local subgroup $M \cong {}^2G_2(3) \cong L_2(8) : 3$, and a non-conjugate subgroup $T \cong L_2(9)$, which contradicts Step 2.

8. *A is not a simple group of Lie type of Lie rank $l > 1$.*
Assume that $A \cong L(q)$ is a group of Lie type of Lie rank $l > 1$ over a field of characteristic p, with $q = p^n$. From Step 6, we have that $r = p$.

Let M, K and U be as in Step 5. Let Φ be a root system associated with A, and $\Pi = \{r_1, ..., r_l\} \subseteq \Phi$ be a fundamental root system. Let X_s denote the root subgroup of A for the root $s \in \Phi$, and $M_1 = M$, $M_2, \ldots, M_l$ be all maximal parabolic subgroups of A containing K. We may assume that $M_i = O_p(M_i)L_i$, where $L_i = H\langle X_{\pm r_j} \mid j \neq i\rangle$ is a

Levi complement of M_i, $F(M_i) = O_p(M_i)$ for every $i \in \{1, \dots, l\}$, and $H \leq K$ is a Cartan subgroup.

We claim that B normalizes M_i for all $i = 1, \dots, l$. We can assert that there exists $x_i \in A$ such that B normalizes $M_i^{x_i}$ for every $i \in \{1, \dots, l\}$ by Step 1. If $a \in A$ and $b \in B$, then $ab = b_1 a_1$ for some $a_1 \in A$ and $b_1 \in B$, and so

$$((M_i^{x_i})^a)^b = ((M_i^{x_i})^{b_1})^{a_1} = (M_i^{x_i})^{a_1} \in M_i^A := \{M_i^a \mid a \in A\}$$

for every $i \in \{1, \dots, l\}$, which implies that B fixes each conjugacy class $M_1^A, M_2^A, \dots, M_l^A$ of maximal parabolic subgroups of A. Consequently, since B normalizes K, if $b \in B$ and $i \in \{1, \dots, l\}$, it follows that $K = K^b \leq M_i^b$, and so $M_i^b = M_j \in M_i^A$ for some $j \in \{1, \dots, l\}$, which implies that $M_i^b = M_i = M_j$ since the subgroups $M_1, M_2, \dots, M_l$ are not pairwise conjugate in A (see [14, Theorem 8.3.3]), and the claim is proved.

Suppose that $B = \langle b \rangle \nleq A$. Then, the element b induces on A some outer automorphism of prime order $p = r$. By order arguments, b cannot be a diagonal automorphism. Moreover, we can consider that b induces on A some field automorphism, because b normalizes all maximal parabolic subgroups of A containing K. Then, B normalizes each root subgroup of A. In particular, B normalizes $\langle X_{\pm r_i} \mid j \neq i \rangle$, and it follows that $[B, \langle X_{\pm r_i} \mid j \neq i \rangle] \leq O_p(M_i) \cap \langle X_{\pm r_i} \mid j \neq i \rangle = 1$ for all $i = 1, \dots, l$, that is, B centralizes $\langle X_{\pm r_i} \mid 1 \leq i \leq l \rangle = \langle X_r \mid r \in \Phi \rangle = A$, a contradiction.

We may assume that $B \leq A$ and so $B \leq N_A(U) = K$. Then, $O_p(M_i)B$ is normal in M_i, and hence, $B \leq O_p(M_i)$ for all i, because $r = p$. For each $i \in \{1, \dots, l\}$, there exists $g_i \in M_i$ such that B permutes with $L_i^{g_i}$. Therefore, $B = O_p(BL_i^{g_i})$ for all i. Hence, we obtain that $\langle L_i^{g_i} \mid 1 \leq i \leq l \rangle$ normalizes B.

By Step 4 and Proposition 2, we may assume that A is isomorphic to a classical simple group of Lie type. We can consider that M_1 is the stabilizer in A of some (isotropic, if A is non-linear) point of the natural projective space with corresponding geometry associated with A. If A is either linear, or orthogonal, or symplectic for $p = 2$, then $O_p(M_1)$ is abelian (see [28, Propositions 4.1.17, 4.1.19, 4.1.20] or [34, Theorems 3.7, 3.10–3.12]). If A is symplectic for $p > 2$ or unitary, then $O_p(M_1)$ is a special p-group and L_1 acts irreducible on $O_p(M_1)/Z(O_p(M_1))$ (see [28, Propositions 4.1.18, 4.1.19] or [34, Theorems 3.8, 3.9]), and hence, $B \leq Z(O_p(M_1))$. In any case, B is normal in M_1. But $O_p(M_2) \leq U \leq M_1$, so $O_p(M_2)$ also normalizes B, and hence, $M_2 = O_p(M_2)L_2^{g_2}$ normalizes B. It follows that B is normal in $\langle M_1, M_2 \rangle = A$, a contradiction. □

Related results to the following corollaries, for products of totally permutable subgroups, were proved in [10, Lemma 2, Corollary 1, Lemma 6]. Corollary 5 next follows now from Theorem 4, and together with Corollary 2(iii), Corollary 1 (for $\mathcal{F} = \mathcal{U}$) and Lemma 1(1) allows us to derive Corollary 6 by arguing as in the reference given.

Corollary 5 *Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \dots, G_r$, for $r \geq 2$, and $G_i \neq 1$ for all $i = 1, \dots, r$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \dots, r\}$, $i \neq j$. Let N be a minimal normal subgroup of G. Then,*

1. *If N is non-abelian, then there exists a unique $i \in \{1, \dots, r\}$ such that $N \leq G_i$. Moreover, G_j centralizes N and $N \cap G_j = 1$ for all $j \in \{1, \dots, r\}$, $j \neq i$.*
2. *If G is a monolithic primitive group, then the unique minimal normal subgroup N is abelian.*

Proof 1. Since N is a non-abelian normal subgroup of G, it follows that $N \leq G^{\mathcal{S}}$, the soluble residual of G. From Corollary 1, it holds that $G^{\mathcal{S}} = G_1^{\mathcal{S}} \dots G_r^{\mathcal{S}}$ and $G_j^{\mathcal{S}} \trianglelefteq G$, for all $j \in \{1, \dots, r\}$. So, either $N \cap G_j^{\mathcal{S}} = 1$ and $[N, G_j^{\mathcal{S}}] = 1$ or $N \leq G_j^{\mathcal{S}}$, for each j. If $N \cap G_j^{\mathcal{S}} = 1$ for all $j \in \{1, \dots, r\}$, then $N \leq G^{\mathcal{S}} = G_1^{\mathcal{S}} \dots G_r^{\mathcal{S}} \leq C_G(N)$, a contradiction. Therefore, there exists $i \in \{1, \dots, r\}$ such that $N \leq G_i$. By Theorem 4, we can deduce that $[N, G_j] \leq F(NG_j) \cap N = 1$ for all $j \in \{1, \dots, r\},\ j \neq i$, and Part 1 follows.

2. If N were non-abelian, it would follow by Part 1 that $G_j \leq C_G(N) = 1$ for all $j \in \{1, \dots, r\},\ j \neq i$, a contradiction. □

Corollary 6 *Let the group $G = AB$ be the product of tcc-permutable subgroups A and B. Then,*

(i) If A is a normal subgroup of G, then B acts u-hypercentrally on A by conjugation (see [19, IV. 6.2]). *In particular, $B^{\mathcal{U}}$ centralizes A.*
(ii) $[A^{\mathcal{U}}, B^{\mathcal{U}}] = 1$.

Example 3 in [1] (also [3, Example 3.6]) shows that if the group $G = AB$ is the product of tcc-permutable subgroups A and B, then A does not necessarily centralize $B^{\mathcal{U}}$.

3 Complete c-permutability and formations

Motivated by the previous research on products of totally permutable subgroups and formations (see [7–10]), it is natural to ask whether those results can be achieved by weakening permutability to cc-permutability. A first approach to this study for products of tcc-permutable subgroups and saturated formations of soluble groups containing $\mathcal{U}$ was carried out in [3]. Our purpose in this section is to analyze whether it is possible to remove the hypotheses of solubility and saturation in the main results of this reference.

Next, we show that the mentioned results from [3] remain valid when any saturated formation containing $\mathcal{U}$ is considered. Taking into account Corollary 5, Lemma 4 and Theorem 4, it is possible now to reformulate the arguments in the proofs of [3, Theorem 1.4, Corollary 1.5] to deduce Theorem 5 and the following Corollary 7. In particular, the proof of Theorem 5 runs as in the one of [3, Theorem 1.4] with suitable changes. Since exact details may be of interest, we include the proof.

Theorem 5 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \dots, G_r$, for $r \geq 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \dots, r\}$, $i \neq j$. Then,*

1. *If $G_i \in \mathcal{F}$ for all $i = 1, \dots, r$, then $G \in \mathcal{F}$.*
2. *If $G \in \mathcal{F}$, then $G_i \in \mathcal{F}$ for all $i = 1, \dots, r$.*

Proof 1. Assume that the result is not true, and let the group $G = G_1 \cdots G_r$ be a counterexample of minimal order. We note that for any normal subgroup N of G, the factor group $G/N = (G_1N/N) \cdots (G_rN/N)$ satisfies the hypotheses of the theorem. Since $\mathcal{F}$ is a saturated formation, G is a primitive group with a unique minimal normal subgroup, say N, and $G/N \in \mathcal{F}$. By Corollary 5, N is abelian and Lemma 4 can be applied. Assume by Lemma 3, and w.l.o.g., that $N \leq G_1 \in \mathcal{F}$ and consider $M = (M \cap G_1)(G_2 \cdots G_r)$ a maximal subgroup of G as in Lemma 4. Let F denote the canonical local definition of $\mathcal{F}$. Since $\mathcal{U} \subseteq \mathcal{F}$, we have that $G_2 \cdots G_r \in \mathrm{F}(p)$ for p the prime divisor of $|N|$. Moreover, the fact that $G_1 \in \mathcal{F}$ implies $M \cap G_1 \in \mathrm{F}(p)$. Therefore, $M = (M \cap G_1)(G_2 \cdots G_r) \in \mathrm{F}(p)$ as $\mathrm{F}(p)$ is a formation and $(M \cap G_1)$ centralizes $G_2 \cdots G_r$. Since $G/C_G(N) = G/N \cong M \in \mathrm{F}(p)$ and $G/N \in \mathcal{F}$, it follows that $G \in \mathcal{F}$, a contradiction which proves Part 1.

2. We argue as in Part 1 and consider $G = G_1 \cdots G_r$ a counterexample of minimal order. We deduce here that G has a unique minimal normal subgroup, say N, and assume w.l.o.g. $N \leq G_1$ by Lemma 3. Assume first that G is primitive. Again, N is abelian, by Corollary 5, and Lemma 4 can be applied. With the notation in this lemma, we have that $G_j \in \mathcal{U} \subseteq \mathcal{F}$ for all $j = 2, \ldots, r$. In addition, $M = (M \cap G_1)(G_2 \cdots G_r) \cong G/N \in \mathcal{F}(p)$ because $G \in \mathcal{F}$. Since $G_2 \cdots G_r$ is a normal nilpotent subgroup of M, it follows from [19, IV.1.14] that $M \cap G_1 \in \mathrm{F}(p)$, which implies $G_1 = N(M \cap G_1) \in \mathcal{F}$, a contradiction.

Consider now the case $N \leq \Phi(G)$, the Frattini subgroup of G. We note that $G_i N/N \in \mathcal{F}$ for all $i = 1, \ldots, r$. Assume that $N \leq G_j$ for some $j \neq 1$. Then for $k = 1, j$, we have that $G_k = NF_k$ with an $\mathcal{F}$-projector F_k of G_k. Since $N \leq G_1 \cap G_j$, N and F_k are tcc-permutable subgroups and Part 1 implies that $G_k = NF_k \in \mathcal{F}$. On the other hand, if $N \not\leq G_j$ for some $j \neq 1$, then $\mathrm{Core}_G(G_j) = 1$, and we can deduce from Corollary 1 (also Theorem 2) that $G_j^{\mathcal{U}} = 1$, that is, $G_j \in \mathcal{U} \subseteq \mathcal{F}$. Consequently, it follows that $G_1 \notin \mathcal{F}$ and $G_j \in \mathcal{U}$ for all $j = 2, \ldots, r$. By the hypothesis, we note that the $\mathcal{F}$-projector F_1 of G_1 permutes with $G_j^{n_j}$ for some $n_j \in N$ for each $j = 2, \ldots, r$. Therefore, F_1 permutes with $\langle G_2^{n_2}, \ldots, G_r^{n_r} \rangle$ and $G = NF_1\langle G_2^{n_2}, \ldots, G_r^{n_r} \rangle = F_1\langle G_2^{n_2}, \ldots, G_r^{n_r} \rangle$ since $N \leq \Phi(G)$. Moreover, F_1 and $G_j^{n_j}$ for each $j = 2, \ldots, r$, are tcc-permutable subgroups by Lemma 1(1), which implies by Corollary 1 (also Theorem 2) that $F_1^{\mathcal{U}}$ is normalized by $\langle G_2^{n_2}, \ldots, G_r^{n_r} \rangle$ and then $F_1^{\mathcal{U}}$ is normal in G. If $F_1^{\mathcal{U}} \neq 1$, then $N \leq F_1^{\mathcal{U}} \leq F_1$, which implies that $G_1 = NF_1 = F_1 \in \mathcal{F}$, a contradiction. Therefore, $F_1 \in \mathcal{U}$. Consequently, $G/N = NF_1G_2 \cdots G_r/N \in \mathcal{U}$ by Part 1, which implies $G \in \mathcal{U}$ since $N \leq \Phi(G)$, and obviously, $G_i \in \mathcal{U} \subseteq \mathcal{F}$ for all $i = 1, \ldots, r$, the final contradiction. □

Lemma 7 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Let the group $G = AB$ be the product of tcc-permutable subgroups A and B. Then,*

1. *If $B \in \mathcal{F}$, then $G^{\mathcal{F}} = A^{\mathcal{F}}$.*
2. *$A^{\mathcal{F}}$ and $B^{\mathcal{F}}$ are normal subgroups of G.*

Proof 1. Assume that the result is false, and let G be a counterexample of minimal order. From Theorem 5, we can assert that $1 \neq A^{\mathcal{F}} \leq G^{\mathcal{F}}$. We claim that $G^{\mathcal{F}} = A^{\mathcal{F}}N$ for all minimal normal subgroup N of G and $\mathrm{Core}_G(A^{\mathcal{F}}) = 1$. Let N be a minimal normal subgroup of G. It is easy to check that $G/N = (AN/N)(BN/N)$ satisfies the hypotheses of the theorem. By the choice of G, we have that $(G/N)^{\mathcal{F}} = (AN/N)^{\mathcal{F}}$. This implies that $G^{\mathcal{F}}N = A^{\mathcal{F}}N$, and consequently, $G^{\mathcal{F}} = A^{\mathcal{F}}(G^{\mathcal{F}} \cap N)$. Then, we can deduce that $N \leq G^{\mathcal{F}}$, $N \not\leq A^{\mathcal{F}}$, and $G^{\mathcal{F}} = A^{\mathcal{F}}N$, which proves the claim.

On the other hand, $A^{\mathcal{U}}$ is normal in G, by Corollary 1 (also Theorem 2), and $A^{\mathcal{F}} \trianglelefteq A^{\mathcal{U}}$, which implies that $A^{\mathcal{F}}$ is subnormal in G. Therefore, $A^{\mathcal{F}}$ is normalized by any minimal normal subgroup N and then $A^{\mathcal{F}} \trianglelefteq G^{\mathcal{F}}$.

Moreover, from Theorem 4, we have that $[A, B] \leq F(G)$. If $F(G) = 1$, then $A^{\mathcal{F}}$ is normal in G, which is not possible. Consequently, we may consider an abelian minimal normal subgroup N. It follows that $(G^{\mathcal{F}})' \leq A^{\mathcal{F}}$ and $(G^{\mathcal{F}})' \leq \mathrm{Core}_G(A^{\mathcal{F}}) = 1$, that is, $G^{\mathcal{F}}$ is abelian.

By Lemma 1(2), there exists an $\mathcal{F}$-projector L of A such that $LB \leq G$ and, by Theorem 5, $LB \in \mathcal{F}$. Then, $A = A^{\mathcal{F}}L$ and $G = A^{\mathcal{F}}LB = G^{\mathcal{F}}LB$. Since $G^{\mathcal{F}}$ is an abelian group, by [6, 4.2.7], there exists an $\mathcal{F}$-projector H of G such that LB is contained in H. Now, we use [6, 4.1.18] to assert that H is an $\mathcal{F}$-normalizer of G. By [6, 4.2.17], it follows that $G^{\mathcal{F}} \cap H = 1$. Finally, $G^{\mathcal{F}} = A^{\mathcal{F}}(LB \cap G^{\mathcal{F}}) = A^{\mathcal{F}}$, the final contradiction.

2. Let q be a prime dividing $|B|$. By Lemma 1(2), there exists a Sylow q-subgroup Q of B such that $AQ \leq G$. From 1, we have that $(AQ)^{\mathcal{F}} = A^{\mathcal{F}}$. So, Q normalizes $A^{\mathcal{F}}$ and the result is proved. □

Theorem 5 and Lemma 7 allow us to argue as in the proof of [3, Corollary 1.5] to deduce the following corollary; we include the proof for completeness.

Corollary 7 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then,*

1. *$G_i^{\mathcal{F}} \trianglelefteq G$ for all $i = 1, \ldots, r$.*
2. *$G^{\mathcal{F}} = G_1^{\mathcal{F}} \cdots G_r^{\mathcal{F}}$.*

Proof From Lemma 7, we deduce that $G_i^{\mathcal{F}}$ for all $i = 1, \ldots, r$, and $K := G_1^{\mathcal{F}} \cdots G_r^{\mathcal{F}}$ are normal subgroups of G. We note that

$$G/G^{\mathcal{F}} = (G_1G^{\mathcal{F}}/G^{\mathcal{F}}) \cdots (G_rG^{\mathcal{F}}/G^{\mathcal{F}})$$

satisfies the hypotheses of the result, and then Theorem 5(2) implies that $K \leq G^{\mathcal{F}}$. By considering now $G/K = (G_1K/K) \cdots (G_rK/K)$, it follows that $G^{\mathcal{F}} \leq K$ from Theorem 5(1). Consequently, $G^{\mathcal{F}} = K$, and we are done. □

In [3, Corollary 3.1(i), Remark 3.2], the behavior of $\mathcal{F}$-projectors, for a saturated formation of soluble groups $\mathcal{F}$ containing $\mathcal{U}$, in products of tcc-permutable subgroups, is analyzed. We point out here that this result and the remark hold analogously, by using Theorem 5, if the hypothesis of solubility is omitted. To be more precise, we state the following.

Corollary 8 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Let the group $G = AB$ be the product of tcc-permutable subgroups A and B. Then, there exist $\mathcal{F}$-projectors X of A and Y of B such that X is permutable with Y. In this case, XY is an $\mathcal{F}$-projector of G.*

Remark Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Let the group $G = G_1 \cdots G_r$ be the product of the pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. The existence of $\mathcal{F}$-projectors X_i of G_i for each $i = 1, \ldots, r$, such that $X_1, \ldots, X_r$ are pairwise permutable remains an open question. Although in this case, $X_1 \cdots X_r$ would be an $\mathcal{F}$-projector of G.

Nevertheless, we give examples now showing that none of the statements in Theorem 5 remains true for arbitrary non-saturated formations containing $\mathcal{U}$ even in the universe of soluble groups.

Example 1 We consider the set of all prime numbers $\mathbb{P}$ and define a mapping $f : \mathbb{P} \longrightarrow$ { classes of groups } by setting $f(5) = (1, Z_2, Z_4, Z_3)$ and $f(p)$ to be the class of abelian groups of exponent dividing $p - 1$ for all $p \neq 5$. Let $\mathcal{F}$ be the class of all soluble groups G such that $\mathrm{Aut}_G(S) \in f(p)$ for all p-chief factors S of G and for all primes p dividing the order of G. By [19, IV. 1.3], it follows that $\mathcal{F}$ is a formation of soluble groups, and clearly also $\mathcal{U} \subseteq \mathcal{F}$.

Now, we consider the example constructed in [1, Example 3]: Let $V = \langle a, b\rangle \cong Z_5 \times Z_5$ and $Z_6 \cong C = \langle \alpha, \beta\rangle \leq \mathrm{Aut}(V)$ given by

$$a^{\alpha} = a^{-1},\ b^{\alpha} = b^{-1};\quad a^{\beta} = b,\ b^{\beta} = a^{-1}b^{-1}.$$

Let $G = [V]C$ be the corresponding semidirect product of V with C. Set $A = \langle \alpha\rangle$ and $B = V\langle \beta\rangle$. Then, $G = AB$ is the product of tcc-permutable subgroups A and B. Observe that A and B are $\mathcal{F}$-groups. But $G \notin \mathcal{F}$, because $G/C_G(V) \cong Z_3 \times Z_2 \notin f(5)$. This shows that Theorem 5(1) is not valid if the formation under consideration is not assumed to be saturated.

We modify the construction of the formation $\mathcal{F}$ by considering $f(5) = (1, Z_2, Z_4, Z_6)$. It holds now that $G, A \in \mathcal{F}$ but $B \notin \mathcal{F}$ because $B/C_B(V) \cong Z_3 \notin f(5)$, which shows the necessity for the formation to be saturated in order to prove Theorem 5(2).

However, in some special cases, it may be possible to obtain a positive answer, as the next result shows.

Proposition 3 *Let $\mathcal{F}$ be a formation. Consider $\mathcal{F} \circ \mathcal{U} = (G : G^{\mathcal{U}} \in \mathcal{F})$, the formation product of $\mathcal{F}$ with $\mathcal{U}$ (which contains $\mathcal{U}$) (see* [19, IV. 1.7]*). Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. If $G_i \in \mathcal{F} \circ \mathcal{U}$ for all $i \in \{1, \ldots, r\}$, then $G \in \mathcal{F} \circ \mathcal{U}$.*

Proof Since $G_i \in \mathcal{F} \circ \mathcal{U}$, it follows that $G_i^{\mathcal{U}} \in \mathcal{F}$ for all i. We deduce from Corollary 1 and Corollary 6 that $G^{\mathcal{U}} = G_1^{\mathcal{U}} \cdots G_r^{\mathcal{U}}$ with $[G_i^{\mathcal{U}}, G_j^{\mathcal{U}}] = 1$ for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Therefore, $G^{\mathcal{U}} \in \mathcal{F}$ because $\mathcal{F}$ is a formation. Hence, $G \in \mathcal{F} \circ \mathcal{U}$, and we are done. □

Remarks 1. Easy constructions show that in general the formations of the form $\mathcal{F} \circ \mathcal{U}$ as in the previous proposition are not saturated; for instance, the formation $\mathcal{A} \circ \mathcal{U}$, for the formation $\mathcal{F} = \mathcal{A}$ of all abelian groups, is not saturated.

2. An example in [8, Remark 1] shows a formation of the form $\mathcal{F} \circ \mathcal{U}$, for a formation $\mathcal{F}$, and a group in $\mathcal{F} \circ \mathcal{U}$ which is a totally permutable product with factors not in $\mathcal{F} \circ \mathcal{U}$; in particular, the converse of Proposition 3 is not true.

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