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On the rigidity of hypersurfaces into space forms

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Abstract Our purpose is to study the rigidity of complete hypersurfaces immersed into a Riemannian space form. In this setting, first we use a classical characterization of the Euclidean sphere $\mathbb{S}^{n+1}$ due to Obata (J Math Soc Jpn 14:333–340, 1962) in order to prove that a closed orientable hypersurface Σ^n immersed with null second-order mean curvature in $\mathbb{S}^{n+1}$ must be isometric to a totally geodesic sphere $\mathbb{S}^n$, provided that its Gauss mapping is contained in a closed hemisphere. Furthermore, as suitable applications of a maximum principle at the infinity for complete noncompact Riemannian manifolds due to Yau (Indiana Univ Math J 25:659–670, 1976), we establish new characterizations of totally geodesic hypersurfaces in the Euclidean and hyperbolic spaces. We also obtain a lower estimate of the index of minimum relative nullity concerning complete noncompact hypersurfaces immersed in such ambient spaces.

Keywords Space forms · Complete hypersurfaces · Totally geodesic hypersurfaces · Gauss mapping · Higher order mean curvatures · Index of minimum relative nullity

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1 Introduction

The study of the behavior of the Gauss mapping plays an important role in order to obtain rigidity results concerning complete hypersurfaces immersed into a space form; for instance, it was proved, independently, by De Giorgi in [13] and Simons in [20] that if the image of the Gauss mapping of a compact minimal hypersurface M^n in the Euclidean sphere $\mathbb{S}^{n+1}$ lies in an open hemisphere, then M^n must be a great hypersphere of $\mathbb{S}^{n+1}$. A few years late, Nomizu and Smyth [16] have shown a similar result for a closed orientable hypersurface $x : M^n \looparrowright \mathbb{S}^{n+1}$ with constant mean curvature. We observe that such results are natural extensions of the classical Bernstein's theorem established in [6] for a complete minimal surface M^2 into the 3-dimensional Euclidean space $\mathbb{R}^3$. The same question was treated by Alencar et al. [3] for the case of null high-order mean curvature H_r, with the additional hypothesis that H_{r-1} does not change sign.

On the other hand, on the middle of the last century, many geometers tried to prove that a compact Riemannian manifold with constant scalar curvature is isometric to a standard sphere provided that it carries a nontrivial conformal vector field. But this result is not true according to a counterexample exhibited by Ejiri [11]. Meanwhile, some characterizations of the Euclidean sphere were obtained related to this problem. Among them, we point out a classical one due to Obata [17], which proves that a complete Riemannian manifold (M^n, g) is isometric to a standard sphere $\mathbb{S}^n(r)$, provided that there exists a nontrivial solution for the PDE $\nabla^2\rho = -\rho g$, where $\nabla^2\rho$ stands for the Hessian of ρ.

In this paper, by using such Obata's theorem jointly with some suitable formulas related to the height and support functions of a hypersurface, we extend the above-mentioned results of De Giorgi and Simons for the context of compact orientable hypersurfaces $x : M^n \looparrowright \mathbb{S}^{n+1}$ with null second-order mean curvature. More precisely, we obtain the following theorem:

Theorem 1 *Let $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ be a closed orientable hypersurface such that $H_2 = 0$. If the image of the Gauss mapping of Σ^n is contained into a closed hemisphere of the Euclidean sphere $\mathbb{S}^{n+1}$, then Σ^n is a totally geodesic sphere of $\mathbb{S}^{n+1}$.*

Afterward, we deal with hypersurfaces of the hyperbolic space $\mathbb{H}^{n+1}$. First, we recall the following extension of Hopf's theorem on a complete noncompact Riemannian manifold Σ^n due to Yau [21]: a subharmonic (or superharmonic) function whose gradient has integrable norm on Σ^n must actually be harmonic. More recently, Camargo et al. [9] extended Yau's result concerning a complete noncompact oriented hypersurface Σ^n immersed in a space form, with bounded second fundamental form. In this setting, they showed that if a smooth function f defined on Σ^n is such that ∇f has integrable norm and $L_r f$ does not change sign on Σ^n, then $L_r f = 0$ on Σ^n (for the details about the L_r operators, see Sect. 2).

Here, we use such analytical machinery in order to obtain a rigidity theorem for hypersurfaces immersed in the hyperbolic space $\mathbb{H}^{n+1}$. For this, we consider the Lorentz model of $\mathbb{H}^{n+1}$ obtained by furnishing the hyperquadric $\{p \in \mathbb{L}^{n+2}; \langle p, p\rangle = -1,\ p_{n+2} > 0\}$ with the Riemannian metric induced by the Lorentz metric of the Minkowski space $\mathbb{L}^{n+2}$. Moreover, we denote by $a^\top$ the tangential component of a vector $a \in \mathbb{L}^{n+2}$ with respect to an immersion $x : \Sigma^n \looparrowright \mathbb{H}^{n+1} \hookrightarrow \mathbb{L}^{n+2}$, and along this paper, $\mathcal{L}^1(\Sigma)$ stands for the space of Lebesgue integrable functions on Σ^n. In this setting, we get the following result:

Theorem 2 *Let $x : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ be a complete hypersurface immersed in $\mathbb{H}^{n+1}$ with bounded nonnegative mean curvature H and such that $H_2 = 0$. Suppose that Σ^n lies between two hyperspheres of $\mathbb{H}^{n+1}$ determined by a spacelike vector $a \in \mathbb{L}^{n+2}$. If $|a^\top| \in \mathcal{L}^1(\Sigma)$, then Σ^n is a totally geodesic hypersphere of $\mathbb{H}^{n+1}$.*

The proofs of Theorems 1 and 2 are presented in Sect. 3. Furthermore, in Sect. 4, we establish rigidity results in the Euclidean space (cf. Theorems 3 and 4; see also Corollaries 1 and 2) and, in Sect. 5, we obtain a lower estimate to the index of minimum relative nullity concerning complete noncompact hypersurfaces Σ^n immersed either in $\mathbb{R}^{n+1}$ or in $\mathbb{H}^{n+1}$ (cf. Theorems 5 and 6).

2 Preliminaries

In this section, we present some known results that we use in order to prove our theorems. Throughout this paper, we denote by Q_c^{n+1} a Riemannian space form of constant sectional curvature $c \in \{-1, 0, 1\}$, and $x : \Sigma^n \looparrowright Q_c^{n+1}$ stands for an immersed hypersurface in Q_c^{n+1}.

If we let A denote the corresponding shape operator, then, at each $p \in \Sigma^n$, A restricts to a self-adjoint linear map $A_p : T_p\Sigma \to T_p\Sigma$. For $0 \le r \le n$, let $S_r(p)$ denotes the rth elementary symmetric function on the eigenvalues of A_p; in this way, one gets n smooth functions $S_r : \Sigma^n \to \mathbb{R}$, such that

$$\det(tI - A) = \sum_{k=0}^{n}(-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by convention. If $p \in \Sigma^n$ and $\{e_k\}$ are a basis of $T_p\Sigma$ formed by eigenvectors of A_p, with corresponding eigenvalues $\{\lambda_k\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \dots, X_n]$ is the rth elementary symmetric polynomial on the indeterminates $X_1, \dots, X_n$.

Also, we define the rth mean curvature H_r of Σ^n, $0 \le r \le n$, by

$$\binom{n}{r} H_r = S_r.$$

We observe that $H_0 = 1$, while H_1 is the usual mean curvature H of Σ^n.

For $0 \le r \le n$, one defines the rth Newton transformation P_r on Σ^n by setting $P_0 = I$ (the identity operator) and, for $1 \le r \le n$, via the recurrence relation

$$P_r = S_r I - AP_{r-1}. \tag{2.1}$$

On the other hand, given $f \in C^\infty(\Sigma)$, for each $0 \le r \le n$, the second-order differential operator L_r is defined as follows

$$L_r f = \mathrm{tr}(P_r \nabla^2 f).$$

It is important to note that this operator is divergence type provided that we have a hypersurface $\Sigma^n \looparrowright Q_c^{n+1}$, where Q_c^{n+1} is a space form. This fact was proved by Rosenberg [18] and it reads as follows

$$L_r f = \mathrm{div}(P_r \nabla f).$$

Moreover, for a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ and $f \in C^\infty(\Sigma)$, it follows from the properties of the Hessian that

$$L_r(\varphi \circ f) = \varphi'(f) L_r f + \varphi''(f)\langle P_r \nabla f, \nabla f\rangle. \tag{2.2}$$

When we are dealing with the Euclidean space $\mathbb{R}^{n+1}$ or the Euclidean sphere $\mathbb{S}^{n+1}$, we may fix a Euclidean vector a; whereas for the hyperbolic space $\mathbb{H}^{n+1}$, we may fix a vector a of the Minkowski space $\mathbb{L}^{n+2}$. We recall that the *height function* l_a and the *support function* f_a of an immersion $x : \Sigma^n \looparrowright Q_c^{n+1}$ are defined, respectively, according to

$$l_a = \langle x, a\rangle \quad \text{and} \quad f_a = \langle N, a\rangle,$$

where N stands for the Gauss mapping of x. In this setting, we have the following split

$$a = a^{\top} + f_a N + c\, l_a x, \tag{2.3}$$

where $a^{\top}$ is the orthogonal projection of a over the tangent bundle $T\Sigma$.

Moreover, based on the paper due to Reilly [19], Rosenberg [18] showed the following identities related with the action of L_r on these functions:

$$L_r l_a = (r+1)S_{r+1} f_a - c(n-r)S_r l_a \tag{2.4}$$

and

$$L_r f_a = -\left(S_1 S_{r+1} - (r+2)S_{r+2}\right) f_a + c(r+1)S_{r+1} l_a - \langle \nabla S_{r+1}, a^{\top}\rangle. \tag{2.5}$$

In particular, letting $r = 0$, we deduce the following well-known relations:

$$\Delta l_a = nH f_a - cnl_a \tag{2.6}$$

and

$$\Delta f_a = -|A|^2 f_a + cnHl_a - n\langle \nabla H, a^{\top}\rangle. \tag{2.7}$$

To close this section, we point out the next inequality that can be found in [2] or [7], which is valid for an immersion $x : \Sigma^n \looparrowright Q_c^{n+1}$ with S_2 constant:

$$S_1^2(|\nabla A|^2 - |\nabla S_1|^2) \geq 2S_2|\nabla A|^2. \tag{2.8}$$

In particular, if $S_2 \geq 0$, then $|\nabla A|^2 - |\nabla S_1|^2 \geq 0$.

3 Proofs of Theorems 1 and 2

Let us begin with the proof of Theorem 1.

Proof First, one notices the existence of an unit vector $a \in \mathbb{R}^{n+2}$ such that the associated support function $f_a = \langle N, a\rangle$ does not change sign, where N is an unit normal vector field globally defined on Σ^n. On the other hand, since $S_2 = 0$, we use Gauss equation

$$n(n-1)(R-1) = n^2H^2 - |A|^2 = 2S_2$$

to conclude that the scalar curvature of Σ^n is identically one. Now, we use formula (2.5) to infer

$$L_1 f_a = 3S_3 f_a.$$

If $S_1 = 0$, we have $|A| \equiv 0$, which gives the desired result. Otherwise, we deduce from previous equation that

$$S_1 L_1(f_a) = 3S_1 S_3 f_a. \tag{3.1}$$

On the other hand, using Proposition 3.1 of [7], we get

$$L_1 S_1 = \left(|\nabla A|^2 - |\nabla S_1|^2\right) + 3 S_1 S_3 + (n-1) S_1^2.$$

If f_a is identically null, we can apply Theorem 1 of [16], due to Nomizu and Smyth, to deduce that Σ^n is totally geodesic. Otherwise, we can multiply the last identity by f_a to obtain

$$f_a L_1 S_1 = \left(|\nabla A|^2 - |\nabla S_1|^2\right) f_a + 3 S_1 S_3 f_a + (n-1) S_1^2 f_a$$

Now, we use the symmetry of L_1 to infer

$$\int_\Sigma S_1 L_1 f_a \mathrm{d}\Sigma = \int_\Sigma f_a L_1 S_1 \mathrm{d}\Sigma.$$

Therefore, we deduce

$$3 \int_\Sigma S_1 S_3 f_a \mathrm{d}\Sigma = \int_\Sigma \left(|\nabla A|^2 - |\nabla S_1|^2\right) f_a \mathrm{d}\Sigma + 3 \int_\Sigma S_1 S_3 f_a \mathrm{d}\Sigma + (n-1) \int_\Sigma S_1^2 f_a \mathrm{d}\Sigma.$$

From where we conclude

$$\int_\Sigma \left(|\nabla A|^2 - |\nabla S_1|^2\right) f_a \mathrm{d}\Sigma + (n-1) \int_\Sigma S_1^2 f_a \mathrm{d}\Sigma = 0.$$

By using (2.8), we have

$$\left(|\nabla A|^2 - |\nabla S_1|^2\right) f_a = 0 \tag{3.2}$$

and

$$H^2 f_a = 0. \tag{3.3}$$

On the other hand, since $H_2 = 0$, we have that $|A|^2 = n^2 H^2$. Consequently, from (3.3), we get

$$|f_a A|^2 = n^2 H^2 f_a^2 = 0. \tag{3.4}$$

At this point, we recall the next relation involving l_a and f_a, which can be found in Alías [4]:

$$\nabla^2 l_a = -l_a g + f_a A, \tag{3.5}$$

where g stands for the Riemannian metric of Σ^n.

Hence, combining (3.4) and (3.5), one obtains the following:

$$\nabla^2 l_a = -l_a g. \tag{3.6}$$

Now, one observes that if l_a is constant, then Σ^n is totally umbilical. But, $S_2 = 0$ gives that it must be, in fact, totally geodesic. Otherwise, we may apply Obata's theorem [17] in order to conclude that Σ^n is isometric to a standard sphere. Once more, $S_2 = 0$ gives that Σ^n is totally geodesic, which completes the proof of Theorem 1. □

Before to start the proof of Theorem 2, we recall that hyperspheres of the hyperbolic space $\mathbb{H}^{n+1}$ can be realized in Minkowski's model as the following level sets:

$$L_\tau = \{p \in \mathbb{H}^{n+1}; \langle p, a\rangle = \tau\},$$

for some spacelike vector $a \in \mathbb{L}^{n+2}$, where $\tau^2 > \langle a, a\rangle$ (see e.g. [5], Sect. 3).

Proof Initially, we observe that, since we are supposing that Σ^n lies between two hyperspheres $L_{\underline{\tau}}$ and $L_{\overline{\tau}}$ of $\mathbb{H}^{n+1}$ determined by a spacelike vector $a \in \mathbb{L}^{n+2}$, the height function $l_a = \langle x, a \rangle$ is bounded on Σ^n. Moreover, the hypothesis under the second-order mean curvature of Σ^n assures us that $S_2 = 0$ on Σ^n. Consequently, since $H \geq 0$ on Σ^n, from Lemma 1.1 and equation (1.3) in [14] we have that P_1 is a positive semi-definite operator. Thus, using (2.2) and (2.4), we deduce

$$L_1 l_a^2 = 2(n-1)S_1 l_a^2 + 2\langle P_1 \nabla l_a \, , \nabla l_a \rangle \geq 0. \tag{3.7}$$

On the other hand, we note that the second fundamental form A of Σ^n is bounded and

$$|\nabla l_a^2| = 2|l_a||a^\top| \in \mathcal{L}^1(\Sigma).$$

Thus, we are in position to apply Corollary 1 of [9], to get that $L_1 l_a^2 = 0$ on Σ^n. Consequently, from equation (3.7), $S_1 l_a = 0$ on Σ^n. Now, from (2.6), we obtain

$$\Delta l_a^2 = 2n l_a^2 + 2|\nabla l_a|^2. \tag{3.8}$$

Reasoning as before, we see that l_a vanishes identically on Σ^n, and therefore, this allows us to conclude that Σ^n is a totally geodesic hypersphere of $\mathbb{H}^{n+1}$. □

4 Rigidity theorems in the Euclidean space

Initially, we observe that a paraboloid has positive Gaussian curvature, it is contained in a semispace, and its Gauss mapping covers only an open hemisphere of $\mathbb{S}^2$ determined by such semispace. We also observe that a cylinder over a plane curve inside of a slice, has null Gaussian curvature, lies between two parallel planes orthogonal to a fixed vector $a \in \mathbb{R}^3$, but its Gauss mapping is not contained in a closed hemisphere determined by a. However, by applying a classical result due to Huber [15] concerning parabolic surfaces, we obtain the following rigidity result in the three-dimensional Euclidean space $\mathbb{R}^3$ related to complete surfaces with nonnegative Gaussian curvature:

Theorem 3 *Let $x : \Sigma^2 \looparrowright \mathbb{R}^3$ be a complete surface with nonnegative Gaussian curvature K_Σ and such that its mean curvature H does not change sign. Suppose that Σ^2 lies between two parallel planes of $\mathbb{R}^3$ which are orthogonal to an unit vector $a \in \mathbb{R}^3$ and that the image of its Gauss mapping lies in a closed hemisphere of $\mathbb{S}^2$ determined by a. Then, Σ^2 is a plane of $\mathbb{R}^3$ orthogonal to a.*

Proof Since we are supposing that the mean curvature H does not change sign, we can choose an orientation for Σ^2 in such away that $H \geq 0$. We note that the hypothesis under the Gauss mapping of Σ^2 also guarantees that f_a does not change sign. On the other hand, since Σ^2 lies between two parallel planes determined by a, there exists constants α and β such that $\alpha \leq l_a \leq \beta$. Thus, if $f_a \leq 0$, we have that $l_a - \alpha$ is a nonnegative function on Σ^2 and, from (2.6), we obtain

$$\Delta(l_a - \alpha) = 2H f_a \leq 0.$$

However, the quoted result of Huber [15] assures that complete surfaces of nonnegative Gaussian curvature must be parabolic. Therefore, l_a is constant on Σ^2, and hence, Σ^2 is a plane orthogonal to a.

Now, if $f_a \geq 0$, we see that $\beta - l_a$ is a nonnegative function on Σ^2, and taking into account once more formula (2.6), we get

$$\Delta(\beta - l_a) = -2Hf_a \leq 0.$$

Therefore, reasoning as before, we also conclude that Σ^2 is a plane orthogonal to a. □

Corollary 1 *Let* $\Sigma^2(u) = \{(x, y, u(x, y)) : (x, y) \in \mathbb{R}^2\} \subset \mathbb{R}^3$ *be a complete graph of a smooth function* $u : \mathbb{R}^2 \to \mathbb{R}$ *with nonnegative Gaussian curvature and such that its mean curvature does not change sign. If* u *is bounded, then* $\Sigma^2(u)$ *is a plane of* $\mathbb{R}^3$.

Proof Since the support function f_{e_3} has strict sign on $\Sigma^2(u)$, the image of the Gauss mapping of $\Sigma^2(u)$ lies in an open hemisphere of $\mathbb{S}^2$ determined by e_3. Therefore, the result follows from Theorem 3. □

Now, we observe that the examples presented in the beginning of this section also satisfy the same conditions of the next theorem, up to $|a^\top| \in \mathcal{L}^1(\Sigma)$. But, using Yau's result [21] mentioned in the introduction, we prove the following rigidity result in the $(n+1)$-dimensional Euclidean space:

Theorem 4 *Let* $x : \Sigma^n \looparrowright \mathbb{R}^{n+1}$ *be a complete hypersurface such that its mean curvature* H *does not change sign. Suppose that there exists an unit vector* $a \in \mathbb{R}^{n+1}$ *such that one of the following conditions is satisfied:*

(a) Σ^n *has scalar curvature* R *bounded from below and the image of its Gauss mapping lies in an open hemisphere of* $\mathbb{S}^n$ *determined by* a;
(b) Σ^n *lies between two hyperplanes which are orthogonal to* a *and the image of its Gauss mapping lies in a closed hemisphere of* $\mathbb{S}^n$ *determined by* a.

If $|a^\top| \in \mathcal{L}^1(\Sigma)$, *then* Σ^n *is a hyperplane of* $\mathbb{R}^{n+1}$ *orthogonal to* a.

Proof First, let us suppose that Σ^n has scalar curvature R bounded from below and that the image of the Gauss mapping of Σ^n lies in an open hemisphere of $\mathbb{S}^n$ determined by a. In this case, the support function f_a has strict sign on Σ^n. Moreover, from (2.6), we deduce

$$\Delta l_a = nHf_a.$$

Consequently, Δl_a does not change sign on Σ^n.

On the other hand, taking into account that $|\nabla l_a| = |a^\top| \in \mathcal{L}^1(\Sigma)$, we may apply the result due to Yau [21], which was described in the introduction, to deduce that l_a is harmonic. Hence, H vanishes identically on Σ^n. In particular, we have $|A|^2 = -n(n-1)R$ and consequently, $|A|$ is bounded on Σ^n. This allows us to conclude that $|\nabla f_a| \in \mathcal{L}^1(\Sigma)$. Indeed,

$$|\nabla f_a| = \left|A\left(a^\top\right)\right| \leq |A||a^\top| \in \mathcal{L}^1(\Sigma).$$

Furthermore, from (2.7), we get

$$\Delta f_a = -|A|^2 f_a.$$

Then, Δf_a also does not change sign on Σ^n. Thus, by applying again Yau's result, we conclude that f_a is harmonic. Hence, A vanishes identically on Σ^n, that is, Σ^n is totally geodesic. Therefore, Σ^n is a hyperplane of $\mathbb{R}^{n+1}$ orthogonal to a; otherwise $|a^\top|$ does not belong to $\mathcal{L}^1(\Sigma)$, which finishes the proof of the first assertion.

Now, let us suppose that Σ^n lies between two hyperplanes which are orthogonal to a and that the image of the Gauss mapping of Σ^n lies in a closed hemisphere of $\mathbb{S}^n$ determined by

a. In such situation, analogously to the previous one, we also conclude that l_a is harmonic. From where, since

$$\frac{1}{2}\Delta l_a^2 = l_a \Delta l_a + |\nabla l_a|^2,$$

we verify that $\Delta l_a^2 \geq 0$ on Σ^n.

On the other hand, since Σ^n lies between two hyperplanes which are orthogonal to a, we have that $|l_a| \leq C_1$. This gives

$$|\nabla l_a^2| = 2|l_a||a^\top| \leq C_2|a^\top| \in \mathcal{L}^1(\Sigma).$$

Hence, we deduce that l_a^2 is also harmonic and consequently, ∇l_a vanishes identically on Σ^n. Therefore, Σ^n is a hyperplane of $\mathbb{R}^{n+1}$ orthogonal to a and we complete the proof of Theorem 4. □

Corollary 2 *Let $\Sigma^n(u) = \{(x, u(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$ be a complete graph of a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ such that its mean curvature does not change sign. Suppose that either the scalar curvature R of $\Sigma^n(u)$ or u is bounded. If $|\nabla u| \in \mathcal{L}^1(\mathbb{R}^n)$, then $\Sigma^n(u)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

Proof It is well known that the unit normal vector field

$$N = \frac{1}{\sqrt{1+|\nabla u|^2}}(-\nabla u, 1)$$

defines a Gauss mapping for $\Sigma^n(u)$. Letting $a = e_{n+1}$, we have $|a^\top|^2 = 1 - f_a^2$. Hence, we obtain

$$|a^\top|^2 = \frac{|\nabla u|^2}{1+|\nabla u|^2}.$$

Therefore, $|\nabla u| \in \mathcal{L}^1(\mathbb{R}^n)$ assures that $|a^\top| \in \mathcal{L}^1(\Sigma^n(u))$, and hence, the result follows from Theorem 4. □

5 Lower estimates for the index of relative nullity

Let $x : \Sigma^n \looparrowright Q_c^{n+1}$ be a hypersurface immersed in a space form Q_c^{n+1}, with second fundamental form A. According to [10], for $p \in \Sigma^n$, we define the *space of relative nullity* $\Delta(p)$ of Σ^n at p by

$$\Delta(p) = \{v \in T_p\Sigma; v \in \ker(A_p)\},$$

where $\ker(A_p)$ denotes the kernel of A_p. The *index of relative nullity* $\nu(p)$ of Σ^n at p is the dimension of $\Delta(p)$, that is,

$$\nu(p) = \dim(\Delta(p)),$$

and the *index of minimum relative nullity* ν_0 of Σ^n is defined by

$$\nu_0 = \min_{p\in\Sigma} \nu(p).$$

Now, we are in position to prove the following extension of Theorem 3 to the case of the rth mean curvatures:

Theorem 5 *Let $x : \Sigma^n \looparrowright \mathbb{R}^{n+1}$ be a complete hypersurface with bounded second fundamental form A and such that, for some $0 \le r \le n-2$, H_{r+1} and H_{r+2} do not change sign. Suppose that the image of the Gauss mapping of Σ^n lies in an open hemisphere of $\mathbb{S}^n$ determined by an unit vector $a \in \mathbb{R}^{n+1}$. If $|a^\top| \in \mathcal{L}^1(\Sigma)$, then the index of minimum relative nullity ν_0 of Σ^n is at least $n-r$. Moreover, if H_r does not vanish on Σ^n, then through every point of Σ^n, there passes an $(n-r)$-hyperplane of $\mathbb{R}^{n+1}$ totally contained in Σ^n.*

Proof From (2.4) we have

$$L_r l_a = (r+1)\binom{n}{r+1} H_{r+1} f_a.$$

Thus, the hypothesis that H_{r+1} does not change sign on Σ^n and that the image of the Gauss mapping of Σ^n lies in an open hemisphere of $\mathbb{S}^n$ determined by a assure that $L_r l_a$ also does not change sign on Σ^n. Consequently, since we are supposing that A is bounded and $|a^\top| \in \mathcal{L}^1(\Sigma)$, from Corollary 1 of [9], we conclude that $H_{r+1} = 0$ on Σ^n. With analogous arguments, from the calculus of $L_{r+1} l_a$, we also get that $H_{r+2} = 0$ on Σ^n. Thus, from Proposition 2.3(c) of [8], we see that $H_j = 0$ for all $j \ge r+1$, and hence, $\nu_0 \ge n-r$.

Now, suppose that H_r does not vanish on Σ^n. By Theorem 5.3 of [10] (see also [12]), the distribution $p \mapsto \Delta(p)$ of minimal relative nullity of Σ^n is smooth and integrable with complete leaves, totally geodesic in Σ^n and in $\mathbb{R}^{n+1}$. Therefore, the result follows from the characterization of complete totally geodesic submanifolds of $\mathbb{R}^{n+1}$ as hyperplanes of suitable dimension. □

In what follows, let $a \in \mathbb{L}^{n+2}$ be an unit timelike vector. The level set given by

$$\mathcal{L}_0 = \{p \in \mathbb{S}_1^{n+1}; \langle p, a\rangle = 0\}$$

defines a round sphere of radius one which is a totally geodesic hypersurface of the de Sitter space $\mathbb{S}_1^{n+1}$. According to the terminology established in [1], we will refer to that sphere as the *equator* of $\mathbb{S}_1^{n+1}$ determined by a. This equator divides $\mathbb{S}_1^{n+1}$ into two connected components, the *chronological future* which is given by

$$\{p \in \mathbb{S}_1^{n+1}; \langle p, a\rangle < 0\},$$

and the *chronological past*, given by

$$\{p \in \mathbb{S}_1^{n+1}; \langle p, a\rangle > 0\}.$$

In order to establish our last result, we also recall that a hypersurface Σ^n immersed in a space form is said to be *r-minimal* if H_{r+1} vanishes identically on Σ^n. In this setting, we can reason as in the proof of Theorem 5 (working with the support function f_a instead of the height function l_a, and taking into account the characterization of complete totally geodesic submanifolds of $\mathbb{H}^{n+1}$) to get the following

Theorem 6 *Let $x : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ be a r-minimal complete noncompact hypersurface with bounded second fundamental form A and such that H_{r+2} does not change sign. Suppose that the image of the Gauss mapping of Σ^n lies in the chronological future (or past) of the equator of $\mathbb{S}_1^{n+1}$ determined by an unit timelike vector $a \in \mathbb{L}^{n+2}$. If $|a^\top| \in \mathcal{L}^1(\Sigma)$, then the index of minimum relative nullity ν_0 of Σ^n is at least $n-r$. Moreover, if H_r does not vanish on Σ^n, then through every point of Σ^n, there passes an $(n-r)$-dimensional hyperbolic space $\mathbb{H}^{n-r} \hookrightarrow \mathbb{H}^{n+1}$ totally contained in Σ^n.*

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