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A vanishing result for strictly p-convex domains

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Abstract In view of Andreotti and Grauert (Bull Soc Math France 90:193–259, 1962) vanishing theorem for q-complete domains in $\mathbb{C}^n$, we reprove a vanishing result by Sha (Invent Math 83(3):437–447, 1986), and Wu (Indiana Univ Math J 36(3):525–548, 1987), for the de Rham cohomology of strictly p-convex domains in $\mathbb{R}^n$ in the sense of Harvey and Lawson (The foundations of p-convexity and p-plurisubharmonicity in riemannian geometry. arXiv:1111.3895v1 [math.DG]). Our proof uses the L^2-techniques developed by Hörmander (An introduction to complex analysis in several variables, 3rd edn. North-Holland Publishing Co, Amsterdam 1990), and Andreotti and Vesentini (Inst Hautes Études Sci Publ Math 25:81–130, 1965).

Keywords p-Convexity in the sense of Harvey and Lawson · de Rham cohomology · Vanishing theorems

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0 Introduction

A weaker condition than holomorphic convexity for domains in $\mathbb{C}^n$ has been introduced by Andreotti and Grauert [1], defining *q-complete domains* as domains in $\mathbb{C}^n$ admitting a proper exhaustion function whose Levi form has $n - p + 1$ positive eigenvalues.

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In a recent series of foundational papers, [6,7], and references therein, Harvey and Lawson raise the interest on generalizations of the concept of convexity for Riemannian manifolds, proving many important results for *p-convex* manifolds: namely, starting with a Riemannian manifold (X, g), they ask whether it admits an exhaustion function whose Hessian is positive definite or satisfies weaker positive conditions.

Interpolating between the classical notions of convex functions and pluri-sub-harmonic functions, in [7], they define the class of *p-pluri-sub-harmonic functions* in terms of the positivity of the minors of their Hessian form, and they study *p-convex domains*, which can be regarded as domains in $\mathbb{R}^n$ endowed with a smooth p-pluri-sub-harmonic proper exhaustion function.

The notions of *geometric pluri-sub-harmonicity* and *geometric convexity*, introduced and studied by Harvey and Lawson [6], is closely related to holomorphic convexity and q-completeness in the sense of Andreotti and Grauert [1].

In the complex case, holomorphic convexity and, more in general, q-completeness provide vanishing theorems for the Dolbeault cohomology ([8], respectively [1,2]).

We are concerned in studying vanishing results for strictly p-convex domains in $\mathbb{R}^n$ in the sense of F. R. Harvey and H. B. Lawson. More precisely, we give a proof of the following result.

Theorem 3.1 *Let X be a strictly p-convex domain in $\mathbb{R}^n$. Then, $H^k_{dR}(X; \mathbb{R}) = \{0\}$ for every $k \geq p$.*

As pointed out to us by Harvey and Lawson, the above result was already known, as a consequence of [9, Theorem 1] by Sha, and [10, Theorem 1] by Wu, see also [7, Proposition 5.7]: more precisely, they prove, using Morse theory, that the existence of a smooth proper strictly p-pluri-sub-harmonic exhaustion function has consequences on the homotopy type of the domain.

In spite of this, our proof differs in the techniques, which are inspired by Andreotti and Vesentini [2]: in particular, the L^2-techniques used in our proof could be hopefully applied in a wider context, a fact which we would like to investigate further in future work.

The organization of the paper is as follows. In Sect. 1, we recall the main definitions introduced in [6,7] and the results proven by Andreotti and Grauert in [1]. In Sect. 2, we prove some useful estimates, which will be used in Sect. 3 to prove Theorem 3.1.

1 The notion of *p*-convexity by Harvey and Lawson

Following Harvey and Lawson [6,7], firstly, we recall point-wise definitions of p-positive symmetric endomorphisms; then, we will turn to manifolds, and finally, we will recall the notion of p-pluri-sub-harmonic (exhaustion) functions and (strictly) p-convex domains.

1.1 *p*-Positive (sections of) symmetric endomorphisms

Let $(V, \langle \cdot \mid \cdot\cdot \rangle)$ be an n-dimensional real inner product space. Let $G : V \to V^*$ denote the isomorphism defined as $G(v) := \langle v \mid \cdot \rangle$.
Let $\mathrm{Sym}^2(V)$ denote the space of symmetric elements of $(V \otimes V)^*$; namely, $A \in \mathrm{Sym}^2(V)$ if and only if $A(v \otimes w) = A(w \otimes v)$, for any $v, w \in V$. By means of the inner product $\langle \cdot \mid \cdot\cdot \rangle$, the space $\mathrm{Sym}^2(V)$ is isomorphic to the space of the $\langle \cdot \mid \cdot\cdot \rangle$-symmetric endomorphisms of V: given $A \in \mathrm{Sym}^2(V)$, we denote by $G^{-1}A \in \mathrm{Hom}(V, V)$ the corresponding $\langle \cdot \mid \cdot\cdot \rangle$-symmetric endomorphism.

The endomorphism $G^{-1}A \in \mathrm{Hom}(V, V)$ extends to $D^{[p]}_{G^{-1}A} \in \mathrm{Hom}(\wedge^p V, \wedge^p V)$; namely, on a simple vector $v_{i_1} \wedge \cdots \wedge v_{i_p} \in \wedge^p V$, the endomorphism $D^{[p]}_{G^{-1}A}$ acts as

$$D^{[p]}_{G^{-1}A}\left(v_{i_1} \wedge \cdots \wedge v_{i_p}\right) := \sum_{\ell=1}^{p} v_{i_1} \wedge \cdots \wedge v_{i_{\ell-1}} \wedge G^{-1}A\left(v_{i_\ell}\right) \wedge v_{i_{\ell+1}} \wedge \cdots \wedge v_{i_p} .$$

Observe that $D^{[p]}_{G^{-1}A} \in \mathrm{Hom}(\wedge^p V, \wedge^p V)$ is a symmetric endomorphism with respect to the scalar product on $\wedge^p V$ induced by $\langle \cdot \mid \cdot\cdot \rangle$.

Finally, given a $\langle \cdot \mid \cdot\cdot \rangle$-symmetric endomorphism $E \in \mathrm{Hom}(V, V)$, let $\mathrm{sgn}(E)$ denote the number of non-negative eigenvalues of E.

Notice that, given $A \in \mathrm{Sym}^2(V)$, and given two inner products on V inducing, respectively, the isomorphisms G_1 and G_2, then there holds $\mathrm{sgn}\left(G_1^{-1}A\right) = \mathrm{sgn}\left(G_2^{-1}A\right)$; it is also important to notice that, for $p > 1$, it might hold $\mathrm{sgn}\left(D^{[p]}_{G_1^{-1}A}\right) \neq \mathrm{sgn}\left(D^{[p]}_{G_2^{-1}A}\right)$, since the eigenvalues of $D^{[p]}_{G^{-1}A}$ are of the form

$$\lambda_{i_1} + \cdots + \lambda_{i_p} \qquad \text{for} \quad i_1, \ldots, i_p \in \{1, \ldots, n\} \text{ s.t. } i_1 < \cdots < i_p ,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $G^{-1}A$.

Definition 1.1 [6,7]

- Let V be a $\mathbb{R}$-vector space endowed with an inner product $\langle \cdot \mid \cdot\cdot \rangle$. Denote the space of *p-positive forms of* kth *branch* on V as

$$\mathcal{P}^{(k)}_p(V, \langle \cdot \mid \cdot\cdot \rangle) := \left\{A \in \mathrm{Sym}^2(V) \;:\; \mathrm{sgn}\left(D^{[p]}_{G^{-1}A}\right) \geq \binom{n}{p} - k + 1\right\} .$$

- Let (X, g) be a Riemannian manifold. Define the space of *p-positive sections of* kth *branch* of the bundle $\mathrm{Sym}^2(TX)$ of symmetric endomorphisms of TX as

$$\mathcal{P}^{(k)}_p(X, g) := \left\{A \in \mathrm{Sym}^2(TX) \;:\; \forall x \in X,\; A_x \in \mathcal{P}^{(k)}_p(T_xX, g_x)\right\}.$$

1.2 p-Pluri-sub-harmonic functions

In order to introduce an exhaustion of a given Riemannian manifold, we focus on special p-positive symmetric 2-forms, those arising from the Hessian of smooth functions.

Thus, let (X, g) be a Riemannian manifold, and let u be a smooth real-valued function on X. Let ∇ denote the Levi-Civita connection of the Riemannian metric g, and let

$$\mathrm{Hess}\, u(V, W) := V\, W\, u - (\nabla_V W)\, u ,$$

where V and W are smooth sections of the tangent bundle TX. Thus, $\mathrm{Hess}\, u(x) \in \mathrm{Sym}^2(T_xX)$, for any $x \in X$.

Definition 1.2 [6] Let (X, g) be a Riemannian manifold.

- The space

$$\mathrm{PSH}^{(k)}_p(X, g) := \left\{u \in \mathcal{C}^\infty(X; \mathbb{R}) \;:\; \mathrm{Hess}\, u \in \mathcal{P}^{(k)}_p(X, g)\right\} ,$$

  is called the space of *p-pluri-sub-harmonic functions of* kth *branch* on X.

- The space

$$\operatorname{int}\left(\mathrm{PSH}_p^{(k)}(X,\ g)\right) := \left\{u \in \mathcal{C}^\infty(X;\mathbb{R}) \ : \ \operatorname{Hess} u \in \operatorname{int}\left(\mathcal{P}_p^{(k)}\right)(X,\ g)\right\},$$

(where $\operatorname{int}\left(\mathcal{P}^{(k)}\right)_p(X,\ g)$ denotes the interior of $\mathcal{P}^{(k)}{}_p(X,\ g)$) is called the space of *strictly p-pluri-sub-harmonic functions of* kth *branch* on X.

1.3 (Strictly) p-convexity

We are now ready to recall the concept of p-convexity, which is central in [7]. Let $(X,\ g)$ be a Riemannian manifold. Let $K \subseteq X$ be a compact set. The *p-convex hull* of K is given by

$$\widetilde{K}^{\mathrm{PSH}_p^{(1)}(X,g)} := \left\{x \in X \ : \ \forall \phi \in \mathrm{PSH}_p^{(1)}(X,\ g),\ \phi(x) \leq \max_{y\in K} \phi(y)\right\}.$$

Definition 1.3 [6] Let $(X,\ g)$ be a Riemannian manifold. Then, X is called *p-convex*; if for any compact set $K \subseteq X$, then $\widetilde{K}^{\mathrm{PSH}_p^{(1)}(X,g)}$ is relatively compact in X.

Define the *p-core* of X, [6, Definition 4.1], as

$$\operatorname{Core}_p(X,\ g) := \Big\{x \in X \ : \ \text{for all } u \in \mathrm{PSH}_p^{(1)}(X,\ g), \\ \operatorname{Hess} u(x) \notin \operatorname{int}\left(\mathcal{P}^{(1)}\right)_p(T_xX,\ g_x)\Big\}.$$

Definition 1.4 [6] Let $(X,\ g)$ be a Riemannian manifold. Then, X is called *strictly p-convex* if (i) $\operatorname{Core}_p(X,\ g) = \varnothing$ and, (ii) for any compact set $K \subseteq X$, then $\widetilde{K}^{\mathrm{PSH}_p^{(1)}(X,g)}$ is relatively compact in X.

1.4 (Strictly) p-convexity and (strictly) p-pluri-sub-harmonic exhaustion functions

The following correspondences come from [6].

Theorem 1.5 [6, Theorem 4.4, Theorem 4.8] *Let $(X,\ g)$ be a Riemannian manifold. Then X is p-convex (respectively, strictly p-convex) if and only if X admits a smooth proper exhaustion function $u \in \mathrm{PSH}_p^{(1)}(X,\ g)$ [respectively, $u \in \operatorname{int}(\mathrm{PSH}_p^{(1)}(X,\ g))$].*

1.5 The p-convexity and the q-completeness

All along the definitions of the previous section, the special case that we had in mind is the following classical construction in Complex Analysis.

In [1], Andreotti and Grauert pointed out the following concept.

Definition 1.6 [1] Let $D \subseteq \mathbb{C}^n$ be a domain, and let ϕ be a smooth real-valued function on D. The function ϕ is called *p-pluri-sub-harmonic* (respectively, *strictly p-pluri-sub-harmonic*) if and only if, for any $z \in D$, the Hermitian form defined, for $\xi =: (\xi^a)_{a\in\{1,\ldots,n\}} \in \mathbb{C}^n$, as

$$L(\phi)_z(\xi) := \sum_{a,b=1}^n \frac{\partial^2 \phi}{\partial z^a \partial \bar{z}^b}(z)\, \xi^a\, \overline{\xi^b},$$

has $n - p + 1$ non-negative (respectively, positive) eigenvalues.

Andreotti and Grauert [1], studied domains of $\mathbb{C}^n$ admitting strictly q-pluri-sub-harmonic exhaustion functions (the so-called *q-complete domains*), proving a vanishing theorem for the higher-degree Dolbeault cohomology groups of such domains; then Andreotti and Vesentini [2], reproved the same result extending the L^2-techniques by Hörmander [8].

Thus, in the same vein as A. Andreotti and H. Grauert, we would consider domains X in $\mathbb{R}^n$ endowed with an exhaustion function $u \in \mathcal{C}^\infty(X;\mathbb{R})$ whose Hessian is in int $\left(\mathcal{P}_p^{(1)}\right)(X,g)$, proving a vanishing result for the higher-degree de Rham cohomology groups for strictly p-convex domains in the sense of F. R. Harvey and H. B. Lawson.

2 Vanishing of the de Rham cohomology for strictly p-convex domains

Let X be an oriented Riemannian manifold of dimension n, and denote by g its Riemannian metric and by vol its volume. The Riemannian metric g induces, for every $x \in X$, a point-wise scalar product $\langle \cdot \,|\cdot\cdot\rangle_{g_x} : \wedge^\bullet T_x^*X \times \wedge^\bullet T_x^*X \to \mathbb{R}$.

Fix $\phi \in \mathcal{C}^0(X;\mathbb{R})$ a continuous function. For every $\varphi,\ \psi \in \mathcal{C}_c^\infty(X;\wedge^\bullet T^*X)$, let

$$\langle \varphi \mid \psi \rangle_{L^2_\phi} := \int_X \langle \varphi | \psi \rangle_{g_x} \exp(-\phi) \text{ vol } \in \mathbb{R} ,$$

and, for $k \in \mathbb{N}$, define $L^2_\phi\left(X;\wedge^k T^*X\right)$ as the completion of the space $\mathcal{C}_c^\infty\left(X;\wedge^k T^*X\right)$ of smooth k-forms with compact support, with respect to the metric induced by $\|\cdot\|_{L^2_\phi} := \langle \cdot \mid \cdot \rangle_{L^2_\phi}$. Therefore, the space $L^2_\phi\left(X;\wedge^k T^*X\right)$ is a Hilbert space, endowed with the scalar product $\langle \cdot \mid \cdot\cdot\rangle_{L^2_\phi}$, and $\mathcal{C}_c^\infty\left(X;\wedge^k T^*X\right)$ is dense in $L^2_\phi\left(X;\wedge^k T^*X\right)$. For any $k \in \mathbb{N}$, let $L^2_{\text{loc}}\left(X;\wedge^k T^*X\right)$ denote the space of k-forms f whose restriction $f\lfloor_K$ to every compact set $K \subseteq X$ belongs to $L^2\left(K;\wedge^k T^*X\right)$.

For every $\phi_1,\ \phi_2 \in \mathcal{C}^0(X;\mathbb{R})$, the operator

$$\mathrm{d}\colon L^2_{\phi_1}\left(X;\wedge^\bullet T^*X\right) \dashrightarrow L^2_{\phi_2}\left(X;\wedge^{\bullet+1} T^*X\right)$$

is densely defined and closed; denote by

$$\mathrm{d}^*_{\phi_2,\,\phi_1}\colon L^2_{\phi_2}\left(X;\wedge^{\bullet+1} T^*X\right) \dashrightarrow L^2_{\phi_1}\left(X;\wedge^{\bullet} T^*X\right)$$

its adjoint, which is a densely defined closed operator.

Recall that on a domain X in $\mathbb{R}^n$, fixed $k \in \mathbb{N}$, $s \in \mathbb{N}$, and $\phi \in \mathcal{C}^\infty(X;\mathbb{R})$, the Sobolev space $W^{s,2}_\phi\left(X;\wedge^k T^*X\right)$ is the space of k-forms $f :=: \widetilde{\sum}_{|I|=k} f_I\, \mathrm{d}x^I$ such that $\frac{\partial^{\ell_1+\cdots+\ell_n} f_I}{\partial^{\ell_1}x^1\ldots\partial^{\ell_n}x^n} \in L^2_\phi\left(X;\wedge^k T^*X\right)$ for every multi-index $(\ell_1,\ldots,\ell_n) \in \mathbb{N}^n$ such that $\ell_1+\cdots+\ell_n \leq s$ and for every strictly increasing multi-index I such that $|I| = k$. The space $W^{s,2}_{\text{loc}}\left(X;\wedge^k T^*X\right)$ is defined as the space of k-forms f whose restriction $f\lfloor_K$ to every compact set $K \subseteq X$ belongs to $W^{s,2}\left(K;\wedge^k T^*X\right)$.

As a matter of notation, the symbol $\widetilde{\sum}_{|I|=k}$ denotes the sum over the strictly increasing multi-indices $I :=: (i_1,\ldots,i_k) \in \mathbb{N}^k$ (that is, the multi-indices such that $0 < i_1 < \cdots < i_k$) of length k. Given I_1 and I_2 two multi-indices of length k, let sign $\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$ be the sign of the permutation $\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$ if I_1 is a permutation of I_2 and zero otherwise.

2.1 Some preliminary computations

Let X be a domain in $\mathbb{R}^n$, that is, an open connected subset of $\mathbb{R}^n$ endowed with the metric and the volume induced, respectively, by the Euclidean metric and the standard volume of $\mathbb{R}^n$.

For $\phi_1,\ \phi_2 \in \mathcal{C}^\infty(X;\mathbb{R})$, consider $\mathrm{d}\colon L^2_{\phi_1}\left(X;\wedge^{k-1}T^*X\right) \dashrightarrow L^2_{\phi_2}\left(X;\wedge^{k}T^*X\right)$. The following lemma gives an explicit expression of the adjoint $\mathrm{d}^*_{\phi_2,\,\phi_1}\colon L^2_{\phi_2}\left(X;\wedge^{k}T^*X\right) \dashrightarrow L^2_{\phi_1}\left(X;\wedge^{k-1}T^*X\right)$ (compare, e.g., with [3, §8.2.1], [5, Lemma O.2] in the complex case).

Lemma 2.1 *Let X be a domain in $\mathbb{R}^n$. Let $\phi_1,\ \phi_2 \in \mathcal{C}^\infty(X;\mathbb{R})$ and consider*

$$\mathrm{L}^2_{\phi_1}\left(X;\wedge^{k-1}T^*X\right) \underset{\mathrm{d}^*_{\phi_2,\,\phi_1}}{\overset{\mathrm{d}}{\rightleftarrows}} \mathrm{L}^2_{\phi_2}\left(X;\wedge^{k}T^*X\right) .$$

Let

$$v \;:=:\; \widetilde{\sum_{|I|=k}} v_I\,\mathrm{d}x^I \in L^2_{\phi_2}\left(X;\wedge^k T^*X\right)$$

and suppose that $v \in \mathrm{dom}\,\mathrm{d}^*_{\phi_2,\,\phi_1}$. Then

$$\begin{aligned}\mathrm{d}^*_{\phi_2,\,\phi_1} v &= \exp(\phi_1)\ \mathrm{d}^*_{0,\,0}\left(\exp(-\phi_2)\ v\right)\\ &= \widetilde{\sum_{|J|=k-1}}\left(-\exp(\phi_1)\ \widetilde{\sum_{|I|=k}}\sum_{\ell=1}^{n}\mathrm{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix}\ \frac{\partial\left(v_I\exp(-\phi_2)\right)}{\partial x^\ell}\right)\mathrm{d}x^J .\end{aligned}$$

Proof By definition of $\mathrm{d}^*_{\phi_2,\,\phi_1}$, for every $u \in \mathrm{dom}\,\mathrm{d}$, one has $\langle \mathrm{d}u \mid v\rangle_{L^2_{\phi_2}} = \left\langle u \,\middle|\, \mathrm{d}^*_{\phi_2,\,\phi_1} v\right\rangle_{L^2_{\phi_1}}$. Hence, consider

$$u \;:=:\; \widetilde{\sum_{|J|=k-1}} u_J\,\mathrm{d}x^J \;\in\; \mathcal{C}^\infty_c\left(X;\wedge^{k-1}T^*X\right) ,$$

and compute

$$\mathrm{d}u \;=\; \widetilde{\sum_{\substack{|J|=k-1\\ |I|=k}}}\sum_{\ell=1}^{n}\mathrm{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix}\frac{\partial u_J}{\partial x^\ell}\,\mathrm{d}x^I .$$

The statement follows by computing

$$\begin{aligned}\langle \mathrm{d}u \mid v\rangle_{L^2_{\phi_2}} &= \int_X \widetilde{\sum_{\substack{|J|=k-1\\ |I|=k}}}\sum_{\ell=1}^{n}\mathrm{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix}\frac{\partial u_J}{\partial x^\ell}\,v_I\ \exp(-\phi_2)\ \mathrm{vol}\\ &= -\int_X \widetilde{\sum_{\substack{|J|=k-1\\ |I|=k}}}\sum_{\ell=1}^{n}\mathrm{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix}\frac{\partial\left(v_I\ \exp(-\phi_2)\right)}{\partial x^\ell}\,u_J\,\mathrm{vol}\end{aligned}$$

and

$$\left\langle u \,\middle|\, \mathrm{d}^*_{\phi_2,\,\phi_1} v\right\rangle_{L^2_{\phi_1}} \;=\; \int_X \widetilde{\sum_{|J|=k-1}}\left(\mathrm{d}^*_{\phi_2,\,\phi_1} v\right)_J\ u_J\ \exp(-\phi_1)\ \mathrm{vol} ,$$

where $\mathrm{d}^*_{\phi_2,\phi_1} v =: \widetilde{\sum}_{|J|=k-1} \left(\mathrm{d}^*_{\phi_2,\phi_1} v\right)_J \, \mathrm{d}x^J$. □

For any fixed $\phi \in \mathcal{C}^\infty(X;\mathbb{R})$ and for any $j \in \{1,\ldots,n\}$, define the operator

$$\delta_j^\phi \colon \mathcal{C}^\infty(X;\mathbb{R}) \to \mathcal{C}^\infty(X;\mathbb{R}) \ ,$$

where

$$\delta_j^\phi(f) \ := \ -\exp(\phi) \ \frac{\partial\left(f \ \exp(-\phi)\right)}{\partial x^j} \ = \ \frac{\partial \phi}{\partial x^j} \cdot f - \frac{\partial f}{\partial x^j} \ .$$

The following lemma states that δ_j^ϕ is the adjoint of $\frac{\partial}{\partial x^j}$ in $L^2_\phi\left(X; \wedge^0 T^* X\right)$ and computes the commutator between δ_j^ϕ and $\frac{\partial}{\partial x^k}$ (compare with, e.g., [8, pages 83–84]).

Lemma 2.2 *Let X be a domain in $\mathbb{R}^n$. Let $\phi \in \mathcal{C}^\infty(X;\mathbb{R})$ and $j \in \{1,\ldots,n\}$, and consider the operator $\delta_j^\phi \colon \mathcal{C}^\infty(X;\mathbb{R}) \to \mathcal{C}^\infty(X;\mathbb{R})$. Then:*

- *for every $w_1, w_2 \in \mathcal{C}^\infty_c(X;\mathbb{R})$,*

$$\int_X w_1 \cdot \frac{\partial w_2}{\partial x^k} \ \exp(-\phi) \ \mathrm{vol} \ = \ \int_X \delta_k^\phi(w_1) \cdot w_2 \ \exp(-\phi) \ \mathrm{vol} \ ;$$

- *for any $k \in \{1,\ldots,n\}$, the following commutation formula holds in* $\mathrm{End}\left(\mathcal{C}^\infty_c(X;\mathbb{R})\right)$*:*

$$\left[\delta_j^\phi, \ \frac{\partial}{\partial x^k}\right] \ = \ -\frac{\partial^2 \phi}{\partial x^j \, \partial x^k} \cdot \ .$$

Finally, we prove the following estimate, which will be used in the proof of Theorem 3.1 (we refer to [8, Sect. 4.2], or, e.g., [5, Lemma O.3] and [3, Sect. 8.3.1] for its complex counterpart).

Proposition 2.3 *Let X be a domain in $\mathbb{R}^n$ and $\phi, \ \psi \in \mathcal{C}^\infty(X;\mathbb{R})$. Consider*

$$\mathrm{L}^2_{\phi-2\psi}\left(X;\wedge^{k-1}T^*X\right) \underset{\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}}{\overset{\mathrm{d}}{\rightleftarrows}} \mathrm{L}^2_{\phi-\psi}\left(X;\wedge^{k}T^*X\right) \underset{\mathrm{d}^*_{\phi,\,\phi-\psi}}{\overset{\mathrm{d}}{\rightleftarrows}} \mathrm{L}^2_{\phi}\left(X;\wedge^{k+1}T^*X\right) \ .$$

Then, for any $\eta :=: \widetilde{\sum}_{|I|=k} \eta_I \, \mathrm{d}x^I \in \mathcal{C}^\infty_c\left(X; \wedge^k T^* X\right)$, one has

$$\int_X \widetilde{\sum_{\substack{|J|=k-1\\|I_1|=k\\|I_2|=k}}} \sum_{\ell_1,\ell_2=1}^{n} \mathrm{sign}\begin{pmatrix}\ell_1 J\\ I_1\end{pmatrix} \mathrm{sign}\begin{pmatrix}\ell_2 J\\ I_2\end{pmatrix} \frac{\partial^2 \phi}{\partial x^{\ell_1}\,\partial x^{\ell_2}} \, \eta_{I_1}\,\eta_{I_2} \ \exp(-\phi) \ \mathrm{vol}$$

$$\leq \ C \cdot \left(\left\| \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi} \eta \right\|^2_{L^2_{\phi-2\psi}} + \|\mathrm{d}\eta\|^2_{L^2_\phi} + \int_X \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \left| \frac{\partial \psi}{\partial x^\ell} \right|^2 \ |\eta_I|^2 \ \exp(-\phi) \ \mathrm{vol} \right) ,$$

where $C :=: C(k,n) \in \mathbb{N}$ is a constant depending just on k and n.

Proof It is straightforward to compute

$$\mathrm{d}\eta \ = \ \widetilde{\sum_{\substack{|I|=k\\|H|=k+1}}} \sum_{\ell=1}^{n} \mathrm{sign}\begin{pmatrix}\ell I\\ H\end{pmatrix} \frac{\partial \eta_I}{\partial x^\ell} \, \mathrm{d}x^H$$

and, using Lemma 2.1,

$$\begin{aligned} d^*_{\phi-\psi,\,\phi-2\psi}\eta &= -\exp(-\psi) \widetilde{\sum_{\substack{|J|=k-1\\|I|=k}}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix} \left(\frac{\partial \eta_I}{\partial x^\ell} - \frac{\partial(\phi-\psi)}{\partial x^\ell}\eta_I\right) dx^J \\ &= \exp(-\psi) \widetilde{\sum_{\substack{|J|=k-1\\|I|=k}}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix} \left(\delta^\phi_\ell(\eta_I) - \frac{\partial\psi}{\partial x^\ell}\eta_I\right) dx^J . \end{aligned}$$

For every J such that $|J| = k-1$, the previous equality gives

$$\widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix} \delta^\phi_\ell(\eta_I) = \exp(\psi) \left(d^*_{\phi-\psi,\,\phi-2\psi}\eta\right)_J + \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix} \frac{\partial\psi}{\partial x^\ell}\eta_I ,$$

where $d^*_{\phi-\psi,\,\phi-2\psi}\eta =: \widetilde{\sum}_{|J|=k-1} \left(d^*_{\phi-\psi,\,\phi-2\psi}\eta\right)_J dx^J$.

By the arithmetic mean–geometric mean inequality, one gets

$$\begin{aligned} &\int_X \widetilde{\sum_{|J|=k-1}} \left| \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix} \delta^\phi_\ell(\eta_I)\right|^2 \exp(-\phi)\ \mathrm{vol} \\ &\le 2\int_X \widetilde{\sum_{|J|=k-1}} \left(\left| \left(d^*_{\phi-\psi,\,\phi-2\psi}\eta\right)_J\right|^2 \exp(2\psi) \right. \\ &\quad \left. + \left| \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix} \frac{\partial\psi}{\partial x^\ell}\eta_I \right|^2 \right) \exp(-\phi)\ \mathrm{vol} \\ &\le C \left(\left\| d^*_{\phi-\psi,\,\phi-2\psi}\eta \right\|^2_{L^2_{\phi-2\psi}} + \int_X \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \left|\frac{\partial\psi}{\partial x^\ell}\right|^2 \cdot |\eta_I|^2 \exp(-\phi)\ \mathrm{vol} \right), \end{aligned} \tag{1}$$

where $C := C(k,n) \in \mathbb{N}$ depends on k and n only.

Now, using Lemma 2.2, one computes

$$\begin{aligned} &\int_X \widetilde{\sum_{|J|=k-1}} \left| \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix} \delta^\phi_\ell(\eta_I)\right|^2 \exp(-\phi)\ \mathrm{vol} \\ &= \widetilde{\sum_{|J|=k-1}} \widetilde{\sum_{\substack{|I_1|=k\\|I_2|=k}}} \sum_{\ell_1,\,\ell_2=1}^{n} \operatorname{sign}\begin{pmatrix}\ell_1 J\\ I_1\end{pmatrix} \operatorname{sign}\begin{pmatrix}\ell_2 J\\ I_2\end{pmatrix} \int_X \delta^\phi_{\ell_1}(\eta_{I_1}) \cdot \delta^\phi_{\ell_2}(\eta_{I_2}) \exp(-\phi)\ \mathrm{vol} \\ &= \widetilde{\sum_{\substack{|J|=k-1\\|I_1|=k\\|I_2|=k}}} \sum_{\ell_1,\,\ell_2=1}^{n} \operatorname{sign}\begin{pmatrix}\ell_1 J\\ I_1\end{pmatrix} \operatorname{sign}\begin{pmatrix}\ell_2 J\\ I_2\end{pmatrix} \\ &\quad \times \int_X \left(\frac{\partial \eta_{I_1}}{\partial x^{\ell_2}} \frac{\partial \eta_{I_2}}{\partial x^{\ell_1}} + \frac{\partial^2\phi}{\partial x^{\ell_1}\partial x^{\ell_2}} \eta_{I_1}\eta_{I_2} \right) \exp(-\phi)\ \mathrm{vol}. \end{aligned} \tag{2}$$

Now, note that

$$
\begin{aligned}
|\mathrm{d}\eta|^2 &= \widetilde{\sum_{|H|=k+1}} \left| \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix} \ell I \\ H \end{pmatrix} \frac{\partial \eta_I}{\partial x^\ell} \right|^2 \\
&= \widetilde{\sum_{|H|=k+1}} \left(\widetilde{\sum_{\substack{|I_1|=k \\ |I_2|=k}}} \sum_{\ell_1,\,\ell_2=1}^{n} \operatorname{sign}\begin{pmatrix} \ell_1 I_1 \\ H \end{pmatrix} \operatorname{sign}\begin{pmatrix} \ell_2 I_2 \\ H \end{pmatrix} \frac{\partial \eta_{I_1}}{\partial x^{\ell_1}} \frac{\partial \eta_{I_2}}{\partial x^{\ell_2}} \right) \\
&= \widetilde{\sum_{\substack{|I_1|=k \\ |I_2|=k}}} \sum_{\ell_1,\,\ell_2=1}^{n} \operatorname{sign}\begin{pmatrix} \ell_1 I_1 \\ \ell_2 I_2 \end{pmatrix} \frac{\partial \eta_{I_1}}{\partial x^{\ell_1}} \frac{\partial \eta_{I_2}}{\partial x^{\ell_2}} \\
&= \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \left| \frac{\partial \eta_I}{\partial x^\ell} \right|^2 - \widetilde{\sum_{\substack{|J|=k-1 \\ |I_1|=k \\ |I_2|=k}}} \sum_{\ell_1,\,\ell_2=1}^{n} \operatorname{sign}\begin{pmatrix} \ell_1 J \\ I_1 \end{pmatrix} \operatorname{sign}\begin{pmatrix} \ell_2 J \\ I_2 \end{pmatrix} \frac{\partial \eta_{I_1}}{\partial x^{\ell_2}} \frac{\partial \eta_{I_2}}{\partial x^{\ell_1}} . \quad (3)
\end{aligned}
$$

Hence, in view of (3), (2), (1), we get

$$
\begin{aligned}
&\int_X \widetilde{\sum_{\substack{|J|=k-1 \\ |I_1|=k \\ |I_2|=k}}} \sum_{\ell_1,\,\ell_2=1}^{n} \operatorname{sign}\begin{pmatrix} \ell_1 J \\ I_1 \end{pmatrix} \operatorname{sign}\begin{pmatrix} \ell_2 J \\ I_2 \end{pmatrix} \frac{\partial^2 \phi}{\partial x^{\ell_1}\,\partial x^{\ell_2}} \eta_{I_1}\,\eta_{I_2} \exp(-\phi) \operatorname{vol} \\
&\leq \int_X \left(\widetilde{\sum_{\substack{|J|=k-1 \\ |I_1|=k \\ |I_2|=k}}} \sum_{\ell_1,\,\ell_2=1}^{n} \operatorname{sign}\begin{pmatrix} \ell_1 J \\ I_1 \end{pmatrix} \operatorname{sign}\begin{pmatrix} \ell_2 J \\ I_2 \end{pmatrix} \frac{\partial^2 \phi}{\partial x^{\ell_1}\,\partial x^{\ell_2}} \eta_{I_1}\,\eta_{I_2} \right. \\
&\left. + \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \left| \frac{\partial \eta_I}{\partial x^\ell} \right|^2 \right) \exp(-\phi) \operatorname{vol} \\
&= \int_X \left(\widetilde{\sum_{|J|=k-1}} \left| \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix} \ell J \\ I \end{pmatrix} \delta_\ell^\phi(\eta_I) \right|^2 + \widetilde{\sum_{|H|=k+1}} \left| (\mathrm{d}\eta)_H \right|^2 \right) \exp(-\phi) \operatorname{vol} \\
&\leq C \cdot \left(\left\| \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi} \eta \right\|^2_{L^2_{\phi-2\psi}} + \|\mathrm{d}\eta\|^2_{L^2_\phi} + \int_X \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \left| \frac{\partial \psi}{\partial x^\ell} \right|^2 |\eta_I|^2 \exp(-\phi) \operatorname{vol} \right) ,
\end{aligned}
$$

concluding the proof. □

Remark 2.4 The argument in the proof of Proposition 2.3 actually proves the following stronger estimate, which will be used in the regularization process in Theorem 3.1.

Let X be a domain in $\mathbb{R}^n$ and $\phi,\ \psi \in \mathcal{C}^\infty(X; \mathbb{R})$. Consider

$$\mathrm{L}^2_{\phi-2\psi}\left(X;\wedge^{k-1}T^*X\right) \underset{\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}}{\overset{\mathrm{d}}{\rightleftarrows}} \mathrm{L}^2_{\phi-\psi}\left(X;\wedge^{k}T^*X\right) \underset{\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}}{\overset{\mathrm{d}}{\rightleftarrows}} \mathrm{L}^2_{\phi}\left(X;\wedge^{k+1}T^*X\right) .$$

Then, for any $\eta := \widetilde{\sum}_{|I|=k}\eta_I\, \mathrm{d}x^I \in \mathcal{C}^\infty_c\left(X;\wedge^k T^*X\right)$, one has

$$\int_X \left(\widetilde{\sum_{\substack{|J|=k-1\\ |I_1|=k\\ |I_2|=k}}} \sum_{\ell_1,\,\ell_2=1}^{n} \operatorname{sign}\begin{pmatrix}\ell_1 J\\ I_1\end{pmatrix} \operatorname{sign}\begin{pmatrix}\ell_2 J\\ I_2\end{pmatrix} \frac{\partial^2\phi}{\partial x^{\ell_1}\,\partial x^{\ell_2}}\,\eta_{I_1}\,\eta_{I_2} \right.$$

$$\left. + \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \left|\frac{\partial \eta_I}{\partial x^\ell}\right|^2 \right) \exp(-\phi)\ \mathrm{vol}$$

$$\leq\ C\cdot\left(\left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta\right\|^2_{L^2_{\phi-2\psi}} + \|\mathrm{d}\eta\|^2_{L^2_\phi} + \int_X \widetilde{\sum_{|I|=k}}\sum_{\ell=1}^{n}\left|\frac{\partial\psi}{\partial x^\ell}\right|^2 |\eta_I|^2 \exp(-\phi)\ \mathrm{vol} \right) ,$$

where $C := C(k,n) \in \mathbb{N}$ is a constant depending just on k and n.

3 Proof of the main theorem

We are ready to prove the following vanishing theorem for the higher-degree de Rham cohomology groups of a strictly p-convex domain in $\mathbb{R}^n$ (for a different proof, involving Morse theory, compare [9, Theorem 1] by Sha, and [10, Theorem 1] by Wu, see also [7, Proposition 5.7]).

Theorem 3.1 *Let X be a strictly p-convex domain in $\mathbb{R}^n$. Then $H^k_{dR}(X;\mathbb{R}) = \{0\}$ for every $k \geq p$.*

Proof We are going to prove that every d-closed k-form $\eta \in \mathcal{C}^\infty\left(X;\wedge^k T^*X\right)$ is d-exact, namely, there exists $\alpha \in \mathcal{C}^\infty\left(X;\wedge^{k-1}T^*X\right)$ such that $\eta = \mathrm{d}\alpha$; the statement of the theorem is a direct consequence of this result. Let us split the proof in the following steps.

`Step 1`—*Definitions of the weight functions and other notations*. Being X a strictly p-convex domain in $\mathbb{R}^n$, by Harvey and Lawson's [6, Theorem 4.8] (see also [7, Theorem 5.4]), there exists a smooth proper strictly p-pluri-sub-harmonic exhaustion function

$$\rho\ \in\ \operatorname{int}\left(\mathrm{PSH}^{(1)}_p(X,\ g)\right) \cap \mathcal{C}^\infty(X;\mathbb{R})\ ,$$

where g is the metric on X induced by the Euclidean metric on $\mathbb{R}^n$.
For every $m \in \mathbb{N}$, consider the compact set

$$K^{(m)}\ :=\ \{x \in X\ :\ \rho(x) \leq m\}\ ,$$

and define

$$L^{(m)}\ :=\ \min_{K^{(m)}} \lambda^{[k]}_1\ >\ 0\ ,$$

where, for every $x \in X$, the real numbers $\lambda_1^{[k]}(x) \leq \cdots \leq \lambda_{\binom{n}{k}}^{[k]}(x)$ are the ordered eigenvalues of $D^{[k]}_{g^{-1}\mathrm{Hess}\,\rho(x)} \in \mathrm{Hom}\left(\wedge^k T_x X,\ \wedge^k T_x X\right)$, and $\lambda_1(x) \leq \cdots \leq \lambda_n(x)$ are the ordered eigenvalues of $g^{-1}\mathrm{Hess}\,\rho(x) \in \mathrm{Hom}\,(T_x X,\ T_x X)$; indeed, note that, for every $x \in X$,

$$\lambda_1^{[k]}(x) \;=\; \lambda_1(x) + \cdots + \lambda_k(x) \;\geq\; \lambda_1(x) + \cdots + \lambda_p(x) \;>\; 0\;,$$

being ρ strictly p-pluri-sub-harmonic and that the function $X \ni x \mapsto \lambda_1^{[k]}(x) \in \mathbb{R}$ is continuous.

Fix $\{\rho_\nu\}_{\nu\in\mathbb{N}} \subset \mathcal{C}^\infty_c\,(X;\mathbb{R})$ such that (i) $0 \leq \rho_\nu \leq 1$ for every $\nu \in \mathbb{N}$, and (ii) for every compact set $K \subseteq X$, there exists $\nu_0 :=: \nu_0(K) \in \mathbb{N}$ such that $\rho_\nu\lfloor_K = 1$ for every $\nu \geq \nu_0$.

Then, we can choose $\psi \in \mathcal{C}^\infty\,(X;\mathbb{R})$ such that for every $\nu \in \mathbb{N}$,

$$|\mathrm{d}\rho_\nu|^2 \;\leq\; \exp\,(\psi)\;\;.$$

For every $m \in \mathbb{N}$, set

$$\gamma^{(m)} \;:=\; \max_{K^{(m)}}\left(C \cdot |\mathrm{d}\psi|^2 + \exp\,(\psi)\right)\;,$$

where $C :=: C(n,k)$ is the constant in Proposition 2.3.

Fix $\chi \in \mathcal{C}^\infty\,(\mathbb{R};\mathbb{R})$ such that $(i)\chi' > 0$, $(ii)\chi'' > 0$, and $(iii)\chi'\lfloor_{(-\infty,\,m]} > \frac{\gamma^{(m)}}{L^{(m)}}$, for every $m \in \mathbb{N}$. Define

$$\phi \;:=\; \chi \circ \rho\;:$$

then, $\phi \in \mathrm{int}\left(\mathrm{PSH}^{(1)}_p\,(X,\ g)\right) \cap \mathcal{C}^\infty\,(X;\mathbb{R})$; furthermore

$$\frac{\partial^2\phi}{\partial x^{\ell_1}\partial x^{\ell_2}} \;=\; \chi'' \circ \rho \cdot \frac{\partial\rho}{\partial x^{\ell_1}} \cdot \frac{\partial\rho}{\partial x^{\ell_2}} + \chi' \circ \rho \cdot \frac{\partial^2\rho}{\partial x^{\ell_1}\partial x^{\ell_2}}\;.$$

Choose $\mu \in \mathcal{C}^\infty\,(X;\mathbb{R})$ such that, for every $m \in \mathbb{N}$,

$$\chi' \circ \rho\lfloor_{K^{(m)}} \cdot L^{(m)} \;\geq\; \mu\lfloor_{K^{(m)}} \;\geq\; \gamma^{(m)}\;.$$

Step 2—*For every $\eta \in \mathcal{C}^\infty_c\left(X;\wedge^k T^*X\right)$, it holds $\|\eta\|^2_{L^2_{\phi-\psi}} \leq C\cdot\left(\left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta\right\|^2_{L^2_{\phi-2\psi}} + \|\mathrm{d}\eta\|^2_{L^2_\phi}\right)$.* Since

$$D^{[k]}_{g^{-1}\mathrm{Hess}\,\rho} \;=\; \left(\widetilde{\sum_{|J|=k-1}}\sum_{\ell_1,\,\ell_2=1}^{n} \mathrm{sign}\begin{pmatrix}\ell_1 J\\ I_1\end{pmatrix}\,\mathrm{sign}\begin{pmatrix}\ell_2 J\\ I_2\end{pmatrix}\,\frac{\partial^2\rho}{\partial x^{\ell_1}\partial x^{\ell_2}}\right)_{I_1,I_2}$$
$$\in \mathrm{Hom}\left(\wedge^k TX,\ \wedge^k TX\right)\;,$$

one estimates

$$
\begin{aligned}
&\widetilde{\sum_{\substack{|J|=k-1\\|I_1|=k\\|I_1|=k}}} \sum_{\ell_1,\ell_2=1}^{n} \operatorname{sign}\begin{pmatrix}\ell_1 J\\ I_1\end{pmatrix} \operatorname{sign}\begin{pmatrix}\ell_2 J\\ I_2\end{pmatrix} \frac{\partial^2\phi}{\partial x^{\ell_1}\partial x^{\ell_2}}\, \eta_{I_1}\, \eta_{I_2} \\
&= \widetilde{\sum_{\substack{|J|=k-1\\|I_1|=k\\|I_1|=k}}} \sum_{\ell_1,\ell_2=1}^{n} \operatorname{sign}\begin{pmatrix}\ell_1 J\\ I_1\end{pmatrix} \operatorname{sign}\begin{pmatrix}\ell_2 J\\ I_2\end{pmatrix} \chi''\circ\rho\cdot \frac{\partial\rho}{\partial x^{\ell_1}}\, \frac{\partial\rho}{\partial x^{\ell_2}}\, \eta_{I_1}\, \eta_{I_2} \\
&\quad + \widetilde{\sum_{\substack{|J|=k-1\\|I_1|=k\\|I_1|=k}}} \sum_{\ell_1,\ell_2=1}^{n} \operatorname{sign}\begin{pmatrix}\ell_1 J\\ I_1\end{pmatrix} \operatorname{sign}\begin{pmatrix}\ell_2 J\\ I_2\end{pmatrix} \chi'\circ\rho\cdot \frac{\partial^2\rho}{\partial x^{\ell_1}\partial x^{\ell_2}}\, \eta_{I_1}\, \eta_{I_2} \\
&= \widetilde{\sum_{|J|=k-1}} \chi''\circ\rho\cdot \left| \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix} \frac{\partial\rho}{\partial x^{\ell}}\, \eta_I \right|^2 \\
&\quad + \chi'\circ\rho\cdot \widetilde{\sum_{\substack{|J|=k-1\\|I_1|=k\\|I_1|=k}}} \sum_{\ell_1,\ell_2=1}^{n} \operatorname{sign}\begin{pmatrix}\ell_1 J\\ I_1\end{pmatrix} \operatorname{sign}\begin{pmatrix}\ell_2 J\\ I_2\end{pmatrix} \frac{\partial^2\rho}{\partial x^{\ell_1}\partial x^{\ell_2}}\, \eta_{I_1}\, \eta_{I_2} \\
&\geq \chi'\circ\rho\cdot\lambda_1^{[k]}(x)\cdot \widetilde{\sum_{|I|=k}} |\eta_I|^2 \\
&\geq \mu\cdot \widetilde{\sum_{|I|=k}} |\eta_I|^2 \ .
\end{aligned}
$$

Hence, using Proposition 2.3, we get that, for every $\eta \in \mathcal{C}_c^\infty\left(X; \wedge^k T^*X\right)$,

$$
\begin{aligned}
\|\eta\|^2_{L^2_{\phi-\psi}} &= \int_X \widetilde{\sum_{|I|=k}} |\eta_I|^2\, \exp\left(-\left(\phi-\psi\right)\right)\, \mathrm{vol} \\
&\leq \int_X \widetilde{\sum_{|I|=k}} \left(\mu - C\cdot \sum_{\ell=1}^{n} \left| \frac{\partial\psi}{\partial x^{\ell}} \right|^2 \right) \cdot |\eta_I|^2\, \exp\left(-\phi\right)\, \mathrm{vol} \\
&\leq \int_X \left(\widetilde{\sum_{\substack{|J|=k-1\\|I_1|=k\\|I_2|=k}}} \sum_{\ell_1,\ell_2=1}^{n} \operatorname{sign}\begin{pmatrix}\ell_1 J\\ I_1\end{pmatrix} \operatorname{sign}\begin{pmatrix}\ell_2 J\\ I_2\end{pmatrix} \frac{\partial^2\phi}{\partial x^{\ell_1}\,\partial x^{\ell_2}}\, \eta_{I_1}\, \eta_{I_2} \right. \\
&\qquad \left. - C\cdot \widetilde{\sum_{|I|=k}} \sum_{\ell=1}^{n} \left| \frac{\partial\psi}{\partial x^{\ell}} \right|^2 |\eta_I|^2 \right) \exp\left(-\phi\right)\, \mathrm{vol} \\
&\leq C\cdot \left(\left\| \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta \right\|^2_{L^2_{\phi-2\psi}} + \|\mathrm{d}\eta\|^2_{L^2_{\phi}} \right) ,
\end{aligned}
$$

where $C :=: C(k, n) \in \mathbb{N}$ is the constant in Proposition 2.3, depending just on k and n.

Step 3—$\mathcal{C}^{\infty}_{\mathrm{c}}\left(X;\, \wedge^k T^* X\right)$ *is dense in* $\left(\operatorname{dom} \mathrm{d} \cap \operatorname{dom} \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi},\ \|\cdot\|_{L^2_{\phi-\psi}} + \left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi} \cdot\right\|_{L^2_{\phi-2\psi}} + \|\mathrm{d}\cdot\|_{L^2_{\phi}}\right)$. Consider

$$\mathrm{L}^2_{\phi-2\psi}\left(X;\wedge^{k-1}T^*X\right) \underset{\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}}{\overset{\mathrm{d}}{\rightleftarrows}} \mathrm{L}^2_{\phi-\psi}\left(X;\wedge^{k}T^*X\right) \underset{\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}}{\overset{\mathrm{d}}{\rightleftarrows}} \mathrm{L}^2_{\phi}\left(X;\wedge^{k+1}T^*X\right) .$$

Fix $\eta \in \operatorname{dom} \mathrm{d} \cap \operatorname{dom} \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi} \subseteq L^2_{\phi-\psi}\left(X;\, \wedge^k T^* X\right)$. Firstly, we prove that $\{\rho_\nu\, \eta\}_{\nu\in\mathbb{N}} \subset \operatorname{dom} \mathrm{d} \cap \operatorname{dom} \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi} \subseteq L^2_{\phi-\psi}\left(X;\, \wedge^k T^* X\right)$ (where $\{\rho_\nu\}_{\nu\in\mathbb{N}} \subset \mathcal{C}^\infty_c\left(X;\mathbb{R}\right)$ has been defined in *Step 1*) is a sequence of functions having compact support and converging to η in the graph norm $\|\cdot\|_{L^2_{\phi-\psi}} + \left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi} \cdot\right\|_{L^2_{\phi-2\psi}} + \|\mathrm{d}\cdot\|_{L^2_{\phi}}$. Indeed,

$$\begin{aligned} \left|\mathrm{d}\left(\rho_\nu\, \eta\right) - \rho_\nu\, \mathrm{d}\eta\right|^2 \exp\left(-\phi\right) &= |\eta|^2 \cdot \left|\mathrm{d}\rho_\nu\right|^2 \exp\left(-\phi\right) \\ &\leq |\eta|^2 \exp\left(-\left(\phi-\psi\right)\right) \ \in\ L^2\left(X;\, \wedge^k T^* X\right) , \end{aligned}$$

hence, by Lebesgue's dominated convergence theorem, $\left\|\mathrm{d}\left(\rho_\nu\, \eta\right) - \rho_\nu\, \mathrm{d}\eta\right\|_{L^2_\phi} \to 0$ as $\nu \to +\infty$. Furthermore, for every $\nu \in \mathbb{N}$, note that $\rho_\nu\, \eta \in \operatorname{dom} \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}$, since the map

$$L^2_{\phi-2\psi}\left(X;\, \wedge^{k-1} T^* X\right) \supseteq \operatorname{dom} \mathrm{d} \ni u \mapsto \left\langle \rho_\nu\, \eta \mid \mathrm{d}u \right\rangle_{L^2_{\phi-\psi}} \in \mathbb{R}$$

is continuous, being

$$\begin{aligned} \left\langle \rho_\nu\, \eta \mid \mathrm{d}u \right\rangle_{L^2_{\phi-\psi}} &= \left\langle \eta \mid \mathrm{d}\left(\rho_\nu\, u\right) \right\rangle_{L^2_{\phi-\psi}} - \left\langle \eta \mid \mathrm{d}\rho_\nu \wedge u \right\rangle_{L^2_{\phi-\psi}} \\ &= \left\langle \rho_\nu\, \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi} \eta \mid u \right\rangle_{L^2_{\phi-2\psi}} - \left\langle \eta \mid \mathrm{d}\rho_\nu \wedge u \right\rangle_{L^2_{\phi-\psi}} , \end{aligned}$$

hence, by the Riesz representation theorem, there exists $\tilde{\eta} =: \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\left(\rho_\nu\, \eta\right) \in L^2_{\phi-2\psi}\left(X;\, \wedge^{k-1} T^* X\right)$ such that, for every $u \in \operatorname{dom} \mathrm{d} \subseteq L^2_{\phi-2\psi}\left(X;\, \wedge^{k-1} T^* X\right)$, it holds $\left\langle \rho_\nu\, \eta \mid \mathrm{d}u \right\rangle_{L^2_{\phi-\psi}} = \left\langle \tilde{\eta} \mid u \right\rangle_{L^2_{\phi-2\psi}}$. Lastly, note that, for every $u \in \operatorname{dom} \mathrm{d} \subseteq L^2_{\phi-2\psi}\left(X;\, \wedge^{k-1} T^* X\right)$,

$$\begin{aligned} &\left|\left\langle \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\left(\rho_\nu\, \eta\right) - \rho_\nu\, \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\, \eta \mid u \right\rangle_{L^2_{\phi-2\psi}}\right| \\ &= \left|\left\langle \rho_\nu\, \eta \mid \mathrm{d}u \right\rangle_{L^2_{\phi-\psi}} - \left\langle \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi} \eta \mid \rho_\nu\, u \right\rangle_{L^2_{\phi-2\psi}}\right| \\ &= \left|\left\langle \eta \mid \mathrm{d}\rho_\nu \wedge u \right\rangle_{L^2_{\phi-\psi}}\right| \\ &\leq \|\eta\|_{L^2_{\phi-\psi}} \cdot \left\|\mathrm{d}\rho_\nu \wedge u\right\|_{L^2_{\phi-\psi}} , \end{aligned}$$

hence, by Lebesgue's dominated convergence theorem, $\left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\left(\rho_\nu\, \eta\right) - \rho_\nu\, \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi} \eta\right\|_{L^2_{\phi-2\psi}} \to 0$ as $\nu \to +\infty$. This shows that $\rho_\nu\, \eta \to \eta$ as $\nu \to +\infty$ with respect to the graph norm.

Hence, we may suppose that $\eta \in \operatorname{dom} \mathrm{d} \cap \operatorname{dom} \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi} \subseteq L^2_{\phi-\psi}\left(X;\, \wedge^k T^* X\right)$ has compact support. Let $\{\Phi_\varepsilon\}_{\varepsilon\in\mathbb{R}} \subseteq \mathcal{C}^\infty\left(\mathbb{R}^n;\mathbb{R}\right)$ be a family of positive mollifiers, that is,

$\Phi_\varepsilon := \varepsilon^{-n}\,\Phi\left(\frac{\cdot}{\varepsilon}\right)$, where $(i)\Phi \in \mathcal{C}^\infty_{\mathrm{c}}\left(\mathbb{R}^n;\mathbb{R}\right)$, $(ii)\int_{\mathbb{R}^n}\Phi\mathrm{vol}_{\mathbb{R}^n} = 1$, $(iii)\lim_{\varepsilon\to 0}\Phi_\varepsilon = \delta$, where δ is the Dirac delta function, and $(iv)\Phi \geq 0$.

Consider the convolution $\{\eta * \Phi_\varepsilon\}_{\varepsilon\in\mathbb{R}} \subset \mathcal{C}^\infty_c\left(X;\wedge^k T^*X\right)$; we prove that $\eta * \Phi_\varepsilon \to \eta$ as $\varepsilon \to 0$ with respect to the graph norm. Clearly, $\|\eta - \eta * \Phi_\varepsilon\|_{L^2_{\phi-\psi}} \to 0$ as $\varepsilon \to 0$. Since $\mathrm{d}\left(\eta * \Phi_\varepsilon\right) = \mathrm{d}\eta * \Phi_\varepsilon$, one has that $\|\mathrm{d}\left(\eta * \Phi_\varepsilon\right) - \mathrm{d}\eta\|_{L^2_\phi} \to 0$ as $\varepsilon \to 0$. Lastly, write

$$\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi} \;=\; \exp\left(-\psi\right)\,\left(\mathrm{d}^*_{0,\,0} + A_{\phi-\psi,\,\phi-2\psi}\right)\;,$$

where $\mathrm{d}^*_{0,\,0}$ is a differential operator with constant coefficients, and $A_{\phi-\psi,\,\phi-2\psi}$ is a differential operator of order zero defined, for every $v \in L^2_{\phi-\psi}\left(X;\wedge^k T^*X\right)$, as

$$A_{\phi-\psi,\,\phi-2\psi}\left(v\right) \;:=\; \widetilde{\sum_{\substack{|J|=k-1\\ |I|=k}}}\sum_{\ell=1}^{n}\mathrm{sign}\begin{pmatrix}\ell J\\ I\end{pmatrix}\frac{\partial\left(\phi-\psi\right)}{\partial x^\ell}\cdot\eta\mathrm{d}x^J\;;$$

hence

$$\begin{aligned}&\left(\mathrm{d}^*_{0,\,0} + A_{\phi-\psi,\,\phi-2\psi}\right)\left(\eta * \Phi_\varepsilon\right)\\ &\quad= \left(\left(\mathrm{d}^*_{0,\,0} + A_{\phi-\psi,\,\phi-2\psi}\right)(\eta)\right) * \Phi_\varepsilon - \left(A_{\phi-\psi,\,\phi-2\psi}\,\eta\right) * \Phi_\varepsilon + A_{\phi-\psi,\,\phi-2\psi}\left(\eta * \Phi_\varepsilon\right)\\ &\quad\to \left(\mathrm{d}^*_{0,\,0} + A_{\phi-\psi,\,\phi-2\psi}\right)(\eta)\end{aligned}$$

as $\varepsilon \to 0$ in $L^2_{\phi-2\psi}\left(X;\wedge^{k-1}T^*X\right)$; having η compact support, it follows that $\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\left(\eta * \Phi_\varepsilon\right) \to \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\left(\eta\right)$ as $\varepsilon \to 0$ in $L^2_{\phi-2\psi}\left(X;\wedge^{k-1}T^*X\right)$.

`Step 4`—*If* $\|\eta\|^2_{L^2_{\phi-\psi}} \leq C\cdot\left(\left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta\right\|^2_{L^2_{\phi-2\psi}} + \|\mathrm{d}\eta\|^2_{L^2_\phi}\right)$ *holds for every* $\mathcal{C}^\infty_{\mathrm{c}}\left(X;\,\wedge^k T^*X\right)$, *then it holds for every* $\eta \in \mathrm{dom}\,\mathrm{d}\cap\mathrm{dom}\,\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}$. Let $\eta \in \mathrm{dom}\,\mathrm{d}\cap\mathrm{dom}\,\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}$. By *Step 3*, take $\left\{\eta_j\right\}_{j\in\mathbb{N}} \subset \mathcal{C}^\infty_c\left(X;\wedge^k T^*X\right)$ such that $\eta_j \to \eta$ as $j \to +\infty$ in the graph norm. Since, for every $j \in \mathbb{N}$, one has $\left\|\eta_j\right\|^2_{L^2_{\phi-\psi}} \leq C\cdot\left(\left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta_j\right\|^2_{L^2_{\phi-2\psi}} + \left\|\mathrm{d}\eta_j\right\|^2_{L^2_\phi}\right)$, and since $\left\|\eta_j - \eta\right\|_{L^2_{\phi-\psi}} \to 0$, $\left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta_j - \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta\right\|_{L^2_{\phi-2\psi}} \to 0$ and $\left\|\mathrm{d}\eta_j - \mathrm{d}\eta\right\|_{L^2_\phi} \to 0$ as $j \to +\infty$, we get that also $\|\eta\|^2_{L^2_{\phi-\psi}} \leq C\cdot\left(\left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta\right\|^2_{L^2_{\phi-2\psi}} + \|\mathrm{d}\eta\|^2_{L^2_\phi}\right)$.

`Step 5`—*Existence of a solution in* $\mathrm{L}^2_{\mathrm{loc}}\left(\mathrm{X};\wedge^{\mathrm{k}}\mathrm{T}^*\mathrm{X}\right)$. We prove here that the operator

$$\mathrm{d}\colon L^2_{\phi-2\psi}\left(X;\wedge^{k-1}T^*X\right) \dashrightarrow \ker\left(\mathrm{d}\colon L^2_{\phi-\psi}\left(X;\wedge^k T^*X\right) \dashrightarrow L^2_\phi\left(X;\wedge^{k+1}T^*X\right)\right)$$

is surjective, hence, for every $\eta \in \ker\left(\mathrm{d}\colon L^2_{\phi-\psi}\left(X;\wedge^k T^*X\right) \dashrightarrow L^2_\phi\left(X;\wedge^{k+1}T^*X\right)\right)$, the equation $\mathrm{d}\alpha = \eta$ has a solution α in $L^2_{\phi-\psi}\left(X;\wedge^{k-1}T^*X\right) \subseteq L^2_{\mathrm{loc}}\left(X;\wedge^{k-1}T^*X\right)$.

We recall (see, e.g., [8, Lemma 4.1.1]) that given two Hilbert spaces $\left(H_1,\,\langle\cdot\,|\cdot\cdot\rangle_{L^2_{H_1}}\right)$ and $\left(H_2,\,\langle\cdot\,|\cdot\cdot\rangle_{L^2_{H_2}}\right)$, and a densely defined closed operator $T\colon H_1 \dashrightarrow H_2$, whose adjoint is $T^*\colon H_2 \dashrightarrow H_1$, if $F \subseteq H_2$ is a closed subspace such that $\mathrm{im}\,T \subseteq F$, then the following conditions are equivalent:

(i) $\mathrm{im}\,T = F$;

(ii) there exists $C > 0$ such that, for every $y \in \operatorname{dom} T^* \cap F$,

$$\|y\|_{L^2_{H_2}} \;\leq\; C \cdot \left\|T^* y\right\|_{L^2_{H_1}} \;.$$

Hence, consider

$$\mathrm{d}\colon L^2_{\phi-2\psi}\left(X; \wedge^{k-1} T^* X\right) \dashrightarrow L^2_{\phi-\psi}\left(X; \wedge^k T^* X\right)$$

and

$$L^2_{\phi-\psi}\left(X; \wedge^k T^*X\right) \;\supseteq\; F := \ker\left(\mathrm{d}\colon L^2_{\phi-\psi}\left(X; \wedge^k T^*X\right) \dashrightarrow L^2_{\phi}\left(X; \wedge^{k+1} T^*X\right)\right)$$
$$\supseteq \operatorname{im}\left(\mathrm{d}\colon L^2_{\psi-2\psi}\left(X; \wedge^{k-1} T^*X\right) \dashrightarrow L^2_{\phi-\psi}\left(X; \wedge^k T^*X\right)\right) \;.$$

By *Step 4*, for every $\eta \in \operatorname{dom} \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi} \cap F \subseteq \operatorname{dom} \mathrm{d} \cap \operatorname{dom} \mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}$, it holds that

$$\|\eta\|^2_{L^2_{\phi-\psi}} \;\leq\; C\, \left\|\mathrm{d}^*_{\phi-\psi,\,\phi-2\psi}\eta\right\|^2_{L^2_{\phi-2\psi}} \;,$$

from which it follows that

$$F \;=\; \operatorname{im}\left(\mathrm{d}\colon L^2_{\psi-2\psi}\left(X; \wedge^{k-1} T^*X\right) \dashrightarrow L^2_{\phi-\psi}\left(X; \wedge^k T^*X\right)\right) \;.$$

`Step 6`—*Sobolev regularity of the solutions with compact support*. We prove that, for every $\alpha \in L^2\left(X; \wedge^{k-1} T^*X\right)$ with compact support, if $\mathrm{d}\alpha \in L^2\left(X; \wedge^k T^*X\right)$ and $\mathrm{d}^*_{0,\,0}\alpha \in L^2\left(X; \wedge^{k-2} T^*X\right)$, then $\alpha \in W^{1,2}\left(X; \wedge^{k-1} T^*X\right)$. Indeed, take $\{\Phi_\varepsilon\}_{\varepsilon\in\mathbb{R}}$ a family of positive mollifiers and, for every $\varepsilon \in \mathbb{R}$, consider $\alpha * \Phi_\varepsilon \in \mathcal{C}^\infty_c\left(X; \wedge^{k-1} T^*X\right)$; by Remark 2.4 with $\phi := 0$ and $\psi := 0$, we get that, for any multi-index I such that $|I| = k-1$ and for any $\ell \in \{1, \ldots, n\}$,

$$\int_X \left|\frac{\partial\left(\alpha_I * \Phi_\varepsilon\right)}{\partial x^\ell}\right|^2 \operatorname{vol} \;\leq\; C \cdot \left(\left\|\mathrm{d}^*_{0,\,0}\left(\alpha * \Phi_\varepsilon\right)\right\|^2_{L^2} + \left\|\mathrm{d}\left(\alpha * \Phi_\varepsilon\right)\right\|^2_{L^2}\right) \;,$$

where $C :=: C(k, n)$ is a constant depending just on k and n; since, for every multi-index I such that $|I| = k-1$, and for every $\ell \in \{1, \ldots, n\}$, it holds that $\lim_{\varepsilon\to 0} \int_X \left|\frac{\partial(\alpha_I*\Phi_\varepsilon)}{\partial x^\ell} - \frac{\partial\alpha_I}{\partial x^\ell}\right|^2 \operatorname{vol} = \lim_{\varepsilon\to 0}\left\|\mathrm{d}^*_{0,\,0}\left(\alpha * \Phi_\varepsilon\right) - \mathrm{d}^*_{0,\,0}\alpha\right\|_{L^2} = \lim_{\varepsilon\to 0}\left\|\mathrm{d}(\alpha * \Phi_\varepsilon) - \mathrm{d}\alpha\right\|_{L^2} = 0$, we get that

$$\int_X \left|\frac{\partial\alpha_I}{\partial x^\ell}\right|^2 \operatorname{vol} \;\leq\; C \cdot \left(\left\|\mathrm{d}^*_{0,\,0}\alpha\right\|^2_{L^2} + \left\|\mathrm{d}\alpha\right\|^2_{L^2}\right) \;,$$

proving the claim.

`Step 7`—*Regularization of the solution*. By *Step 5*, if $\eta \in \mathcal{C}^\infty\left(X; \wedge^k T^*X\right)$ is such that $\mathrm{d}\eta = 0$, then the equation $\mathrm{d}\alpha = \eta$ has a solution $\alpha \in L^2_{\mathrm{loc}}\left(X; \wedge^{k-1} T^*X\right)$; we prove that actually $\alpha \in \mathcal{C}^\infty\left(X; \wedge^{k-1} T^*X\right)$.

Note that we may suppose that the solution $\alpha \in L^2_{\mathrm{loc}}\left(X; \wedge^{k-1} T^*X\right)$ satisfies

$$\alpha \;\in\; (\ker \mathrm{d})^{\perp_{L^2_{\mathrm{loc}}\left(X;\wedge^{k-1}T^*X\right)}} \;=\; \overline{\operatorname{im} \mathrm{d}^*_{0,\,0}} \;=\; \operatorname{im} \mathrm{d}^*_{0,\,0} \;\subseteq\; \ker \mathrm{d}^*_{0,\,0} \;;$$

hence, α satisfies the system of differential equation

$$\begin{cases} \mathrm{d}\alpha = \eta \\ \mathrm{d}^*_{0,\,0}\alpha = 0 \end{cases} .$$

We prove, by induction on $s \in \mathbb{N}$, that $\alpha \in W^{s,2}_{\mathrm{loc}}\left(X; \wedge^{k-1} T^* X\right)$ for every $s \in \mathbb{N}$. Indeed, we have by *Step 5* that $\alpha \in W^{0,2}_{\mathrm{loc}}\left(X; \wedge^{k-1} T^* X\right) = L^2_{\mathrm{loc}}\left(X; \wedge^{k-1} T^* X\right)$. Suppose now that $\alpha \in W^{s,2}_{\mathrm{loc}}\left(X; \wedge^{k-1} T^* X\right)$ and prove that $\alpha \in W^{s+1,2}_{\mathrm{loc}}\left(X; \wedge^{k-1} T^* X\right)$. Clearly, $\eta \in \mathcal{C}^\infty\left(X; \wedge^k T^* X\right) \subseteq W^{\sigma,2}_{\mathrm{loc}}\left(X; \wedge^k T^* X\right)$ for every $\sigma \in \mathbb{N}$. Take K a compact subset of X, and choose $\widehat{\chi} \in \mathcal{C}^\infty_c(X; \mathbb{R})$ such that $\operatorname{supp}\widehat{\chi} \supset K$. For any multi-index $L :=: (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$ such that $\ell_1 + \cdots + \ell_n = s$, being

$$\mathrm{d}\left(\widehat{\chi} \cdot \frac{\partial^s \alpha}{\partial^{\ell_1} x^1 \ldots \partial^{\ell_n} x^n}\right) = \mathrm{d}\widehat{\chi} \wedge \frac{\partial^s \alpha}{\partial^{\ell_1} x^1 \ldots \partial^{\ell_n} x^n} + \widehat{\chi} \cdot \frac{\partial^s \eta}{\partial^{\ell_1} x^1 \ldots \partial^{\ell_n} x^n} \in L^2\left(K; \wedge^k T^* X\right)$$

and

$$\mathrm{d}^*_{0,0}\left(\widehat{\chi} \cdot \frac{\partial^s \alpha}{\partial^{\ell_1} x^1 \ldots \partial^{\ell_n} x^n}\right) = - \widetilde{\sum_{\substack{|J|=k-1 \\ |I|=k}}} \sum_{\ell=1}^{n} \operatorname{sign}\begin{pmatrix} \ell J \\ I \end{pmatrix} \frac{\partial \widehat{\chi}}{\partial x^\ell} \cdot \frac{\partial^s \alpha_I}{\partial^{\ell_1} x^1 \ldots \partial^{\ell_n} x^n} \mathrm{d}x^J$$
$$\in L^2\left(K; \wedge^{k-2} T^* X\right)$$

we get that $\widehat{\chi} \cdot \frac{\partial^s \alpha}{\partial^{\ell_1} x^1 \ldots \partial^{\ell_n} x^n} \in W^{1,2}\left(K; \wedge^{k-1} T^* X\right)$, that is, $\alpha \in W^{s+1,2}\left(K; \wedge^{k-1} T^* X\right)$. Hence, $\alpha \in W^{s+1,2}_{\mathrm{loc}}\left(X; \wedge^{k-1} T^* X\right)$. Since $W^{\sigma,2}_{\mathrm{loc}}\left(X; \wedge^{k-1} T^* X\right) \hookrightarrow \mathcal{C}^m\left(X; \wedge^{k-1} T^* X\right)$ for every $0 \le m < \sigma - \frac{n}{2}$, see [4, Corollary 7.11], we get that $\alpha \in \mathcal{C}^\infty\left(X; \wedge^{k-1} T^* X\right)$, concluding the proof of the theorem. □

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References

1. Andreotti, A., Grauert, H.: Théorème de finitude pour la cohomologie des espaces complexes. Bull. Soc. Math. France **90**, 193–259 (1962)
2. Andreotti, A., Vesentini, E.: Carleman estimates for the Laplace–Beltrami equation on complex manifolds, Inst. Hautes Études Sci. Publ. Math. **25**, 81–130 (1965); Erratum to: Carleman estimates for the Laplace–Beltrami equation on complex manifolds, ibidem **27**, 153–155 (1965)
3. Della Sala, G., Saracco, A., Simioniuc, A., Tomassini, G.: Lectures on complex analysis and analytic geometry, Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], 3, Edizioni della Normale, Pisa (2006)
4. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order, Reprint of the 1998 edition. Classics in Mathematics, Springer, Berlin (2001)
5. Gunning, R.C.: Introduction to Holomorphic Functions of Several Variables, vol. I, Function Theory, The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA (1990)
6. Harvey, F.R., Lawson Jr., H.B.: Geometric plurisubharmonicity and convexity—an introduction. arXiv:1111.3875v1 [math.DG]
7. Harvey, F.R., Lawson Jr., H.B.: The foundations of p-convexity and p-plurisubharmonicity in riemannian geometry. arXiv:1111.3895v1 [math.DG]
8. Hörmander, L.: An Introduction to Complex Analysis in Several Variables, 3rd edn. North-Holland Mathematical Library, 7. North-Holland Publishing Co, Amsterdam (1990)
9. Sha, J.-P.: p-Convex Riemannian manifolds. Invent. Math. **83**(3), 437–447 (1986)
10. Wu, H.: Manifolds of partially positive curvature. Indiana Univ. Math. J. **36**(3), 525–548 (1987)