Annali di Matematica (2014) 193:961–973
DOI 10.1007/s10231-012-0309-3

Pencils on separating $(M-2)$-curves

Marc Coppens

Received: 12 June 2012 / Accepted: 22 November 2012 / Published online: 18 January 2013

Abstract A separating $(M-2)$-curve is a smooth geometrically irreducible real projective curve X such that $X(\mathbb{R})$ has $g-1$ connected components and $X(\mathbb{C}) \setminus X(\mathbb{R})$ is disconnected. Let T_g be a Teichmüller space of separating $(M-2)$-curves of genus g. We consider two partitions of T_g, one by means of a concept of special type, the other one by means of the separating gonality. We show that those two partitions are very closely related to each other. As an application, we obtain the existence of real curves having isolated real linear systems g^1_{g-1} for all $g \geq 4$.

Keywords Real curve · Linear pencil · Separating gonality · Special type · Teichmüller space

Mathematics Subject Classification (2000) 14H05 · 14H51 · 14P99

1 Introduction

Let X be a smooth real projective curve of genus g. We assume X is complete and geometrically irreducible; hence, the set $X(\mathbb{C})$ of complex points is in a natural way of a compact Riemann surface of genus g. Let $X(\mathbb{R})$ be the set of real points and assume it is not empty. Let $C_1, \cdots, C_s$ be the connected components of $X(\mathbb{R})$. It is well known that $s \leq g+1$ (Harnack's inequality). Let $f : X \to \mathbb{P}^1$ be a morphism of degree k. It is known that the parity of the fibers (counted with multiplicities) of $f|_{C_i} : C_i \to \mathbb{P}^1(\mathbb{R})$ is constant. In particular, in case this parity is odd then $f(C_i) = \mathbb{P}^1(\mathbb{R})$. In our paper [6], we considered the following problem.

M. Coppens
Departement IBW, Thomas Moore Kempen,
Kleinhoefstraat 4, 2440 Geel, Belgium

M. Coppens (✉)
Departement of Mathematics, K.U. Leuven,
Celestijnenlaan 200B, 3001 Leuven, Belgium
e-mail: marc.coppens@khk.be

Problem Fix $k, s' \leq s$ and s' components $C_{i_1}, \cdots, C_{i_{s'}}$ of $X(\mathbb{R})$. Does there exist a morphism $f : X \to \mathbb{P}^1$ of degree k such that f has odd parity on C_j for $j \in \{i_1, \cdots, i_{s'}\}$ and $f(C_j) \neq \mathbb{P}^1(\mathbb{R})$ for $j \notin \{i_1, \cdots, i_{s'}\}$.

Of course, $s - s' \equiv 0 \pmod 2$ is a necessary condition and in [6, Proposition 1] it is proved that in case $k = g + 1$ this condition is also sufficient. However, in case $k = g$ then this condition is not sufficient because of the following example mentioned in [6, Example 3]. A real curve X of genus 3 with $s = 2$ and such that $X(\mathbb{R})$ disconnects $X(\mathbb{C})$ is isomorphic to a smooth plane real curve of degree 4 having two nested ovals (C_1 in the inner part of C_2). Taking $k = 3$, $s' = 1$ and $i_1 = 1$, then for each morphism $f : X \to \mathbb{P}^1$ of degree 3 having odd parity on C_1 one has $f(C_2) = \mathbb{P}^1(\mathbb{R})$.

A real curve X such that $X(\mathbb{R})$ disconnects $X(\mathbb{C})$ is called separating, and it is shown in [6, Theorem 1.A] that the condition $s - s' \equiv 0 \pmod 2$ is sufficient for an affirmative answer to the problem in case $k = g$ and X is not separating. In [3, Example 5.9] as a second example, one finds separating curves of genus 4 with $s = 3$ such that there exist components C_1 and C_2 of $X(\mathbb{R})$ such that for each morphism $f : X \to \mathbb{P}^1$ of degree 4 having odd parity on C_1 and C_2 one has $f(C_3) = \mathbb{P}^1(\mathbb{R})$ (C_3 is the other component of $X(\mathbb{R})$ different from C_1 and C_2). The argument makes use of the description of a canonically embedded curve of genus 4 in $\mathbb{P}^3$ as the intersection of a cubic and a quadric surface. In both examples, we have $s = g - 1$. Classically, a real curve X satisfying $s = g + 1$ is called an M-curve, and in the literature, a real curve satisfying $s = g + 1 - a$ is also called an $(M - a)$-curve. So both examples are separating $(M - 2)$-curves. In Theorem 3.1, we prove that for all $g \geq 3$ there exists a separating $(M - 2)$-curve X having components $C_1, \cdots, C_{g-1}$ of $X(\mathbb{R})$ such that, if $f : X \to \mathbb{P}^1$ is a morphism of degree g having odd parity on $C_2, \cdots, C_{g-1}$ then $f(C_1) = \mathbb{P}^1(\mathbb{R})$ (in this statement, the numbering of the components of $X(\mathbb{R})$ is important). We say such a curve is of special type. Theorem 3.1 is a direct consequence of Proposition 3.2. In Proposition 3.2, we prove a more geometric statement related to this concept: the existence of a canonically embedded separating $(M - 2)$-curve X possessing a strong kind of linking between the connected components of $X(\mathbb{R})$.

We prove a stronger statement. Let T_g be the Teichmüller space parameterizing separating $(M - 2)$-curves of genus g. In case $t \in T_g$ then we write X_t to denote the corresponding real curve. This space T_g is a real connected manifold of dimension $3g - 3$. We say a property P holds for a general separating $(M - 2)$-curve if there exists a non-empty open subset U of T_g such that P holds for all curves X_t with $t \in U$ (roughly speaking: the curves satisfying property P have the maximal $3g - 3$ moduli). From Corollary 4.8, it follows that for $g \geq 4$ both properties "being of simple type" and "not being of simple type" do hold for a general separating $(M - 2)$-curve of genus g (in case $g = 3$ all separating $(M - 2)$-curves are of special type). Let $T_{g,s}$ (resp. $T_{g,ns}$) be the set of points $t \in T_g$ such that X_t is of special type (resp. X_t is not of special type). So we have a partition $T_g = T_{g,s} \cup T_{g,ns}$. In Lemma 2.6, we show $T_{g,s}$ is closed, hence $T_{g,ns}$ is open. This partition turns out to be closely related to another very natural parition of T_g.

In case a real curve X has a morphism $f : X \to \mathbb{P}^1$ with $X(\mathbb{R}) = f^{-1}(\mathbb{P}^1(\mathbb{R}))$ then X is separating. Such morphism is called a separating morphism. In [4], we introduce the separating gonality $\mathrm{sepgon}(X)$ of a separating real curve X: it is the minimal degree such that there exists a separating morphism $f : X \to \mathbb{P}^1$. For a separating $(M - 2)$-curve X trivially one has $\mathrm{sepgon}(X) \geq g - 1$. On the other hand, from [7] it follows $\mathrm{sepgon}(X) \leq g$ and in [4] it is proved that both possibilities $g - 1$ and g do occur. Let $T_{g,g}$ (resp. $T_{g,g-1}$) be the set of points $t \in T_g$ such that $\mathrm{sepgon}(X_t) = g$ (resp. $\mathrm{sepgon}(X_t) = g - 1$). So we obtain a second partition $T_g = T_{g,g} \cup T_{g,g-1}$ and the relation between both partitions is given by the fact that the closure $\overline{T_{g,ns}}$ of $T_{g,ns}$ is equal to $T_{g,g-1}$ (see Corollary 4.7). It follows that $T_{g,g} = T_{g,s} \setminus (T_{g,s} \cap \overline{T_{g,ns}})$ is a non-empty open subset of T_g.

The fibers of a separating morphism $f : X \to \mathbb{P}^1$ of degree $g-1$ correspond to a linear system g^1_{g-1} on X. Complete linear systems of degree $g-1$ and dimension at least one on X are parameterized by a subscheme W^1_{g-1} of the Jacobian $J(X)$ and in case X is not hyperelliptic then all components of $W^1_{g-1}(\mathbb{C})$ have dimension $g-4$. Linear systems g^1_{g-1} corresponding to separating morphisms of degree $g-1$ on a separating $(M-2)$-curve X are parameterized by a dense open subset of some irreducible components of $W^1_{g-1}(\mathbb{R})$. In case X is a general non-special separating $(M-2)$-curve then all such components have real dimension $g-4$. If X is a special separating $(M-2)$-curve with $\mathrm{sepgon}(X) = g-1$ then our results imply $X = X_t$ for some $t \in T_{g,s} \cap \overline{T_{g,ns}}$. In Corollary 4.5, we prove this intersection is non-empty and in Proposition 4.1, we prove such X has finitely many g^1_{g-1} associated with separated morphisms of degree $g-1$. In particular for such curve, $W^1_{g-1}(\mathbb{R})$ has isolated points (see Corollary 4.9). In case $g \geq 5$ this is remarkable when compared with $\dim(W^1_{g-1}(\mathbb{C})) = g-4$. The finiteness follows from the following remarkable fact proved in Proposition 4.1. If X is an $(M-2)$-curve of special type, then a linear system g^1_{g-1} on X corresponding to a separated morphism $f : X \to \mathbb{P}^1$ is half canonical.

2 Preliminaries and notations

A *real curve* X is a one-dimensional geometrically connected projective variety defined over the field $\mathbb{R}$ of the real numbers. Using a base extension $\mathbb{R} \subset \mathbb{C}$, we obtain a complex curve $X_{\mathbb{C}}$. Its set of closed points is denoted by $X(\mathbb{C})$ and it is called the space of complex points on X. Complex conjugation related to $\mathbb{R} \subset \mathbb{C}$ defines a complex conjugation on $X(\mathbb{C})$, for $P \in X(\mathbb{C})$ we write $\overline{P}$ to denote the complex conjugated point. On X itself (considered as a scheme) there are two types of closed points according to the residue field being $\mathbb{R}$ or $\mathbb{C}$. In case the residue field is $\mathbb{R}$ then we say it is a *real point* on X. The set of real points is denoted by $X(\mathbb{R})$, and there exists a natural inclusion $X(\mathbb{R}) \subset X(\mathbb{C})$. In case the residue field is $\mathbb{C}$, then the closed point on X corresponds to two conjugated points $P, \overline{P}$ on $X(\mathbb{C}) \setminus X(\mathbb{R})$. Such closed point on X is denoted by $P + \overline{P}$ and it is called a *non-real point* on X. The real projective line $Proj(\mathbb{R}[X_0, X_1])$ is denoted by $\mathbb{P}^1$. A linear system of dimension r and degree d on a smooth real curve X is denoted by g^r_d. It is a projective space of linearly equivalent real divisors on X.

In case $X_{\mathbb{C}}$ is a smooth (resp. stable) complex curve, we call X a smooth (resp. stable) real curve. The moduli functor of stable curves of genus g is not representable, hence there is no universal family. Instead, we make use of the so-called suited families of stable curves.

Definition 2.1 Let X be a real stable curve of genus g. A *suited family of stable curves of genus g for* X is a projective morphism $\pi : \mathcal{C} \to S$ defined over $\mathbb{R}$ such that

1. S is smooth, geometrically irreducible and quasi-projective.
2. Each geometric fiber of π is a stable curve of genus g.
3. For each $s \in S(\mathbb{C})$, the Kodaira–Spencer map $T_s(S) \to Ext^1(\Omega_{\pi^{-1}(s)}, \mathcal{O}_{\pi^{-1}(s)})$ is surjective (here $\Omega_{\pi^{-1}(s)}$ is the sheaf of Kähler differentials).
4. There exists $s_0 \in S(\mathbb{R})$ such that $\pi^{-1}(s_0) \cong X$ over $\mathbb{R}$.

In case X is smooth, we also assume π is a smooth morphism.

In [4, Lemma 4], it is explained such suited families do exist. Let X be a smooth real curve and let $\pi : \mathcal{C} \to S$ be a suited family for X. Let $k \in \mathbb{Z}$ with $k \geq 2$. There exists a quasi-projective morphism $\pi_k : H_k(\pi) \to S$ representing morphisms of degree k from fibers of π to $\mathbb{P}^1$ (see [10, Section 4.c]). Let $f : X \to \mathbb{P}^1$ be a morphism of degree k. It defines an invertible sheaf $L = f^*(\mathcal{O}_{\mathbb{P}^1}(1))$ of degree k on X. The morphism f induces an exact sequence $0 \to T_X \to f^*(T_{\mathbb{P}^1}) \to N_f \to 0$ (N_f is defined by this exact sequence) and since $T_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$ this exact sequence looks like

$$0 \to T_X \to L^{\otimes 2} \to N_f \to 0$$

The morphism f corresponds to a point $[f]$ on $H_k(\pi)$ and from Horikawa's deformation theory of holomorphic maps (see [11], see also [14, 3.4.2]), it follows $T_{[f]}(H_k(\pi))$ is canonically identified with $H^0(X, N_f)$ and since $H^1(X, N_f) = 0$ it follows $H_k(\pi)$ is smooth of dimension $2k + 2g - 2$. Moreover, $T_{s_0}(S)$ is isomorphic to $H^1(X, T_X)$ and the connecting homomorphism $H^0(X, N_f) \to H^1(X, T_X)$ associated with the exact sequence is identified with the tangent map $d_{[f]}(\pi_k) : T_{[f]}(H_k(\pi)) \to T_{s_0}(S)$. In particular, $d_{[f]}(\pi_k)$ is surjective in case $H^1(X, L^{\otimes 2}) = 0$. Hence the condition $H^1(X, L^{\otimes 2}) = 0$ implies $\pi_k^{-1}(s_0)$ has dimension $2k - g + 1$ and it is smooth at $[f]$. In [5], we introduced the topological degree of f. Choose an orientation on $\mathbb{P}^1(\mathbb{R})$. For each component C of $X(\mathbb{R})$ (this is a smooth real manifold diffeomorphic to S^1), we consider the restriction $f|_C : C \to \mathbb{P}^1(\mathbb{R})$ and we fix an orientation on C such that $\deg(f|_C) \geq 0$. We say f is *of topological degree* $(d_1, \cdots, d_s)$ with $d_1 \geq \cdots \geq d_s \geq 0$ if there is a numbering $C_1, \cdots, C_s$ of all components of $X(\mathbb{R})$ such that $\deg(f|_{C_i}) = d_i$. In families of morphisms from smooth real curves to $\mathbb{P}^1$ this topological degree is constant, hence it is constant on connected components of $H_k(\pi)(\mathbb{R})$.

Let X be a smooth real curve. In case $X(\mathbb{R}) \neq \emptyset$ then it is a disjoint union of $s = s(X)$ connected components diffeomorphic to a circle. In case $X(\mathbb{C}) \setminus X(\setminus\mathbb{R})$ is not connected, it has two connected components and X is called a *separating real curve*. For a separating real curve, one has $1 \leq s \leq g - 1$ and $s \equiv g + 1 \pmod 2$. In case $s = g + 1 - a$ then X is called an *$(M - a)$-curve*. The following definitions are already mentioned in the introduction.

Definition 2.2 A separating $(M - 2)$-curve X is of *special type* if there exists a component C of $X(\mathbb{R})$ such that for each morphism $f : X \to \mathbb{P}^1$ of degree g having odd parity on each connected component $C' \neq C$ of $X(\mathbb{R})$ one has $f(C) = \mathbb{P}^1(\mathbb{R})$. If no such component C exists then we say X is *not of special type*.

Definition 2.3 A morphism $f : X \to \mathbb{P}^1$ is called *a separating morphism* if $f^{-1}(\mathbb{P}^1(\mathbb{R})) = X(\mathbb{R})$.

In case X has a separating morphism then X is a separating real curve.

Definition 2.4 The *separating gonality* $\mathrm{sepgon}(X)$ of a separating real curve X is the minimal degree k such that there exists a separating morphism $f : X \to \mathbb{P}^1$ of degree k.

As already mentioned in the introduction, in case X is a separating $(M - 2)$-curve then $\mathrm{sepgon}(X)$ is either g or $g + 1$. As mentioned in the introduction, we write T_g to denote a Teichmüller space parameterizing separating $(M - 2)$-curves and we obtain two partitions $T_g = T_{g,s} \cup T_{g,ns}$ and $T_g = T_{g,g} \cup T_{g,g-1}$. Remember T_g is a smooth real manifold of dimension $3g - 3$, and it has a universal family $t_g : \mathcal{X}_g \to T_g$. For each separating real $(M - 2)$-curve X_0, there exists $t_0 \in T_g$ such that $t_g^{-1}(t_0) \cong X_0$. Moreover, if $\pi : \mathcal{C}_g \to S$ is a suited family of curves for X_0 and $s_0 \in S(\mathbb{R})$ with $\pi^{-1}(s_0) \cong X_0$, then there exist neighborhoods U (resp. V) of t_0 (resp. s_0) in T_g (resp. $S(\mathbb{R})$) and a diffeomorphism $U \to V$ such that, if $u \in U$ maps to $v \in V$ then $t_g^{-1}(u) \cong \pi^{-1}(v)$.

Lemma 2.5 *Let X be a separating $(M-2)$-curve, let $C_1, \cdots, C_{g-1}$ be the connected components of $X(\mathbb{R})$ and assume $f : X \to \mathbb{P}^1$ is a covering of degree g having odd parity on $C_1, \cdots, C_{g-2}$. Then $f(C_{g-1}) \neq \mathbb{P}^1(\mathbb{R})$ unless $f|_{C_{g-1}}$ is an unramified covering $C_{g-1} \to \mathbb{P}^1(\mathbb{R})$ of degree 2.*

Proof First of all, the morphism f has even parity on C_{g-1} (because of the necessary condition involving s and s' for the problem mentioned in the introduction). Since each fiber above a point x of $\mathbb{P}^1(\mathbb{R})$ contains a point of C_i for $1 \leq i \leq g-2$, it contains at most 2 points of C_{g-1} (counted with multiplicities), and there cannot be a ramification point on C_{g-1} of index more than two. If there is a ramification point x_0 on C_{g-1} of index two then close to $f(x_0)$, there exists $x' \in \mathbb{P}^1(\mathbb{R})$ such that $f^{-1}(x')$ contains a non-real point. It follows $f^{-1}(x')$ cannot contain a point of C_{g-1} hence $f(C_{g-1}) \neq \mathbb{P}^1(\mathbb{R})$. Hence, $f(C_{g-1}) = \mathbb{P}^1(\mathbb{R})$ implies f has no ramification point on C_{g-1}, hence $f|_{C_{g-1}}$ is an unramified covering $C_{g-1} \to \mathbb{P}^1(\mathbb{R})$ of degree two. □

Remark In the situation of the previous lemma, if $f(C_{g-1}) = \mathbb{P}^1(\mathbb{R})$ it follows $f^{-1}(\mathbb{P}^1(\mathbb{R})) = X(\mathbb{R})$, hence f is a separating morphism of degree g. In that case f has topological degree $(2, 1, \cdots, 1)$. In case $f(C_{g-1}) \neq \mathbb{P}^1(\mathbb{R})$ it has topological degree $(1, \cdots, 1, 0)$.

Lemma 2.6 *$T_{g,ns} \subset T_g$ is open and (hence) $T_{g,s} \subset T_g$ is closed.*

Proof We are going to prove that $T_{g,ns} \subset T_g$ is open. Let $t \in T_{g,ns}$ and let $X = t_g^{-1}(t)$. Let $\pi : \mathcal{C} \to S$ be a suited family for X and $s \in S(\mathbb{R})$ such that $\pi^{-1}(s) \cong X$. It is enough to prove there exists a classical open neighborhood U of s in $S(\mathbb{R})$ such that for all $s' \in U$ the curve $\pi^{-1}(s')$ is a separating $(M-2)$-curve not of special type. It is well known that points in $S(\mathbb{R})$ close to s do correspond to separating $(M-2)$-curves, so we only have to show they are also of non-special type.

Choose a component C of X. Since the curve is not of special type, there exists a covering $f : X \to \mathbb{P}^1$ of degree g such that it has topological degree $(1, \cdots, 1, 0)$ and $f(C) \neq \mathbb{P}^1(\mathbb{R})$. Consider $\pi_g : H_g(\pi) \to S$ with $H_g(\pi)$ parameterizing morphisms of degree g from fibers of π to $\mathbb{P}^1$ and now let H be the connected component of $H_g(\pi)(\mathbb{R})$ containing [f]. From the deformation theory of Horikawa, we know H is smooth of dimension $4g-2$. Moreover, f corresponds to an invertible sheaf L of degree g, therefore $H^1(X, L^{\otimes 2}) = 0$, hence the description of the tangent map of π_g at [f] implies this tangent map has maximal rank. So the image of a neighborhood of [f] on H contains a neighborhood U of s in S. Intersecting those neighborhoods for all choices of C (again denoted by U), we obtain for each $s' \in U$ and for each component C' of $\pi^{-1}(s')(\mathbb{R})$ the existence of a morphism $f' : \pi^{-1}(s') \to \mathbb{P}^1$ of topological degree $(1, \cdots, 1, 0)$ having even parity on C', hence $f'(C') \neq \mathbb{P}^1(\mathbb{R})$ because of Lemma 2.5. This means $\pi^{-1}(s')$ is not of special type. □

Lemma 2.7 *$T_{g,g-1} \subset T_g$ is closed and (hence) $T_{g,g} \subset T_g$ is open.*

Proof Let X_0 be a curve corresponding to a point on the closure of $T_{g,g-1}$. Then X_0 is the limit of a family of separating $(M-2)$-curves X_t $(t > 0)$ having a separating morphism $f_t : X_t \to \mathbb{P}^1$ of degree $g-1$. Since $X_t(\mathbb{R})$ has $g-1$ components such morphism has to be of topological type $(1, \cdots, 1)$. Therefore, the fiber of f_t over a real point of $\mathbb{P}^1$ is of type $P_1 + \cdots + P_{g-1}$ with P_i belonging to different components of $X_t(\mathbb{R})$. The limit of such divisor on X_0 is of the same type and belongs to a complete linear system of dimension at least 1. So it defines a complete linear system g^r_{g-1} for some $r \geq 1$ having odd degree on each component C of $X_0(\mathbb{R})$. In case $r > 1$ then for P_1, P'_1 on the same component C of $X_0(\mathbb{R})$, there should exist $D \in g^r_{g-1}$ containing $P_1 + P'_1$. Since D should contain a point of each component of $X_0(\mathbb{R})$, this is impossible. So $r = 1$. In case D would have a base point (say P_1) then for P'_1 general on the same component, there should exist $D \in g^1_{g-1}$ containing $P_1 + P'_1$ giving the same contradiction. So g^1_{g-1} corresponds to a base point free linear system having odd degree on each component of $X_0(\mathbb{R})$, so it defines a separating morphism $f_0 : X_0 \to \mathbb{P}^1$ of degree $g - 1$. □

3 Existence of separating $(M - 2)$-curves of special type

Theorem 3.1 *For each $g \geq 3$ there exists a separating $(M - 2)$-curve X of special type.*

This theorem is an immediate corollary of the next proposition. This proposition shows that the components of the real locus of a canonically embedded real curve can be strongly linked with each other. Therefore, the proposition describes the geometric reason for the existence of separating $(M-2)$-curves of special type. It would be interesting to obtain more information concerning the way the components of the real locus of a canonically embedded real curve can be linked.

For a curve X embedded in some projective space $\mathbb{P}$ and an effective divisor E on X, we denote $\langle E \rangle$ for the linear span: it is the intersection of hyperplanes H of $\mathbb{P}$ such that $H.X \geq E$ (and it is $\mathbb{P}$ in case such hyperplane does not exist).

Proposition 3.2 *For all $g \geq 3$ there is a canonically embedded $(M - 2)$-curve $X \subset \mathbb{P}^{g-1}$ having real components $C_1, \cdots, C_{g-1}$ of $X(\mathbb{R})$ such that*

1. *for all $P_i \in C_i$ $(1 \leq i \leq g - 1)$ one has* $\dim(\langle P_1, \cdots, P_{g-1} \rangle) = g - 2$
2. *for all $P_i \in C_i$ $(2 \leq i \leq g - 1)$ and for each real hyperplane $H \subset \mathbb{P}^{g-1}$ containing $\langle P_2, \cdots, P_{g-1} \rangle$ one has $H \cap C_1 \neq \emptyset$.*

Proof of Theorem 3.1 Let X be as described in Proposition 3.2. Take $P_i \in C_i$ $(2 \leq i \leq g - 1)$ and consider $|K_X - (P_2 + \cdots + P_{g-1})|$. From (1) in Proposition 3.2 we have $\dim(\langle P_2, \cdots, P_{g-1} \rangle) = g - 3$ hence $\dim(|K_X - (P_2 + \cdots + P_{g-1}|) = 1$ ($|K_X - (P_2 + \cdots + P_{g-1})|$ is the linear system induced by the pencil of hyperplanes in $\mathbb{P}^{g-1}$ containing $\langle P_2, \cdots, P_{g-1} \rangle$, it is denoted by g^1_g). Since K_X has even degree on each component of $X(\mathbb{R})$ it follows g^1_g has odd degree on C_i for $2 \leq i \leq g$ and even degree on C_1. From (2) in Proposition 3.2, it follows each divisor $D \in g^1_g$ contains some point of C_1, hence it contains a divisor of degree 2 with support on C_1. This proves each divisor of g^1_g is of the type $D = P'_1 + P''_1 + P'_2 + \cdots + P'_{g-1}$ with $P'_i \in C_i$ for $1 \leq i \leq g - 1$ and $P''_1 \in C_1$.

Assume P'_i is a base point of g^1_g for some $2 \leq i \leq g - 1$, then no divisor of g^1_g can contain another point of C_i. This is impossible hence P'_i is not a base point for $2 \leq i \leq g - 1$. Assume, for example, P''_1 is a base point for g^1_g then $\dim |P'_1 + P'_2 + \cdots + P'_{g-1}| = 1$. Then, the geometric version of the Riemann–Roch theorem (see for example, [8, p. 248]) implies $\dim\langle P'_1, \cdots, P'_{g-1} \rangle = g - 3$ contradicting (1) in Proposition 3.2. So g^1_g is base point free and it defines a covering $f : X \to \mathbb{P}^1$ having odd degree on C_i for $2 \leq i \leq g - 1$ and such that C_1 also dominates $\mathbb{P}^1(\mathbb{R})$. From the description of the divisors of g^1_g it follows all fibers of f over $\mathbb{P}^1(\mathbb{R})$ are totally real, hence X is a separating curve.

Conversely, if $f : X \to \mathbb{P}^1$ is a morphism of degree g having odd parity on C_i for $2 \leq i \leq g - 1$, then for a real fiber E of f one has $|K_X - E| \neq \emptyset$ and $|K_X - E|$ has odd parity on $C_2, \cdots, C_{g-1}$. Since $\deg(K_X - E) = g - 2$ each divisor of $|K_X - E|$ is of type $P_2 + \cdots + P_{g-1}$ with $P_i \in C_i$ for $2 \leq i \leq g - 1$. So f corresponds to $|K_X - (P_2 + \cdots + P_{g-1})|$ and we already proved $f(C_1) = \mathbb{P}^1(\mathbb{R})$. This shows X is of special type. □

For a curve X satisfying properties (1) and (2) of Proposition 3.2, we found $|K_X - (P_2 + \cdots + P_{g-1})|$ with $P_i \in C_i$ $(2 \leq i \leq g-1)$ defines a covering $\pi : X \to \mathbb{P}^1$ such that C_i dominates $\mathbb{P}^1(\mathbb{R})$ for $1 \leq i \leq g-1$. In particular, π is not ramified at some real point of X. Since $\deg(\pi|_{C_1}) = 2$, it also implies condition (2) of Proposition 3.2 is equivalent to: for all $P_i \in C_i$ $(2 \leq i \leq g-1)$ and for all real hyperplanes $H \subset \mathbb{P}^{g-1}$ containing $\langle P_2, \cdots, P_{g-1}\rangle$ one has H intersects C_1 transversally at 2 points. In the proof, we are going to use this (at first sight stronger) statement.

Proof of Proposition 3.2 We are going to prove for all $g \geq 3$ the existence of a canonically embedded smooth real curve $X \subset \mathbb{P}^{g-1}$ of genus g such that $X(\mathbb{R})$ has $g-1$ connected components $C_1, \cdots, C_{g-1}$ and satisfying the following two properties

(P1) For all $P_i \in C_i$ $(1 \leq i \leq g-1)$ one has $\dim\big(\langle P_1, \cdots, P_{g-1}\rangle\big) = g-2$.

(P2) For all $P_i \in C_i$ $(2 \leq i \leq g-1)$ each hyperplane $H \subset \mathbb{P}^{g-1}$ containing $\langle P_2, \cdots, P_{g-1}\rangle$ intersects C_1 transversally at two points.

In the first part of the proof, we prove the existence of X for the (already known) case $g = 3$. The arguments used to prove this case will be generalized in the second part of the proof in order to obtain a proof by induction on g. In both parts of the proof, we are going to use the following fact. Let Γ_0 be a canonically embedded non-hyperelliptic real singular curve having an isolated real node S as its only singularity and such that $\Gamma_0(\mathbb{R}) \setminus \{S\}$ has n connected components. There exists a real algebraic deformation $\pi : \mathfrak{X} \to I$ with I a small neighborhood of 0 in $[0, +\infty[\subset \mathbb{R}$ such that $\pi^{-1}(0) = \Gamma_0$ and for $t > 0$ the curve $X_t = \pi^{-1}(t)$ is a smooth real complete curve of genus g such that $X_t(\mathbb{R})$ has $n+1$ connected components (see for example, [13, Section 7], it can be shown directly by using part of Construction II in [5]). We can assume for all $t \in I$ the curve X_t is not hyperelliptic. Using the relative dualizing sheaf for this deformation we can assume it is a family of canonically embedded real curves in $\mathbb{P}^{g-1}$.

First part of the proof Let X_0 be a real hyperelliptic curve of genus 2. It has a unique real component $C_{0,1}$ and $C_{0,1}$ dominates $\mathbb{P}^1(\mathbb{R})$ for the hyperelliptic covering (see [9, Section 6]). Take $Q + \overline{Q}$ general on X_0 (hence $Q \in X_0(\mathbb{C}) \setminus X_0(\mathbb{R})$) and consider the real linear system $|K_{X_0} + (Q + \overline{Q})|$ on X_0. Since all real divisors in g^1_2 on X_0 consist of 2 real points we have $Q + \overline{Q} \notin g^1_2$.

In both parts of the proof, we use the following general fact concerning smooth complex curves M of genus $g \geq 2$. Let P and Q be two different points on M with $\dim|P + Q| = 0$ (this is always the case if M is not hyperelliptic) and consider the linear system $|K_M + P + Q|$. This is a base point free linear system on M, and it defines a morphism $\phi : M \to \mathbb{P}^g$ such that the image $\Gamma \subset \mathbb{P}^g$ of M is the nodal curve of arithmetic genus $g+1$ obtained from M by identifying P and Q to become an ordinary node $S = \phi(P) = \phi(Q)$ of Γ and Γ is embedded by the dualizing sheaf ω_Γ (this is well known, an argument can be found in [4, Lemma 5]).

Applying this argument using $|K_{X_0} + (Q + \overline{Q})|$, we obtain a canonically embedded real singular curve $\Gamma_0 \subset \mathbb{P}^2$ of degree 4, birationally equivalent to X_0. The singular point S on Γ_0 is an isolated point on $\Gamma_0(\mathbb{R})$ and projection with center S on a real line $\mathbb{P}^1 \subset \mathbb{P}^2$ induces a real covering $X_0 \to \mathbb{P}^1$ corresponding to the g^1_2 on X_0. The real locus $X_0(\mathbb{R})$ corresponds to the unique connected component $C_{0,1}$ of $\Gamma_0(\mathbb{R}) \setminus \{S\}$. Since $S \notin C_{0,1}$ one has

(P1') For all $P \in C_{0,1}$ one has $\dim\langle P, S\rangle = 1$.
Moreover, if $H \subset \mathbb{P}^2$ is a real line containing S, then H induces a divisor on X_0 belonging to $g^1_2 + Q + \overline{Q}$. This divisor is real, hence it contains two different points of $X_0(\mathbb{R})$. On Γ_0 one has

(P2') Each real line $H \subset \mathbb{P}^2$ with $S \in H$ intersects $C_{0,1}$ transversally at 2 points.

We obtain a real family $\pi : \mathfrak{X} \to I \subset [0, +\infty[\subset \mathbb{R}$ of canonically embedded real curves of genus 3 in $\mathbb{P}^2$ such that $\pi^{-1}(0) = \Gamma_0$ and for $t > 0$ the curve $\pi^{-1}(t) = X_t$ is smooth such that $X_t(\mathbb{R})$ has 2 connected components. Let $C_{t,1}$ be the connected component of $X_t(\mathbb{R})$ specializing to $C_{0,1}$ and let $C_{t,2}$ be the connected component of $X_t(\mathbb{R})$ specializing to $\{S\}$. Let $\mathcal{C}_i$ be the union of those components $C_{t,i}$ (including S in case $i = 2$). For the classical topology on $\mathfrak{X}(\mathbb{C})$ those are closed subsets. Consider the fibered product $\mathcal{C}_1 \times_I \mathcal{C}_2$ and its subset $\mathcal{Z}$ defined by $(P_1, P_2) \in \mathcal{Z}$ if and only if $\dim\langle P_1, P_2\rangle = 0$ (i.e., $P_1 = P_2$). This is a closed subset in $\mathcal{C}_1 \times_I \mathcal{C}_2$ and since the natural map $\mathcal{C}_1 \times_I \mathcal{C}_2 \to I$ is proper, it follows the image Z of $\mathcal{Z}$ in I is closed. Because of (P1') one has $0 \notin Z$. Shrinking I we can assume $\mathcal{Z} = \emptyset$. Let $G_{\mathbb{R}}$ be the Grassmannian of real lines in $\mathbb{P}^2$ and define $\mathcal{I} \subset \mathcal{C}_2 \times G_{\mathbb{R}}$ by $(P, L) \in \mathcal{I}$ if and only if $P \in L$. Let $\mathcal{Z}' \subset \mathcal{I}$ be defined by $(P, L) \in \mathcal{Z}'$ if and only if L does not intersect $C_{\pi(P),1}$ transversally. Since $\mathcal{Z}' \subset \mathcal{C}_2 \times G_{\mathbb{R}}$ is closed and the induced map $\mathcal{C}_2 \times G_{\mathbb{R}} \to I$ is proper it follows the image Z' of $\mathcal{Z}'$ in I is closed. Because of (P2') one has $0 \notin Z'$. Shrinking I we can assume $\mathcal{Z}' = \emptyset$.

Take $t_0 \neq 0$ and let $X = X_{t_0} \subset \mathbb{P}^2$. It is a canonically embedded real curve of genus 3 and $X(\mathbb{R})$ has two connected components $C_i = C_{t_0,i}$ $(i = 1, 2)$. Let $P_i \in C_i$ for $i = 1, 2$ then $(P_1, P_2) \notin \mathcal{Z} = \emptyset$, hence $\dim\langle P_1, P_2\rangle = 1$. This implies (P1) for this curve X. Let $P_2 \in C_2$ and let L be a real line in $\mathbb{P}^2$ with $P_2 \in L$. Then $(P_2, L) \in \mathcal{I}$. Choose a family $(P_{t,2}, L_t)_{t\geq 0}$ in $\mathcal{I}$ with $(P_{t_0}, L_{t_0}) = (P_2, L)$. Then $P_{0,2} = S$ hence L_0 intersects $C_{0,1}$ transversally at 2 points. Since $\mathcal{Z}' = \emptyset$ it follows all intersections of L_t and $C_{1,t}$ $(t \geq 0)$ are transversal. Since $\bigcup_{t\geq 0}\{t\} \times L_t$ and $\mathcal{C}_1$ are closed in the classical topology of $I \times \mathbb{P}^2$ it follows L intersects C_1 transversally at 2 points. This implies (P2) for this curve X. □

Second part of the proof Repeating the arguments of the first part of the proof, we are going to finish the proof by induction on the genus. Assume $X_0 \subset \mathbb{P}^{g-1}$ is a canonically embedded smooth real curve of some genus $g \geq 3$ satisfying properties (P1) and (P1). Take $Q + \overline{Q}$ general on X_0 (by assumption already X_0 is not hyperelliptic hence $\dim |Q + \overline{Q}| = 0$). Using $|K_{X_0} + (Q + \overline{Q})|$, which is a real linear system on X_0, we obtain the canonically embedded real singular curve $\Gamma_0 \subset \mathbb{P}^g$ having a unique singular point S. This singular point is an isolated point on $\Gamma_0(\mathbb{R})$. Choosing a real hyperplane $\mathbb{P}^{g-1} \subset \mathbb{P}^g$ not containing S then projection with center S on $\mathbb{P}^{g-1}$ induces a canonical embedding $X_0 \subset \mathbb{P}^{g-1}$ defined over $\mathbb{R}$. Let $C_{0,i}$ $(1 \leq i \leq g-1)$ be the connected component of $\Gamma_0(\mathbb{R}) \setminus \{S\}$ corresponding to the component C_i of $X_0(\mathbb{R})$. As before assumptions (P1) and (P2) imply

(P1') For each $P_{0,i} \in C_{0,i}$ $(1 \leq i \leq g-1)$ one has $\dim\big(\langle P_{0,1}, \cdots, P_{0,g-1}, S\rangle\big) = g - 1$.
(P2') For each $P_{0,i} \in C_{0,i}$ $(2 \leq i \leq g-1)$ each real hyperplane H in $\mathbb{P}^g$ containing $\langle P_{0,2}, \cdots, P_{0,g-1}, S\rangle$ intersects $C_{0,1}$ transversally at two points.

Consider a real deformation $\pi : \mathfrak{X} \subset I \times \mathbb{P}^g \to I \subset [0, +\infty[\subset \mathbb{R}$ of canonically embedded real curves of genus $g + 1$ with $\pi^{-1}(0) = \Gamma_0 \subset \mathbb{P}^g$ and for $t \neq 0$ one has $X_t = \pi^{-1}(t)$ is a smooth real curve of genus $g + 1$ such that $X_t(\mathbb{R})$ has g connected components. For $1 \leq i \leq g-1$ and $t \neq 0$ let $C_{t,i}$ be the component specializing to $C_{0,i}$ and let $C_{t,g}$ be the component specializing to $\{S\}$. For $1 \leq i \leq g$ let $\mathcal{C}_i$ be the union of those components $C_{t,i}$ (including S in case $i = g$). Let $\prod_{i=1,I}^{g} \mathcal{C}_i$ be the set of g-uples $(P_1, \cdots, P_g)$ with $P_i \in \mathcal{C}_i$ and $\pi(P_i) = \pi(P_j)$ for $i \neq j$ and let $\mathcal{Z} \subset \prod_{i=1,I}^{g} \mathcal{C}_i$ be defined by $(P_1, \cdots, P_g) \in \mathcal{Z}$ if and only if $\dim\big(\langle P_1, \cdots, P_g\rangle\big) < g - 1$. Let $\mathcal{I} \subset \prod_{i=2,I}^{g} \mathcal{C}_i \times G_{\mathbb{R}}$ (now $G_{\mathbb{R}}$ is the Grassmannian of real linear subspaces of dimension $g - 2$ in $\mathbb{P}^g$) be defined by $(P_2, \cdots, P_g, H) \in \mathcal{I}$ if and only if $P_i \in \mathcal{C}_i, \pi(P_i) = \pi(P_j)$ for $i \neq j$ and $P_i \in H$ and let $\mathcal{Z}' \subset \mathcal{I}$ be defined by $(P_2, \cdots, P_g, H) \in \mathcal{Z}'$ if and only if H does not intersect $C_{t,1}$

transversally ($t=\pi(P_i)$). From (P1') and (P2') it follows, by shrinking I, we can assume $\mathcal{Z}$ and $\mathcal{Z}'$ being empty. Then, taking $t_0 \neq 0$ and $X = X_{t_0} \subset \mathbb{P}^g$, we obtain a canonically embedded smooth real curve X of genus g such that $X(\mathbb{R})$ has g connected components $C_i = C_{t_0,i}$. As in the previous case, the arguments imply this curve X satisfies (P1) and (P2). $\square$

Condition 1 in Proposition 3.2 implies for $P_i \in C_i$ $(1 \leq i \leq g-1)$ one has $\dim |P_1+\cdots+P_{g-1}| = 0$. This implies $\mathrm{sepgon}(X) \neq g-1$, hence we proved the existence of separating $(M-1)$-curves of special type of separating gonality g. As mentioned in the introduction, we are going to prove that in case $t \in T_{g,s}$ corresponds to a curve X_t with separating gonality $g-1$ then t is not an inner point of $T_{g,s}$. This indicates that it is natural to include the use of the separating gonality in the deformation argument used in the proof of Proposition 3.2 (i.e., to use condition 1 to prove Theorem 3.1).

4 The relation between special type and the separating gonality

We start by proving the following remarkable fact concerning separating morphisms of degree $g-1$ on separating $(M-2)$-curves of special type.

Proposition 4.1 *Let X be a real separating $(M-2)$-curve of special type of genus $g \geq 3$ satisfying* $\mathrm{sepgon}(X) = g-1$*, then each g^1_{g-1} on X having odd degree on each component of $X(\mathbb{R})$ is half canonical. In particular, X has only finitely many linear systems g^1_{g-1} associated with separated morphisms of degree $g-1$.*

Proof We assume X is canonically embedded in $\mathbb{P}^{g-1}$ (as a matter of fact, X cannot be hyperelliptic (see [9, Section 6]) and for an effective divisor E on X we write $\langle E\rangle$ to denote its linear span in $\mathbb{P}^{g-1}$. Let $C_1, \cdots, C_{g-1}$ be the connected components of $X(\mathbb{R})$ and assume for each covering $f: X \to \mathbb{P}^1$ of degree g having degree 1 on C_i for $2 \leq i \leq g-1$ one has $f(C_1) = \mathbb{P}^1(\mathbb{R})$. Let $h: X \to \mathbb{P}^1$ be a separating morphism of degree $g-1$ and let E be a real fiber of h (hence $E = Q_1 + \cdots Q_{g-1}$ for $Q_i \in C_i$ for $1 \leq i \leq g-1$). Because of the Riemann–Roch theorem $\dim |K_X - E| \geq 1$ and $|K_X - E|$ has odd parity on each C_i $(1 \leq i \leq g-1)$. Since $\deg(K_X - E) = g-1$ each real divisor of $|K_X - E|$ is again of type $Q_1 + \cdots Q_{g-1}$ with $Q_i \in C_i$ for $1 \leq i \leq g-1$. Choose $P_1 \in C_1$ and let $P_1 + P_2 + \cdots + P_{g-1}$ be a real fiber of h and $P_1 + Q_2 + \cdots + Q_{g-1} \in |K_X - E|$ (here $P_i, Q_i \in C_i$ for $2 \leq i \leq g-1$). In case $P_1 + P_2 + \cdots + P_{g-1} \neq P_1 + Q_2 + \cdots + Q_{g-1}$ we can assume without loss of generality that $P_{g-1} \neq Q_{g-1}$. Assume X is canonically embedded and assume $Q_{g-1} \in \langle P_1 + \cdots P_{g-2}\rangle$. Since $P_{g-1} \in \langle P_1 + \cdots P_{g-2}\rangle$ it follows $\dim(\langle P_1 + \cdots P_{g-1} + Q_{g-1}\rangle) = g-3$, and therefore $\dim(|P_1 + Q_2 + \cdots + Q_{g-2}|) = 1$. Hence, there would exist a g^1_{g-2} on X having odd degree on $C_1, \cdots, C_{g-2}$. Since $|P_1 + Q_2 + \cdots + Q_{g-2}|$ has odd parity on each C_i for $1 \leq i \leq g-2$ and because of the existence of one more component C_{g-1} this is impossible. This proves $Q_{g-1} \notin \langle P_1 + \cdots + P_{g-2}\rangle$ and therefore $\dim |P_1 + \cdots + P_{g-2} + Q_{g-1}| = 0$. Since $2P_1 + P_2 + \cdots + P_{g-2} + Q_{g-1} \in |K_X - (Q_2 + \cdots + Q_{g-2} + P_{g-1})|$ one obtains $\dim |2P_1 + P_2 + \cdots + P_{g-2} + Q_{g-1}| = 1$ and P_1 is not a base point of $|2P_1 + P_2 + \cdots + P_{g-2} + Q_{g-1}|$. A morphism $f: X \to \mathbb{P}^1$ associated with the base point free linear system defined by $|2P_1 + P_2 + \cdots + P_{g-2} + Q_{g-1}|$ is ramified at $P_1 \in C_1$ and there is no other point of C_1 at that fiber. This implies the existence of a fiber containing no point of C_1, hence $f(C_1) \neq \mathbb{P}^1(\mathbb{R})$ and therefore the existence of a divisor $D \in |2P_1 + P_2 + \cdots + P_{g-2} + Q_{g-1}|$ with $\mathrm{Supp}(D) \cap C_1 = \emptyset$. In case the linear system $|2P_1 + P_2 + \cdots + P_{g-2} + Q_{g-1}|$ has no base point the morphism f has degree g and it has odd degree on $C_2, \cdots, C_{g-1}$ and therefore

$f(C_1) \neq \mathbb{P}^1(\mathbb{R})$ contradicting our assumptions. We are going to show that by deforming $(Q_2, \cdots, Q_{g-2}, P_{g-1})$ on $C_2 \times \cdots \times C_{g-2} \times C_{g-1}$ we obtain such contradiction.

Consider the closed subset $Z \subset X^{(g)}(\mathbb{R}) \times C_2 \times \cdots \times C_{g-1}$ defined by $(D', Q'_2, \cdots, Q'_{g-2}, P'_{g-1}) \in Z$ if and only if $D' \in |K_X - (Q'_2 + \cdots + Q'_{g-2} + P'_{g-1})|$. Consider the morphisms $p_1 : Z \to C_2 \times \cdots \times C_{g-1}$ and $p_2 : Z \to X^{(g)}(\mathbb{R})$ induced by projection. Since $\dim(|Q'_2 + \cdots + Q'_{g-2} + P'_{g-1}|) = 0$ for all $(Q'_2, \cdots, Q'_{g-2}, P'_{g-1}) \in C_2 \times \cdots \times C_{g-1}$, it follows from the Riemann–Roch theorem that $p_1^{-1}(Q'_1, \cdots, Q'_{g-1}, P'_{g-1}) \cong \mathbb{P}^1(\mathbb{R})$, in particular p_1 is a locally trivial $\mathbb{P}^1(\mathbb{R})$-bundle. Let $d_0 = (Q_2, \cdots, Q_{g-2}, P_{g-1})$, we proved there exists $(d_0, D) \in p_1^{-1}(d_0)$ such that $D \notin X(\mathbb{R})^{(g)}$. Since $X(\mathbb{R})^{(g)}$ is closed in $X^{(g)}(\mathbb{R})$ there exists a classical neighborhood V of D in $X^{(g)}(\mathbb{R})$ such that $V \cap X(\mathbb{R})^{(g)} = \emptyset$. Let $S = \{d \in C_2 \times \cdots \times C_{g-1} : p_2(p_1^{-1}(d)) \cap V = \emptyset\}$ and assume $d_0 \in \overline{S}$. Take a neighborhood U of d_0 in $C_2 \times \cdots \times C_{g-1}$, such that $p_1^{-1}(U)$ is homeomorphic to $\mathbb{P}^1(\mathbb{R}) \times U$ and $p_1|_{p_1^{-1}(U)}$ is identified with the projection $\mathbb{P}^1(\mathbb{R}) \times U \to U$. The closure of $p_1^{-1}(S \cap U)$ in $X^{(g)} \times U$ is identified with $\mathbb{P}^1(\mathbb{R}) \times (\overline{S \cap U})$ (here $\overline{S \cap U}$ is the closure of $S \cap U$ in U) hence $p_1^{-1}(d_0)$ belongs to the closure of $p_1^{-1}(S \times U)$. But $p_2^{-1}(V)$ is a neighborhood of (D, d_0) in Z hence $p_2^{-1}(V) \cap p_1^{-1}(S \cap U) \neq \emptyset$. Of course this contradicts the definition of S, hence $d_0 \notin \overline{S}$. Hence, there exists a neighborhood U of d_0 in $C_2 \times \cdots \times C_{g-1}$ such that for all $d' = (Q'_2, \cdots, Q'_{g-2}, P'_{g-1}) \in U$ one has $p_2(p_1^{-1}(d')) \cap V \neq \emptyset$, hence there exists a divisor $D' \in V$ with $D' \in |K_X - (Q'_2 + \cdots + Q'_{g-2} + P'_{g-1})|$. In particular, $|K_X - (Q'_2 + \cdots + Q'_{g-2} + P'_{g-1}|$ contain a divisor D' containing a non-real point in its support. Since $|K_X - (Q'_2 + \cdots + Q'_{g-2} + P'_{g-1})|$ has odd parity on $C_2, \cdots, C_{g-1}$ and even parity on C_1 it follows $\mathrm{Supp}(D') \cap C_1 = \emptyset$. In case $|K_X - (Q'_2 + \cdots + Q'_{g-2} + P'_{g-1})|$ would contain a base point for all $d' \in U$, using terminology from [1], it would imply $\dim((W^1_{g-1} + W^0_1)(\mathbb{R})) \geq g - 2$. Since $W^1_{g-1} = g - 4$ (X is not hyperelliptic, so we can apply Martens' Theorem, see [1]) this is impossible. So we can assume $|K_X - (Q'_2 + \cdots + Q'_{g-2} + P'_{g-1})|$ is base point free. But then it corresponds to a covering $f' : X \to \mathbb{P}^1$ of degree g having odd degree on $C_2, \cdots, C_{g-1}$ and $f'(C_1) \neq \mathbb{P}^1(\mathbb{R})$. This contradicts the assumptions on X. This proves $|K_X - (P_1 + \cdots + P_{g-1})| = |P_1 + \cdots + P_{g-1}|$ and so $P_1 + \cdots + P_{g-1}$ is a half-canonical divisor. From parity considerations, we also obtain $\dim |P_1 + \cdots + P_{g-1}| < 2$ for such divisor, implying the finiteness of linear systems g^1_{g-1} associated with separating morphisms on a real separating $(M-2)$-curve of special type. □

In [6, Example 3], it is noted that each separating $(M-2)$-curve of genus 3 is of special type. It follows from the previous proposition this is not the case for genus $g \geq 4$.

Corollary 4.2 *Let g be an integer at least 4. There exist real separating $(M-2)$-curves of genus g not of special type.*

Proof From [5], we know there exists a dividing $(M-2)$-curve X such that $\mathrm{sepgon}(X) = g - 1$. Assume X is of special type. Let $\pi : \mathcal{X} \to S$ be a suited family for X and $s_0 \in S(\mathbb{R})$ with $X = \pi^{-1}(s_0)$. Let $\pi_{g-1} : \mathcal{H} \to S$ be the parameter space parameterizing morphisms of degree $g-1$ from fibers of π to $\mathbb{P}^1$. From deformation theory of Horikawa, it follows $\mathcal{H}$ is smooth of dimension $4g-4$. Such morphism corresponds to a linear system g^1_{g-1}, let $\mathcal{H}_h(\mathbb{C})$ be the subset of $\mathcal{H}(\mathbb{C})$ corresponding to half canonical linear systems g^1_{g-1}. This is a closed subset of $\mathcal{H}(\mathbb{C})$ of dimension $3g-1$, and it is invariant under complex conjugation, so $\mathcal{H}_h(\mathbb{C})$ are the complex points of a closed subset $\mathcal{H}_h \subset \mathcal{H}$ defined over $\mathbb{R}$ and we find $\dim(\mathcal{H}_h(\mathbb{R})) \leq 3g-1$, in particular for each $f \in \mathcal{H}_h(\mathbb{R})$ one has $U \cap \mathcal{H}(\mathbb{R}) \neq \mathcal{H}_h(\mathbb{R})$. By assumption, there exists $[f] \in \pi_{g-1}^{-1}(s_0)(\mathbb{R})$ such that $f : X \to \mathbb{P}^1$ is a separating morphism. From Proposition 4.1, it follows $[f] \in \mathcal{H}_h(\mathbb{R})$. Hence, f deforms to a separating morphism $[f']$ that is not half canonical. By Proposition 4.1, this is defined on a fiber X' of π not of special type. □

The previous proof also implies the following fact.

Corollary 4.3 *$T_{g,s} \cap T_{g,g-1} \subset \overline{T_{g,ns}}$ in case $g \geq 4$.*

We now prove the strong relation between both partitions of T_g.

Theorem 4.4 *Let X be a real separating $(M-2)$-curve of genus g not of special type, then* $\mathrm{sepgon}(X) = g-1$, *hence $T_{g,ns} \subset T_{g,g-1}$.*

Proof Assume X is a separating $(M-2)$-curve of genus g not of special type. We can assume that there exists a separating real morphism $f : X \to \mathbb{P}^1$ of degree g, otherwise clearly $\mathrm{sepgon}(X) = g-1$. For each component C_i of $X(\mathbb{R})$ one has $f|_{C_i} : C_i \to \mathbb{P}^1(\mathbb{R})$ is a covering of some degree $d_i \geq 1$ and $\sum_{i=1}^{g-1} d_i = g$. It follows $d_i = 1$ except for one value $d_i = 2$, we can assume $d_1 = 2$ and $d_2 = \cdots = d_{g-1} = 1$. The morphism f corresponds to a linear systems $g = g^1_g$ and $|K_X - g^1_g| \neq \emptyset$. Since $\deg(K_X - g^1_g) = g-2$ and $|K_X - g^1_g|$ has odd degree on $C_2, \cdots, C_{g-1}$ it follows $|K_X - g^1_g| = \{Q_2 + \cdots + Q_{g-1}\}$ for some $Q_i \in C_i$. By assumption X is not special, hence there exists $Q'_i \in C_i$ for $2 \leq i \leq g$ such that $|K_X - (Q'_2 + \cdots + Q'_{g-1})|$ defines a $g^1_g = g'$ on X such that g' corresponds to a non-separating morphism $f' : X \to \mathbb{P}^1$, hence C_i for $2 \leq i \leq g-1$ dominates $\mathbb{P}^1(\mathbb{R})$ but C_1 does not. This implies g' contains a real divisor $P' + \overline{P'} + P'_2 + \cdots + P'_{g-1}$ with $P' + \overline{P'}$ a non-real point of X. Take a path $\gamma : [0,1] \to C_2 \times \cdots \times C_{g-1}$ with $\gamma(0) = (Q_2, \cdots, Q_{g-1})$ and $\gamma(1) = (Q'_2, \cdots, Q'_{g-1})$. Let $\gamma(t) = (Q_2(t), \cdots, Q_{g-1}(t)$ and $g^1_g(t) = |K_X - (Q_2(t) + \cdots + Q_{g-1}(t))|$. In case $g^1_g(t)$ is base point free for all $t \in I$ we can find a family of real morphisms $f_t : X \to \mathbb{P}^1$ with $f_0 = f$ and $f_1 = f'$. Since the topological degree of f (resp. f') is $(2, 1, \cdots, 1)$ (resp. $(1, \cdots, 1, 0)$) and this discrete invariant should be constant in this family, we obtain a contradiction. So there exists $t_0 \in I$ such that $g^1_g(t_0)$ has a base point. Moreover, for $t < t_0$ we can assume $g^1_g(t)$ defines a separating morphism $f_t : X \to \mathbb{P}^1$. By continuity it follows each divisor on $g^1_g(t_0)$ is of type $\overline{P_1} + \overline{P'_1} + \overline{P_2} + \cdots + \overline{P_{g-1}}$ with $\overline{P_1}, \overline{P'_1} \in C_1$ and $\overline{P_i} \in C_i$ for $2 \leq i \leq g-1$. Assume $\overline{P_2}$ is a fixed point of $g^1_g(t_0)$ then for $\overline{P'_2} \in C_2 \setminus \{\overline{P_2}\}$ there is no divisor in $g^1_g(t_0)$ containing $\overline{P'_2}$, a contradiction. So we find $\overline{P_i}$ is not a fixed point for $2 \leq i \leq g-1$, hence we can assume $\overline{P'_1}$ is a fixed point. But then we find $\dim|\overline{P_1} + \overline{P_2} + \cdots + \overline{P'_{g-1}}| = 1$, hence $g^1_g(t_0) - \overline{P'_1}$ defines a separating morphism $f_0 : X \to \mathbb{P}^1$ of degree $g-1$. This proves $\mathrm{sepgon}(X) = g-1$. □

Corollary 4.5 *Let $g \geq 3$. There exist separating $(M-2)$-curves X of special type such that* $\mathrm{sepgon}(X) = g-1$.

Proof From Lemma 2.6 it follows $T_{g,ns}$ is an open subset of T_g and it follows from Corollary 4.2 that $T_{g,ns} \neq \emptyset$. It is already proved in Theorem 3.1 that $T_{g,ns} \neq T_g$ (indeed, $T_{g,ns} \neq \emptyset$). Since T_g is connected it follows $T_{g,ns}$ is not closed. On the other hand, we just proved $T_{g,ns} \subset T_{g,g-1}$ and it is proved in Lemma 2.7 that $T_{g,g-1}$ is closed. Hence, $T_{g,ns} \neq T_{g,g-1}$ and therefore $T_{g,s} \cap T_{g,g-1} \neq \emptyset$. □

The proof of this corollary implies the following inclusion.

Corollary 4.6 $\overline{T_{g,ns}} \cap T_{g,s} \subset T_{g,g-1}$

Together with Corollary 4.3, this implies

Corollary 4.7 $\overline{T_{g,ns}} = T_{g,g-1}$.

Corollary 4.8 *Let $g \geq 4$. There exist general separating $(M-2)$-curves of genus g of special type and general separating $(M-2)$-curves of non-special type.*

Proof From Corollary 4.2, it follows that there exist separating $(M-2)$-curves of genus g of non-special type. Then from Lemma 2.6, we know there exist general separating $(M-2)$-curves of genus g of non-special type. In [4], it is proved that $T_{g,g} \neq \emptyset$ (this is also obtained from Proposition 3.2). Such curve does not belong to $\overline{T_{g,ns}}$ hence $T_g \setminus \overline{T_{g,ns}} \subset T_{g,s}$ is an open non-empty subset of T_g, and it parameterizes general separating $(M-2)$-curves of special type. □

The previous result also implies the following remarkable corollary.

Corollary 4.9 *There exist dividing $(M-2)$-curves X of genus $g \geq 4$ such that $W^1_{g-1}(X)(\mathbb{R})$ has an isolated point.*

Proof Again let X be a dividing $(M-2)$-curve of special type having separable gonality $g-1$. From Corollary 4.5, we know X does exist. A separating morphism $f : X \to \mathbb{P}^1$ of degree $g-1$ corresponds to a complete base point free g^1_{g-1} on X, hence it belongs to a connected component of $W^1_{g-1}(X)(\mathbb{R})$ and each g'^1_{g-1} close to g^1_{g-1} is also base point free, complete and induces a separating morphism. But from Proposition 4.1, it follows g'^1_{g-1} has to be half canonical. Since a curve has only finitely many half-canonical linear systems, it follows g^1_{g-1} corresponds to an isolated point of $W^1_{g-1}(X)(\mathbb{R})$. □

This corollary is in sharp contrast (in case $g \geq 5$) to the fact that the dimension of each component of $W^1_{g-1}(X_{\mathbb{C}})$ is at least $g-4$. In the final remark, we explain that it seems to indicate difficulties in studying the real gonality of real curves.

Remark In his paper [2], E. Ballico considers an upper bound for the real gonality of real curves. In moving families of real curves X, some components of $W^1_d(\mathbb{R})$ existing on general curves can vanish at "transition" curves (meaning curves having such components, but not on all curves of some neighborhood in the moduli space; this terminology is not used in loc. cit.). In his arguments, the author proves that having such a transition curve using degree $[(g+3)/2]$ (this is the gonality of a general complex curve of genus g), then there is a real pencil of degree at most $[g+3)/2]+3$ that propagates to all nearby real curves of it. How to finish the argument to conclude that it propagates on a dense set of the moduli space of real curves is not clear to me (it seems to me there is no argument in loc. cit.). As a matter of fact, the previous corollary shows that such components of $W^1_d(\mathbb{R})$ can vanish in isolated points at those transition curves. In particular, those transition curves do not need to have a singular locus of $W^1_d(X_{\mathbb{C}})$ of dimension at least 1; the basic tool in loc. cit. is the study of complex curves having a singular locus of some W^1_d of dimension at least one (or more). The previous corollary is the most extreme case showing what could go wrong in the argument from [2]. On the other hand, it is clear that the arguments coming from [2] had much influence on the present paper.

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