On unbounded composition operators in L^2-spaces

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Abstract Fundamental properties of unbounded composition operators in L^2-spaces are studied. Characterizations of normal and quasinormal composition operators are provided. Formally normal composition operators are shown to be normal. Composition operators generating Stieltjes moment sequences are completely characterized. The unbounded counterparts of the celebrated Lambert's characterizations of subnormality of bounded composition operators are shown to be false. Various illustrative examples are supplied.

Keywords Composition operator in L^2-space · Normal operator · Quasinormal operator · Formally normal operator · Subnormal operator · Operator generating Stieltjes moment sequences

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1 Introduction

Composition operators (in L^2-spaces over σ-finite spaces), which play an essential role in Ergodic Theory, turn out to be interesting objects of Operator Theory. The questions of boundedness, normality, quasinormality, subnormality, hyponormality etc. of such operators have been answered (cf. [8–10,16,17,19,20,23,30–32,34,42–44,54,56]; see also [15,18,33, 45,47] for particular classes of composition operators). This means that the theory of bounded composition operators on L^2-spaces is well-developed.

The literature on unbounded composition operators in L^2-spaces is meager. So far, only the questions of seminormality, k-expansivity, and complete hyperexpansivity have been studied (cf. [11,24]). Very little is known about other properties of unbounded composition operators. To the best of our knowledge, there is no paper concerning the issue of subnormality of such operators. It is a difficult question mainly because Lambert's criterion for subnormality of bounded operators (cf. [29]) is no longer valid for unbounded ones. In the present paper, we show that the unbounded counterparts of the celebrated Lambert's characterizations of subnormality of bounded composition operators given in [31] fail to hold. This is achieved by proving that a composition operator satisfies the requirements of Lambert's characterizations if and only if it generates Stieltjes moment sequences (see Definition 2.3, Theorem 10.4). Thus, knowing that there exists a non-subnormal composition operator which generates Stieltjes moment sequences (see [25, Theorem 4.3.3]), we obtain the above-mentioned result (see Sect. 11). We point out that there exists a non-subnormal formally normal operator which generates Stieltjes moment sequences (for details see [7, Section 3.2]). This is never the case for composition operators because, as shown in Theorem 9.4, each formally normal composition operator is normal, and as such subnormal. We refer the reader to [48–51] for the foundations of the theory of unbounded subnormal operators (for the bounded case see [14,21]).

The above discussion makes plain the importance of the question of when C^∞-vectors of a composition operator form a dense subset of the underlying L^2-space. This and related topics are studied in Sect. 4. In Sect. 3, we collect some necessary facts on composition operators. Illustrative examples are gathered in Sect. 5. In Sect. 6, we address the question of injectivity of composition operators. In Sect. 7, we describe the polar decomposition of a composition operator. Next, in Sects. 8 and 9, we characterize normal, quasinormal and formally normal composition operators. Finally, in Sect. 10, we investigate composition operators which generate Stieltjes moment sequences. We conclude the paper with two appendices. In "Appendix A," we gather particular properties of L^2-spaces exploited throughout the paper. "Appendix B" is mostly devoted to the operator of conditional expectation which plays an essential role in our investigations.

Caution. All measure spaces being considered in this paper, except for "Appendices A and B", are assumed to be σ-finite.

2 Preliminaries

Denote by $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{R}_+$ the sets of complex numbers, real numbers and nonnegative real numbers, respectively. We write $\mathbb{Z}_+$ for the set of all nonnegative integers and $\mathbb{N}$ for the set of all positive integers. The characteristic function of a subset Δ of a set X will be denoted by χ_Δ. We write $\Delta \bigtriangleup \Delta' = (\Delta \backslash \Delta') \cup (\Delta' \backslash \Delta)$ for subsets Δ and Δ' of X. Given a sequence $\{\Delta_n\}_{n=1}^\infty$ of subsets of X and a subset Δ of X such that $\Delta_n \subseteq \Delta_{n+1}$ for every $n \in \mathbb{N}$, and $\Delta = \bigcup_{n=1}^\infty \Delta_n$, we write $\Delta_n \nearrow \Delta$ (as $n \to \infty$). Denote by $\mathrm{card}(X)$ the cardinal number of X. If X is a topological space, then $\mathfrak{B}(X)$ stands for the σ-algebra of Borel subsets of X.

Let A be an operator in a complex Hilbert space $\mathcal{H}$ (all operators considered in this paper are linear). Denote by $\mathcal{D}(A)$, $\mathcal{N}(A)$, $\mathcal{R}(A)$, $\bar{A}$ and A^* the domain, the kernel, the range, the closure and the adjoint of A (in case they exist). If A is closed and densely defined, then A has a (unique) *polar decomposition* $A = U|A|$, where U is a partial isometry on $\mathcal{H}$ such that $\mathcal{N}(U) = \mathcal{N}(A)$ and $|A|$ is the square root of A^*A (cf. [3, Section 8.1]). Set $\mathcal{D}^\infty(A) = \bigcap_{n=0}^\infty \mathcal{D}(A^n)$. Members of $\mathcal{D}^\infty(A)$ are called C^∞-vectors of A. Denote by $\|\cdot\|_A$ the *graph norm* of A, that is,

$$\|f\|_A^2 := \|f\|^2 + \|Af\|^2, \quad f \in \mathcal{D}(A).$$

Given $n \in \mathbb{Z}_+$, we define the norm $\|\cdot\|_{A,n}$ on $\mathcal{D}(A^n)$ by

$$\|f\|_{A,n}^2 := \sum_{j=0}^n \|A^j f\|^2, \quad f \in \mathcal{D}(A^n).$$

Clearly, for every $n \in \mathbb{N}$, $(\mathcal{D}(A^n), \|\cdot\|_{A^n})$ and $(\mathcal{D}(A^n), \|\cdot\|_{A,n})$ are inner product spaces (with standard inner products). A vector subspace $\mathcal{E}$ of $\mathcal{D}(A)$ is called a *core* for A if $\mathcal{E}$ is dense in $\mathcal{D}(A)$ with respect to the graph norm of A. Denote by I the identity operator on $\mathcal{H}$.

By applying Propositions 2.1 and 3.2, one may obtain a criterion for closedness of a linear combination of composition operators.

Proposition 2.1 *Let $A_1, \ldots, A_n$ be closed operators in $\mathcal{H}$ ($n \in \mathbb{N}$). Then $\sum_{j=1}^n A_j$ is closed if and only if there exists $c \in \mathbb{R}_+$ such that*

$$\sum_{j=1}^n \|A_j f\|^2 \leqslant c \left(\|f\|^2 + \left\| \sum_{j=1}^n A_j f \right\|^2 \right), \quad f \in \bigcap_{j=1}^n \mathcal{D}(A_j). \tag{2.1}$$

Proof Define the vector space $\mathcal{X} = \bigcap_{j=1}^n \mathcal{D}(A_j)$ and the norm $\|\cdot\|_*$ on $\mathcal{X}$ by $\|f\|_*^2 = \|f\|^2 + \sum_{j=1}^n \|A_j f\|^2$ for $f \in \mathcal{X}$. Since the operators $A_1, \ldots, A_n$ are closed, we deduce that $(\mathcal{X}, \|\cdot\|_*)$ is a Hilbert space. Recall that $A := \sum_{j=1}^n A_j$ is closed if and only if $(\mathcal{X}, \|\cdot\|_A)$ is a Hilbert space. Since the identity map from $(\mathcal{X}, \|\cdot\|_*)$ to $(\mathcal{X}, \|\cdot\|_A)$ is continuous, we conclude from the inverse mapping theorem that $(\mathcal{X}, \|\cdot\|_A)$ is a Hilbert space if and only if (2.1) holds for some $c \in \mathbb{R}_+$. $\qquad\square$

A densely defined operator N in $\mathcal{H}$ is said to be *normal* if N is closed and $N^*N = NN^*$ (or equivalently if and only if $\mathcal{D}(N) = \mathcal{D}(N^*)$ and $\|Nf\| = \|N^*f\|$ for all $f \in \mathcal{D}(N)$, see [55, Proposition, p. 125]). We say that a densely defined operator A in $\mathcal{H}$ is *formally normal* if $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $\|Af\| = \|A^*f\|$ for all $f \in \mathcal{D}(A)$ (cf. [2,12]). A densely defined operator A in $\mathcal{H}$ is called *hyponormal* if $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $\|A^*f\| \leqslant \|Af\|$ for all $f \in \mathcal{D}(A)$ (cf. [27,35,53]). Clearly, a closed densely defined operator A in $\mathcal{H}$ is normal if and only if both operators A and A^* are hyponormal. Obviously normality implies formal normality and formal normality implies hyponormality. It is well-known that none of these implications can be reversed in general. We say that a densely defined operator S in $\mathcal{H}$ is *subnormal* if there exist a complex Hilbert space $\mathcal{K}$ and a normal operator N in $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding), $\mathcal{D}(S) \subseteq \mathcal{D}(N)$ and $Sf = Nf$ for all $f \in \mathcal{D}(S)$.

The members of the next class are related to subnormal operators. A closed densely defined operator A in $\mathcal{H}$ is said to be *quasinormal* if A commutes with the spectral measure $E_{|A|}$ of $|A|$, that is, $E_{|A|}(\Delta)A \subseteq AE_{|A|}(\Delta)$ for all $\Delta \in \mathfrak{B}(\mathbb{R}_+)$ (cf. [4,49]). In view of [49, Proposition 1], a closed densely defined operator A in $\mathcal{H}$ is quasinormal if and only if $U|A| \subseteq |A|U$, where $A = U|A|$ is the polar decomposition of A. This combined with [3, Theorem 8.1.5]

shows that if A is a normal operator, then A is quasinormal and $\mathcal{N}(A) = \mathcal{N}(A^*)$. In turn, quasinormality together with the inclusion $\mathcal{N}(A^*) \subseteq \mathcal{N}(A)$ characterizes normality. This result can be found in [52]. For the reader's convenience, we include its proof.

Theorem 2.2 *An operator A in $\mathcal{H}$ is normal if and only if A is quasinormal and $\mathcal{N}(A^*) \subseteq \mathcal{N}(A)$. Moreover, if A is normal, then $\mathcal{N}(A) = \mathcal{N}(A^*)$.*

Proof In view of the above discussion, it is enough to prove the sufficiency. First, we show that if A is quasinormal and $A = U|A|$ is its polar decomposition, then $U|A| = |A|U$. Indeed, by [49, Proposition 1], $U|A| \subseteq |A|U$. Taking adjoints, we get $U^*|A| \subseteq |A|U^*$, which implies that $U^*(\mathcal{D}(|A|)) \subseteq \mathcal{D}(|A|)$. Hence, if $f \in \mathcal{D}(|A|U)$, then $U^*Uf \in \mathcal{D}(|A|)$. Since $I - U^*U$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(|A|)$, we conclude that $f = U^*Uf + (I - U^*U)f \in \mathcal{D}(|A|)$. This shows that $\mathcal{D}(|A|U) \subset \mathcal{D}(U|A|)$, which implies that $U|A| = |A|U$.

Now suppose that A is quasinormal and $\mathcal{N}(A^*) \subseteq \mathcal{N}(A)$. Since the operators $P := UU^*$ and $P^\perp := (I - P)$ are the orthogonal projections of $\mathcal{H}$ onto $\overline{\mathcal{R}(A)}$ and $\mathcal{N}(A^*)$, respectively, we infer from the inclusion $\mathcal{N}(A^*) \subset \mathcal{N}(A)$ that

$$\mathcal{R}(P^\perp) \subset \mathcal{N}(A) = \mathcal{N}(|A|) \subset \mathcal{D}(|A|^2). \tag{2.2}$$

It follows from $U|A| = |A|U$ and $A^* = |A|U^*$ that

$$AA^* = U|A|^2U^* = |A|^2P. \tag{2.3}$$

We will show that

$$|A|^2P = |A|^2. \tag{2.4}$$

Indeed, if $f \in \mathcal{H}$, then, by (2.2) and the equality $f = Pf + P^\perp f$, we see that $Pf \in \mathcal{D}(|A|^2)$ if and only if $f \in \mathcal{D}(|A|^2)$. This implies that $\mathcal{D}(|A|^2P) = \mathcal{D}(|A|^2)$. Using (2.2) again, we see that $|A|^2f = |A|^2Pf$ for every $f \in \mathcal{D}(|A|^2)$. Hence, the equality (2.4) is valid. Combining (2.3) with (2.4), we get $AA^* = A^*A$.

The "moreover" part is well-known and easy to prove. $\qquad\qquad\square$

Recall that quasinormal operators are subnormal (see [4, Theorem 1] and [49, Theorem 2]). The reverse implication does not hold in general. Clearly, subnormal operators are hyponormal, but not reversely. It is worth pointing out that formally normal operators may not be subnormal (cf. [13,40,46]).

A finite complex matrix $[c_{i,j}]_{i,j=0}^n$ is said to be *nonnegative* if

$$\sum_{i,j=0}^n c_{i,j}\alpha_i\bar{\alpha}_j \geqslant 0, \quad \alpha_0, \ldots, \alpha_n \in \mathbb{C}.$$

If this is the case, then we write $[c_{i,j}]_{i,j=0}^n \geqslant 0$. A sequence $\{\gamma_n\}_{n=0}^\infty$ of real numbers is said to be a *Stieltjes* moment sequence if there exists a positive Borel measure ρ on $\mathbb{R}_+$ such that

$$\gamma_n = \int\limits_{\mathbb{R}_+} s^n \mathrm{d}\rho(s), \quad n \in \mathbb{Z}_+.$$

A sequence $\{\gamma_n\}_{n=0}^\infty \subseteq \mathbb{R}$ is said to be *positive definite* if for every $n \in \mathbb{Z}_+$, $[\gamma_{i+j}]_{i,j=0}^n \geqslant 0$. By the Stieltjes theorem (see [1, Theorem 6.2.5]), we have

a sequence $\{\gamma_n\}_{n=0}^\infty \subseteq \mathbb{R}$ is a Stieltjes moment sequence if and only if

the sequences $\{\gamma_n\}_{n=0}^\infty$ and $\{\gamma_{n+1}\}_{n=0}^\infty$ are positive definite. $\tag{2.5}$

Definition 2.3 We say that an operator S in $\mathcal{H}$ *generates Stieltjes moment sequences* if $\mathcal{D}^\infty(S)$ is dense in $\mathcal{H}$ and $\{\|S^n f\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $f \in \mathcal{D}^\infty(S)$.

It is well-known that if S is subnormal, then $\{\|S^n f\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $f \in \mathcal{D}^\infty(S)$ (see [7, Proposition 3.2.1]; see also Proposition 2.4 below). Hence, if $\mathcal{D}^\infty(S)$ is dense in $\mathcal{H}$ and S is subnormal, then S generates Stieltjes moment sequences. It turns out that the converse implication does not hold in general (see [7, Section 3.2]).

The following can be proved analogously to [7, Proposition 3.2.1] by using (2.5).

Proposition 2.4 *If S is a subnormal operator in $\mathcal{H}$, then the following two assertions hold*:

(i) $\left[\|S^{i+j} f\|^2\right]_{i,j=0}^n \geqslant 0$ *for all* $f \in \mathcal{D}(S^{2n})$ *and* $n \in \mathbb{Z}_+$,
(ii) $\left[\|S^{i+j+1} f\|^2\right]_{i,j=0}^n \geqslant 0$ *for all* $f \in \mathcal{D}(S^{2n+1})$ *and* $n \in \mathbb{Z}_+$.

For the reader's convenience, we state a theorem which is occasionally called the Mittag-Leffler theorem (cf. [41, Lemma 1.1.2]).

Theorem 2.5 *Let $\{\mathcal{E}_n\}_{n=0}^\infty$ be a sequence of Banach spaces such that for every $n \in \mathbb{Z}_+$, $\mathcal{E}_{n+1}$ is a vector subspace of $\mathcal{E}_n$, $\mathcal{E}_{n+1}$ is dense in $\mathcal{E}_n$ and the embedding map of $\mathcal{E}_{n+1}$ into $\mathcal{E}_n$ is continuous. Then, $\bigcap_{n=0}^\infty \mathcal{E}_n$ is dense in each space $\mathcal{E}_k$, $k \in \mathbb{Z}_+$.*

3 Basic properties of composition operators

From now on, except for "Appendices A and B", $(X, \mathscr{A}, \mu)$ always stands for a σ-finite measure space. We shall abbreviate the expressions "almost everywhere with respect to μ" and "for μ-almost every x" to "a.e. $[\mu]$" and "for μ-a.e. x", respectively. As usual, $L^2(\mu) = L^2(X, \mathscr{A}, \mu)$ denotes the Hilbert space of all square integrable complex functions on X. The norm and the inner product of $L^2(\mu)$ are denoted by $\|\cdot\|$ and $\langle\cdot, -\rangle$, respectively. Let ϕ be an $\mathscr{A}$-*measurable* transformation[1] of X, that is, $\phi^{-1}(\Delta) \in \mathscr{A}$ for all $\Delta \in \mathscr{A}$. Denote by $\mu \circ \phi^{-1}$ the positive measure on $\mathscr{A}$ given by $\mu \circ \phi^{-1}(\Delta) = \mu(\phi^{-1}(\Delta))$ for all $\Delta \in \mathscr{A}$. We say that ϕ is *nonsingular* if $\mu \circ \phi^{-1}$ is absolutely continuous with respect to μ. It is easily seen that if ϕ is nonsingular, then the mapping $C_\phi \colon L^2(\mu) \supseteq \mathcal{D}(C_\phi) \to L^2(\mu)$ given by

$$\mathcal{D}(C_\phi) = \{f \in L^2(\mu) \colon f \circ \phi \in L^2(\mu)\} \text{ and } C_\phi f = f \circ \phi \text{ for } f \in \mathcal{D}(C_\phi), \quad (3.1)$$

is well-defined and linear. Such an operator is called a *composition operator* induced by ϕ; the transformation ϕ will be referred to as a *symbol* of C_ϕ. Note that if the operator C_ϕ given by (3.1) is well-defined, then the transformation ϕ is nonsingular.

Convention. For the remainder of this paper, whenever C_ϕ is mentioned the transformation ϕ is assumed to be nonsingular.

If ϕ is nonsingular, then by the Radon-Nikodym theorem there exists a unique (up to sets of measure zero) $\mathscr{A}$-measurable function $\mathsf{h}_\phi \colon X \to [0, \infty]$ such that

$$\mu \circ \phi^{-1}(\Delta) = \int_\Delta \mathsf{h}_\phi \mathrm{d}\mu, \quad \Delta \in \mathscr{A}. \quad (3.2)$$

Here and later on ϕ^n stands for the n-fold composition of ϕ with itself if $n \geqslant 1$ and ϕ^0 for the identity transformation of X. We also write $\phi^{-n}(\Delta) := (\phi^n)^{-1}(\Delta)$ for $\Delta \in \mathscr{A}$ and $n \in \mathbb{Z}_+$.

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[1] By a transformation of X we understand a map from X to X.

Note that $h_{\phi^0} = 1$ a.e. $[\mu]$. It is clear that the composition $\phi_1 \circ \ldots \circ \phi_n$ of finitely many nonsingular transformations $\phi_1, \ldots, \phi_n$ of X is a nonsingular transformation and

$$C_{\phi_n} \ldots C_{\phi_1} \subseteq C_{\phi_1 \circ \ldots \circ \phi_n}, \quad n \in \mathbb{N}. \tag{3.3}$$

Now we construct an $\mathscr{A}$-measurable transformation ϕ of X such that ϕ is not nonsingular while ϕ^2 is nonsingular.

Example 3.1 Set $X = \{0\} \cup \{1\} \cup [2, 3]$. Let $\mathscr{A} = \{\Delta \cap X \colon \Delta \in \mathfrak{B}(\mathbb{R}_+)\}$. Define the finite Borel measure μ on X by

$$\mu(\Delta) = \chi_\Delta(0) + \chi_\Delta(1) + m(\Delta \cap [2, 3]), \quad \Delta \in \mathscr{A},$$

where m stands for the Lebesgue measure on $\mathbb{R}$. Let ϕ be an $\mathscr{A}$-measurable transformation of X given by $\phi(0) = 2, \phi(1) = 1$ and $\phi(x) = 1$ for $x \in [2, 3]$. Since $\mu(\{2\}) = 0$ and $(\mu \circ \phi^{-1})(\{2\}) = 1$, we see that ϕ is not nonsingular. However, ϕ^2 is nonsingular because $\phi^2(x) = 1$ for all $x \in X$ and $\mu(\{1\}) > 0$.

Suppose that ϕ is a nonsingular transformation of X. In view of the measure transport theorem ([22, Theorem C, p. 163]), we have

$$\int_X |f \circ \phi|^2 \mathrm{d}\mu = \int_X |f|^2 h_\phi \mathrm{d}\mu \text{ for every } \mathscr{A}\text{-measurable function } f \colon X \to \mathbb{C}. \tag{3.4}$$

This implies that

$$\mathcal{D}(C_\phi) = L^2((1 + h_\phi)\mathrm{d}\mu), \quad \|f\|_{C_\phi}^2 = \int_X |f|^2 (1 + h_\phi)\mathrm{d}\mu, \tag{3.5}$$

$$\mathcal{D}(C_\phi^n) = L^2\left(\left(\sum_{j=0}^n h_{\phi^j}\right)\mathrm{d}\mu\right), \quad \|f\|_{C_\phi,n}^2 = \int_X |f|^2 \left(\sum_{j=0}^n h_{\phi^j}\right)\mathrm{d}\mu, \quad n \in \mathbb{Z}_+. \tag{3.6}$$

Moreover, if $\phi_1, \ldots, \phi_n$ are nonsingular transformations of X $(n \in \mathbb{N})$, then

$$\mathcal{D}(C_{\phi_n} \ldots C_{\phi_1}) = L^2\left((1 + \sum_{j=1}^n h_{\phi_1 \circ \cdots \circ \phi_j})\mathrm{d}\mu\right). \tag{3.7}$$

The following proposition is somewhat related to [17, p. 664] and [11, Lemma 6.1].

Proposition 3.2 *Let ϕ be a nonsingular transformation of X. Then C_ϕ is a closed operator and*

$$\overline{\mathcal{D}(C_\phi)} = \chi_{\mathsf{F}_\phi} L^2(\mu) \text{ with } \mathsf{F}_\phi = \{x \in X \colon h_\phi(x) < \infty\}. \tag{3.8}$$

Moreover, the following conditions are equivalent:

(i) *C_ϕ is densely defined,*
(ii) *$h_\phi < \infty$ a.e. $[\mu]$,*
(iii) *the measure $\mu \circ \phi^{-1}$ is σ-finite.*

Proof Applying (3.5), we get $C_\phi = \overline{C_\phi}$ and $\overline{\mathcal{D}(C_\phi)} \subseteq \chi_{\mathsf{F}_\phi} L^2(\mu)$. To prove the opposite inclusion $\chi_{\mathsf{F}_\phi} L^2(\mu) \subseteq \overline{\mathcal{D}(C_\phi)}$, take $f \in L^2(\mu)$ such that $f|_{X \setminus \mathsf{F}_\phi} = 0$ a.e. $[\mu]$, and set $X_n = \{x \in X \colon h_\phi(x) \leqslant n\}$ for $n \in \mathbb{N}$. Noting that $X_n \nearrow \mathsf{F}_\phi$ as $n \to \infty$, we see that $\int_X |\chi_{X_n} f|^2 (1 + h_\phi)\mathrm{d}\mu < \infty$ for all $n \in \mathbb{N}$, and $\lim_{n \to \infty} \int_X |f - \chi_{X_n} f|^2 \mathrm{d}\mu = 0$, which completes the proof of (3.8).

(i)⇔(ii) Employ (3.8).

(ii)⇔(iii) Apply (3.2) and the assumption that μ is σ-finite. □

Corollary 3.3 *Suppose that $\phi_1, \ldots, \phi_n$ are nonsingular transformations of X and $\lambda_1, \ldots, \lambda_n$ are nonzero complex numbers ($n \in \mathbb{N}$). Then, $\sum_{j=1}^{n} \lambda_j C_{\phi_j}$ is densely defined if and only if C_{ϕ_k} is densely defined for every $k = 1, \ldots, n$.*

Proof By (3.5), $\mathcal{D}(\sum_{j=1}^{n} \lambda_j C_{\phi_j}) = L^2((1 + \sum_{j=1}^{n} \mathsf{h}_{\phi_j})\mathrm{d}\mu)$, and thus the "if" part follows from Proposition 3.2 and Lemma 12.1. The "only if" part is obvious. □

4 Products of composition operators

First we give necessary and sufficient conditions for a product of composition operators to be densely defined.

Proposition 4.1 *Let $\phi_1, \ldots, \phi_n$ be nonsingular transformations of X ($2 \leqslant n < \infty$). Then, the following assertions hold*:

(i) $C_{\phi_n} \ldots C_{\phi_1}$ *is a closable operator,*

(ii) $C_{\phi_n} \ldots C_{\phi_1}$ *is densely defined if and only if $C_{\phi_1 \circ \ldots \circ \phi_k}$ is densely defined for every $k = 1, \ldots, n$,*

(iii) *if $C_{\phi_{n-1}} \ldots C_{\phi_1}$ is densely defined, then*

$$C_{\phi_1 \circ \ldots \circ \phi_k} = \overline{C_{\phi_k} \ldots C_{\phi_1}}, \quad k = 1, \ldots, n, \tag{4.1}$$

(iv) *if $C_{\phi_1 \circ \ldots \circ \phi_n}$ is densely defined, then so is the operator C_{ϕ_n},*

(v) *if $C_{\phi_n} \ldots C_{\phi_1}$ is densely defined, then so are the operators $C_{\phi_1}, \ldots, C_{\phi_n}$.*

Proof (i) Apply (3.3) and Proposition 3.2.

(ii) To prove the "if" part, assume that $C_{\phi_1 \circ \ldots \circ \phi_k}$ is densely defined for $k = 1, \ldots, n$. It follows from Proposition 3.2 that $\mathsf{h}_{\phi_1 \circ \ldots \circ \phi_k} < \infty$ a.e. $[\mu]$ for $k = 1, \ldots, n$. Applying (3.7) and Lemma 12.1 to $\rho_1 \equiv 1$ and $\rho_2 = 1 + \sum_{j=1}^{n} \mathsf{h}_{\phi_1 \circ \ldots \circ \phi_j}$ we get $\overline{\mathcal{D}(C_{\phi_n} \ldots C_{\phi_1})} = L^2(\mu)$. The "only if" part follows from (3.3) and the fact that the operators $C_{\phi_k} \ldots C_{\phi_1}, k = 1, \ldots, n$, are densely defined.

(iii) It follows from (ii) and Proposition 3.2 that $h := \sum_{j=1}^{n-1} \mathsf{h}_{\phi_1 \circ \ldots \circ \phi_j} < \infty$ a.e. $[\mu]$. Set $Y = \{x \in X : \mathsf{h}_{\phi_1 \circ \ldots \circ \phi_n}(x) < \infty\}$ and $\mathscr{A}_Y = \{\Delta \in \mathscr{A} : \Delta \subseteq Y\}$. Equip $\mathcal{D}(C_{\phi_1 \circ \ldots \circ \phi_n})$ with the graph norm of $C_{\phi_1 \circ \ldots \circ \phi_n}$ and note that the mapping

$$\Theta : \mathcal{D}(C_{\phi_1 \circ \ldots \circ \phi_n}) \ni f \longmapsto f|_Y \in L^2(Y, \mathscr{A}_Y, (1 + \mathsf{h}_{\phi_1 \circ \ldots \circ \phi_n})\mathrm{d}\mu)$$

is a well-defined unitary isomorphism (use (3.5)). It follows from Lemma 12.1 that $L^2(Y, \mathscr{A}_Y, (1 + h + \mathsf{h}_{\phi_1 \circ \ldots \circ \phi_n})\mathrm{d}\mu)$ is dense in $L^2(Y, \mathscr{A}_Y, (1 + \mathsf{h}_{\phi_1 \circ \ldots \circ \phi_n})\mathrm{d}\mu)$. Since, by (3.3) and (3.7), $\Theta(\mathcal{D}(C_{\phi_n} \ldots C_{\phi_1})) = L^2(Y, \mathscr{A}_Y, (1 + h + \mathsf{h}_{\phi_1 \circ \ldots \circ \phi_n})\mathrm{d}\mu)$, we deduce that $\overline{C_{\phi_n} \ldots C_{\phi_1}} = C_{\phi_1 \circ \ldots \circ \phi_n}$. Applying the previous argument to the systems $(C_{\phi_1}, \ldots, C_{\phi_k}), k \in \{1, \ldots, n-1\}$, we obtain (4.1).

(iv) It is sufficient to discuss the case of $n = 2$. Suppose that $C_{\phi_1 \circ \phi_2}$ is densely defined. In view of Proposition 3.2, the measure $\mu \circ (\phi_1 \circ \phi_2)^{-1}$ is σ-finite. Since $\mu \circ (\phi_1 \circ \phi_2)^{-1} = (\mu \circ \phi_2^{-1}) \circ \phi_1^{-1}$, we see that the measure $\mu \circ \phi_2^{-1}$ is σ-finite as well. Applying Proposition 3.2 again, we conclude that C_{ϕ_2} is densely defined.

(v) Apply (ii) and (iv). □

Corollary 4.2 *If C_ϕ^{n-1} is densely defined for some $n \in \mathbb{N}$, then $\overline{C_\phi^n} = C_{\phi^n}$.*

The following is an immediate consequence of (3.7) and Corollary 12.4.

Proposition 4.3 *If $\phi_1, \ldots, \phi_m$ and $\psi_1, \ldots, \psi_n$ are nonsingular transformations of X, then $\mathcal{D}(C_{\phi_m} \ldots C_{\phi_1}) \subseteq \mathcal{D}(C_{\psi_m} \ldots C_{\psi_1})$ if and only if there exists $c \in \mathbb{R}_+$ such that $\sum_{j=1}^m \mathsf{h}_{\psi_1 \circ \ldots \circ \psi_j} \leqslant c\big(1 + \sum_{j=1}^n \mathsf{h}_{\phi_1 \circ \ldots \circ \phi_j}\big)$ a.e. $[\mu]$.*

Now we give necessary and sufficient conditions for a product of composition operators to be closed.

Proposition 4.4 *Let $\phi_1, \ldots, \phi_n$ be nonsingular transformations of X ($2 \leqslant n < \infty$). Then, the following three conditions are equivalent:*

(i) $C_{\phi_n} \ldots C_{\phi_1} = C_{\phi_1 \circ \ldots \circ \phi_n}$,
(ii) $\mathcal{D}(C_{\phi_1 \circ \ldots \circ \phi_n}) \subseteq \mathcal{D}(C_{\phi_n} \ldots C_{\phi_1})$,
(iii) *there exists $c \in \mathbb{R}_+$ such that $\sum_{j=1}^{n-1} \mathsf{h}_{\phi_1 \circ \ldots \circ \phi_j} \leqslant c(1 + \mathsf{h}_{\phi_1 \circ \ldots \circ \phi_n})$ a.e. $[\mu]$.*

Moreover, any of the conditions (i) *to* (iii) *implies that*

(iv) $C_{\phi_n} \ldots C_{\phi_1}$ *is closed.*

If $C_{\phi_{n-1}} \ldots C_{\phi_1}$ is densely defined, then all the conditions (i) *to* (iv) *are equivalent.*

Proof The equivalence of (i) and (ii) is a direct consequence of (3.3). The equivalence of (ii) and (iii) follows from Proposition 4.3. That (i) implies (iv) follows from Proposition 3.2. Finally, if the product $C_{\phi_{n-1}} \ldots C_{\phi_1}$ is densely defined, then (iv) implies (i) due to Proposition 4.1 (iii). □

Corollary 4.5 *If ϕ is a nonsingular transformation of X, then the following assertions hold for all $n \in \mathbb{N}$:*

(i) C_{ϕ^n} *is densely defined if and only if $\mathsf{h}_{\phi^n} < \infty$ a.e. $[\mu]$,*
(ii) C_ϕ^n *is densely defined if and only if $\sum_{j=1}^n \mathsf{h}_{\phi^j} < \infty$ a.e. $[\mu]$,*
(iii) $C_\phi^n = C_{\phi^n}$ *if and only if there exists $c \in \mathbb{R}_+$ such that $\mathsf{h}_{\phi^k} \leqslant c(1 + \mathsf{h}_{\phi^n})$ a.e. $[\mu]$ for $k = 1, \ldots, n$.*

Proof Use Propositions 3.2, 4.1 (ii) and 4.4 (for (ii) see also [24, p. 515]). □

Corollary 4.6 *If ϕ is a nonsingular transformation of X and $\overline{\mathcal{D}(C_\phi^m)} = L^2(\mu)$ for some $m \in \mathbb{N}$, then there exists a sequence $\{X_n\}_{n=1}^\infty \subseteq \mathscr{A}$ such that*

(i) $X_n \nearrow X$ *as $n \to \infty$,*
(ii) $\mu(X_n) < \infty$ *for all $n \in \mathbb{N}$,*
(iii) $\sum_{j=1}^m \mathsf{h}_{\phi^j}(x) \leqslant n$ *for μ-a.e. $x \in X_n$ and $n \in \mathbb{N}$.*

The question of when C^∞-vectors of an operator A in a Hilbert space $\mathcal{H}$ form a dense subspace of $\mathcal{H}$ is of independent interest (cf. [28,39]). If every power of A is densely defined, then one could expect that $\mathcal{D}^\infty(A)$ is dense in $\mathcal{H}$. This is the case for any closed densely defined operator (even in a Banach space), the resolvent set of which is nonempty[2]. As shown below, this is also the case for composition operators. However, this seems to be not true in

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[2] This can be deduced from the fact that the intersection of ranges of all powers of a bounded operator which has dense range is dense in the underlying space.

general. Dropping the assumption of closedness, we can provide a simple counterexample. Indeed, take an infinite dimensional separable Hilbert space $\mathcal{H}$. Then, there exists a dense subset $\{e_n : n \in \mathbb{Z}_+\}$ of $\mathcal{H}$ which consists of linearly independent vectors. Let A be the operator in $\mathcal{H}$ whose domain is the linear span of $\{e_n : n \in \mathbb{N}\}$ and $Ae_j = e_{j-1}$ for every $j \in \mathbb{N}$. Since $\{e_n : n \geqslant k\}$ is dense in $\mathcal{H}$ for every $k \in \mathbb{Z}_+$, we deduce that the operator A^n is densely defined for every $n \in \mathbb{Z}_+$. However, $\mathcal{D}^\infty(A) = \{0\}$.

Theorem 4.7 *If ϕ is a nonsingular transformation of X, then the following conditions are equivalent*:

(i) $\mathcal{D}(C_\phi^n)$ *is dense in $L^2(\mu)$ for every $n \in \mathbb{N}$,*
(ii) $\mathcal{D}^\infty(C_\phi)$ *is dense in $L^2(\mu)$,*
(iii) $\mathcal{D}^\infty(C_\phi)$ *is a core for C_ϕ^n for every $n \in \mathbb{Z}_+$,*
(iv) $\mathcal{D}^\infty(C_\phi)$ *is dense in $(\mathcal{D}(C_\phi^n), \| \cdot \|_{C_\phi,n})$ for every $n \in \mathbb{Z}_+$.*

Proof The implications (iv)$\Rightarrow$(iii), (iii)$\Rightarrow$(ii) and (ii)$\Rightarrow$(i) are obvious.

(i)$\Rightarrow$(iv) In view of Corollary 4.5 (ii), $0 \leqslant h_{\phi^n} < \infty$ a.e. $[\mu]$ for all $n \in \mathbb{N}$. Given $n \in \mathbb{Z}_+$ we denote by $\mathcal{H}_n$, the inner product space $(\mathcal{D}(C_\phi^n), \| \cdot \|_{C_\phi,n})$. It follows from (3.6) that $\mathcal{H}_n$ is a Hilbert space which coincides with $L^2((\sum_{j=0}^n h_{\phi^j}) d\mu)$. Hence, in view of Lemma 12.1, $\mathcal{H}_{n+1}$ is a dense subspace of $\mathcal{H}_n$. Clearly, the embedding map of $\mathcal{H}_{n+1}$ into $\mathcal{H}_n$ is continuous. Applying Theorem 2.5 to the sequence $\{\mathcal{H}_n\}_{n=0}^\infty$, we conclude that $\mathcal{D}^\infty(C_\phi) = \bigcap_{i=0}^\infty \mathcal{H}_i$ is dense in $\mathcal{D}(C_\phi^n)$ with respect to the norm $\| \cdot \|_{C_\phi,n}$ for every $n \in \mathbb{Z}_+$. This completes the proof. □

Regarding Theorem 4.7, we mention the following surprising fact which can be deduced from [39, Theorem 4.5] by using Theorem 2.5 and [39, Corollaries 1.2 and 1.4].

Theorem 4.8 *Let A be an unbounded selfadjoint operator in a complex Hilbert space $\mathcal{H}$ and let $\mathfrak{N}$ be a (possibly empty) subset of $\mathbb{N} \setminus \{1\}$ such that $\mathbb{N} \setminus \mathfrak{N}$ is infinite. Then, there exists a closed symmetric operator T in $\mathcal{H}$ such that $T \subseteq A$, $\mathcal{D}^\infty(T)$ is dense in $\mathcal{H}$ and for every $k \in \mathbb{N}$, $\mathcal{D}^\infty(T)$ is a core for T^k if and only if $k \in \mathbb{N} \setminus \mathfrak{N}$.*

5 Examples

We begin by showing that Corollary 4.2 is no longer true if the assumption that C_ϕ^{n-1} is densely defined is dropped.

Example 5.1 We will demonstrate that there is a nonsingular transformation ϕ such that C_ϕ is densely defined, C_{ϕ^j} and C_ϕ^j are not densely defined for every $j \in \{2, 3, \ldots\}$, and $\overline{C_\phi^3} \subsetneq C_{\phi^3}$ (however, by Corollary 4.2, $\overline{C_\phi^2} = C_{\phi^2}$). For this, we will re-examine Example 4.2 given in [24]. Suppose that $\{a_i\}_{i=0}^\infty$, $\{b_i\}_{i=0}^\infty$ and $\{c_{i,j}\}_{i,j=0}^\infty$ are disjoint sets of distinct elements. Set $X = \{a_i\}_{i=0}^\infty \cup \{b_i\}_{i=0}^\infty \cup \{c_{i,j}\}_{i,j=0}^\infty$ and $\mathscr{A} = 2^X$. Let μ be a unique σ-finite measure on $\mathscr{A}$ determined by

$$\mu(\{x\}) = \begin{cases} 1 & \text{if } x = a_i \quad \text{for some } i \in \mathbb{Z}_+, \\ \frac{1}{2^{i+1}} & \text{if } x = b_i \quad \text{for some } i \in \mathbb{Z}_+, \\ \frac{1}{2^{j+1}} & \text{if } x = c_{i,j} \text{ for some } i, j \in \mathbb{Z}_+. \end{cases}$$

Define a nonsingular transformation ϕ of X by

$$\phi(x) = \begin{cases} a_{i+1} & \text{if } x = a_i & \text{for some } i \in \mathbb{Z}_+, \\ a_0 & \text{if } x = b_i & \text{for some } i \in \mathbb{Z}_+, \\ b_i & \text{if } x = c_{i,j} & \text{for some } i, j \in \mathbb{Z}_+. \end{cases}$$

Then $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$, and thus by Proposition 3.2, the operator C_ϕ is densely defined. Since $\mathsf{h}_{\phi^2}(a_0) = \infty$, we infer from Proposition 3.2 that C_{ϕ^2} is not densely defined. It follows from (3.7) that $\mathcal{D}(C_\phi^3) = L^2((1 + \mathsf{h}_\phi + \mathsf{h}_{\phi^2} + \mathsf{h}_{\phi^3})\mathrm{d}\mu)$. This and $\mathsf{h}_{\phi^2}(a_0) = \infty$ imply that $f(a_0) = 0$ for every $f \in \mathcal{D}(C_\phi^3)$. Since the convergence in the graph norm is stronger than the pointwise convergence, we deduce that $f(a_0) = 0$ for every $f \in \mathcal{D}(\overline{C_\phi^3})$. As $\mathcal{D}(C_{\phi^3}) = L^2((1 + \mathsf{h}_{\phi^3})\mathrm{d}\mu)$ (cf. (3.5)) and $\mathsf{h}_{\phi^3}(a_0) = 0$ (because $\phi^{-3}(\{a_0\}) = \varnothing$), we see that $\chi_{\{a_0\}} \in \mathcal{D}(C_{\phi^3}) \setminus \mathcal{D}(\overline{C_\phi^3})$. Finally, arguing as above and using the fact that $\mathsf{h}_{\phi^{j+2}}(a_j) = \infty$ for every $j \in \mathbb{Z}_+$, we conclude that C_{ϕ^j} is not densely defined for every $j \in \{2, 3, \ldots\}$. As a consequence, C_ϕ^j is not densely defined for every $j \in \{2, 3, \ldots\}$.

The composition operator C_ϕ constructed in Example 5.1 is densely defined, and its square is not densely defined; however, $\dim \mathcal{D}(C_\phi^n) = \infty$ for all $n \in \mathbb{N}$ (because $\chi_{\{a_i\}} \in \mathcal{D}(C_\phi^n)$ for all $i \geqslant n - 1$). In fact, there are more pathological examples.

Example 5.2 It was proved in [26, Theorem 4.2] that there exists a hyponormal weighted shift S on a rootless and leafless directed tree with positive weights whose square has trivial domain. By [25, Lemma 4.3.1], S is unitarily equivalent to a composition operator C. As a consequence, C is injective and hyponormal, and $\mathcal{D}(C^2) = \mathcal{D}^\infty(C) = \{0\}$ (see also [6] for a recent construction).

Regarding Proposition 4.1, we note that it may happen that the operators C_{ϕ_1} and C_{ϕ_2} are densely defined, while the operators $C_{\phi_1 \circ \phi_2}$ and $C_{\phi_2} C_{\phi_1}$ are not (even if $\phi_1 = \phi_2$, see Example 5.1). Below, we will show that for some ϕ_1 and ϕ_2 the composition operator $C_{\phi_1 \circ \phi_2}$ is densely defined (even bounded), while C_{ϕ_1} is not.

Example 5.3 Set $X = \mathbb{Z}_+$ and $\mathscr{A} = 2^X$. Let μ be the counting measure on X and let ϕ_1 and ϕ_2 be the nonsingular transformations of X given by $\phi_1(2n) = n$, $\phi_1(2n + 1) = 0$ and $\phi_2(n) = 2n$ for $n \in \mathbb{Z}_+$. Then, $\phi_1 \circ \phi_2$ is the identity transformation of X, and hence, $C_{\phi_1 \circ \phi_2}$ is the identity operator on $L^2(\mu)$. However, since $\mu(\phi_1^{-1}(\{0\})) = \infty$, the measure $\mu \circ \phi_1^{-1}$ is not σ-finite, and thus by Proposition 3.2, the operator C_{ϕ_1} is not densely defined.

Our next aim is to provide examples showing that the equality $C_\phi^n = C_{\phi^n}$ which appears in Corollary 4.5 (iii) does not hold in general even if $\mathcal{D}^\infty(C_\phi)$ is dense in $L^2(\mu)$ (which is not the case for the operator given in Example 5.1).

Example 5.4 Set $X = \mathbb{N}$ and $\mathscr{A} = 2^X$. Let μ be a counting measure on X and let $\{J_n\}_{n=1}^\infty$ be a partition of X. Define a nonsingular transformation ϕ of X by $\phi(x) = \min J_{n^2}$ for $x \in J_n$ and $n \in \mathbb{N}$. Set $\mathbb{N}_s = \{n^2 : n \in \mathbb{N}\}$ and note that

$$X = \{1\} \sqcup \bigsqcup_{q \in \mathbb{N} \setminus \mathbb{N}_s} \{q^{2^n} : n \in \mathbb{Z}_+\}, \tag{5.1}$$

where all terms in (5.1) are pairwise disjoint (they are equivalence classes under the equivalence relation $\sim$ given by: $p \sim q$ if and only if $p^{2^m} = q^{2^n}$ for some $m, n \in \mathbb{Z}_+$). Since

$\mathsf{h}_{\phi^j}(x) = \mathrm{card}(\phi^{-j}(\{x\}))$ for $x \in X$ and $j \in \mathbb{N}$, we infer from (5.1) that for all $j \in \mathbb{N}$ and $x \in X$ (m appearing below varies over the set of integers)

$$\mathsf{h}_{\phi^j}(x) = \begin{cases} \mathrm{card}(J_1) & \text{if } x = \min J_1, \\ \mathrm{card}(J_{q^{2m-j}}) & \text{if } x = \min J_{q^{2m}} \text{ with } q \in \mathbb{N} \setminus \mathbb{N}_s \text{ and } m \geqslant j, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

By (5.1), (5.2), Proposition 3.2 and Theorem 4.7, the following are equivalent:

- $\mathrm{card}(J_k) < \aleph_0$ for every $k \in \mathbb{N}$,
- C_ϕ is densely defined,
- C_ϕ^n is densely defined for some $n \in \mathbb{N}$,
- C_ϕ^n is densely defined for every $n \in \mathbb{N}$,
- $\mathcal{D}^\infty(C_\phi)$ is dense in $L^2(\mu)$.

The above, combined with (5.2) and Proposition 4.4, implies that if C_ϕ is densely defined, then for every integer $n \geqslant 2$, C_ϕ^n is closed if and only if there exists $c \in \mathbb{R}_+$ such that

$$\mathrm{card}(J_{q^{2s}}) \leqslant c, \quad s = 0, \dots, n-2, \ q \in \mathbb{N} \setminus \mathbb{N}_s,$$
$$\mathrm{card}(J_{q^{2s+1}}) \leqslant c\big(1 + \mathrm{card}(J_{q^{2s}})\big), \quad s \in \mathbb{Z}_+, \ q \in \mathbb{N} \setminus \mathbb{N}_s.$$

Using this and an induction argument, one can prove that if C_ϕ is densely defined, then either C_ϕ^n is closed for every integer $n \geqslant 1$, or C_ϕ^n is not closed for every integer $n \geqslant 2$. Summarizing, if we choose a partition $\{J_i\}_{i=1}^\infty$ of X such that J_n is finite for every $n \in \mathbb{N}$, and $\sup\{\mathrm{card}(J_q) : q \in \mathbb{N} \setminus \mathbb{N}_s\} = \aleph_0$ (which is possible), then $\mathcal{D}^\infty(C_\phi)$ is dense in $L^2(\mu)$ and C_ϕ^n is not closed for every integer $n \geqslant 2$. On the other hand, if $\kappa \geqslant 2$ is any fixed integer and a partition $\{J_i\}_{i=1}^\infty$ of X is selected so that J_1 is finite and $\mathrm{card}(J_{q^{2n}}) = \kappa^n$ for all $n \in \mathbb{Z}_+$ and $q \in \mathbb{N} \setminus \mathbb{N}_s$ (which is also possible), then $\mathcal{D}^\infty(C_\phi)$ is dense in $L^2(\mu)$ and C_ϕ^n is closed and unbounded for every $n \in \mathbb{N}$.

6 Injectivity of C_ϕ

In this section, we provide necessary and sufficient conditions for a composition operator to be injective. The following set plays an important role in our considerations.

$$\mathsf{N}_\phi = \{x \in X : \mathsf{h}_\phi(x) = 0\}.$$

The following description of the kernel of C_ϕ follows immediately from (3.4).

Proposition 6.1 *If $\phi \colon X \to X$ is nonsingular, then $\mathcal{N}(C_\phi) = \chi_{\mathsf{N}_\phi} L^2(\mu)$.*

Proposition 6.2 *Let ϕ be a nonsingular transformation of X. Consider the following four conditions:*

(i) $\mathcal{N}(C_\phi) = \{0\}$,
(ii) $\mu(\mathsf{N}_\phi) = 0$,
(iii) $\chi_{\mathsf{N}_\phi} \circ \phi = \chi_{\mathsf{N}_\phi}$ *a.e.* $[\mu]$,
(iv) $\mathcal{N}(C_\phi) \subseteq \mathcal{N}(C_\phi^*)$.

Then the conditions (i), (ii) *and* (iii) *are equivalent. Moreover, if C_ϕ is densely defined, then the conditions* (i) *to* (iv) *are equivalent.*

Proof (i)⇔(ii) Apply Proposition 6.1 and the σ-finiteness of μ.

(ii)⇒(iii) Since ϕ is nonsingular, we have $\mu(N_\phi) = 0$ and $\mu(\phi^{-1}(N_\phi)) = 0$, which implies that $\mu(N_\phi \bigtriangleup \phi^{-1}(N_\phi)) = 0$. The latter is equivalent to (iii).

(iii)⇒(ii) By the measure transport theorem, we have

$$\mu(N_\phi) = \int_X \chi_{N_\phi} d\mu = \int_X \chi_{N_\phi} \circ \phi d\mu = \int_X \chi_{N_\phi} h_\phi d\mu = 0.$$

Now suppose that C_ϕ is densely defined.

(i)⇒(iv) Obvious.

(iv)⇒(ii) Let $\{X_n\}_{n=1}^\infty$ be as in Corollary 4.6 (with $m = 1$).

Then, by (3.4), we see that $\chi_{X_n}, \chi_{N_\phi \cap X_n} \in \mathcal{D}(C_\phi)$ and $\|C_\phi(\chi_{N_\phi \cap X_n})\|^2 = \int_{N_\phi \cap X_n} h_\phi d\mu = 0$ for all $n \in \mathbb{N}$, which together with our assumption that $\mathcal{N}(C_\phi) \subseteq \mathcal{N}(C_\phi^*)$ yields

$$0 = \langle \chi_{N_\phi \cap X_n}, C_\phi \chi_{X_n} \rangle = \int_{N_\phi \cap X_n} \chi_{X_n} \circ \phi d\mu = \mu(N_\phi \cap X_n \cap \phi^{-1}(X_n))$$

for all $n \in \mathbb{N}$. Since $N_\phi \cap X_n \cap \phi^{-1}(X_n) \nearrow N_\phi$ as $n \to \infty$, the continuity of measure implies that $\mu(N_\phi) = 0$. This completes the proof. □

Corollary 6.3 *If C_ϕ is hyponormal, then $\mathcal{N}(C_\phi) = \{0\}$.*

Proof It follows from the definition of hyponormality that $\mathcal{N}(C_\phi) \subseteq \mathcal{N}(C_\phi^*)$. This and Proposition 6.2 complete the proof. □

Corollary 6.4 *If C_ϕ is formally normal, then*

$$\mathcal{D}(C_\phi) \cap \mathcal{N}(C_\phi^*) = \{0\}.$$

Proof If $f \in \mathcal{D}(C_\phi) \cap \mathcal{N}(C_\phi^*)$, then $\|C_\phi f\| = \|C_\phi^* f\| = 0$, which means that $f \in \mathcal{N}(C_\phi)$. Applying Corollary 6.3 completes the proof. □

It turns out that composition of h_ϕ with ϕ is positive a.e. $[\mu]$ (see also the proof of [23, Corollary 5]).

Proposition 6.5 *If $\phi\colon X \to X$ is nonsingular, then $h_\phi \circ \phi > 0$ a.e. $[\mu]$.*

Proof Note that $\mu(\phi^{-1}(N_\phi)) = \int_X \chi_{N_\phi} \circ \phi d\mu = \int_X \chi_{N_\phi} h_\phi d\mu = 0$. This combined with $\phi^{-1}(N_\phi) = \{x \in X\colon h_\phi(\phi(x)) = 0\}$ completes the proof. □

Corollary 6.6 *If ϕ is a nonsingular transformation of X and $h_\phi \circ \phi = h_\phi$ a.e. $[\mu]$, then $\mathcal{N}(C_\phi) = \{0\}$.*

Proof Apply Propositions 6.1 and 6.5. □

7 The polar decomposition

Given an $\mathscr{A}$-measurable function $u\colon X \to \mathbb{C}$, we denote by M_u the operator of multiplication by u in $L^2(\mu)$ defined by

$$\mathcal{D}(M_u) = \{f \in L^2(\mu)\colon u \cdot f \in L^2(\mu)\},$$
$$M_u f = u \cdot f, \quad f \in \mathcal{D}(M_u).$$

The operator M_u is a normal operator (cf. [3, Section 7.2].

The polar decomposition of C_ϕ can be explicitly described as follows.

Proposition 7.1 *Suppose that the composition operator C_ϕ is densely defined and $C_\phi = U|C_\phi|$ is its polar decomposition. Then,*

(i) $|C_\phi| = M_{\mathsf{h}_\phi^{1/2}}$,

(ii) *the initial space of U is given by* [3]

$$\overline{\mathcal{R}(|C_\phi|)} = \left\{ \mathsf{h}_\phi^{1/2} f \colon f \in L^2(\mathsf{h}_\phi \mathrm{d}\mu) \right\}, \tag{7.1}$$

(iii) *the final space of U is given by*

$$\overline{\mathcal{R}(C_\phi)} = \{ f \circ \phi \colon f \in L^2(\mathsf{h}_\phi \mathrm{d}\mu) \}, \tag{7.2}$$

(iv) *the partial isometry U is given by* [4]

$$Ug = \frac{g \circ \phi}{(\mathsf{h}_\phi \circ \phi)^{1/2}}, \quad g \in L^2(\mu), \tag{7.3}$$

(v) *the adjoint U^* of U is given by*

$$U^*g = \mathsf{h}_\phi^{1/2} \cdot V^{-1}Pg, \quad g \in L^2(\mu),$$

where $V \colon L^2(\mathsf{h}_\phi \mathrm{d}\mu) \to \overline{\mathcal{R}(C_\phi)}$ is a unitary operator defined by $Vf = f \circ \phi$ for $f \in L^2(\mathsf{h}_\phi \mathrm{d}\mu)$ and P is the orthogonal projection of $L^2(\mu)$ onto $\overline{\mathcal{R}(C_\phi)}$.

Proof (i) We will show that $C_\phi^* C_\phi \subseteq M_{\mathsf{h}_\phi}$. Let $\{X_n\}_{n=1}^\infty$ be as in Corollary 4.6 (with $m = 1$). Take $f \in \mathcal{D}(C_\phi^* C_\phi)$ and fix $n \in \mathbb{N}$. By (3.5), $\chi_\Delta \in \mathcal{D}(C_\phi)$ whenever $\Delta \in \mathscr{A}$ and $\Delta \subseteq X_n$. Thus, for every such Δ, we have

$$\int_\Delta C_\phi^* C_\phi f \mathrm{d}\mu = \langle C_\phi^* C_\phi f, \chi_\Delta \rangle = \langle C_\phi f, C_\phi \chi_\Delta \rangle \overset{(3.2)}{=} \int_\Delta f \mathsf{h}_\phi \mathrm{d}\mu.$$

Since both functions $(C_\phi^* C_\phi f)\chi_{X_n}$ and $(f\mathsf{h}_\phi)\chi_{X_n}$ are in $L^1(\mu)$, we deduce that $C_\phi^* C_\phi f = f\mathsf{h}_\phi$ a.e. $[\mu]$ on X_n. This and $X_n \nearrow X$ give $C_\phi^* C_\phi f = f\mathsf{h}_\phi$ a.e. $[\mu]$. As a consequence, we have $C_\phi^* C_\phi \subseteq M_{\mathsf{h}_\phi}$. Since both are selfadjoint operators, they are equal. Thus $|C_\phi| = M_{\mathsf{h}_\phi}^{1/2} = M_{\mathsf{h}_\phi^{1/2}}$.

(ii) By [3, Section 8.1] and Proposition 6.1, we have

$$\overline{\mathcal{R}(|C_\phi|)} = \mathcal{N}(|C_\phi|)^\perp = \mathcal{N}(C_\phi)^\perp = \chi_{X \setminus \mathsf{N}_\phi} L^2(\mu), \tag{7.4}$$

which as easily seen gives (7.1).

(iii) By (3.4) and (ii), the mapping $W \colon \overline{\mathcal{R}(|C_\phi|)} \to L^2(\mu)$ given by

$$W(\mathsf{h}_\phi^{1/2} f) = f \circ \phi, \quad f \in L^2(\mathsf{h}_\phi \mathrm{d}\mu), \tag{7.5}$$

is a well-defined isometry. Using (i), we verify that $W|_{\mathcal{R}(|C_\phi|)} = U|_{\mathcal{R}(|C_\phi|)}$, which implies that $\overline{\mathcal{R}(C_\phi)} = \mathcal{R}(U) = \mathcal{R}(W)$. Hence, (iii) holds and, by (7.5), we have

$$U^*(f \circ \phi) = \mathsf{h}_\phi^{1/2} f, \quad f \in L^2(\mathsf{h}_\phi \mathrm{d}\mu). \tag{7.6}$$

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[3] Note that the mapping $L^2(\mathsf{h}_\phi \mathrm{d}\mu) \ni f \mapsto \mathsf{h}_\phi^{1/2} f \in L^2(\mu)$ is an isometry.

[4] Recall that $\mathsf{h}_\phi \circ \phi > 0$ a.e. $[\mu]$ (cf. Proposition 6.5).

(iv) Applying the measure transport theorem to the restriction of ϕ to the full μ-measure set on which $\mathsf{h}_\phi \circ \phi$ is positive (cf. Proposition 6.5), we get

$$\int_X \frac{|g \circ \phi|^2}{\mathsf{h}_\phi \circ \phi} \mathrm{d}\mu = \int_{X \setminus \mathsf{N}_\phi} |g|^2 \mathrm{d}\mu, \quad g \in L^2(\mu). \tag{7.7}$$

This and Proposition 6.1 imply that the mapping $\tilde{U} \colon L^2(\mu) \ni g \mapsto \frac{g \circ \phi}{(\mathsf{h}_\phi \circ \phi)^{1/2}} \in L^2(\mu)$ is a contraction such that $\mathcal{N}(\tilde{U}) = \chi_{\mathsf{N}_\phi} L^2(\mu) = \mathcal{N}(|C_\phi|)$. Hence, by (7.4) and (7.7), $\tilde{U}$ is an isometry on $\overline{\mathcal{R}(|C_\phi|)}$. Clearly, by (i), $\tilde{U}|C_\phi|g = C_\phi g$ for $g \in \mathcal{D}(C_\phi) = \mathcal{D}(|C_\phi|)$, which implies that $U = \tilde{U}$.

(v) By (3.4) and (7.2), V is a well-defined unitary operator. If $g \in L^2(\mu)$, then by (iii), $Pg = f \circ \phi$ a.e. $[\mu]$ for some $f \in L^2(\mathsf{h}_\phi \mathrm{d}\mu)$. Thus, by $\mathcal{N}(U^*) = \mathcal{R}(I - P)$ and (7.6), we have

$$U^* g = U^* Pg = U^*(f \circ \phi) = \mathsf{h}_\phi^{1/2} f = \mathsf{h}_\phi^{1/2} \cdot V^{-1} Pg.$$

This completes the proof. □

Regarding Proposition 7.1, we note that the formulas for $|C_\phi|$ and $\overline{\mathcal{R}(C_\phi)}$ are well-known in the case of bounded composition operators (cf. [23, Lemma 1]). The formula (7.3) has appeared in [10, p. 387] in the context of bounded operators without proof.

Corollary 7.2 *Suppose that C_ϕ is densely defined and $g \in L^2(\mu)$. Then g belongs to $\overline{\mathcal{R}(C_\phi)}$ if and only if one of the following equivalent conditions holds[5]:*

(i) *there is an $\mathscr{A}$-measurable function $f \colon X \to \mathbb{C}$ such that $g = f \circ \phi$ a.e. $[\mu]$,*
(ii) *there is a $\phi^{-1}(\mathscr{A})$-measurable function $f \colon X \to \mathbb{C}$ such that $g = f$ a.e. $[\mu]$,*
(iii) *g is $(\phi^{-1}(\mathscr{A}))^\mu$-measurable,*
(iv) *for every Borel set Δ in $\mathbb{C}$ there exists $\Delta' \in \mathscr{A}$ such that*

$$\mu\big(g^{-1}(\Delta) \,\triangle\, \phi^{-1}(\Delta')\big) = 0.$$

In particular, $\overline{\mathcal{R}(C_\phi)} = L^2(\mu|_{(\phi^{-1}(\mathscr{A}))^\mu})$.

Proof Apply (3.4), (7.2), (13.1), (13.2) and Lemma 13.3. □

Corollary 7.3 *If C_ϕ is densely defined, then the map $V \colon L^2(\mathsf{h}_\phi \mathrm{d}\mu) \to \overline{\mathcal{R}(C_\phi)}$ given by $Vf = f \circ \phi$ for $f \in L^2(\mathsf{h}_\phi \mathrm{d}\mu)$ is a well-defined unitary operator such that*

$$\mathcal{D}(C_\phi^*) = \big\{ g \in L^2(\mu) \colon \mathsf{h}_\phi \cdot V^{-1} Pg \in L^2(\mu) \big\},$$
$$C_\phi^* g = \mathsf{h}_\phi \cdot V^{-1} Pg, \quad g \in \mathcal{D}(C_\phi^*), \tag{7.8}$$

where P is the orthogonal projection of $L^2(\mu)$ onto $\overline{\mathcal{R}(C_\phi)} = L^2(\mu|_{(\phi^{-1}(\mathscr{A}))^\mu})$.

Proof If $C_\phi = U|C_\phi|$ is the polar decomposition of C_ϕ, then $C_\phi^* = |C_\phi|U^*$. This, Proposition 7.1 and Corollary 7.2 complete the proof. □

Remark 7.4 Concerning Corollary 7.3, we observe that, in view of (13.3), $\mathsf{E}(g) := \mathsf{E}(g|\phi^{-1}(\mathscr{A})) = Pg$ a.e. $[\mu]$ and thus $C_\phi^* g = \mathsf{h}_\phi \cdot (\mathsf{E}(g) \circ \phi^{-1})$ for every $g \in \mathcal{D}(C_\phi^*)$, where $\mathsf{E}(g) \circ \phi^{-1}$ is understood as in [11, Lemma 6.4].

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[5] See "Appendix B" for definitions and notation.

8 Normality and quasinormality

It turns out that the characterizations of quasinormality and normality of unbounded composition operators take the same forms as those for bounded ones.

Proposition 8.1 *If C_ϕ is densely defined, then C_ϕ is quasinormal if and only if $\mathsf{h}_\phi = \mathsf{h}_\phi \circ \phi$ a.e. $[\mu]$.*

Proof Let $C_\phi = U|C_\phi|$ be the polar decomposition of C_ϕ. Suppose that C_ϕ is quasinormal. Then, by [49, Proposition 1], $U|C_\phi| \subseteq |C_\phi|U$. Let $\{X_n\}_{n=1}^\infty$ be as in Corollary 4.6 (with $m = 1$). Then, by (3.5), $\{\chi_{X_n}\}_{n=1}^\infty \subseteq \mathcal{D}(C_\phi)$, which together with Proposition 7.1 implies that for every $n \in \mathbb{N}$,

$$\chi_{X_n} \circ \phi = U|C_\phi|\chi_{X_n} = |C_\phi|U\chi_{X_n} = \left(\frac{\mathsf{h}_\phi}{\mathsf{h}_\phi \circ \phi}\right)^{1/2} \chi_{X_n} \circ \phi \quad \text{a.e. } [\mu].$$

Since $X_n \nearrow X$ as $n \to \infty$, we conclude that $\mathsf{h}_\phi = \mathsf{h}_\phi \circ \phi$ a.e. $[\mu]$.

For the converse, take $f \in \mathcal{D}(|C_\phi|)$. By (7.3) and $\mathcal{D}(|C_\phi|) = \mathcal{D}(C_\phi)$, we have

$$\mathsf{h}_\phi^{1/2} U f = \left(\frac{\mathsf{h}_\phi}{\mathsf{h}_\phi \circ \phi}\right)^{1/2} f \circ \phi = f \circ \phi \in L^2(\mu).$$

Hence, by Proposition 7.1 (i), $f \in \mathcal{D}(|C_\phi|U)$ and $|C_\phi|Uf = C_\phi f = U|C_\phi|f$. Therefore, $U|C_\phi| \subseteq |C_\phi|U$. Applying [49, Proposition 1] completes the proof. ∎

Proposition 8.2 *If $\overline{\mathcal{D}(C_\phi)} = L^2(\mu)$, then the following are equivalent*:

(i) C_ϕ *is normal*,
(ii) $\mathsf{h}_\phi = \mathsf{h}_\phi \circ \phi$ *a.e.* $[\mu]$ *and* $\mathcal{N}(C_\phi^*) \subseteq \mathcal{N}(C_\phi)$,
(iii) $\mathsf{h}_\phi = \mathsf{h}_\phi \circ \phi$ *a.e.* $[\mu]$ *and* $\mathcal{N}(C_\phi^*) = \{0\}$,
(iv) $\mathsf{h}_\phi = \mathsf{h}_\phi \circ \phi$ *a.e.* $[\mu]$ *and for every* $\Delta \in \mathscr{A}$ *there exists* $\Delta' \in \mathscr{A}$ *such that* $\mu(\Delta \, \triangle \, \phi^{-1}(\Delta')) = 0$.

Moreover, if C_ϕ is normal, then $\mathcal{N}(C_\phi) = \{0\}$ and $\mathsf{h}_\phi > 0$ a.e. $[\mu]$.

Proof (i) $\Rightarrow$(iii) Since normal operators are always quasinormal, we infer from Proposition 8.1 that $\mathsf{h}_\phi = \mathsf{h}_\phi \circ \phi$ a.e. $[\mu]$. Clearly, $\mathcal{N}(C_\phi) = \mathcal{N}(C_\phi^*)$. That $\mathcal{N}(C_\phi^*) = \{0\}$ follows from Corollary 6.6.
(iii)$\Rightarrow$(ii) Evident.
(ii)$\Rightarrow$(i) This is a direct consequence of Proposition 8.1 and Theorem 2.2.
(iii)$\Leftrightarrow$(iv) Since $\mathcal{N}(C_\phi^*) = \{0\}$ if and only if $\mathcal{R}(C_\phi)$ is dense in $L^2(\mu)$, it suffices to apply Corollary 7.2, Lemma 13.2 and (13.1).

The "moreover" part follows from the above and Proposition 6.5. ∎

9 Formal normality

In this section, we show that formally normal composition operators are normal. The proof of this result relies on a characterization of composition operators for which $\mathsf{h}_{\phi^2} = \mathsf{h}_\phi^2$ a.e. $[\mu]$. We also provide an alternative proof which depends heavily on the fact that quasinormal formally normal operators are normal.

We begin by proving a result which is of measure-theoretic nature. We refer the reader to "Appendix B" for the definition and basic properties of $\mathsf{E}(\cdot|\phi^{-1}(\mathscr{A}))$ (which makes sense if $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$). For brevity, we write $\mathsf{E}(\cdot) = \mathsf{E}(\cdot|\phi^{-1}(\mathscr{A}))$.

Lemma 9.1 *If ϕ is a nonsingular transformation of X such that $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$, then the following two conditions are equivalent for every $n \in \mathbb{N}$:*

(i) $\mathsf{h}_{\phi^{n+1}} = \mathsf{h}_{\phi^n} \cdot \mathsf{h}_\phi$ *a.e.* $[\mu]$,
(ii) $\mathsf{E}(\mathsf{h}_{\phi^n}) = \mathsf{h}_{\phi^n} \circ \phi$ *a.e.* $[\mu|_{\phi^{-1}(\mathscr{A})}]$.

Proof (i)$\Rightarrow$(ii) Note that

$$\int_{\phi^{-1}(\Delta)} \mathsf{E}(\mathsf{h}_{\phi^n}) \mathrm{d}\mu = \int_{\phi^{-1}(\Delta)} \mathsf{h}_{\phi^n} \mathrm{d}\mu = \mu((\phi^{-n}(\phi^{-1}(\Delta))))$$

$$= \mu((\phi^{-(n+1)}(\Delta))) = \int_\Delta \mathsf{h}_{\phi^{n+1}} \mathrm{d}\mu = \int_\Delta \mathsf{h}_{\phi^n} \cdot \mathsf{h}_\phi \mathrm{d}\mu$$

$$= \int_X (\chi_\Delta \circ \phi)(\mathsf{h}_{\phi^n} \circ \phi) \mathrm{d}\mu = \int_{\phi^{-1}(\Delta)} \mathsf{h}_{\phi^n} \circ \phi \mathrm{d}\mu, \quad \Delta \in \mathscr{A},$$

which, by the uniqueness assertion in the Radon-Nikodym theorem, implies (ii).

Arguing as above, we can prove the reverse implication. $\qquad\square$

The next two lemmas are key ingredients of the proof of Theorem 9.4. The first lemma shows that $\mathsf{h}_{\phi^2} = \mathsf{h}_\phi^2$ a.e. $[\mu]$ if and only if C_ϕ is "formally normal" on its range. This result is of some independent interest.

Lemma 9.2 *Suppose that C_ϕ is densely defined. Then, the following two conditions are equivalent:*

(i) C_ϕ^2 *is densely defined,* $\mathcal{D}(C_\phi^2) \subseteq \mathcal{D}(C_\phi^* C_\phi)$ *and* $\|C_\phi^2 f\| = \|C_\phi^* C_\phi f\|$ *for every* $f \in \mathcal{D}(C_\phi^2)$,
(ii) $\mathsf{h}_{\phi^2} = \mathsf{h}_\phi^2$ *a.e.* $[\mu]$.

Proof (i)$\Rightarrow$(ii) Take $f \in \mathcal{D}(C_\phi^2)$. Then, by Proposition 7.1(i), we have

$$\int_X |f|^2 \mathsf{h}_\phi^2 \mathrm{d}\mu = \|M_{\mathsf{h}_\phi} f\|^2 = \|C_\phi^* C_\phi f\|^2 = \|C_{\phi^2} f\|^2 = \int_X |f|^2 \mathsf{h}_{\phi^2} \mathrm{d}\mu. \tag{9.1}$$

Let $\{X_n\}_{n=1}^\infty$ be as in Corollary 4.6 (with $m = 2$). Then, $\{\chi_{X_n}\}_{n=1}^\infty \subseteq \mathcal{D}(C_\phi^2)$ and

$$\int_\Delta \mathsf{h}_\phi^2 \mathrm{d}\mu \overset{(9.1)}{=} \int_\Delta \mathsf{h}_{\phi^2} \mathrm{d}\mu < \infty, \quad \Delta \in \mathscr{A}, \ \Delta \subseteq X_n, \ n \in \mathbb{N},$$

which implies that $\mathsf{h}_\phi^2 = \mathsf{h}_{\phi^2}$ a.e. $[\mu]$ on X_n for every $n \in \mathbb{N}$. Hence, (ii) holds.

(ii)$\Rightarrow$(i) Since, by Proposition 3.2, $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$, we infer from (ii) and Corollary 4.5 that C_ϕ^2 is densely defined. Now we take $f \in \mathcal{D}(C_\phi^2)$. Then, by (3.6), $\int_X |f \mathsf{h}_\phi|^2 \mathrm{d}\mu = \int_X |f|^2 \mathsf{h}_{\phi^2} \mathrm{d}\mu < \infty$, which means that $f \in \mathcal{D}(M_{\mathsf{h}_\phi}) = \mathcal{D}(C_\phi^* C_\phi)$. Arguing as in (9.1), we obtain (i). $\qquad\square$

It is worth pointing out that implication (i)$\Rightarrow$(ii) of Lemma 9.2 is no longer true if we drop the assumption that C_ϕ^2 is densely defined. To see this, it is enough to consider a nonzero densely defined composition operator whose square has trivial domain and to apply Corollary 4.5. For examples of such operators, we refer the reader to recent articles [26] and [6].

Lemma 9.3 *If ϕ is nonsingular transformation of X, then the following conditions are equivalent*:

(i) C_ϕ *is normal*,

(ii) C_ϕ *is formally normal and* $\overline{\mathcal{D}(C_\phi^2)} = L^2(\mu)$.

Proof (i)$\Rightarrow$(ii) Evident (since powers of normal operators are normal, cf. [3]).
(ii)$\Rightarrow$(i) First we will show that

$$\mathcal{D}(C_\phi) \cap \overline{\mathcal{R}(C_\phi)} = \mathcal{D}(C_\phi^*) \cap \overline{\mathcal{R}(C_\phi)}. \tag{9.2}$$

Indeed, if $g \in \mathcal{D}(C_\phi^*) \cap \overline{\mathcal{R}(C_\phi)}$, then by (7.2) there exists $f \in L^2(\mathsf{h}_\phi \mathrm{d}\mu)$ such that $g = f \circ \phi$ a.e. $[\mu]$. It follows from Corollary 7.3 that $\mathsf{h}_\phi f = \mathsf{h}_\phi V^{-1} g \in L^2(\mu)$. This combined with the fact that (ii) implies condition (i) of Lemma 9.2 leads to

$$\int\limits_X |g \circ \phi|^2 \mathrm{d}\mu = \int\limits_X |f \circ \phi^2|^2 \mathrm{d}\mu = \int\limits_X |f|^2 \mathsf{h}_{\phi^2} \mathrm{d}\mu = \int\limits_X |f\mathsf{h}_\phi|^2 \mathrm{d}\mu < \infty,$$

which means that $g \in \mathcal{D}(C_\phi)$. This yields (9.2).

Let P be the orthogonal projection of $L^2(\mu)$ onto $\overline{\mathcal{R}(C_\phi)}$. We will prove that

$$P\mathcal{D}(C_\phi) \subseteq \mathcal{D}(C_\phi). \tag{9.3}$$

Indeed, take $f \in \mathcal{D}(C_\phi)$. Since $(I - P)f \in \mathcal{N}(C_\phi^*)$ and $\mathcal{D}(C_\phi) \subseteq \mathcal{D}(C_\phi^*)$, we get $Pf \in \mathcal{D}(C_\phi^*) \cap \overline{\mathcal{R}(C_\phi)}$. Hence, by (9.2), $Pf \in \mathcal{D}(C_\phi)$, which proves (9.3).

It follows from (9.3) and Corollary 6.4 that

$$\mathcal{D}(C_\phi) \subseteq \left(\mathcal{D}(C_\phi) \cap \mathcal{N}(C_\phi^*)\right) \oplus \left(\mathcal{D}(C_\phi) \cap \overline{\mathcal{R}(C_\phi)}\right) = \mathcal{D}(C_\phi) \cap \overline{\mathcal{R}(C_\phi)},$$

which together with $\overline{\mathcal{D}(C_\phi)} = L^2(\mu)$ imply that $\overline{\mathcal{R}(C_\phi)} = L^2(\mu)$. Therefore, by (9.2), $\mathcal{D}(C_\phi) = \mathcal{D}(C_\phi^*)$, which completes the proof. $\square$

Now we show that the assumption $\overline{\mathcal{D}(C_\phi^2)} = L^2(\mu)$ can be dropped without affecting the conclusion of Lemma 9.3.

Theorem 9.4 *Let ϕ be a nonsingular transformation of X. Then, C_ϕ is normal if and only if C_ϕ is formally normal.*

Proof It suffices to prove the "if" part. Suppose C_ϕ is formally normal. Let $\{X_n\}_{n=1}^\infty \subseteq \mathscr{A}$ be as in Corollary 4.6 (with $m = 1$). Take $\Delta \in \mathscr{A}$. Since $\{\chi_{X_n \cap \Delta}\}_{n=1}^\infty \subseteq \mathcal{D}(C_\phi)$, we get (see also Remark 7.4)

$$\int\limits_{X_n \cap \Delta} \mathsf{h}_\phi \mathrm{d}\mu \overset{(3.4)}{=} \|C_\phi(\chi_{X_n \cap \Delta})\|^2 = \|C_\phi^*(\chi_{X_n \cap \Delta})\|^2$$

$$\overset{(7.8)}{=} \int\limits_X \mathsf{h}_\phi^2 \cdot |V^{-1}\mathsf{E}(\chi_{X_n \cap \Delta})|^2 \mathrm{d}\mu$$

$$= \int\limits_X (\mathsf{h}_\phi \circ \phi)(\mathsf{E}(\chi_{X_n \cap \Delta}))^2 \mathrm{d}\mu, \quad n \in \mathbb{N}.$$

Using (13.6) and Lebesgue's monotone convergence theorem, we obtain

$$\int\limits_\Delta \mathsf{h}_\phi \mathrm{d}\mu = \int\limits_X (\mathsf{h}_\phi \circ \phi)(\mathsf{E}(\chi_\Delta))^2 \mathrm{d}\mu, \quad \Delta \in \mathscr{A},$$

which yields

$$\int_{\phi^{-1}(\varDelta)} \mathsf{h}_\phi \mathrm{d}\mu = \int_{\phi^{-1}(\varDelta)} \mathsf{h}_\phi \circ \phi \mathrm{d}\mu, \quad \varDelta \in \mathscr{A}.$$

This in turn implies that $\mathsf{E}(\mathsf{h}_\phi) = \mathsf{h}_\phi \circ \phi$ a.e. $[\mu|_{\phi^{-1}(\mathscr{A})}]$. By Lemma 9.1, $\mathsf{h}_{\phi^2} = \mathsf{h}_\phi^2$ a.e. $[\mu]$. Since $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$, we see that $\mathsf{h}_\phi + \mathsf{h}_{\phi^2} < \infty$ a.e. $[\mu]$. Using Corollary 4.5 (ii), we get $\overline{\mathcal{D}(C_\phi^2)} = L^2(\mu)$. Applying Lemma 9.3 completes the proof.   □

Remark 9.5 The "if" part of Theorem 9.4 can be proved using a different more advanced way. Indeed, assume that C_ϕ is formally normal. Then, by the polarization formula, we have

$$\int_X f\bar{g}\mathsf{h}_\phi \mathrm{d}\mu = \langle C_\phi f, C_\phi g\rangle = \langle C_\phi^* f, C_\phi^* g\rangle$$

$$\overset{(7.8)}{=} \int_X \mathsf{h}_\phi^2 \big(V^{-1}\mathsf{E}(f)\big)\big(\overline{V^{-1}\mathsf{E}(g)}\big)\mathrm{d}\mu$$

$$= \int_X (\mathsf{h}_\phi \circ \phi)\,\mathsf{E}(f)\,\overline{\mathsf{E}(g)}\mathrm{d}\mu, \quad f, g \in \mathcal{D}(C_\phi). \tag{9.4}$$

By Propositions 3.2 and 6.1, and Corollary 6.3, we can assume that $0 < \mathsf{h}_\phi(x) < \infty$ for all $x \in X$. Let $\{X_n\}_{n=1}^\infty \subseteq \mathscr{A}$ be as in Corollary 4.6 (with $m = 1$). Set $Y_n = \{x \in X_n : \mathsf{h}_\phi(x) \geqslant 1/n\}$ for $n \in \mathbb{N}$. Clearly, $Y_n \nearrow X$ as $n \to \infty$. Take $\varDelta \in \mathscr{A}$. Since $\{\chi_{Y_n}\}_{n=1}^\infty, \left\{\mathsf{h}_\phi^{-1} \cdot \chi_{Y_n \cap \varDelta}\right\}_{n=1}^\infty \subseteq \mathcal{D}(C_\phi)$, we can substitute $f = \mathsf{h}_\phi^{-1} \cdot \chi_{Y_n \cap \varDelta}$ and $g = \chi_{Y_n}$ into (9.4). What we get is

$$\mu(Y_n \cap \varDelta) = \int_X (\mathsf{h}_\phi \circ \phi)\,\mathsf{E}(\mathsf{h}_\phi^{-1} \cdot \chi_{Y_n \cap \varDelta})\,\mathsf{E}(\chi_{Y_n})\mathrm{d}\mu$$

$$\overset{(13.5)}{=} \int_{Y_n \cap \varDelta} \frac{\mathsf{h}_\phi \circ \phi}{\mathsf{h}_\phi}\,\mathsf{E}(\chi_{Y_n})\mathrm{d}\mu, \quad n \in \mathbb{N}.$$

Using (13.6) and Lebesgue's monotone convergence theorem, we obtain

$$\mu(\varDelta) = \int_\varDelta \frac{\mathsf{h}_\phi \circ \phi}{\mathsf{h}_\phi}\mathrm{d}\mu, \quad \varDelta \in \mathscr{A},$$

which implies that $\mathsf{h}_\phi \circ \phi = \mathsf{h}_\phi$ a.e. $[\mu]$. By Proposition 8.1, C_ϕ is quasinormal. Since quasinormal formally normal operators are normal (cf. [49, Corollary 4]; see also [52]), the proof is complete.

10 Generating Stieltjes moment sequences

We begin by proving two lemmas which are main tools in the proof of Theorem 10.4 below.

Lemma 10.1 *Suppose ϕ is a nonsingular transformation of X and $\{\mathcal{E}_n\}_{n=1}^\infty$ is a sequence of subsets of $L^2(\mu)$ satisfying the following three conditions*:

(i) $\mathcal{E}_n$ *fulfills* (12.5), $\mathcal{E}_n \subseteq \mathcal{D}(C_\phi^n)$ *and* $\overline{\mathcal{E}_n} = L^2(\mu)$ *for all* $n \in \mathbb{N}$,

(ii) $\left[\|C_\phi^{i+j} f\|^2\right]_{i,j=0}^n \geqslant 0$ *for all* $f \in \mathcal{E}_{2n}$ *and* $n \in \mathbb{N}$,

(iii) $\left[\|C_\phi^{i+j+1} f\|^2\right]_{i,j=0}^n \geqslant 0$ *for all* $f \in \mathcal{E}_{2n+1}$ *and* $n \in \mathbb{N}$.

Then the following three assertions hold:

(a) $\{h_{\phi^n}(x)\}_{n=0}^{\infty}$ *is a Stieltjes moment sequence for μ-a.e. $x \in X$,*
(b) $C_{\phi}^n = C_{\phi^n}$ *for every $n \in \mathbb{N}$,*
(c) $\mathcal{D}^{\infty}(C_{\phi})$ *is a core for C_{ϕ}^n for every $n \in \mathbb{Z}_+$.*

Proof (a) By (i) and Corollary 4.5 (ii), there is no loss of generality in assuming that $0 \leqslant h_{\phi^n}(x) < \infty$ for all $x \in X$ and $n \in \mathbb{Z}_+$. Using (3.4), we obtain

$$\int_X \left| \sum_{i,j=0}^{n} \alpha_i \bar{\alpha}_j h_{\phi^{i+j}} \right| |f|^2 d\mu < \infty, \quad f \in \mathcal{D}(C_{\phi}^{2n}), \ \{\alpha_i\}_{i=0}^{n} \subseteq \mathbb{C}, \ n \in \mathbb{Z}_+. \quad (10.1)$$

If $\{\alpha_i\}_{i=0}^{n} \subseteq \mathbb{C}$, then by (i) and (ii) we have

$$0 \leqslant \sum_{i,j=0}^{n} \|C_{\phi}^{i+j} f\|^2 \alpha_i \bar{\alpha}_j = \int_X \left(\sum_{i,j=0}^{n} \alpha_i \bar{\alpha}_j h_{\phi^{i+j}} \right) |f|^2 d\mu, \quad f \in \mathcal{E}_{2n}, \ n \in \mathbb{N}.$$

Combining (i), (10.1) and Corollary 12.6 (with $\mathcal{E} = \mathcal{E}_{2n}$), we see that

$$\sum_{i,j=0}^{n} \alpha_i \bar{\alpha}_j h_{\phi^{i+j}} \geqslant 0 \text{ a.e. } [\mu] \text{ for all } n \in \mathbb{N} \text{ and } \{\alpha_i\}_{i=0}^{n} \subseteq \mathbb{C}.$$

Let Q be a countable dense subset of $\mathbb{C}$. Then, there exists a set $\Delta_0 \in \mathscr{A}$ such that $\mu(X \setminus \Delta_0) = 0$ and $\sum_{i,j=0}^{n} \alpha_i \bar{\alpha}_j h_{\phi^{i+j}}(x) \geqslant 0$ for all $n \in \mathbb{N}$, $\{\alpha_i\}_{i=0}^{n} \subseteq Q$ and $x \in \Delta_0$. As Q is dense in $\mathbb{C}$, we conclude that $[h_{\phi^{i+j}}(x)]_{i,j=0}^{n} \geqslant 0$ for all $n \in \mathbb{N}$ and $x \in \Delta_0$. Using (iii) and applying a similar reasoning as above, we infer that there exists a set $\Delta_1 \in \mathscr{A}$ such that $\mu(X \setminus \Delta_1) = 0$ and $[h_{\phi^{i+j+1}}(x)]_{i,j=0}^{n} \geqslant 0$ for all $n \in \mathbb{N}$ and $x \in \Delta_1$. Employing (2.5) yields (a).

(b) By (a), there exists $\Delta \in \mathscr{A}$ such that $\mu(X \setminus \Delta) = 0$, $h_{\phi^0}(x) = 1$ and $\{h_{\phi^n}(x)\}_{n=0}^{\infty}$ is a Stieltjes moment sequence for every $x \in \Delta$. Hence, for every $x \in \Delta$ there exists a Borel probability measure μ_x on $\mathbb{R}_+$ such that $h_{\phi^n}(x) = \int_{\mathbb{R}_+} s^n d\mu_x(s)$ for all $n \in \mathbb{Z}_+$. This yields

$$\left(\sum_{j=0}^{n} h_{\phi^j} \right)(x) = \int_{\mathbb{R}_+} \left(\sum_{j=0}^{n} s^j \right) d\mu_x(s)$$

$$= \int_{[0,1)} \left(\sum_{j=0}^{n} s^j \right) d\mu_x(s) + \int_{[1,\infty)} \left(\sum_{j=0}^{n} s^j \right) d\mu_x(s)$$

$$\leqslant (n+1) \int_{[0,1)} 1 d\mu_x(s) + (n+1) \int_{[1,\infty)} s^n d\mu_x(s)$$

$$\leqslant (n+1)(1 + h_{\phi^n})(x), \quad x \in \Delta, \ n \in \mathbb{N}.$$

Hence, the domains of C_{ϕ}^n and C_{ϕ^n} coincide for all $n \in \mathbb{N}$. By (3.3), this gives (b).

(c) Apply (i) and Theorem 4.7. This completes the proof. □

Lemma 10.2 *Suppose that ϕ is a nonsingular transformation of X satisfying the following two conditions*:

(i) $\mathcal{D}(C_{\phi}^n)$ *is dense in $L^2(\mu)$ for every $n \in \mathbb{N}$,*
(ii) $[\|C_{\phi}^{i+j} f\|^2]_{i,j=0}^{n} \geqslant 0$ *for all $f \in \mathcal{D}(C_{\phi}^{2n})$ and $n \in \mathbb{N}$.*

Then the assertions (a), (b) *and* (c) *of Lemma* 10.1 *hold.*

Proof Set $\mathcal{E}_n = \mathcal{D}(C_\phi^n)$ for $n \in \mathbb{N}$. According to (3.6), each $\mathcal{E}_n$ satisfies (12.5). Substituting $C_\phi f$ for f in (ii) implies that the hypothesis (iii) of Lemma 10.1 is satisfied. Applying Lemma 10.1 completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Corollary 10.3 *If C_ϕ is subnormal and $\overline{\mathcal{D}(C_\phi^n)} = L^2(\mu)$ for all $n \in \mathbb{N}$, then the assertions* (a), (b) *and* (c) *of Lemma* 10.1 *hold.*

Proof Apply Proposition 2.4 and Lemma 10.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

The following theorem completely characterizes composition operators that generate Stieltjes moment sequences. It should be compared with Lambert's characterizations of bounded subnormal composition operators (cf. [31]). In particular, condition (ii) of Theorem 10.4 is the Lambert condition, which in the bounded case is equivalent to subnormality.

Theorem 10.4 *If ϕ is a nonsingular transformation of X, then the following conditions are equivalent*:

(i) C_ϕ *generates Stieltjes moment sequences,*
(ii) $\{h_{\phi^n}(x)\}_{n=0}^\infty$ *is a Stieltjes moment sequence for μ-a.e. $x \in X$,*
(iii) $\overline{\mathcal{D}(C_\phi^k)} = L^2(\mu)$ *for all $k \in \mathbb{N}$, and $\{\mu(\phi^{-n}(\Delta))\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $\Delta \in \mathscr{A}$ such that $\mu(\phi^{-k}(\Delta)) < \infty$ for all $k \in \mathbb{Z}_+$,*
(iv) $h_{\phi^n} < \infty$ *a.e. $[\mu]$ for all $n \in \mathbb{N}$ and $L(p) \geqslant 0$ a.e. $[\mu]$ whenever $p(t) \geqslant 0$ for all $t \in \mathbb{R}_+$, where $L \colon \mathbb{C}[t] \to \mathcal{M}$ is a linear mapping determined by*[6]

$$L(t^n) = h_{\phi^n}, \quad n \in \mathbb{Z}_+;$$

here $\mathbb{C}[t]$ is the set of all complex polynomials in one real variable t and $\mathcal{M}$ is the set of all $\mathscr{A}$-measurable complex functions on X.

Moreover, if (i) *holds, then $C_\phi^n = C_{\phi^n}$ and $\mathcal{D}^\infty(C_\phi)$ is a core for C_ϕ^n for all $n \in \mathbb{Z}_+$.*

Proof (i)⇒(ii) Set $\mathcal{E}_n = \mathcal{D}^\infty(C_\phi)$ for $n \in \mathbb{N}$. By (2.5), (3.6) and Lemma 10.1, we see that the condition (ii) and the "moreover" part hold.

(ii)⇒(i) Take $f \in \mathcal{D}^\infty(C_\phi)$, $n \in \mathbb{Z}_+$ and $\{\alpha_i\}_{i=0}^n \subseteq \mathbb{C}$. Then, by (2.5), we have

$$\sum_{i,j=0}^n \alpha_i \overline{\alpha}_j \|C_\phi^{i+j} f\|^2 \overset{(3.4)}{=} \int_X \left(\sum_{i,j=0}^n \alpha_i \overline{\alpha}_j h_{\phi^{i+j}}(x) \right) |f(x)|^2 d\mu(x) \geqslant 0.$$

Applying the above to $C_\phi f$ in place of f, we deduce that the sequences $\{\|C_\phi^k f\|^2\}_{k=0}^\infty$ and $\{\|C_\phi^{k+1} f\|^2\}_{k=0}^\infty$ are positive definite. Therefore, by (2.5), $\{\|C_\phi^k f\|^2\}_{k=0}^\infty$ is a Stieltjes moment sequence. It follows from Corollary 4.5(ii) and Theorem 4.7 that $\mathcal{D}^\infty(C_\phi)$ is dense in $L^2(\mu)$.

(i)⇒(iii) Evident (because $\chi_\Delta \in \mathcal{D}^\infty(C_\phi)$ for every Δ as in (iii)).

(iii)⇒(i) By Theorem 4.7, the set $\mathcal{D}^\infty(C_\phi)$ is dense in $L^2(\mu)$. Consider a simple $\mathscr{A}$-measurable function $u = \sum_{i=1}^k \alpha_i \chi_{\Delta_i}$, where $\{\alpha_i\}_{i=1}^k$ are positive real numbers and $\{\Delta_i\}_{i=1}^k$ are pairwise disjoint sets in $\mathscr{A}$. Suppose that u is in $\mathcal{D}^\infty(C_\phi)$. Then, by the measure transport theorem, $\{\chi_{\Delta_i}\}_{i=1}^k \subseteq \mathcal{D}^\infty(C_\phi)$ and

$$\|C_\phi^n u\|^2 = \sum_{i,j=1}^k \alpha_i \alpha_j \int_{\Delta_i \cap \Delta_j} h_{\phi^n} d\mu = \sum_{i=1}^k \alpha_i^2 \int_{\Delta_i} h_{\phi^n} d\mu \overset{(3.4)}{=} \sum_{i=1}^k \alpha_i^2 \mu(\phi^{-n}(\Delta_i))$$

---

[6] To make the definition of L correct we have to modify h_{ϕ^n} so that $0 \leqslant h_{\phi^n}(x) < \infty$ for all $x \in X$ and $n \in \mathbb{Z}_+$.

for all $n \in \mathbb{Z}_+$. Hence, by (iii), we have

$$\{\|C_\phi^n u\|^2\}_{n=0}^\infty \text{ is a Stieltjes moment sequence for every simple}$$
$$\text{nonnegative } \mathscr{A}\text{-measurable function } u \in \mathcal{D}^\infty(C_\phi). \tag{10.2}$$

Now take $f \in \mathcal{D}^\infty(C_\phi)$. Then, there exists a sequence $\{u_n\}_{n=1}^\infty$ of simple $\mathscr{A}$-measurable functions $u_n \colon X \to \mathbb{R}_+$ such that $u_n(x) \leqslant u_{n+1}(x) \leqslant |f(x)|$ and $\lim_{k\to\infty} u_k(x) = |f(x)|$ for all $n \in \mathbb{N}$ and $x \in X$. This implies that $\{u_n\}_{n=1}^\infty \subseteq \mathcal{D}^\infty(C_\phi)$ and, by Lebesgue's monotone convergence theorem,

$$\|C_\phi^n f\|^2 = \int_X |f|^2 \mathsf{h}_{\phi^n} \mathrm{d}\mu = \lim_{k\to\infty} \int_X u_k^2 \mathsf{h}_{\phi^n} \mathrm{d}\mu = \lim_{k\to\infty} \|C_\phi^n u_k\|^2, \quad n \in \mathbb{Z}_+.$$

Since the class of Stieltjes moment sequences is closed under the operation of taking pointwise limits (cf. (2.5)), we infer from (10.2) that $\{\|C_\phi^n f\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence.

(ii)⇒(iv) If $p \in \mathbb{C}[t]$ is such that $p(t) \geqslant 0$ for all $t \in \mathbb{R}_+$, then there exist $q_1, q_2 \in \mathbb{C}[t]$ such that $p(t) = t|q_1(t)|^2 + |q_2(t)|^2$ for all $t \in \mathbb{R}$ (see [36, Problem 45, p. 78]). This fact combined with (2.5) implies that $L(p) \geqslant 0$ a.e. $[\mu]$.

(iv)⇒(ii) Let Q be a countable dense subset of $\mathbb{C}$. If $q \in \mathbb{C}[t]$ is a polynomial with coefficients in Q, then the polynomials $p_1 := |q|^2$ and $p_2 := t|q|^2$ are nonnegative on $\mathbb{R}_+$. Hence, $L(p_i) \geqslant 0$ a.e. $[\mu]$ for $i = 1, 2$. Since Q is countable, this implies that there exists $\Delta \in \mathscr{A}$ such that $\mu(X \setminus \Delta) = 0$,

$$0 \leq \mathsf{h}_{\phi^n}(x) < \infty, \sum_{i,j=0}^n \alpha_i \overline{\alpha}_j \mathsf{h}_{\phi^{i+j}}(x) \geqslant 0 \text{ and } \sum_{i,j=0}^n \alpha_i \overline{\alpha}_j \mathsf{h}_{\phi^{i+j+1}}(x) \geqslant 0 \tag{10.3}$$

for all $n \in \mathbb{Z}_+$, $\{\alpha_i\}_{i=0}^n \subseteq Q$ and $x \in \Delta$. As Q is dense in $\mathbb{C}$, we see that (10.3) holds for all $n \in \mathbb{Z}_+$, $\{\alpha_i\}_{i=0}^n \subseteq \mathbb{C}$ and $x \in \Delta$. This and (2.5) complete the proof. $\qquad \square$

11 Conclusion

We close the paper by pointing out that there exists a composition operator generating Stieltjes moment sequences which is not subnormal and even not hyponormal. Such an operator can be constructed on the basis of a weighted shift on a directed tree with one branching vertex (cf. [25, Section 4.3]). In view of Theorem 10.4, any composition operator C_ϕ which generates Stieltjes moment sequences, in particular the aforementioned, satisfies the conditions (ii), (iii) and (iv) of this theorem as well as its "moreover" part (specifically, $\mathcal{D}^\infty(C_\phi)$ is a core for C_ϕ^n for every $n \in \mathbb{Z}_+$, which is considerably more than is required in Definition 2.3). Therefore, none of the Lambert characterizations of subnormality of bounded composition operators (cf. [31]) is valid in the unbounded case. It is worth mentioning that the above example is built over the discrete measure space. However, it can be immediately adapted to the context of measures which are equivalent to the Lebesgue measure on $[0, \infty)$ by applying [24, Theorem 2.7].

12 Appendix A

Here we gather some useful properties of L^2-spaces. The first two lemmas seem to be folklore. For the reader's convenience, we include their proofs.

Lemma 12.1 *Let $(X, \mathscr{A}, \mu)$ be a measure space and let ρ_1, ρ_2 be $\mathscr{A}$-measurable scalar functions on X such that $0 < \rho_i < \infty$ a.e. $[\mu]$ for $i = 1, 2$. Then, $L^2(\rho_1 d\mu) \cap L^2(\rho_2 d\mu)$ is dense[7] in $L^2(\rho_i d\mu)$ for $i = 1, 2$.*

Proof Since $L^2(\rho_1 d\mu) \cap L^2(\rho_2 d\mu) = L^2((\rho_1 + \rho_2) d\mu)$, we can assume that $0 < \rho_2(x) \leqslant \rho_1(x) < \infty$ for all $x \in X$. Take $\Delta \in \mathscr{A}$ such that $\chi_\Delta \in L^2(\rho_2 d\mu)$. Set $\Delta_n = \{x \in \Delta : \rho_1(x) \leqslant n \text{ and } \frac{1}{n} \leqslant \rho_2(x)\}$ for $n \in \mathbb{N}$. Note that $\{\chi_{\Delta_n}\}_{n=1}^\infty \subseteq L^2(\rho_1 d\mu)$. Since $\Delta_n \nearrow \Delta$ as $n \to \infty$, we see that $\{\chi_{\Delta_n}\}_{n=1}^\infty$ converges to χ_Δ in $L^2(\rho_2 d\mu)$. Applying [38, Theorem 3.13] completes the proof.   □

Note that Lemma 12.1 is no longer true if one of the density functions ρ_1 and ρ_2 takes the value ∞ on a set of positive measure μ (even if $\rho_2 \leqslant \rho_1$). Employing Lemma 12.1 and the Radon-Nikodym theorem, we get the following.

Corollary 12.2 *Let $(X, \mathscr{A}, \mu_1)$ and $(X, \mathscr{A}, \mu_2)$ be σ-finite measure spaces. If the measures μ_1 and μ_2 are mutually absolutely continuous, then $L^2(\mu_1) \cap L^2(\mu_2)$ is dense in $L^2(\mu_i)$ for $i = 1, 2$.*

Corollary 12.2 is no longer true if one of the measures μ_1 and μ_2 is not σ-finite.

Lemma 12.3 *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and ρ_1, ρ_2 be $\mathscr{A}$-measurable scalar functions on X such that $0 < \rho_1 \leqslant \infty$ a.e. $[\mu]$ and $0 \leqslant \rho_2 \leqslant \infty$ a.e. $[\mu]$. Then, the following two conditions are equivalent*:

(i) $\int_X |f|^2 \rho_2 d\mu < \infty$ *for every $\mathscr{A}$-measurable function $f : X \to \mathbb{C}$ such that $\int_X |f|^2 \rho_1 d\mu < \infty$,*
(ii) *there exists $c \in \mathbb{R}_+$ such that $\rho_2 \leqslant c \rho_1$ a.e. $[\mu]$.*

Proof (i)⇒(ii) Without loss of generality we can assume that $\rho_1 < \infty$ a.e. $[\mu]$. We can also assume that $\rho_2 < \infty$ a.e. $[\mu]$ (indeed, otherwise, since $\rho_1 < \infty$ a.e. $[\mu]$ and μ is σ-finite, there exist $\Omega \in \mathscr{A}$ and $k \in \mathbb{N}$ such that $\rho_1(x) \leqslant k$ and $\rho_2(x) = \infty$ for all $x \in \Omega$, and $0 < \mu(\Omega) < \infty$; hence $\int_\Omega \rho_1 d\mu < \infty$ and $\int_\Omega \rho_2 d\mu = \infty$, which is a contradiction). Finally, replacing ρ_2 by $\frac{\rho_2}{\rho_1}$ if necessary, we can assume that $\rho_1(x) = 1$ for all $x \in X$. Now applying the Landau-Riesz summability theorem (cf. [5, Problem G, p. 398]), we obtain (ii). The implication (ii)⇒(i) is obvious.   □

Corollary 12.4 *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and ρ_1, ρ_2 be $\mathscr{A}$-measurable scalar functions on X such that $0 < \rho_i \leqslant \infty$ a.e. $[\mu]$ for $i = 1, 2$. Then, $L^2(\rho_1 d\mu) \subseteq L^2(\rho_2 d\mu)$ if and only if there exists $c \in \mathbb{R}_+$ such that $\rho_2 \leqslant c \rho_1$ a.e. $[\mu]$.*

Implication (i)⇒(ii) of Lemma 12.3 is not true if we drop the assumption that $\rho_1 > 0$ a.e. $[\mu]$. Corollary 12.4 is no longer true if μ is not σ-finite (e.g., $X = \mathbb{N}$, $\mathscr{A} = 2^X$, $\mu(\{1\}) = 1$, $\mu(\{i\}) = \infty$ for $i \geqslant 2$, $\rho_1 \equiv 1$ and $\rho_2(n) = n$ for $n \in X$).
The following lemma generalizes [24, Lemma 2.1].

---

[7] This makes sense because the measures $\rho_1 d\mu$ and $\rho_2 d\mu$ are mutually absolutely continuous.

Lemma 12.5 *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space, $\mathcal{E}$ be a dense subset of $L^2(\mu)$ and $h \colon X \to \mathbb{C}$ be an $\mathscr{A}$-measurable function such that*

$$\int_\Delta |h||f|^2 \mathrm{d}\mu < \infty \text{ and } \int_\Delta h|f|^2 \mathrm{d}\mu \geqslant 0 \text{ for all } f \in \mathcal{E} \text{ and } \Delta \in \mathscr{A}_*, \tag{12.1}$$

where $\mathscr{A}_* = \{\Delta \in \mathscr{A} \colon \mu(\Delta) < \infty\}$. Then, $h \geqslant 0$ a.e. $[\mu]$.

Proof Set $\Xi_f = \{x \in X \colon |f(x)| > 0\}$ for $f \in \mathcal{E}$. First, we will show that

$$h(x) \geqslant 0 \text{ for } \mu\text{-a.e. } x \in \Xi_f \text{ and for every } f \in \mathcal{E}. \tag{12.2}$$

Indeed, fix $f \in \mathcal{E}$ and set $\Xi_{f,k} = \left\{x \in X \colon |f(x)| \geqslant \frac{1}{k}\right\}$ for $k \in \mathbb{N}$. It follows from Chebyshev's inequality that $\Xi_{f,k} \in \mathscr{A}_*$ for $k \in \mathbb{N}$. Applying (12.1), we deduce that

$$\int_{\Xi_{f,k}} |h||f|^2 \mathrm{d}\mu < \infty \text{ and } \int_{\Xi_{f,k} \cap \Delta} h|f|^2 \mathrm{d}\mu \geqslant 0 \text{ for all } \Delta \in \mathscr{A} \text{ and } k \in \mathbb{N}.$$

This implies that $h \geqslant 0$ a.e. $[\mu]$ on $\Xi_{f,k}$ for every $k \in \mathbb{N}$. Since $\Xi_{f,k} \nearrow \Xi_f$ as $k \to \infty$, we conclude that $h \geqslant 0$ a.e. $[\mu]$ on Ξ_f.

Set $\Sigma = \{x \in X \colon h(x) \geqslant 0\}$. Suppose that, contrary to our claim, $\mu(X \setminus \Sigma) > 0$. As μ is σ-finite, there exists a set $\Omega \in \mathscr{A}$ such that $\Omega \subseteq X \setminus \Sigma$ and $0 < \mu(\Omega) < \infty$. This means that $\chi_\Omega \in L^2(\mu)$. Since $\mathcal{E}$ is dense in $L^2(\mu)$, there exists a sequence $\{f_n\}_{n=1}^\infty \subseteq \mathcal{E}$ which converges to χ_Ω in $L^2(\mu)$. Passing to a subsequence if necessary, we can assume that the sequence $\{f_n\}_{n=1}^\infty$ converges a.e. $[\mu]$ to χ_Ω, and thus

$$\lim_{n \to \infty} f_n(x) = \chi_\Omega(x) = 1 \text{ for } \mu\text{-a.e. } x \in \Omega. \tag{12.3}$$

It follows from (12.2) that $\mu(\Omega \cap \Xi_f) = 0$ for every $f \in \mathcal{E}$. Applying this property to $f = f_n$ $(n \in \mathbb{N})$, we see that $\mu\left(\Omega \cap \bigcup_{n=1}^\infty \Xi_{f_n}\right) = 0$, which means that

$$f_n(x) = 0 \text{ for all } n \in \mathbb{N} \text{ and for } \mu\text{-a.e. } x \in \Omega. \tag{12.4}$$

Combining (12.3) with (12.4), we conclude that $\mu(\Omega) = 0$, a contradiction. □

Applying Lemma 12.5 to h and $-h$, we see that this lemma remains valid if "$\geqslant$" is replaced by "$=$".

Corollary 12.6 *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and $\mathcal{E}$ be a dense subset of $L^2(\mu)$ such that*

$$f\chi_\Delta \in \mathcal{E} \text{ for all } f \in \mathcal{E} \text{ and } \Delta \in \mathscr{A}_*. \tag{12.5}$$

If $h \colon X \to \mathbb{C}$ is an $\mathscr{A}$-measurable function such that $\int_X |h||f|^2 \mathrm{d}\mu < \infty$ and $\int_X h|f|^2 \mathrm{d}\mu \geqslant 0$ for all $f \in \mathcal{E}$, then $h \geqslant 0$ a.e. $[\mu]$.

13 Appendix B

In this appendix, we describe (mostly without proofs) some results from measure theory which play an important role in our analysis of composition operators. Let $(X, \mathscr{A}, \mu)$ be a fixed measure space and let $\mathscr{B} \subseteq \mathscr{A}$ be a σ-algebra. We say that $\mathscr{B}$ is *relatively μ-complete* if $\mathscr{A}_0 \subseteq \mathscr{B}$, where $\mathscr{A}_0 = \{\Delta \in \mathscr{A} \colon \mu(\Delta) = 0\}$ (cf. [23]). It is easily seen that the smallest

relatively μ-complete σ-algebra containing $\mathscr{B}$, denoted by $\mathscr{B}^\mu$, coincides with the σ-algebra generated by $\mathscr{B} \cup \mathscr{A}_0$, and that

$$\mathscr{B}^\mu = \{\Delta \in \mathscr{A} \mid \exists \Delta' \in \mathscr{B}: \mu(\Delta \triangle \Delta') = 0\}. \tag{13.1}$$

The $\mathscr{B}^\mu$-measurable functions are described below (cf. [38, Lemma 1, p. 169]).

Lemma 13.1 *A function $f: X \to \mathbb{C}$ is $\mathscr{B}^\mu$-measurable if and only if there exists a $\mathscr{B}$-measurable function $g: X \to \mathbb{C}$ such that $f = g$ a.e. $[\mu]$.*

By the above lemma $L^2(\mu|_{\mathscr{B}})$ is a subset of $L^2(\mu)$ if and only if $\mathscr{B} = \mathscr{B}^\mu$. The question of when $L^2(\mu|_{\mathscr{B}}) = L^2(\mu)$ has a simple answer (σ-finiteness is essential!).

Lemma 13.2 *If μ is σ-finite and $\mathscr{B}$ is relatively μ-complete, then $L^2(\mu|_{\mathscr{B}}) = L^2(\mu)$ if and only if $\mathscr{B} = \mathscr{A}$.*

Proof Suppose that $L^2(\mu|_{\mathscr{B}}) = L^2(\mu)$ and $\Delta \in \mathscr{A} \setminus \mathscr{B}$. Since μ is σ-finite, we may assume that $\mu(\Delta) < \infty$. Then, $\chi_\Delta \in L^2(\mu) \setminus L^2(\mu|_{\mathscr{B}})$, a contradiction. $\qquad\square$

Given a transformation ϕ of X, we set $\phi^{-1}(\mathscr{A}) = \{\phi^{-1}(\Delta): \Delta \in \mathscr{A}\}$.

Lemma 13.3 *Suppose that $\phi: X \to X$ is an $\mathscr{A}$-measurable transformation and $f: X \to \mathbb{C}$ is an arbitrary function. Then f is $(\phi^{-1}(\mathscr{A}))^\mu$-measurable if and only if there exists an $\mathscr{A}$-measurable function $u: X \to \mathbb{C}$ such that $f = u \circ \phi$ a.e $[\mu]$.*

Proof Applying the following well-known fact

> a function $g: X \to \mathbb{C}$ is $\phi^{-1}(\mathscr{A})$-measurable if and only if there
>
> exists an $\mathscr{A}$-measurable function $u: X \to \mathbb{C}$ such that $g = u \circ \phi$, $\qquad$ (13.2)

and Lemma 13.1 completes the proof. $\qquad\square$

Let $P_{\mathscr{B}}$ be the orthogonal projection of $L^2(\mu)$ onto its closed subspace $L^2(\mu|_{\mathscr{B}^\mu})$. Set $\mathscr{B}_* = \{\Delta \in \mathscr{B}: \mu(\Delta) < \infty\}$. It follows from Lemma 13.1 that

> for every $f \in L^2(\mu)$ there exists a unique (up to sets of measure zero)
>
> $\mathscr{B}$-measurable function $\mathsf{E}(f|\mathscr{B}): X \to \mathbb{C}$ such that $P_{\mathscr{B}} f = \mathsf{E}(f|\mathscr{B})$ a.e. $[\mu]$. (13.3)

This and the fact that $\langle \chi_\Delta, f \rangle = \langle \chi_\Delta, P_{\mathscr{B}} f \rangle$ for all $f \in L^2(\mu)$ and $\Delta \in \mathscr{B}_*$ yield

$$\int_\Delta f \, \mathrm{d}\mu = \int_\Delta \mathsf{E}(f|\mathscr{B}) \mathrm{d}\mu, \quad f \in L^2(\mu), \ \Delta \in \mathscr{B}_*. \tag{13.4}$$

Now suppose that $\mu|_{\mathscr{B}}$ is σ-finite. It follows from (13.4) that $\mathsf{E}(f|\mathscr{B}) \geqslant 0$ a.e. $[\mu]$ whenever $f \geqslant 0$ a.e. $[\mu]$. By applying the standard approximation procedure, we see that for every $\mathscr{A}$-measurable function $f: X \to [0, \infty]$ there exists a unique (up to sets of measure zero) $\mathscr{B}$-measurable function $\mathsf{E}(f|\mathscr{B}): X \to [0, \infty]$ such that the equality in (13.4) holds for every $\Delta \in \mathscr{B}$. Thus for every $\mathscr{A}$-measurable function $f: X \to [0, \infty]$ and for every $\mathscr{B}$-measurable function $g: X \to [0, \infty]$ we have

$$\int_X gf \, \mathrm{d}\mu = \int_X g\mathsf{E}(f|\mathscr{B}) \mathrm{d}\mu. \tag{13.5}$$

We call $\mathsf{E}(f|\mathscr{B})$ the *conditional expectation* of f with respect to $\mathscr{B}$ (cf. [37]). Clearly,

$$\text{if } 0 \leqslant f_n \nearrow f \text{ and } f, f_n \text{ are } \mathscr{A}\text{-measurable, then } \mathsf{E}(f_n|\mathscr{B}) \nearrow \mathsf{E}(f|\mathscr{B}), \quad (13.6)$$

where $g_n \nearrow g$ means that for μ-a.e. $x \in X$, the sequence $\{g_n(x)\}_{n=1}^{\infty}$ is monotonically increasing and convergent to $g(x)$.

Concluding "Appendix B", we note that if μ is σ-finite and $\phi\colon X \to X$ is a nonsingular transformation such that $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$ (equivalently, C_ϕ is densely defined), then the measure $\mu|_{\phi^{-1}(\mathscr{A})}$ is σ-finite (cf. Proposition 3.2). Thus we may consider the conditional expectation $\mathsf{E}(\cdot|\phi^{-1}(\mathscr{A}))$ with respect to $\phi^{-1}(\mathscr{A})$.

References

1. Berg, C., Christensen, J.P.R., Ressel, P.: Harmonic Analysis on Semigroups. Springer, Berlin (1984)
2. Biriuk, G., Coddington, E.A.: Normal extensions of unbounded formally normal operators. J. Math. Mech. **12**, 617–638 (1964)
3. Birman, M.Sh., Solomjak, M.Z.: Spectral theory of selfadjoint operators in Hilbert space. D. Reidel Publishing Co., Dordrecht (1987)
4. Brown, A.: On a class of operators. Proc. Am. Math. Soc. **4**, 723–728 (1953)
5. Brown, A., Pearcy, C.: Introduction to operator theory. I. Elements of functional analysis. Graduate texts in mathematics, No. 55. Springer, New York (1977)
6. Budzyński, P.: A note on unbounded hyponormal composition operators in L^2-spaces, submitted
7. Budzyński, P., Jabłoński, Z.J., Jung, I.B., Stochel, J.: Unbounded subnormal weighted shifts on directed trees. J. Math. Anal. Appl. **394**, 819–834 (2012)
8. Budzyński, P., Stochel, J.: Joint subnormality of n-tuples and C_0-semigroups of composition operators on L^2-spaces. Studia Math. **179**, 167–184 (2007)
9. Budzyński, P., Stochel, J.: Joint subnormality of n-tuples and C_0-semigroups of composition operators on L^2-spaces, II. Studia Math. **193**, 29–52 (2009)
10. Burnap, Ch., Jung, I.B., Lambert, A.: Separating partial normality classes with composition operators. J. Operator Theory **53**, 381–397 (2005)
11. Campbell, J.T., Hornor, W.E.: Seminormal composition operators. J. Operator Theory **29**, 323–343 (1993)
12. Coddington, E.A.: Normal extensions of formally normal operators. Pacific J. Math. **10**, 1203–1209 (1960)
13. Coddington, E.A.: Formally normal operators having no normal extension. Canad. J. Math. **17**, 1030–1040 (1965)
14. Conway, J.B.: The theory of subnormal operators. Mathematical Surveys and Monographs, Providence, Rhode Island (1991)
15. Daniluk, A., Stochel, J.: Seminormal composition operators induced by affine transformations. Hokkaido Math. J. **26**, 377–404 (1997)
16. Dibrell, P., Campbell, J.T.: Hyponormal powers of composition operators. Proc. Am. Math. Soc. **102**, 914–918 (1988)
17. Dunford, N., Schwartz, J.T.: Linear operators, Part I. Interscience, New York (1958)
18. Embry, M.R., Lambert, A.: Subnormal weighted translation semigroups. J. Funct. Anal. **24**, 268–275 (1977)
19. Embry-Wardrop, M.R., Lambert, A.: Measurable transformations and centered composition operators. Proc. Royal Irish Acad. **90A**, 165–172 (1990)
20. Embry-Wardrop, M.R., Lambert, A.: Subnormality for the adjoint of a composition operator on L^2. J. Operator Theory **25**, 309–318 (1991)
21. Halmos, P.R.: Normal dilations and extensions of operators. Summa Brasil. Math. **2**, 125–134 (1950)
22. Halmos, P.R.: Measure Theory. Princeton, van Nostrand (1956)
23. Harrington, D., Whitley, R.: Seminormal composition operators. J. Operator Theory **11**, 125–135 (1984)
24. Jabłonski, Z.J.: Hyperexpansive composition operators. Math. Proc. Cambridge Philos. Soc. **135**, 513–526 (2003)
25. Jabłoński, Z.J., Jung, I.B., Stochel, J.: A non-hyponormal operator generating Stieltjes moment sequences. J. Funct. Anal. **262**, 3946–3980 (2012)
26. Jabłoński, Z.J., Jung, I.B., Stochel, J.: A hyponormal weighted shift on a directed tree whose square has trivial domain, submitted

27. Janas, J.: On unbounded hyponormal operators, III. Studia Math. **112**, 75–82 (1994)
28. Kouchekian, S., Thomson, J.E.: A note on density for the core of unbounded Bergman operators. Proc. Am. Math. Soc. **139**, 2067–2072 (2011)
29. Lambert, A.: Subnormality and weighted shifts. J. London Math. Soc. **14**, 476–480 (1976)
30. Lambert, A.: Hyponormal composition operators. Bull. London Math. Soc. **18**, 395–400 (1986)
31. Lambert, A.: Subnormal composition operators. Proc. Am. Math. Soc. **103**, 750–754 (1988)
32. Lambert, A.: Normal extensions of subnormal composition operators. Michigan Math. J. **35**, 443–450 (1988)
33. Mlak, W.: Operators induced by transformations of Gaussian variables. Ann. Polon. Math. **46**, 197–212 (1985)
34. Nordgren, E.: Composition Operators on Hilbert Spaces, Lecture Notes in Math. 693. Springer, Berlin (1978)
35. Ôta, S., Schmüdgen, K.: On some classes of unbounded operators. Integr. Equat. Oper. Th. **12**, 211–226 (1989)
36. Pólya, G., Szegö, G.: Problems and theorems in analysis, Vol. II, Theory of functions, zeros, polynomials, determinants, number theory, geometry, revised and enlarged translation by C. E. Billigheimer of the fourth German edition, Die Grundlehren der Mathematischen Wissenschaften, Band 216. Springer, New York (1976)
37. Rao, M.M.: Conditional Measures and Applications, Pure and Applied Mathematics (Boca Raton) 271. Chapman and Hall/CRC, Boca Raton (2005)
38. Rudin, W.: Real and Complex Analysis. McGraw-Hill, New York (1987)
39. Schmüdgen, K.: On domains of powers of closed symmetric operators. J. Operator Theory **9**, 53–75 (1983)
40. Schmüdgen, K.: A formally normal operator having no normal extension. Proc. Am. Math. Soc. **95**, 503–504 (1985)
41. Schmüdgen, K.: Unbounded Operator Algebras and Representation Theory. Operator Theory: Advances and Applications, 37. Birkhäuser Verlag, Basel (1990)
42. Singh, R.K.: Compact and quasinormal composition operators. Proc. Am. Math. Soc. **45**, 80–82 (1974)
43. Singh, R.K., Manhas, J.S.: Composition Operators on Function Spaces. Elsevier Science Publishers B.V, North-Holland-Amsterdam (1993)
44. Singh, R.K., Veluchamy, T.: Spectrum of a normal composition operators on L^2 spaces. Indian J. Pure Appl. Math. **16**, 1123–1131 (1985)
45. Stochel, J.: Seminormal composition operators on L^2 spaces induced by matrices. Hokkaido Math. J. **19**, 307–324 (1990)
46. Stochel, J.: Moment functions on real algebraic sets. Ark. Mat. **30**, 133–148 (1992)
47. Stochel, J., Stochel, J.B.: Seminormal composition operators on L^2 spaces induced by matrices: the Laplace density case. J. Math. Anal. Appl. **375**, 1–7 (2011)
48. Stochel, J., Szafraniec, F.H.: On normal extensions of unbounded operators. I. J. Operator Theory **14**, 31–55 (1985)
49. Stochel, J., Szafraniec, F.H.: On normal extensions of unbounded operators. II. Acta Sci. Math. (Szeged) **53**, 153–177 (1989)
50. Stochel, J., Szafraniec, F.H.: On normal extensions of unbounded operators. III. Spectral properties. Publ. RIMS, Kyoto Univ. **25**, 105–139 (1989)
51. Stochel, J., Szafraniec, F.H.: The complex moment problem and subnormality: a polar decomposition approach. J. Funct. Anal. **159**, 432–491 (1998)
52. Stochel, J., Szafraniec, F.H.: Unbounded operators and subnormality, in preparation
53. Szafraniec, F.H.: Kato-Protter type inequalities, bounded vectors and the exponential function. Ann. Polon. Math. **51**, 303–312 (1990)
54. Veluchamy, T., Panayappan, S.: Paranormal composition operators. Indian J. Pure Appl. Math. **24**, 257–262 (1993)
55. Weidmann, J.: Linear Operators in Hilbert Spaces. Springer, Berlin (1980)
56. Whitley, R.: Normal and quasinormal composition operators. Proc. Amer. Math. Soc. **70**, 114–118 (1978)