Annali di Matematica (2014) 193:999–1018
DOI 10.1007/s10231-012-0312-8

On L^r hypoellipticity of solutions with compact support of the Cauchy–Riemann equation

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Received: 13 September 2012 / Accepted: 10 December 2012 / Published online: 1 January 2013

Abstract In one complex variable, the existence of a compactly supported solution to the Cauchy–Riemann equation is related to the vanishing of certain integrals of the data; trying to generalize this approach, we find an explicit construction, via convolution, for a compactly supported solution in $\mathbb{C}^n$, which allows us to estimate the L^p norm of the solution. We also investigate the possible generalizations of this method to domains of the form $P \setminus Z$, where P is a polydisc and Z is the zero locus of some holomorphic function.

Keywords Cauchy–Riemann equation · L^p regularity · Compact support · Integral representations

Mathematics Subject Classification (2010) 32A26 · 32W05

1 Introduction

In this paper, we investigate the inhomogeneous Cauchy–Riemann equation

$$\bar{\partial} u = g$$

when g has compact support and belongs to some L^r space. The question is whether it is possible to find a solution u with the same properties, namely compactly supported and in L^r, expressed in terms of a linear bounded integral operator applied to g.

The L^r solvability of the Cauchy–Riemann equation has been discussed by Kerzman for smoothly bounded strongly pseudoconvex domains (see [9]), by Fornaess and Sibony

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in $\mathbb{C}$ with weights and in Runge domains in $\mathbb{C}^2$ (see [6]). Other works on the subject are [1,4,5,8,10,11] and [15]. The problem of controlling the support of the solution is also widely discussed. In one complex variable, the existence of a compactly supported solution in $\mathbb{C}$ is related to the vanishing of some integrals, resemblant of the *moment conditions* which appear in CR geometry:

$$\int_{\mathbb{C}} g(z) z^k \mathrm{d}m_1(z),$$

where $\mathrm{d}m_1$ is the Lebesgue measure on $\mathbb{C}$. If these integrals vanish for every $k \in \mathbb{N}$, then there exists a function u such that $\partial u/\partial \bar{z} = g$ and $\mathrm{supp} u \Subset \{|z| < R\}$ for some R.

It is not hard to generalize this result to domains like punctured discs, as we do in Lemma 3.2.

In higher dimension, it is well known that the existence of a compactly supported solution depends on the vanishing of the cohomology with compact supports; $H_c^{p,q}(\Omega)$ vanishes, for $\Omega \subseteq \mathbb{C}^n$ Stein, if $q < n$. For smooth forms, the existence of a solution compactly supported in a sublevel of some strictly plurisubharmonic exhausting function has also been widely studied, beginning with the work of Andreotti and Grauert ([3], but see also [12] and [13]).

We recall that, for $(0, 1)$-forms, we have the following well-known result: let $\Omega \subseteq \mathbb{C}^n$ be a bounded Stein domain for $n \geq 2$ and ω a $(0, 1)$-form with coefficients in $L_c^r(\Omega)$ such that $\overline{\partial}\omega = 0$. Then there exists a unique $f \in L_c^r(\Omega)$ such that $\overline{\partial} f = \omega$, with $\|f\|_r \leq C\|\omega\|_r$, where C depends only on Ω.

This result leaves the question open for $q > 1$.

We tackle the problem for a very special class of domains, which generalize the punctured disc: we consider the Stein open domain obtained by removing a complex hypersurface from a polydisc $\mathbb{D}^n$. Given $f \in \mathcal{O}(\overline{\mathbb{D}^n})$ with $Z = \{f = 0\}$, we consider the domain $\mathbb{D}^n \setminus Z$: the particular structure of these open sets allows us to give a constructive proof of our results. We will state our results in terms of $(0, q)$-forms, the extension to the (p, q)-forms being obvious.

After fixing the notation and choosing suitable coordinates, we analyse the behaviour of the support of the Cauchy transform with respect to a fixed variable, pointing out the link with the moment conditions in Lemma 3.2.

In Sect. 4, the main tool of this note is defined; the coronas construction consists in applying to a given function a collection of linear convolution operators $K_m^{(i)}$ (see Definition 4.1) which produce a decomposition

$$\varphi = \varphi_1 + \cdots + \varphi_{n-1} + \varphi_n$$

where each $\varphi_j,\ j < n$, satisfies the moment conditions in the variable z_j (see Corollary 4.7).

The last function φ_n is obtained as a remainder of the coronas construction; therefore, we do not have *a priori* any information about the vanishing of the moment integrals for it; this is the goal of Sect. 5, where we introduce some quantities $J_{\mu,l}^{(i)}(\varphi)$ whose vanishing ensures that φ_n satisfies the moment conditions with respect to the variable z_n, as it is shown in Theorem 5.1.

Sections 6, 7 and 8 are devoted to the study of the solutions with compact support in a polydisc; while the $(0, n)$ case follows quite naturally from the previous construction, we need to apply some inductive arguments to treat the case of $(0, q)$-forms, so we make some additional assumptions on the summability of the derivatives of the datum.

We remark that, in the case of the polydisc, the operators constructed in Sect. 4 are linear and continuous, that is, bounded, from L_c^r to L_c^r and preserve any additional regularity or summability. Therefore, we obtain estimates on the L^r norm of the solution for $(0, n)$-forms; for $(0, q)$-forms we can estimate the sum of the norm of the solution and the norms of some derivatives, with a partial gain of regularity.

More precisely, let ω be a generic $(0, q)$-form, and let us write

$$\omega = \sum_{|J|=n-q} \omega_J \mathrm{d}\hat{\bar{z}}_J,$$

where $\mathrm{d}\hat{\bar{z}}_J = \bigwedge_{k \notin J} \mathrm{d}\bar{z}_k$. Let us denote by $\overline{\partial}_j$ the derivative with respect to the variable $\bar{z}_j$; we denote by $\mathcal{W}^q(\mathbb{D}^n)$ the space of $(0, q)$-forms ω with $L^r_c(\mathbb{D}^n)$ coefficients such that

$$(*) \qquad \overline{\partial}_{j_{n-q}} \cdots \overline{\partial}_{j_k} \omega_J \in L^r(\mathbb{C}^n) \qquad k = 1, \ldots, n-q, \quad \forall\, |J| = n-q.$$

The space $\mathcal{W}^q(\mathbb{D}^n)$ can be made into a Banach space with the norm

$$\|\omega\|_W = \sum_J \|\omega_J\|_r + \sum_J \sum_{k=1}^{n-q} \|\overline{\partial}_{j_{n-q}} \cdots \overline{\partial}_{j_k} \omega_J\|_r.$$

In Theorems 6.2, 7.4 and 8.1, we show that, given ω a $(0, q)$-form compactly supported in $\mathbb{C}^n$, with $\overline{\partial}\omega = 0$, with L^r coefficients and satisfying $(*)$, we can find a $(0, q-1)$-form $\beta \in L^r_c(\mathbb{C}^n)$ such that $\overline{\partial}\beta = \omega$, with β satisfying condition $(*)$. The operator associating β with ω is linear and bounded from $\mathcal{W}^q(\mathbb{D}^n)$ to $\mathcal{W}^{q-1}(\mathcal{D}^n)$.

This result in $\mathbb{C}^n$ easily gives the corollary

Corollary 8.4 *Let ω be a $\overline{\partial}$-closed $(0, q)$-form with compact support in $\mathbb{D}^n \setminus Z$ and satisfying conditions $(*)$, and then, for any $k \in \mathbb{N}$, we can find a $(0, q-1)$-form $\beta \in L^r_c(\mathbb{D}^n)$ such that $\overline{\partial}(f^k\beta) = \omega$. Equivalently, we can find a $(0, q-1)$-form $\eta = f^k\beta$ such that $\eta \in L^r_c(\mathbb{D}^n)$, η is 0 on Z up to order k and $\overline{\partial}\eta = \omega$.*

The starting point of this work was an incisive question asked by G. Tomassini and the second author to the first author.

2 Notations

We denote by $\mathbb{D}$ the unit disc in $\mathbb{C}$ and by $\mathbb{D}^n$ its n-fold product, the unit polydisc in $\mathbb{C}^n$. The projection from $\mathbb{C}^n$ onto the j th coordinate will be denoted by π_j.

The standard Lebesgue measure on $\mathbb{C}^n$ will be $\mathrm{d}m_n$, and we will denote by $g *_k h$ the partial convolution in the kth variable:

$$(g *_k h)(z_1, \ldots, z_n) := \int_{\mathbb{C}} g(\cdots, z_{k-1}, \zeta, z_{k+1}, \cdots) h(\cdots, z_{k-1}, z_k - \zeta, z_{k+1}, \cdots) \mathrm{d}m_1(\zeta).$$

If T is a distribution in $\mathbb{C}^n$, we set $\overline{\partial}_j T = \frac{\partial T}{\partial \bar{z}_j}$, $j = 1, \ldots, n$.

Let $J = (j_1, \ldots, j_q)$, $j_k = 1, \ldots, n$, then we define $\hat{z}_J \in \mathbb{C}^{n-q}$ with coordinates in J deleted. For instance, $\hat{z}_k = (\cdots, z_{k-1}, z_{k+1}, \cdots) \in \mathbb{C}^{n-1}$.

3 On the Cauchy transform

Given $\varphi \in \mathcal{D}(\mathbb{C}^n)$ a smooth function with compact support, the functions

$$\zeta \to \varphi(\cdots, z_{k-1}, \zeta, z_{k+1}, \cdots),$$

for $k = 1, \ldots, n$, are still smooth and with compact support, contained in $\pi_k(\text{supp}\varphi)$. The Cauchy transform of φ in the kth variable is

$$G_k(\varphi)(z) = \varphi *_k \frac{1}{\pi z_k} := \int_{\mathbb{C}} \frac{\varphi(\cdots, z_{k-1}, \zeta_k, z_{k+1}, \cdots)}{\pi(\zeta_k - z_k)} \mathrm{d}m_1(\zeta_k)$$

and we know that [2]

Lemma 3.1 *We have, with the above notations,*

$$\overline{\partial}_k G_k(\varphi)(z) = \varphi(z) \qquad \forall z \in \mathbb{C}^n$$

and

$$\|G_k(\varphi)\|_{L^r} \leq \left\| \frac{1}{\pi \zeta_1} \right\|_{L^1(\mathbb{D})} \times \|\varphi\|_{L^r}.$$

So the Cauchy transform extends as a bounded linear operator on $\varphi \in L^r_c(\mathbb{D}^n)$. Moreover, $G_k(\varphi)$ is holomorphic in z_k outside of the support of φ considered as a function of z_k, $\hat{z}_k$ being fixed.

Throughout this note, f will be a given function holomorphic in a neighbourhood of $\overline{\mathbb{D}^n}$, and $Z = Z(f)$ will denote its zero locus.

The set of directions for which there is a complex line with that direction contained in Z is an analytic subset of $\mathbb{CP}^{n-1}$ of dimension $n - 2$; therefore, we can find n linearly independent complex directions not lying in it. So, after a linear change of coordinates, for every $1 \leq k \leq n$, we can find a number N_k such that, given $n - 1$ complex numbers a_j, $j \in \{1, \ldots, n\} \backslash \{k\}$, with $| a_j | < 1$, the number of solutions of

$$f(\cdots, a_{k-1}, z_k, a_{k+1}, \cdots) = 0$$

as an equation in z_k, is less than $N_k + 1$.

Because these solutions are those of an analytic function, there is always a parametrization of them by measurable functions: it is an easy application of [17, Theorem 7.34]; let us denote these solutions by $\{c_{1,k}(a), \ldots, c_{N_k,k}(a)\}$ where the functions $c_{j,k} = c_{j,k}(a)$ are measurable from $\mathbb{C}^{n-1}$ to $\mathbb{C}$.

Let $\varphi \in L^r_c(\mathbb{D}^n \backslash Z)$ and fix $\hat{z}_k \in \mathbb{D}^{n-1}$; denote by $S_\varphi(\hat{z}_k)$ its support as a function of z_k which depends on $\hat{z}_k$. Then, by compactness, there exist numbers $\delta_1, \ldots, \delta_n$ such that $S_\varphi(\hat{z}_k)$ has distance at least δ_k from $c_{j,k}(a)$, $j = 1, \ldots, N_k$, for every $a \in \mathbb{C}^{n-1}$, so there are numbers $r_{j,k} = r_{j,k}(\hat{z}_k) \geq \delta_k > 0$ such that the disc $D(c_{j,k},\ r_{j,k})$ in the z_k variable is *not* in $S_\varphi(\hat{z}_k)$.

However, these discs could intersect without coinciding; suppose that the discs

$$D(c_{j_1,k},\ r_{j_1,k}), \ldots, D(c_{j_h,k},\ r_{j_h,k})$$

form a connected component of the union of all the discs for the variable z_k, then we can suppose that $r_{j_i,k} = \delta_k$ for $i = 1, \ldots, h$. If the discs

$$D(c_{j_1,k},\ \delta_k/3N_k), \ldots, D(c_{j_h,k},\ \delta_k/3N_k)$$

are disjoint, then we are done; otherwise, let us consider a connected component of their union, and let us suppose, without loss of generality, that it coincides with the union. Obviously, the diameter of such a connected component is less than δ_k; therefore, a disc centred in one of the centres with radius δ_k will enclose the whole connected components and, by the definition of δ_k, will still be in the complement of S_φ.

Therefore, we can set all the centres equal to one of them (it is not relevant which one) and take δ_k as a radius. The functions $c_{j,k}$ will still be measurable. The discs will be then either disjoint or coinciding, and their radii will be bounded from below by $\delta_k/3N_k$; we set $\delta = \min\{\delta_1/3N_1, \dots, \delta_n/3N_n\}$.

As we already said, $\varphi *_k (\pi z_k)^{-1} = G_k(\varphi)$ is holomorphic for $z_k \notin \mathbb{D}$ and for $z_k \in D(c_{j,k}, r_{j,k})$.

This will be precised in the next section with the help of the following definitions.

Let $\varphi \in L^r_c(\mathbb{D}^n)$, we define

$$[\varphi]_k(l) = \frac{1}{\pi} \int_{\mathbb{C}} \varphi(\cdots, z_{k-1}, \zeta_k, z_{k+1}, \cdots)\zeta_k^l \mathrm{d}m_1(\zeta_k);$$

let $\varphi \in L^r_c(\mathbb{D}^n \backslash Z)$, we define

$$[\varphi, j]_k(l) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\varphi(\cdots, z_{k-1}, \zeta_k, z_{k+1}, \cdots)}{(\zeta_k - c_{j,k})^{l+1}} \mathrm{d}m_1(\zeta_k).$$

We have the following lemma linking this with $\overline{\partial}$ equation.

Lemma 3.2 *Let $\varphi \in L^r_c(\mathbb{D}^n \backslash Z)$, then the following are equivalent:*

(i) $[\varphi]_k(l) = [\varphi, j]_k(l) = 0$ *for every $l \in \mathbb{N}$ and $1 \le j \le N_k$*
(ii) $G_k(\varphi) \in L^r_c(\mathbb{D}^n \backslash Z)$ $(\Rightarrow \overline{\partial}_k G_k(\varphi) = \varphi)$.

Proof Without loss of generality, we can set $k = 1$; we notice that, by Lemma 3.1, $G_1(\varphi)$ is in $L^r(\mathbb{C}^n)$, so (ii) is equivalent to the compactness of its support. Moreover, we remark that $G_1(\varphi)$ has compact support in $\mathbb{D}^n \setminus Z$ if and only if for almost every $a = (a_2, \dots, a_n) \in \mathbb{C}^{n-1}$ the function $z \mapsto G_1(\varphi)(z, a_2, \dots, a_n)$ has compact support in

$$(\mathbb{D}^n \setminus Z) \cap \{z_2 = a_2, \dots, z_n = a_n\} = \mathbb{D} \setminus \{c_{1,1}(a), \dots, c_{1,N_1}(a)\}.$$

On the other hand, $[\varphi]_1(l)$ and $[\varphi, j]_1(l)$ vanish if and only if the integrals that define them vanish for almost every $z_2, \dots, z_n$. So, we are reduced to the 1 variable case: let then $c_1, \dots, c_N$ be points in $\mathbb{D} \subset \mathbb{C}$ and $\phi \in L^r_c(\mathbb{D} \setminus \{c_1, \dots, c_N\})$; we set $G(z) = G_1(\varphi)(z)$.

If (ii) holds, for any $h \in \mathcal{O}(\mathbb{D} \setminus \{c_1, \dots, c_N\})$ we have

$$\int_{\mathbb{C}} \varphi(z)h(z)\mathrm{d}m_1(z) = \int_{\mathbb{C}} \frac{\partial G(z)}{\partial \bar{z}} h(z)\mathrm{d}m_1(z) = -\int_{\mathbb{C}} G(z)\frac{\partial h(z)}{\partial \bar{z}}\mathrm{d}m_1(z) = 0$$

where we have used Stokes' theorem, as $G(z)$ has compact support. The last integral vanishes because h is holomorphic.

On the other hand, suppose that (i) holds and let $K = \mathrm{supp}\varphi$. Consider $r < 1$ such that $K \Subset \mathbb{D}_r = \{|z| < r\}$ and take z with $|z| > r$; then

$$\begin{aligned} G(z) &= -\frac{1}{z\pi} \int_K \varphi(\zeta) \frac{1}{1 - \frac{\zeta}{z}} \mathrm{d}m_1(z) = -\frac{1}{\pi z} \int_K \varphi(\zeta) \sum_{l \ge 0} \frac{\zeta^l}{z^l} \mathrm{d}m_1(\zeta) \\ &= -\frac{1}{\pi z} \sum_{l \ge 0} z^{-l} \int_K \varphi(\zeta)\zeta^l \mathrm{d}m_1(\zeta) = -\frac{1}{\pi} \sum_{l \ge 0} z^{-l-1} [\varphi]_1(l). \end{aligned}$$

So, $G(z) = 0$ if $|z| > r$, therefore $\mathrm{supp}G(z) \Subset \mathbb{D}$.

Moreover, fix j, $1 \leq j \leq N$; there exists $r_j > 0$ such that the closure of $D(c_j, r_j) = \{|z - c_j| < r_j\}$ does not meet $\operatorname{supp}\varphi(z)$. So, if $|z - c_j| < r_j$, we have

$$G(z) = \frac{1}{\pi}\int_K \varphi(\zeta)\frac{1}{(\zeta - c_j) - (z - c_j)}\mathrm{d}m_1(\zeta) = \frac{1}{\pi}\int_K \varphi(\zeta)\frac{1}{\zeta - c_j}\frac{1}{1 - (z - c_j)/(\zeta - c_j)}\mathrm{d}m_1(\zeta)$$

$$= \frac{1}{\pi}\int_K \varphi(\zeta)\frac{1}{\zeta - c_j}\sum_{l \geq 0}\frac{(z - c_j)^l}{(\zeta - c_j)^l}\mathrm{d}m_1(\zeta) = \frac{1}{\pi}\sum_{l \geq 0}(z - c_j)^l[\varphi, j]_1(l).$$

Therefore, by hypothesis, $G(z) = 0$ if $|z - c_j| < r_j$, so $\operatorname{supp}G(z) \Subset \mathbb{D} \setminus \{c_1, \ldots, c_N\}$. □

Moreover, we have the following relations between the Cauchy transform and the quantities defined above.

Lemma 3.3 *If g and h are L^r functions, compactly supported in $\mathbb{D}^n$, and $g \star_1 \frac{1}{z_1} = h \star_1 \frac{1}{z_1}$ for $z_1 \notin \mathbb{D}$, then $[g]_1(k) = [h]_1(k)$ for every k.*

Proof If $z_1 \notin \mathbb{D}$, we have

$$g \star_1 \frac{1}{z_1} = \int_{\mathbb{D}} g(\zeta_1, \hat{z}_1)\frac{1}{z_1 - \zeta_1}\mathrm{d}m_1(\zeta_1) = \frac{1}{z_1}\int_{\mathbb{D}} g(\zeta_1, \hat{z}_1)\frac{1}{1 - \frac{\zeta_1}{z_1}}\mathrm{d}m_1(\zeta_1)$$

$$= \frac{1}{z_1}\sum_{k \geq 0} z_1^{-k}\int_{\mathbb{D}} g(\zeta_1, \hat{z}_1)\zeta_1^k \mathrm{d}m_1(\zeta_1) = \sum_{k \geq 0}[g]_1(k) z_1^{-k-1}.$$

A similar expansion holds for h, so that

$$h \star_1 \frac{1}{z_1} = \sum_{k \geq 0}[h]_1(k) z_1^{-k-1}.$$

Therefore, given that $(g - h) \star_1 \frac{1}{z_1} = 0$ for $z_1 \notin \mathbb{D}$, we have $[g]_1(k) = [h]_1(k)$ for every k. □

Lemma 3.4 *If g and h are L^r functions, compactly supported in $\mathbb{D}^n$, and there exists $j \geq 1$ such that $g \star_1 \frac{1}{z_1} = h \star_1 \frac{1}{z_1}$ for every $z_1 \in D(c_{j,1}(\hat{z}_1), r_{j,1}(\hat{z}_1))$, then $[g, j]_1(k) = [h, j]_1(k)$ for every k.*

We omit the proof as it can be easily obtained from the previous one.

Finally, we recall a result about the solution with compact support of the equation $\overline{\partial} f = \omega$ when ω is a $(0, 1)$-form with compact support.

Proposition 3.5 *Let $\Omega \subseteq \mathbb{C}^n$, $n \geq 2$, be a bounded Stein domain and ω a $(0, 1)$-form with coefficients in $L^r_c(\Omega)$ such that $\overline{\partial}\omega = 0$. Then there exists a unique $f \in L^r_c(\Omega)$ such that $\overline{\partial} f = \omega$, with $\|f\|_r \leq C\|\omega\|_r$, where C depends only on Ω.*

Proof We notice that if f_1 and f_2 are two compactly supported (distributional) solutions, then the difference $f_1 - f_2$ is $\overline{\partial}$−closed, that is, a holomorphic function, but then $f_1 = f_2$. Moreover, by Serre [16], $H^{0,1}_c(\Omega) = 0$, so there exists at least one distributional solution to $\overline{\partial} T = \omega$, compactly supported in Ω; on the other hand, we know that there is $f \in L^r_c(\mathbb{C}^n)$, solving $\overline{\partial} f = \omega$, given, as described in [14, Theorem 4.1, p. 71], by convolution with the Bochner–Martinelli–Koppelman kernel (the desired estimate of the L^r norm follows by the usual inequalities on convolution).

Therefore, we have $T = f$ and the desired estimate follows. □

4 The coronas construction

Let φ be a function in $L^r_c(\mathbb{D}^n\backslash Z)$ and consider the Cauchy transform $G_1(\varphi)(z)$; for almost every $\hat{z}_1$, $G_1(z)$ is holomorphic in z_1 in the complement of $S(\hat{z}_1)$.

Because $\pi_1(\text{supp}\varphi)$ is compact in $\mathbb{D}$, there exists $D(0,r)$ containing $S(\hat{z}_1)$; let $\delta = (1-r)/3$ and define the corona

$$C_0 = \{z_1 \in \mathbb{D} \ : \ r+\delta <| z_1 |< r+2\delta\} \Subset \mathbb{D}$$

and let $A_0 = m_1(C_0)$.

In the same way, set $\delta_j(\hat{z}_1) = r_{j,1}(\hat{z}_1)/3$ and define

$$C_j(\hat{z}_1) = \{z_1 \in \mathbb{D} \ : \ \delta_j(\hat{z}_1) \leq | z_1 - c_{j,1} | \leq 2\delta_j(\hat{z}_1)\} \Subset \mathbb{D}$$

and set $A_j(\hat{z}_1) = 1/m_1(C_j(\hat{z}_1))$.

Definition 4.1 The outer corona component of φ is the function

$$K_0^{(1)}(\varphi)(z) = A_0 \mathbb{1}_{C_0}(z_1) z_1 G_1(\varphi)(z)$$

and the inner coronas components of φ are the functions

$$K_j^{(1)}(\varphi)(z) = A_j(\hat{z}_1)\mathbb{1}_{C_j(\hat{z}_1)}(z_1)(z_1 - c_{j,1}) G_1(\varphi)(z).$$

Remark 4.2 The outer and inner coronas components of φ are well defined for almost every $\hat{z}_1$, because $\varphi(\cdot, \hat{z}_1)$ is in $L^r(\mathbb{C})$ and has compact support for almost every $\hat{z}_1$. We define exactly the same way the quantities $K_j^{(k)}(\varphi)(z)$ with respect to the variables z_k.

Lemma 4.3 *The operators* $K_m^{(1)}$, $m = 0, \ldots, N_1$, *are linear and well defined from* $L^r_c(\mathbb{C}^n)$ *to* $L^r_c(\mathbb{C}^n)$.

Proof As noted before, $K_m^{(1)}(\varphi)$ is well defined almost everywhere, and it is obviously linear; moreover, it has compact support in $\mathbb{D}$ by definition. We know that, by Lemma 3.1, $\|G_1(\varphi)\|_{L^r(\mathbb{C}^n)} \leq M \,\|\varphi\|_{L^r(\mathbb{C}^n)}$; hence, we have

$$\| K_0^{(1)}(\varphi) \|_r \leq A_0 \left\| \mathbb{1}_{C_0} G_1(\varphi) \right\|_{L^r} \leq A_0 M \,\|\varphi\|_{L^r},$$

where $M := \left\| \frac{1}{\pi z_1} \right\|_{L^1(\mathbb{D})}$.

For $j \geq 1$, $A_j(\hat{z}_1) = 1/m_1(C_j(\hat{z}_1))$, but $m_1(C_j(\hat{z}_1) \geq \delta > 0$ uniformly in $\hat{z}_1 \in \mathbb{D}^{n-1}$ hence $A_j(\hat{z}_1) \leq \delta^{-1} < \infty$, uniformly in $\hat{z}_1 \in \mathbb{D}^{n-1}$. So we get

$$\| K_m^{(1)}(\varphi) \|_r \leq \|A_m(\cdot)\|_{L^\infty(\mathbb{D}^{n-1})} \times \left\| \mathbb{1}_{C_0} G_1(\varphi) \right\|_{L^r} \leq \delta^{-1} M \,\|\varphi\|_{L^r}.$$

So for fixed $\hat{z}_1 \in \mathbb{D}^{n-1}$ $K_m^{(1)}(\varphi)$ has compact support in z_1, and because it operates only in z_1 and φ has compact support in $\mathbb{C}^n$, then $K_m^{(1)}(\varphi)$ has compact support in $\mathbb{C}^n$. □

Remark 4.4 The operator $K_0^{(1)}$ is also bounded from L^r_c to L^r_c, therefore continuous. The operators $K_m^{(1)}$ for $m \geq 1$ are not.

The following results link the quantities $[\varphi]_1(k)$ and $[\varphi, c_{j,1}]_1(k)$ with the corresponding ones for $K_0^{(1)}(\varphi)$ and $K_j^{(1)}(\varphi)$.

Lemma 4.5 *We have*

$$K_0^{(1)}(\varphi)(z) = A_0 \mathbb{1}_{C_0}(z_1) \sum_{k\geq 0} [\varphi]_1(k) z_1^{-k},$$
$$K_j^{(1)}(\varphi)(z) = A_j \mathbb{1}_{C_j(\hat{z}_1)}(z_1) \sum_{k\geq 0} [\varphi, j]_1(k)(z_1 - c_{j,1}(\hat{z}_1))^{k+1},$$

the convergence of the series being uniform in $\overline{C_0}$ or $\overline{C_j}$.

Proof If $|z_1| > r + \delta$ and $(\zeta_1, \hat{z}_1) \in \operatorname{supp}\varphi$, then

$$\frac{|\zeta_1|}{|z_1|} \leq \frac{r}{r+\delta} < 1.$$

so, in particular, if $z_1 \in C_0$, then $|z_1| > |\zeta_1|$. Therefore, if $z_1 \in C_0$, we have

$$G_1(z) = \frac{1}{\pi} \int_{\mathbb{C}} \varphi(\zeta_1, \hat{z}_1) \frac{1}{z_1 - \zeta_1} \mathrm{d}m_1(\zeta_1) = \frac{1}{\pi z_1} \int_{\mathbb{C}} \varphi(\zeta_1, \hat{z}_1) \frac{1}{1 - \frac{\zeta_1}{z_1}} \mathrm{d}m_1(\zeta_1)$$
$$= \frac{1}{\pi z_1} \int_{\mathbb{C}} \varphi(\zeta_1, \hat{z}_1) \sum_{k\geq 0} \frac{\zeta_1^k}{z_1^k} \mathrm{d}m_1(\zeta_1) = \frac{1}{z_1} \sum [\varphi]_1(k) z_1^{-k} = \sum_{k\geq 0} [\varphi_1](k) z_1^{-k-1}$$

So $K_0^{(1)}(\varphi) = A_0 \mathbb{1}_{C_0}(z_1) \sum_{k\geq 0} [\varphi_1](k) z_1^{-k}$, and the convergence is obviously uniform on $\overline{C_0}$.

On the other hand, if $z_1 \in C_j(\hat{z}_1)$ and $(\zeta_1, \hat{z}_1) \in \operatorname{supp}\varphi$, then

$$\frac{|z_1 - c_{j,1}(\hat{z}_1)|}{|\zeta_1 - c_{j,1}(\hat{z}_1)|} \leq \frac{2}{3} < 1,$$

so we have that, for $z_1 \in C_j(\hat{z}_1)$,

$$G_1(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\varphi}{z_1 - \zeta_1} \mathrm{d}m_1(\zeta) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\varphi}{(z_1 - c_{j,1}(\hat{z}_1)) + (c_{j,1}(\hat{z}_1) - \zeta_1)} \mathrm{d}m_1(\zeta_1)$$
$$= \frac{1}{\pi} \int_{\mathbb{C}} \frac{\varphi}{\zeta_1 - c_{j,1}(\hat{z}_1)} \frac{1}{1 - \frac{z_1 - c_{j,1}(\hat{z}_1)}{\zeta_1 - c_{j,1}(\hat{z}_1)}} \mathrm{d}m_1(\zeta_1)$$
$$= \frac{1}{\pi} \int_{C} \frac{\varphi}{\zeta_1 - c_{j,1}(\hat{z}_1)} \sum_{k\geq 0} \frac{(z_1 - c_{j,1}(\hat{z}_1))^k}{(\zeta_1 - c_{j,1}(\hat{z}_1))^k} \mathrm{d}m_1(\zeta_1) = \sum_{k\geq 0} [\varphi, j]_1(k)(z_1 - c_{j,1}(\hat{z}_1))^k.$$

So $K_j^{(1)}(\varphi) = A_j(\hat{z}_1) \mathbb{1}_{C_j(\hat{z}_1)}(z_1) \sum_{k\geq 0} [\varphi, j]_1(k)(z_1 - c_{j,1}(\hat{z}_1))^{k+1}$, and the convergence is obviously uniform on $\overline{C_j}$. □

We set

$$K^{(1)}(\varphi) = \sum_{m=0}^{N_1} K_m^{(1)}(\varphi).$$

Proposition 4.6 *We have* $[K^{(1)}(\varphi)]_1 = [\varphi]_1$ *and* $[K^{(1)}(\varphi), j]_1 = [\varphi, j]_1$.

Proof We divide the proof into several steps.

1. $[K_0^{(1)}(\varphi)]_1(k) = [\varphi]_1(k)$—we calculate

$$H(z) = K_0^{(1)}(\varphi) \star_1 \frac{1}{z_1} = \left(A_0 \mathbb{1}_{C_0}(z_1) \sum_{k \geq 0} [\varphi]_1(k) z_1^{-k} \right) \star_1 \frac{1}{z_1}$$
$$= A_0 \sum_{k \geq 0} [\varphi]_1(k) \left(\mathbb{1}_{C_0}(z_1) z_1^{-k} \star_1 \frac{1}{z_1} \right) = A_0 \sum_{k \geq 0} [\varphi]_1(k) \int_{C_0} \frac{\zeta_1^{-k}}{z_1 - \zeta_1} \mathrm{d}m_1(\zeta_1).$$

If $z_1 \notin \mathbb{D}$, we know that

$$\int_{C_0} \frac{\zeta_1^{-k}}{z_1 - \zeta_1} \mathrm{d}m_1(\zeta_1) = A_0^{-1} z_1^{-k-1}$$

so

$$H(z) = A_0 \sum_{k \geq 0} [\varphi]_1(k) (A_0^{-1} z_1^{-k-1}) = \sum_{k \geq 0} [\varphi]_1(k) z_1^{-k-1}.$$

Then we have that, if $z_1 \notin \mathbb{D}$,

$$H(z) = G_1(z)$$

so, by Lemma 3.3, $[\varphi]_1(k) = [K_0^{(1)}(\varphi)]_1(k)$.

2. $[K_j^{(1)}(\varphi)]_1(k) = 0$ *for* $j > 0$—We calculate

$$H_j(z) = K_j^{(1)}(\varphi) \star_1 \frac{1}{z_1} = \left(A_j(\hat{z}_1) \mathbb{1}_{C_j(\hat{z}_1)}(z_1) \sum_{k \geq 0} [\varphi, j]_1(k) (z_1 - c_{j,1}(\hat{z}_1))^{k+1} \right) \star \frac{1}{z_1}$$
$$= A_j(\hat{z}_1) \sum_{k \geq 0} [\varphi, j]_1(k) \left(\mathbb{1}_{C_j(\hat{z}_1)}(z_1 - c_{j,1}(\hat{z}_1))^{k+1} \star_1 \frac{1}{z_1} \right)$$
$$= A_j(\hat{z}_1) \sum_{k \geq 0} [\varphi, j]_1(k) \int_{C_j} \frac{(\zeta_1 - c_{j,1}(\hat{z}_1))^{k+1}}{z_1 - \zeta_1} \mathrm{d}m_1(\zeta_1).$$

If $|z_1 - c_{j,1}(\hat{z}_1)| > r_{j,1}(\hat{z}_1)$, then

$$\int_{C_j} \frac{(\zeta_1 - c_{j,1}(\hat{z}_1))^{k+1}}{z_1 - \zeta_1} \mathrm{d}m_1(\zeta_1) = 0$$

for every $k \geq 0$. Therefore, $H_j(z) = 0$, so by Lemma 3.3 $0 = [K_j^{(1)}(\varphi)]_1(k)$.

3. $[K_j^{(1)}(\varphi), j]_1(k) = [\varphi, j]_1(k)$ *for* $j > 0$—by direct computation, using Lemma 4.5, we have

$$[K_j^{(1)}\varphi, j]_1(l) = A_j(\hat{z}_1) \sum_{k \geq 0} [\varphi, j]_1(k) \int_{C_j(\hat{z}_1)} (\zeta_1 - c_{j,1}(\hat{z}_1))^{k+1} (\zeta_1 - c_{j,1}(\hat{z}_1))^{-l-1} \mathrm{d}m_1(\zeta_1)$$
$$= \sum_{k \geq 0} [\varphi, j]_1(k) \delta_{k,l} = [\varphi, j]_1(l).$$

4. $[K_m^{(1)}\varphi, j]_1(k) = 0$ *if* $m \neq j$—by step 2, $H_m(z) = 0$ if $|z_1 - c_{m,1}(\hat{z}_1)| > r_{m,1}(\hat{z}_1)$, so in particular if $z_1 \in D(c_{j,1}(\hat{z}_1), r_{j,1}(\hat{z}_1))$ with $j \neq m$, we have $H(z) = 0$. By Lemma 3.4, it follows that

$$[K_m^{(1)}(\varphi), j]_1(k) = 0$$

if $m \neq j$ and $m \neq 0$.
If $m = 0$, we notice that, if $|z_1| < r$,

$$H(z) = A_0 \sum_{k \geq 0} [\varphi]_1(k) \int_{C_0} \frac{\zeta_1^{-k}}{z_1 - \zeta_1} \mathrm{d}m_1(\zeta_1)$$

and

$$\int_{C_0} \frac{\zeta_1^{-k}}{z_1 - \zeta_1} \mathrm{d}m_1(\zeta_1) = 0$$

for every k, as $|z_1| < r < |\zeta_1|$. So $H(z) = 0$ and by Lemma 3.3 we have that $[K_0^{(1)}(\varphi), j]_1(k) = 0$ for every k.

□

Corollary 4.7 *Let* $\varphi \in L_c^r(\mathbb{D}^n \backslash Z)$, *there are* $\varphi_1, \ldots, \varphi_n$, *all in* $L_c^r(\mathbb{D}^n \backslash Z)$ *and such that*

$$\varphi = \varphi_1 + \cdots + \varphi_n, \ \forall i < n, \ \forall j = 1, \ldots, N_i, \ [\varphi_i]_i = [\varphi_i, j]_i = 0.$$

Proof We set $\varphi_1 := \varphi - K^{(1)}\varphi$, and we notice that $[\varphi_1]_1 = 0$, $[\varphi_1, j]_1 = 0$ for every $j = 1, \ldots, N_1$.

Now, we can repeat this procedure replacing z_1 by z_2 and φ by $K^{(1)}(\varphi)$; we will apply then the operators $K_m^{(2)}$, defined with respect to the variable z_2, with the relative coronas.

We set $\varphi_2 := K^{(1)}\varphi - K^{(2)}K^{(1)}\varphi$ with the property that $[\varphi_2]_2 = 0$, $[\varphi_2, j]_2 = 0$ for every $j = 1, \cdots, N_2$.

Iterating the algorithm we set $\varphi_{n-1} := K^{(n-2)} \cdots K^{(1)}\varphi - K^{(n-1)} \cdots K^{(1)}\varphi$ and

$$\varphi_n := \varphi - \varphi_1 - \cdots - \varphi_{n-1}.$$

By an easy recursion we have

$$\varphi_n = K^{(n-1)} \cdots K^{(1)}\varphi$$

with, of course $\varphi = \varphi_1 + \cdots + \varphi_n$.

So finally, we find a decomposition $\varphi = \varphi_1 + \cdots + \varphi_n$ such that for $i < n$, we have $[\varphi_i]_i = 0$, $[\varphi_i, j]_i = 0$ for every $j = 1, \ldots, N_i$. □

5 Obstructions to a solution with compact support

Let us define the two quantities which tell us when the last term in the decomposition from Corollary 4.7 verifies also

$$\forall j = 1, \ldots, \ N_n, \ [\varphi_n]_n = 0, \ [\varphi_n, j]_n = 0.$$

We note that

$$\varphi_n = K^{(n-1)} \cdots K^{(1)}\varphi$$

and, more precisely, we have

$$\varphi_n = \sum_{m_{n-1}=0}^{N_{n-1}} \cdots \sum_{m_1=0}^{N_1} K^{(n-1)}_{m_{n-1}} \cdots K^{(1)}_{m_1}(\varphi).$$

We set

$$M_{n-1} := \{(m_1, \ldots, m_{n-1}) :: m_j \le N_j\} \subset \mathbb{N}^{n-1};$$
$$\mu = (m_1, \ldots, m_{n-1}) \in M_{n-1},\ I(\mu) := \{k \le n-1 :: m_k = 0\},\ l = (l_1, \ldots, l_{n-1}) \in \mathbb{N}^{n-1}$$

and

$$J^{(0)}_{\mu,l}(\varphi)(k) := \frac{1}{\pi^n} \int_{\mathbb{C}^n} \varphi(\zeta)\zeta_n^k \prod_{i \in I(\mu)} \zeta_i^{l_i} \prod_{j \notin I(\mu)} \mathbb{1}_{C^{(j)}_{m_j}(z,\zeta)}(z_j) \frac{(z_j - c_{m_j,j}(z,\zeta))^{l_j+1}}{(\zeta_j - c_{m_j,j}(z,\zeta))^{-l_j-1}}\, \mathrm{d}m_n(\zeta)$$

$$J^{(j)}_{\mu,l}(\varphi)(k) := \frac{1}{\pi^n} \int_{\mathbb{C}^n} \varphi(\zeta)(\zeta_n - c_{j,n})^{-k-1} \prod_{i \in I(\mu)} \zeta_i^{l_i} \prod_{s \notin I(\mu)} \mathbb{1}_{C^{(s)}_{m_s}(z,\zeta)}(z_s) \times \frac{(z_s - c_{m_s,s}(z,\zeta))^{l_s+1}}{(\zeta_s - c_{m_s,s}(z,\zeta))^{-l_s-1}}\, \mathrm{d}m_n(\zeta);$$

where

$$c_{h,k}(z,\zeta) = c_{h,k}(z_1, \ldots, z_{k-1}, \zeta_{k+1}, \ldots, \zeta_n) \qquad 1 < k < n$$
$$c_{h,1}(z,\zeta) = c_{h,1}(\zeta_2, \ldots, \zeta_n)$$
$$c_{h,n}(z,\zeta) = c_{h,n}(z_1, \ldots, z_{n-1}).$$

and the same notation is used for $\mathbb{1}_{C^{(j)}_k(z,\zeta)}(z_j)$. We have the link:

Theorem 5.1 *Consider $\varphi \in L^r_c(\mathbb{D}^n \backslash Z)$. If $J^{(0)}_{\mu,l}(\varphi) = 0$ for every $\mu \in M_{n-1}$ and $l \in \mathbb{N}^n$, then $[\varphi_n]_n = 0$; given also $j = 1, \ldots, N_n$, if $J^{(j)}_{\mu,l}(\varphi) = 0$ for every $\mu \in M_{n-1}$ and $l \in \mathbb{N}^n$, then $[\varphi_n, j]_n = 0$.*

Proof By direct calculation, using the series expansions given by Lemma 4.5, we have that

$$[K^{(h)}_0(\varphi)]_{h+1}(k) = \frac{1}{\pi} A^{(h)}_0 \mathbb{1}_{C^{(h)}_0}(z_h) \sum_{l \ge 0} z_h^{-l} \int_{\mathbb{C}} [\varphi]_h(l) \zeta_{h+1}^k \mathrm{d}m_1(\zeta_{h+1})$$

$$[K^{(h)}_0(\varphi), m]_{h+1}(k) = \frac{1}{\pi} A^{(h)}_0 \mathbb{1}_{C^{(h)}_0}(z_h) \sum_{l \ge 0} z_h^{-l} \int_{\mathbb{C}} \frac{[\varphi]_h(l)}{(\zeta_{h+1} - c_{m,h+1})^{k+1}} \mathrm{d}m_1(\zeta_{h+1})$$

$$[K^{(h)}_j(\varphi)]_{h+1}(k) = \frac{1}{\pi} A^{(h)}_j \mathbb{1}_{C^{(h)}_j}(z_h) \sum_{l \ge 0} (z_h - c_{j,h})^{l+1} \int_{\mathbb{C}} [\varphi, j]_h(l) \zeta_{h+1}^k \mathrm{d}m_1(\zeta_{h+1})$$

$$[K^{(h)}_j(\varphi), m]_{h+1}(k) = \frac{1}{\pi} A^{(h)}_j \mathbb{1}_{C^{(h)}_j}(z_h) \sum_{l \ge 0} (z_h - c_{j,h})^{l+1} \int_{\mathbb{C}} \frac{[\varphi, j]_h(l)}{(\zeta_{h+1} - c_{m,h+1})^{k+1}} \mathrm{d}m_1(\zeta_{h+1}).$$

Therefore, by induction, we obtain that

$$[K^{(n-1)}_{\mu_{n-1}} \cdots K^{(1)}_{\mu_1} \varphi]_n(l_n) = \prod_{i=1}^{n-1} A^{(i)}_{\mu_i} \prod_{i \in I(\mu)} \mathbb{1}_{C^{(i)}_{\mu_i}}(z_i) \sum_{l' \in \mathbb{N}^{n-1}} \prod_{i \in I(\mu)} z_i^{-l_i} J^{(0)}_{\mu, l' \cup \{l_n\}}(\varphi)$$

and

$$[K^{(n-1)}_{\mu_{n-1}}\cdots K^{(1)}_{\mu_1}\varphi, j]_n(l_n)$$
$$=\prod_{i=1}^{n-1} A^{(i)}_{\mu_i}\prod_{i\in I(\mu)}\mathbb{1}_{C^{(i)}_{\mu_i}}(z_i)\sum_{l'\in\mathbb{N}^{n-1}}\prod_{i\in I(\mu)} z_i^{-l_i} J^{(j)}_{\mu,l'\cup\{l_n\}}(\varphi).$$

So, if $J^{(0)}_{\mu,l}(\varphi)=J^{(j)}_{\mu,l}(\varphi)=0$, all the coefficients vanish, then

$$[\varphi_n]_n(k)=0 \qquad [\varphi_n, j]_n(k)=0$$

as we wanted. □

Definition 5.2 We shall say that $\varphi\in L^r_c(\mathbb{D}^n\backslash Z)$ verifies the structure conditions if $J^{(0)}_{\mu,l}(\varphi)=0$ for every $\mu\in M_{n-1}$ and $l\in\mathbb{N}^n$, and if $J^{(j)}_{\mu,l}(\varphi)=0$ for every $\mu\in M_{n-1}$ and $l\in\mathbb{N}^n$.

6 $(0, n)$-forms on the polydisc

As for now, we do not have a way to deal with the integrals $J^{(m)}_{\mu,l}(k)$ on the domain $\mathbb{D}^n\setminus Z$, so we turn to the much easier case of the polydisc itself. We look first at the problem for $(0, n)$-forms.

Let ω be a $(0, n)$-form with $L^r_c(\mathbb{D}^n)$ coefficients; we can find a function $\varphi\in L^r_c(\mathbb{D}^n)$ such that

$$\omega=\varphi \mathrm{d}\bar{z}_1\wedge\cdots\wedge \mathrm{d}\bar{z}_n.$$

In this case, the operators $K^{(m)}$ coincide with the outer corona components $K^{(m)}_0$, so the obstructions to a solution of compact support are given by the integrals $J^{(0)}_{0,l}(k)$, where the subscript 0 stands for a multi-index of the appropriate length containing only 0s. We have the following result.

Lemma 6.1 *If there is a current T with compact support in $\mathbb{D}^n$ such that $\bar\partial T=\omega$, then we have*

$$\forall l\in\mathbb{N}^{n-1},\ \forall j=1,\ldots,N_n,\ J^{(0)}_{0,l}(\varphi)(j)=0,$$

that is, φ verifies the structure conditions for the polydisc.

Proof Let $\{\rho_\epsilon\}\subset\mathcal{D}(\mathbb{C}^n)$ be a family of functions such that $\rho_\epsilon\to\delta_0$, when $\epsilon\to 0$, in the sense of distributions, with $\operatorname{supp}\rho_\epsilon\subset\{|z|<\epsilon\}$ and $\|\rho_\epsilon\|_1=1$.

We write

$$T=T_1\mathrm{d}\hat{\bar{z}}_1+\cdots+T_n\mathrm{d}\hat{\bar{z}}_n$$

so we have

$$\varphi=\bar\partial_1 T_1+\cdots+\bar\partial_n T_n=t_1+\cdots+t_n$$

where, obviously, every t_h is compactly supported in $\mathbb{D}^n$.

We set $T^\epsilon_h=T_h\star\rho_\epsilon\in\mathcal{D}(\mathbb{C}^n)$; by standard theorems on convolution,

$$\operatorname{supp}(T^\epsilon_h)\subseteq\{z\mid \operatorname{dist}(z,\operatorname{supp}T_h)\le\epsilon\}$$

so, for ϵ small enough, all the regularized functions are compactly supported in $\mathbb{D}^n$ and

$$\overline{\partial}_h T_h^\epsilon = t_h \star \rho_\epsilon = t_h^\epsilon.$$

By Lemma 3.2, we have that

$$[t_h^\epsilon]_h(k) = 0$$

for every $k \in \mathbb{N}$ and $h = 1, \ldots, n$.

Moreover, we have that

$$\varphi^\epsilon = \varphi \star \rho_\epsilon = t_1^\epsilon + \ldots + t_n^\epsilon$$

and $\varphi^\epsilon \to \varphi$ in L^r as $\epsilon \to 0$.

As φ and φ^ϵ are compactly supported in $\mathbb{D}^n$, for ϵ small enough, we can see them as continuous functionals on $L^q_{\mathrm{loc}}(\mathbb{D}^n)$ (where $q^{-1} + r^{-1} = 1$). The convergence $\varphi^\epsilon \to \varphi$ holds also in this sense.

The functions $\zeta_n^k \prod_{i=1}^n \zeta_i^{l_i}$ are in $L^q_{\mathrm{loc}}(\mathbb{D}^n)$ for every $l \in \mathbb{N}^{n-1}$, $k \in \mathbb{N}$; therefore,

$$J_{0,l}^{(0)}(\varphi^\epsilon)(k) \xrightarrow[\epsilon \to 0]{} J_{0,l}^{(0)}(\varphi)(k).$$

Now, consider t_h^ϵ, with $h \leq n-1$; we know that $[t_h^\epsilon]_h(l) = 0$ for every l so we can apply Fubini and get

$$\begin{aligned} J_{0,l}^{(0)}(t_h^\epsilon)(k) &= \frac{1}{\pi^n} \int\limits_{\mathbb{C}^n} t_h^\epsilon(\zeta) \zeta_n^k \prod_{i=1}^n \zeta_i^{l_i} \mathrm{d}m_n(\zeta) \\ &= \frac{1}{\pi^n} \int\limits_{\mathbb{C}^{n-1}} \zeta_n^k \prod_{\substack{i=1 \\ i \neq h}}^n \zeta_i^{l_i} \int\limits_{\mathbb{C}} t_h^\epsilon(\zeta) \zeta_h^{l_h} \mathrm{d}m_1(\zeta_h) \mathrm{d}m_{n-1}(\hat{\zeta}_h) \\ &= \frac{1}{\pi^n} \int\limits_{\mathbb{C}^{n-1}} \zeta_n^k \prod_{\substack{i=1 \\ i \neq h}}^n \zeta_i^{l_i} [t_h^\epsilon]_h(l_h) \mathrm{d}m_{n-1}(\hat{\zeta}_h) = 0; \end{aligned}$$

If $h = n$, it is again an application of Fubini's theorem to show that $J_{0,l}^{(0)}(t_n^\epsilon)(k) = 0$.

By additivity of the integral, it follows that $J_{0,l}^{(0)}(\varphi^\epsilon)(k) = 0$, so letting $\epsilon \to 0$ we obtain the thesis. □

Theorem 6.2 *If ω is a $(0, n)$-form in $L^r_c(\mathbb{D}^n)$ such that there is a $(0, n-1)$ current T, compactly supported in $\mathbb{D}^n$, such that $\overline{\partial} T = \omega$, then we can find a $(0, n-1)$-form $\eta \in L^r_c(\mathbb{D}^n)$ such that $\overline{\partial}\eta = \omega$ and the operator associating η with ω is linear and bounded in the L^r norm.*

Proof Again, we can find a function $\varphi \in L^r_c(\mathbb{D}^n)$ such that

$$\omega = \varphi \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_n.$$

By Corollary 4.7 we can write $\varphi = \varphi_1 + \cdots + \varphi_n$, and by Lemma 3.2, the convolutions

$$f_1 = \varphi_1 \star_1 \frac{1}{\pi z_1}, \ \ldots, \ f_{n-1} = \varphi_{n-1} \star_{n-1} \frac{1}{\pi z_{n-1}}$$

are compactly supported and

$$\overline{\partial}_1 f_1 + \cdots + \overline{\partial}_{n-1} f_{n-1} = \varphi_1 + \cdots + \varphi_{n-1} = \varphi - \varphi_n.$$

Moreover, by Lemma 6.1, φ satisfies the structure conditions, and then, by Theorem 5.1, $[\phi_n]_n(k) = 0$ for every $k \in \mathbb{N}$. So, also

$$f_n = \varphi_n \star_n \frac{1}{\pi z_n}$$

is compactly supported, always by Lemma 3.2.

We set

$$\eta = \sum_{j=1}^{n} (-1)^{j-1} f_j \mathrm{d}\hat{\bar{z}}_j$$

so that

$$\overline{\partial}\eta = \varphi \mathrm{d}\bar{z}$$

and the coefficients of η belong to $L^r_c(\mathbb{D}^n)$.

The dependence of η on ω is clearly linear; moreover, we have that $\|f_j\|_r \leq \gamma \|\varphi_j\|_r$, where γ depends only on the dimension n and on the radii of $\mathbb{D}^n$. We recall that $\|K_0^{(m)}\varphi\|_r \leq A_0 M \|\varphi\|_r$, so $\|f_j\|_r \leq (A_0 M + 1)^j \gamma \|\varphi\|_r$. □

Remark 6.3 We note that

$$\eta = \sum_{j=1}^{n} (-1)^{j-1} f_j \mathrm{d}\hat{\bar{z}}_j = \sum \eta_j \mathrm{d}\hat{\bar{z}}_j$$

is such that $\overline{\partial}_j \eta_j \in L^r_c(\mathbb{D}^n)$ for every j and actually

$$\sum_j \|\overline{\partial}_j \eta_j\|_r \leq C \|\omega\|_r.$$

7 (0, $n-1$)-forms on the polydisc

We turn our attention to $(0, n-1)$-forms. Firstly, we give a refined version of Lemma 6.1.

Proposition 7.1 *Suppose $\varphi \in L^r(\mathbb{C}^n)$ and $T_1, \ldots, T_{n-1}$ are distributions, compactly supported in $\mathbb{D}^n$, such that*

$$\varphi = \overline{\partial}_1 T_1 + \cdots + \overline{\partial}_{n-1} T_{n-1}.$$

Then we can find $\varphi_1, \ldots, \varphi_{n-1} \in L^r(\mathbb{C}^n)$, compactly supported in $\mathbb{D}^n$ such that $\varphi = \varphi_1 + \cdots + \varphi_{n-1}$ and $[\varphi_i]_i(k) = 0$, for every $k \in \mathbb{N}$.

Proof After performing the same regularization as in the proof of Lemma 6.1, we have

$$\frac{1}{\pi^n} \int_{\mathbb{C}^n} t_h^\epsilon(\zeta) a(\zeta_n) \prod_{i=1}^{n-1} \zeta_i^{l_i} \mathrm{d}m_n(\zeta) = 0$$

for every $a(\zeta_n)$ for which the integral is well defined (e.g. $a \in L^1$). This is because h ranges from 1 to $n-1$, so we can isolate the terms $[t_h^\epsilon]_h(l)$ employing only the functions which appear in the product.

Therefore, the function

$$\frac{1}{\pi^n}\int\limits_{\mathbb{C}^{n-1}} t_h^{\epsilon}(\zeta)\prod_{i=1}^{n-1}\zeta_i^{l_i}\,\mathrm{d}m_n(\hat{\zeta}_n)$$

vanishes for almost every z_n, and the same is true for the function

$$\frac{1}{\pi^n}\int\limits_{\mathbb{C}^{n-1}} \varphi^{\epsilon}(\zeta)\prod_{i=1}^{n-1}\zeta_i^{l_i}\,\mathrm{d}m_n(\hat{\zeta}_n)$$

and, letting $\epsilon \to 0$, also for

$$\frac{1}{\pi^n}\int\limits_{\mathbb{C}^{n-1}} \varphi(\zeta)\prod_{i=1}^{n-1}\zeta_i^{l_i}\,\mathrm{d}m_n(\hat{\zeta}_n).$$

By the analogue of Theorem 5.1 in the first $n-1$ coordinates,

$$[K^{(n-2)}\cdots K^{(1)}\varphi]_{n-1}(k) = 0,$$

so defining $\varphi_1,\dots,\varphi_{n-2}$ as in Corollary 4.7 and setting $\varphi_{n-1} = \varphi - \varphi_1 - \cdots - \varphi_{n-2}$, we have that $[\varphi_i]_i(k) = 0$, as requested. □

The following corollary is immediate.

Corollary 7.2 *Let ω be a $(0,n)$-form with $L_c^r(\mathbb{D}^n)$ coefficients, and let T be a current, compactly supported in $\mathbb{D}^n$ such that $\overline{\partial}T = \omega$, with*

$$T = T_1\mathrm{d}\hat{\bar{z}}_1 + \cdots + T_{n-1}\mathrm{d}\hat{\bar{z}}_{n-1},$$

that is, $T = S \wedge \mathrm{d}\bar{z}_n$, for some $(0, n-2)$-current S. Then we can find η with $L^r(\mathbb{C}^n)$ coefficients, compactly supported in $\mathbb{D}^n$, such that $\overline{\partial}\eta = \omega$ and with

$$\eta = \eta_1\mathrm{d}\hat{\bar{z}}_1 + \cdots + \eta_{n-1}\mathrm{d}\hat{\bar{z}}_{n-1}.$$

Remark 7.3 Obviously, we can suppose that the coefficient of $\mathrm{d}\hat{\bar{z}}_k$ in T is zero and obtain that there exists a solution with coefficients in $L^r(\mathbb{C}^n)$ with compact support in $\mathbb{D}^n$ where the coefficient of $\mathrm{d}\hat{\bar{z}}_k$ is zero.

By induction, we can show that if there exists a solution with the coefficients of $\mathrm{d}\hat{\bar{z}}_{k_1},\dots,\mathrm{d}\hat{\bar{z}}_{k_r}$ equal to zero, then we can produce a solution in L^r with the same vanishing coefficients.

We note that the construction of $\varphi_1,\dots,\varphi_{n-1}$ does not involve the n-th coordinate, so $\overline{\partial}_n\varphi$ and $\overline{\partial}_n\varphi_j$ share the same regularity, whatever it is.

We denote by $\mathcal{W}^q(\mathbb{D}^n)$ the space of $(0,q)$-forms ω with $L_c^r(\mathbb{D}^n)$ coefficients such that

$$\omega = \sum_{|J|=n-q}\omega_J\mathrm{d}\hat{\bar{z}}_J$$

and

$$(*)\qquad \overline{\partial}_{j_{n-q}}\cdots\overline{\partial}_{j_k}\omega_J \in L^r(\mathbb{C}^n)\qquad k=1,\dots,n-q,\quad \forall\,|J| = n-q. \tag{7.1}$$

The space $\mathcal{W}^q(\mathbb{D}^n)$ can be made into a Banach space with the norm

$$\|\omega\|_W = \sum_J\|\omega_J\|_r + \sum_J\sum_{k=1}^{n-q}\|\overline{\partial}_{j_{n-q}}\cdots\overline{\partial}_{j_k}\omega_J\|_r.$$

Theorem 7.4 *If ω is a $(0, n-1)$-form in $L^r_c(\mathbb{D}^n)$, $\overline{\partial}\omega = 0$, such that $\omega \in \mathcal{W}^{n-1}(\mathbb{D}^n)$, then we can find a $(0, n-2)$-form $\beta \in \mathcal{W}^{n-2}(\mathbb{D}^n)$ such that $\overline{\partial}\beta = \omega$. The operator associating η with ω is linear and bounded from $\mathcal{W}^{n-1}(\mathbb{D}^n)$ to $\mathcal{W}^{n-2}(\mathbb{D}^n)$.*

Proof We proceed by induction on n; the case $n = 2$ is true. If there exists a distribution T with compact support such that $\overline{\partial}T = \omega$, then by Corollary 7.2, we have

$$\omega_n = \sum_{j=1}^{n-1} \omega_{nj}$$

with $\omega_{nj} \in L^r$ and $[\omega_{nj}]_j(k) = 0$.

We consider the following family of compactly supported $(0, n-2)$-forms in $\mathbb{C}^{n-1}$, depending on the parameter z_n:

$$\psi_{z_n} = \sum_{j=1}^{n-1} \left(\omega_j + (-1)^{n+j} \frac{\partial \omega_{nj}}{\partial \bar{z}_n} \star_j \frac{1}{\pi z_j} \right) \mathrm{d}\hat{\bar{z}}_j.$$

Note that as ψ_{z_n} is thought as a form in $\mathbb{C}^{n-1}$, the notation $\mathrm{d}\hat{\bar{z}}_j$ has to be understood as the exterior product of the differentials $\mathrm{d}\bar{z}_1, \dots, \mathrm{d}\bar{z}_{n-1}$, with $\mathrm{d}\bar{z}_j$ missing.

Now, we have that

$$(\overline{\partial}' \psi_{z_n}) \wedge \mathrm{d}\bar{z}_n = \overline{\partial}\omega = 0$$

where $\overline{\partial}'$ operates in the first $n-1$ coordinates. We note that

$$\frac{\partial}{\partial \bar{z}_j} \left(\omega_j + (-1)^{n+j} \frac{\partial \omega_{nj}}{\partial \bar{z}_n} \star_j \frac{1}{\pi z_j} \right) = \overline{\partial}_j \omega_j + (-1)^{n+j} \overline{\partial}_n \omega_{nj}$$

belongs to $L^r(\mathbb{C}^{n-1})$ for almost all z_n. By inductive hypothesis, we can solve $\overline{\partial}' \xi_{z_n} = \psi_{z_n}$ with compact support (and the result will be in $\mathcal{W}^{n-2}(\mathbb{D}^{n-1})$ for almost all z_n).

We have $\overline{\partial}(\xi_{z_n} \wedge \mathrm{d}\bar{z}_n) = \psi_{z_n} \wedge \mathrm{d}\bar{z}_n$; we define a $(0, n-2)$-form in $\mathbb{C}^n$ with

$$\gamma = \sum_{j=1}^{n-1} (-1)^{j-1} \omega_{nj} \star_j \frac{1}{\pi z_j} \mathrm{d}\hat{\bar{z}}_{jn}.$$

So we have

$$\overline{\partial}\gamma = \omega_n \mathrm{d}\hat{\bar{z}}_n + \sum_{j=1}^{n-1} (-1)^{n+j-3} \frac{\partial \omega_{nj}}{\partial \bar{z}_n} \star_j \frac{1}{\pi z_j} \mathrm{d}\hat{\bar{z}}_j;$$

therefore

$$\overline{\partial}(\gamma + \xi_{z_n} \wedge \mathrm{d}\bar{z}_n) = \omega.$$

The form $\gamma + \xi_{z_n} \wedge \mathrm{d}\bar{z}_n$ has compact support and belongs to $L^r(\mathbb{C}^n)$.

Moreover, by inductive hypothesis, for almost every z_n,

$$\|\xi_{z_n}\|^r_W \leq C \|\psi_{z_n}\|^r_W.$$

Integrating with respect to z_n we get

$$\|\xi_{z_n} \wedge \mathrm{d}\bar{z}_n\|_W \leq C \left(\int \|\psi_{z_n}\|^r_W \mathrm{d}m_1(z_n) \right)^{1/r} \leq (C + (n-1)\|(\pi z_1)^{-1}\|_1)\|\omega\|_W.$$

On the other side,

$$\begin{aligned}\|\gamma_{jn}\| &\leq C_1\|\omega\|_W\\ \|\overline{\partial}_j\gamma_{jn}\| &\leq C_2\|\omega_n\|_r\\ \|\overline{\partial}_n\gamma_{jn}\| &\leq C_3\|\overline{\partial}_n\omega_n\|_r\\ \|\overline{\partial}_j\overline{\partial}_n\gamma_{jn}\| &\leq C_4\|\overline{\partial}_n\omega_n\|_r\end{aligned}$$

so

$$\|\gamma\|_W \leq C'\|\omega\|_W.$$

Summing up we get

$$\|\gamma + \xi_{z_n} \wedge \mathrm{d}\bar{z}_n\|_W \leq \widetilde{C}\|\omega\|_W.$$

All the constants depend only on n and on the radii of the polydisc. □

Remark 7.5 Looking with some attention to the construction of the solution we performed in the proof of the previous theorem, we can notice that where ω depends on some parameters with a given regularity or summability, the constructed solution η would depend on the same parameter, with the same regularity or summability.

8 The general case on the polydisc

Let ω be a generic $(0, q)$-form, and let us write

$$\omega = \sum_{|J|=n-q} \omega_J \mathrm{d}\hat{\bar{z}}_J.$$

In the previous two sections, we treated the existence of compactly supported solutions with L^r estimates for the Cauchy–Riemann equation with datum ω when $q = n$ or $q = n-1$. Moreover, by Proposition 3.5, we know the answer also when $q = 1$. Therefore, we have a complete answer for $n = 1, 2, 3$. We now proceed to state and prove the general result, covering also the case $2 \leq q \leq n-2$.

Theorem 8.1 *If ω is a $(0, q)$-form, with $1 \leq q \leq n-1$, in $L^r_c(\mathbb{D}^n)$, $\overline{\partial}\omega = 0$, such that $\omega \in \mathcal{W}^q(\mathbb{D}^n)$, then we can find a $(0, q-1)$-form $\beta \in \mathcal{W}^{q-1}(\mathbb{D}^n)$ such that $\overline{\partial}\beta = \omega$. The operator associating β with ω is linear and bounded from $\mathcal{W}^q(\mathbb{D}^n)$ to $\mathcal{W}^{q-1}(\mathbb{D}^n)$.*

Proof Again, we note that we already know the result when $q = 1$ or $q = n-1$, so we will prove it for $2 \leq q \leq n-2$. Following Hörmander [7, Chapter 2], we can write

$$\omega = g \wedge \mathrm{d}\bar{z}_n + h$$

where g, h do not contain $\mathrm{d}\bar{z}_n$.

We can look at h as a family of $(0, q)$-forms in $\mathbb{C}^{n-1}$, depending on the complex parameter z_n; similarly, g can be understood as a family of $(0, q-1)$-forms.

We denote by $\overline{\partial}_{\mathbb{C}^{n-1}}$ the $\overline{\partial}$ operator in the first $n-1$ variables, that is,

$$\overline{\partial}_{\mathbb{C}^{n-1}}\psi = \sum_{n\notin I}\sum_{k\notin I\cup\{n\}} \overline{\partial}_k\psi_I \mathrm{d}\bar{z}_k \wedge \mathrm{d}\bar{z}_I.$$

If ψ does not contain $\mathrm{d}\bar{z}_n$, then $\overline{\partial}'\psi = \overline{\partial}_{\mathbb{C}^{n-1}}\psi$.

We proceed by induction on the dimension, and we prove the following:

$I_n.1$ The statement of the theorem holds in $\mathbb{C}^n$, and the estimates on the $\mathcal{W}^q$ norms depend only on the dimension and the radii of the polydisc;

$I_n.2$ if the coefficients of ω depend on a parameter $z_{n+1} \in \mathbb{D}$ in such a way that $\omega \in \mathcal{W}^q(\mathbb{D}^{n+1})$, then also $\beta \in \mathcal{W}^{q-1}(\mathbb{D}^{n+1})$ and $\|\beta\|_W \leq C\|\omega\|_W$, with C depending only on the radii of the polydisc and the dimension.

We note that the case $q = 1$ of $I_n.1$ holds by Proposition 3.5; also, the case $q = 1$ of $I_n.2$ holds as well, because the solution can be constructed by convolution with the Bochner–Martinelli–Koppelman kernel, so we can apply the usual results concerning the derivatives of a convolution and Young inequality to obtain the desired estimates. In particular, given that for $n = 2$ the only case is $q = 1$, we have proved that $I_2.1$ and $I_2.2$ hold true.

We also know from Theorem 7.4 that $I_n.1$ holds in the case $q = n - 1$, and in Remark 7.5 we note that $I_n.2$ holds in that case as well.

We assume $I_{n-1}.1$ and $I_{n-1}.2$ to hold for $1 \leq q \leq n-1$. As we just noted, we need only to show $I_n.1$ and $I_n.2$ for $2 \leq q \leq n-2$.

Reduction. We note that $\bar\partial_{\mathbb{C}^{n-1}} h = 0$; therefore, h is a family of $\bar\partial-$closed $(0, q)$-forms in $\mathbb{C}^{n-1}$ depending on the parameter z_n. Moreover, by assumption, $\bar\partial_n h_I \in L^r_c(\mathbb{C}^n)$. We denote by U_t the $(n-1)-$dimensional open set $\mathbb{D}^n \cap \{z_n = t\}$, and we note that U_t is still a polydisc, hence Stein, for every t for which it is non-empty.

As a well-known consequence of Serre's duality (see [16]), we have $H^q_c(U_t, \mathcal{O}) = 0$, if $2 \leq q \leq n-2$; therefore, we can find a family T of $(0, q-1)-$currents in $\mathbb{C}^{n-1}$ such that $\bar\partial_{\mathbb{C}^{n-1}} T = h$ for almost every z_n. Then, by $I_{n-1}.2$, we can find a family H with $H \in L^r_c(\mathbb{D}^n)$ (and therefore $H_{z_n} \in L^r_c(U_{z_n})$ for almost every z_n) and with $H \in \mathcal{W}^{q-1}(\mathbb{D}^n)$.

Moreover, as H_{z_n} depends linearly on h by $I_{n-1}.1$, if $h_{z_n} = 0$, then also $H_{z_n} = 0$. Therefore, H is compactly supported in $\mathbb{D}^n$.

Now,

$$\bar\partial H = \bar\partial_{\mathbb{C}^{n-1}} H + \sum_I \bar\partial_n H_I \mathrm{d}\bar z_n \wedge \mathrm{d}\bar z_I = h + \sum_I \bar\partial_n H_I \mathrm{d}\bar z_n \wedge \mathrm{d}\bar z_I$$

so

$$\omega - \bar\partial H = g' \wedge \mathrm{d}\bar z_n$$

where g' does not contain $\mathrm{d}\bar z_n$. Moreover, as ω and $\bar\partial H$ are in $L^r_c(\mathbb{D}^n)$, also g' is. Further, we observe that

$$(\bar\partial_{\mathbb{C}^{n-1}} g') \wedge \mathrm{d}\bar z_n = \bar\partial(\omega - \bar\partial H) = \bar\partial\omega = 0$$

and finally, for z_n fixed, g' is a $(0, q-1)$-form in $\mathbb{C}^{n-1}$, belonging to $\mathcal{W}^{q-1}(\mathbb{D}^{n-1})$.

Solution. We have reduced ourselves to solve $\bar\partial G = g' \wedge \mathrm{d}\bar z_n$, but as $\bar\partial_{\mathbb{C}^{n-1}} g' = 0$, we can, by the same argument used in the reduction, obtain a family G' of $(0, q-2)$ forms in $\mathbb{C}^{n-1}$ such that $\bar\partial_{\mathbb{C}^{n-1}} G' = g'$, by $I_{n-1}.2$.

Again, by the same reasoning, $G' \in L^r_c(\mathbb{D}^n)$, and if we set $G = G' \wedge \mathrm{d}\bar z_n$, we obtain a $(0, q-1)$-form $G \in \mathcal{W}^{q-1}(\mathbb{D}^n)$ such that $\bar\partial G = g' \wedge \mathrm{d}\bar z_n$.

So, $\beta = G + H$ is the solution we looked for. Moreover, the norms $\|H\|_W$ and $\|G\|_W$ are controlled, respectively, by $\|h\|_W$ and $\|\omega\|_W + \|\bar\partial H\|_W$, which is controlled by

$$\|\omega\|_W + \|H\|_W \leq \|\omega\|_W + \|h\|_W \leq 2\|\omega\|_W.$$

This shows $I_n.1$.

To show $I_n.2$ it is enough to notice that all our operations are constructive and preserve the regularity (or summability) of an extra parameter; the case $q = n - 1$ of $I_n.2$ was already noted in Remark 7.5. □

Remark 8.2 We have to separate the proof for $(0, n-1)$-forms from the general case because in the former Serre's duality tells us only that $H_c^{n-1}(U_t, \mathcal{O})$ is equal to the topological dual of $H^0(U_t, \Omega^{n-1})$ (we recall that U_t is an open set in $\mathbb{C}^{n-1}$), in general not vanishing, so the induction does not work there.

Remark 8.3 We note that, in the proof of Theorem 8.1, we never actually used the fact that our domain is the polydisc. Indeed, if we had the analogues of Theorems 6.2 and 7.4 for the domain $\mathbb{D}^n \setminus Z$ in every dimension, then we could apply the same proof to get Theorem 8.1 for $\mathbb{D}^n \setminus Z$, with exactly the same statement.

As a corollary of the previous results, we obtain the following.

Corollary 8.4 *Let ω be a $\overline{\partial}$-closed $(0, q)$-form with compact support in $\mathbb{D}^n \setminus Z$ and satisfying conditions 7.1; then, for any $k \in \mathbb{N}$, we can find a $(0, q-1)$-form $\beta \in L_c^r(\mathbb{D}^n)$ such that $\overline{\partial}(f^k\beta) = \omega$. Equivalently, we can find a $(0, q-1)$-form $\eta = f^k\beta$ such that $\eta \in L_c^r(\mathbb{D}^n)$, η is 0 on Z up to order k and $\overline{\partial}\eta = \omega$.*

Proof The $(0, q)$-form $\phi := \omega/f^k$ is still $\overline{\partial}-$closed and satisfies 7.1; hence, we have a $(0, q-1)$-form $\beta \in L_c^r(\mathbb{D}^n)$ such that $\overline{\partial}\beta = \phi$. So $\eta = f^k\beta$ verifies all the requirements. □

Remark 8.5 We note that the solution operators we constructed are defined from $\mathcal{W}^q(\mathbb{D}^n)$ to $\mathcal{W}^{q-1}(\mathbb{D}^n)$ which means that we have a partial gain of regularity: every coefficient of the datum is supposed to have derivatives in $n - q$ variables in L^r, while the solution we find has derivatives in L^r for $n - q + 1$ variables.

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