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The p-Laplace operator with the nonlocal Robin boundary conditions on arbitrary open sets

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Abstract Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set with boundary $\partial\Omega$, $1 < p < \infty$ and let $f \in L^q(\Omega)$ for some $q > N > 1$. In the first part of the article, we show that weak solutions of the quasi-linear elliptic equation $-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + a(x)|u|^{p-2}u = f$ in Ω with the nonlocal Robin type boundary conditions formally given by $|\nabla u|^{p-2}\partial u/\partial \nu + b(x)|u|^{p-2}u + \Theta_p(u) = 0$ on $\partial\Omega$ belong to $L^\infty(\Omega)$. In the second part, assuming that Ω has a finite measure, we prove that for every $p \in (1,\infty)$, a realization of the operator Δ_p in $L^2(\Omega)$ with the above-mentioned nonlocal Robin boundary conditions generates a nonlinear order-preserving semigroup $(S_\Theta(t))_{t\geq 0}$ of contraction operators in $L^2(\Omega)$ if and only if $\partial\Omega$ is admissible (in the sense of the relative capacity) with respect to the $(N-1)$-dimensional Hausdorff measure $\mathscr{H}^{N-1}|_{\partial\Omega}$. We also show that this semigroup is ultracontractive in the sense that, for every $u_0 \in L^q(\Omega)$ $(q \geq 2)$ one has $S_\Theta(t)u_0 \in L^\infty(\Omega)$ for every $t > 0$. Moreover, $\|S_\Theta(t)$ satisfies the following $(L^q - L^\infty)$-Hölder type estimate: there is a constant $C \geq 0$ such that for every $t > 0$ and $u_0, v_0 \in L^q(\Omega)$ $(q \geq 2)$,

$$\|S_\Theta(t)u_0 - S_\Theta(t)v_0\|_{\infty,\Omega} \leq C|\Omega|^\beta t^{-\delta}\|u_0 - v_0\|_{q,\Omega}^\gamma,$$

where β, δ, and γ are explicit constants depending on N, p, and q only.

Keywords The p-Laplace operator · Nonlocal Robin boundary conditions on arbitrary open sets · Bounded weak solutions · A priori estimates · Relative capacity · Nonlinear ultracontractive semigroups · Hölder type estimates

Mathematics Subject Classification 35J92 · 35A15 · 35B65 · 35K55 · 47H20

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1 Introduction

Before we present the problems considered in this article, we first clarify the notion of the normal derivative and the nonlocal boundary conditions on arbitrary open sets. Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set with boundary $\partial\Omega$. Since $\partial\Omega$ may be so bad such that no normal vector can be defined, we will use the following generalized version of a normal derivative in the weak sense introduced in [7]. Let μ be a Borel measure on $\partial\Omega$ and let $F : \Omega \to \mathbb{R}^N$ be a measurable function. If there exists a function $f \in L^1_{loc}(\mathbb{R}^N)$ such that

$$\int_\Omega F\nabla\varphi \,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x + \int_{\partial\Omega} \varphi \,\mathrm{d}\mu \tag{1.1}$$

for all $\varphi \in C^1_c(\overline{\Omega})$, then we say that μ is the *normal measure* of F that we denoted by $\mathsf{N}^\star(F) := \mu$. If $\mathsf{N}^\star(F)$ exists, then it is unique and $d\,\mathsf{N}^\star(\psi F) = \psi d\,\mathsf{N}^\star(F)$ for all $\psi \in C^1(\overline{\Omega})$. If $p \in (1,\infty)$, $u \in W^{1,1}_{loc}(\Omega)$ and $\mathsf{N}^\star(|\nabla u|^{p-2}\nabla u)$ exists, then we will denote by $\mathsf{N}_p(u) := \mathsf{N}^\star(|\nabla u|^{p-2}\nabla u)$ the p-generalized normal measure of $|\nabla u|^{p-2}\nabla u$. The derivative $d\,\mathsf{N}_p(u)/d\sigma$ (where σ denotes the restriction to $\partial\Omega$ of the $(N-1)$-dimensional Hausdorff measure $\mathscr{H}^{N-1}$) will be called the p-generalized normal derivative of u. To justify this definition, let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^1, ν the outer normal to $\partial\Omega$ and let σ be the surface measure on $\partial\Omega$. If $u \in C^1(\overline{\Omega})$ is such that there are $f \in L^1_{loc}(\mathbb{R}^N)$ and $g \in L^1(\partial\Omega,\sigma)$ with

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi \,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x + \int_{\partial\Omega} g\varphi \,\mathrm{d}\sigma \tag{1.2}$$

for all $\varphi \in C^1(\overline{\Omega})$, then $g = |\nabla u|^{p-2}\partial u/\partial\nu$ and hence, $d\,\mathsf{N}_p(u)/d\sigma = |\nabla u|^{p-2}\partial u/\partial\nu$.

Throughout the remainder of this article, if $\Omega \subset \mathbb{R}^N$ is an arbitrary open set with boundary $\partial\Omega$, then without any mention, $\sigma = \mathscr{H}^{N-1}|_{\partial\Omega}$ (which coincides with the surface measure if Ω has a Lipschitz continuous boundary). Moreover, if $u \in W^{1,1}_{loc}(\Omega)$ is such that there exist $f \in L^1_{loc}(\mathbb{R}^N)$ and $g \in L^1(\partial\Omega,\sigma)$ satisfying (1.2), then we will simply denote $d\,\mathsf{N}_p(u)/d\sigma = |\nabla u|^{p-2}\partial u/\partial\nu$.

For a measurable function u on $\partial\Omega$ and $p \in (1,\infty)$, we let

$$[u]^p_{1,p} := \iint_{\partial\Omega\times\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+p-2}} \,\mathrm{d}\sigma_x \,\mathrm{d}\sigma_y.$$

We let the Besov type space

$$\mathbb{B}^p(\partial\Omega,\sigma) := \{u \in L^p(\Omega) : \; [u]^p_{1,p} < \infty\}$$

be equipped with the norm

$$\|u\|_{\mathbb{B}^p(\partial\Omega,\sigma)} := \left(\|u\|^p_{p,\partial\Omega} + [u]^p_{1,p}\right)^{1/p}.$$

Throughout the remainder of this paper, we let

$$k_p(x,y) := |x-y|^{N+p-2}.$$

Let $(\mathbb{B}^p(\partial\Omega,\sigma))^*$ denote the dual of the reflexive Banach space $\mathbb{B}^p(\partial\Omega,\sigma)$. We define a (nonlocal) operator $\Theta_p : \; \mathbb{B}^p(\partial\Omega,\sigma) \to (\mathbb{B}^p(\partial\Omega,\sigma))^*$ as follows: for $u, v \in \mathbb{B}^p(\partial\Omega,\sigma)$, we set

$$\langle \Theta_p(u), v\rangle := \iint\limits_{\partial\Omega\times\partial\Omega} \frac{|u(x)-u(y)|^{p-2}}{k_p(x,y)}(u(x)-u(y))(v(x)-v(y))\, \mathrm{d}\sigma_x\, \mathrm{d}\sigma_y, \quad (1.3)$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between $\mathbb{B}^p(\partial\Omega,\sigma)$ and $(\mathbb{B}^p(\partial\Omega,\sigma))^*$.

In the first part of this article, given an arbitrary open set $\Omega \subset \mathbb{R}^N$ with boundary $\partial\Omega$, $p \in (1,\infty)$ and $f \in L^q(\Omega)$ for some $q \in [1,\infty]$, we study the existence of bounded weak solutions (see Definition 3.1 below) of the quasi-linear elliptic equation with the nonlocal (see Subsection 5.1) Robin type boundary conditions formally given by

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + a(x)|u|^{p-2}u = f & \text{in } \Omega \\ |\nabla u|^{p-2}\partial u/\partial \nu + b(x)|u|^{p-2}u + \Theta_p(u) = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

Here, the nonnegative functions a, b belong to $L^\infty(\Omega)$ and $L^\infty(\partial\Omega)$, respectively. We show that for arbitrary open sets, if $q > N > 1$, then weak solutions of Eq. (1.4) belong to $L^\infty(\Omega)$ and we also provide the a priori estimates of the solutions and the difference of solutions. The local Robin boundary conditions, that is, the second line in Eq. (1.4) without the term $\Theta_p(u)$, have been investigated in [6,7,14] and the references therein. In [6,7], replacing the measure σ by an upper d-Ahlfors measure (for some $d \in (0,N)$), the authors have shown that weak solutions of the corresponding local problem are bounded provided that Ω has the extension property of Sobolev functions (see Definition 4.5 below). The case of variable exponents, that is, $p = p(x)$, has been investigated in [5]. The local problem on general domains is included in [14] where the authors have also obtained that weak solutions are bounded by using some Moser type iterations. The associated linear problem, that is, $p = 2$, without any regularity assumption on Ω has been considered in [1,2,12,13,35]. In [34], the authors have considered the nonlocal problem as in Eq. (1.4), but they have also replaced the measure σ by an upper d-Ahlfors measure μ and have shown that weak solutions of the corresponding problem are bounded under the restriction that Ω has the extension property of Sobolev functions. In [32,33], it has been said that on a bounded domain with a Lipschitz continuous boundary, weak solutions of Eq. (1.4) are uniformly continuous on Ω. Unfortunately, the proofs of the main results in [32] are incorrect. Some interesting spectral properties of the linear ($p = 2$) nonlocal Robin boundary conditions on Lipschitz domains are included in [20]. In this article, we obtain that weak solutions of Eq. (1.4) are bounded without any regularity assumption on Ω. This improves the results obtained in [34] since we do not assume any regularity assumption on the open set, and for "bad open sets", the measure σ is not always an upper d-Ahlfors measure (for any $d \in (0,N)$).

Of concern in the second part of the paper is the following first-order Cauchy problem

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \Delta_p u(t,x) & t>0,\ x\in\Omega \\ |\nabla u(t,x)|^{p-2}\partial u(t,x)/\partial\nu + b(x)|u(t,x)|^{p-2}u(t,x) + \Theta_p(u(t,x)) = 0 & t>0,\ x\in\partial\Omega \\ u(0,x) = u_0 & x\in\Omega, \end{cases} \quad (1.5)$$

where u_0 is a given function in $L^2(\Omega)$ and we assume that Ω has a finite measure. We show that for every $p \in (1,\infty)$, (1.5) corresponds to a well-posed Cauchy problem in $L^2(\Omega)$ if and only if the relatively open set $\Gamma_0 \subseteq \partial\Omega$, on which the measure σ is locally finite, is $\mathrm{Cap}_{p,\Omega}$-admissible (see Definition 4.3 below) with respect to σ. Then, assuming that the solution of (1.5) is given by $u(t) = S_\Theta(t)u_0$ where $(S_\Theta(t))_{t\ge 0}$ is a strongly continuous order-preserving nonlinear (linear if $p = 2$) semigroup of contraction operators on $L^2(\Omega)$. We show that there is a constant $C \geq 0$ such that for every $u_0, v_0 \in L^q(\Omega)$ ($q \geq 2$) and $t > 0$,

$$\|S_\Theta(t)u_0 - S_\Theta(t)v_0\|_{\infty,\Omega} \leq C|\Omega|^\beta t^{-\delta}\|u_0 - v_0\|_{q,\Omega}^\gamma, \tag{1.6}$$

where

$$\beta := \frac{N-1}{N}\left[1 - \left(\frac{q}{q-2+p}\right)^N\right], \quad \delta := \frac{1}{p-2}\left[1 - \left(\frac{q}{q-2+p}\right)^N\right]$$
$$\text{and } \gamma := \left(\frac{q}{q-2+p}\right)^N.$$

The estimate (1.6) shows in particular that the semigroup $(S_\Theta(t))_{t\geq 0}$ is ultracontractive in the sense that it maps $L^q(\Omega)$ into $L^\infty(\Omega)$. Similar results for the linear local Robin boundary conditions on arbitrary domains are obtained in [2,14,35].

We outline the plan of the paper as follows. In Sect. 2, we introduce the Maz'ya space that plays an important role here. The main results obtained in this paper are based on some properties of this space. Section 3 concerns the study of the elliptic problem. We show that weak solutions of (1.4) belong to $L^\infty(\Omega)$ provided that $f \in L^q(\Omega)$ with $q > N > 1$, and we also give an a priori estimate of the solutions. In Sect. 4, we introduce the relative p-capacity and the notion of admissible subsets of $\partial\Omega$. We also give a large class of admissible sets and some examples of nonadmissible sets. Finally, in Sect. 5, we characterize the well-posedness of Eq. (1.5). If it is well posed, we show that its unique solution is given in terms of a strongly continuous nonlinear semigroup of contraction operators on $L^2(\Omega)$ that is order preserving, nonexpansive on $L^\infty(\Omega)$, and ultracontractive.

2 The Maz'ya space

Let $\Omega \subset \mathbb{R}^N$ be an open set with boundary $\partial\Omega$ and let $p \in [1, \infty)$. We recall that the measure $\sigma = \mathscr{H}^{N-1}|_{\partial\Omega}$. It is well known that σ is a regular Borel measure on $\partial\Omega$ but is not a Radon measure, that is, compact subsets of $\partial\Omega$ may have infinite σ-measure.

We denote by $W^{1,p}(\Omega)$ the first-order Sobolev space endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|\nabla u\|_{p,\Omega}^p + \|u\|_{p,\Omega}^p\right)^{1/p}.$$

The following important inequality is due to Maz'ya [25, Section 3.6, p.189].

Theorem 2.1 (Maz'ya) *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set with $N > 1$. Then, there is a constant $C = C(N) > 0$ such that for every $u \in W^{1,1}(\Omega) \cap C_c(\overline{\Omega})$,*

$$\|u\|_{\frac{N}{N-1},\Omega} \leq C(N)\left(\|\nabla u\|_{1,\Omega} + \|u\|_{1,\partial\Omega}\right). \tag{2.1}$$

In the inequality (2.1), the constant $C(N)$ is exactly the so called isoperimetric constant. As a corollary of the Maz'ya' theorem, we have the following two results.

Corollary 2.2 *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set with $N > 1$ and let $1 \leq p < \infty$. Then, there exists a constant $C(N, p) > 0$ such that for every $u \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$,*

$$\|u\|_{\frac{pN}{N-1},\Omega} \leq C(N, p)\left(\|\nabla u\|_{p,\Omega}^p + \|u\|_{p,\Omega}^p + \|u\|_{p,\partial\Omega}^p\right)^{1/p}. \tag{2.2}$$

Proof Let $1 \leq p < \infty$ and let $u \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$. Then, $u^p \in W^{1,1}(\Omega) \cap C_c(\overline{\Omega})$. It follows from (2.1) that

$$\begin{aligned}
\|u\|^p_{\frac{pN}{N-1},\Omega} = \|u^p\|_{\frac{N}{N-1},\Omega} &\leq C(N) \left(\|\nabla(u^p)\|_1 + \|u^p\|_{1,\partial\Omega}\right) \\
&\leq C(N) \left(p\|u^{p-1}\nabla u\|_1 + \|u\|^p_{p,\partial\Omega}\right) \\
&\leq C(N) \left(p\|u^{p-1}\|_{p',\Omega}\|\nabla u\|_{p,\Omega} + \|u\|^p_{p,\partial\Omega}\right) \\
&\leq C(N) \left(\frac{p}{p'}\|u^{p-1}\|^{p'}_{p',\Omega} + \|\nabla u\|^p_{p,\Omega} + \|u\|^p_{p,\partial\Omega}\right) \\
&\leq C(N,p) \left(\|u\|^p_{p,\Omega} + \|\nabla u\|^p_{p,\Omega} + \|u\|^p_{p,\partial\Omega}\right).
\end{aligned}$$

We have shown (2.2), and the proof is finished. □

Corollary 2.3 ([26, Corollary 2.11.2]) *Let $\Omega \subset \mathbb{R}^N$ be an open set of finite measure with $N > 1$ and let $1 \leq p < \infty$. Then, there is a constant $C = C(N, p, \Omega) > 0$ such that for every $u \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$,*

$$\|u\|^p_{\frac{pN}{N-1},\Omega} \leq C \left(\|\nabla u\|^p_{p,\Omega} + \|u\|^p_{p,\partial\Omega}\right). \tag{2.3}$$

Throughout the remainder of this paper, for $1 \leq p < \infty$ and $N > 1$, we let $p^\star := pN/(N-1)$.

Now, we let the Maz'ya spaces $W^1_{p,p}(\Omega, \partial\Omega)$ and $\mathscr{W}^1_{p,p}(\Omega, \partial\Omega)$ to be respectively the abstract completion of

$$W_\sigma := \left\{ u \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega}) : \int_{\partial\Omega} |u|^p \, d\sigma < \infty \right\}$$

with respect to the norm

$$\|u\|_{W^1_{p,p}(\Omega,\partial\Omega)} = \left(\|\nabla u\|^p_{p,\Omega} + \|u\|^p_{p,\Omega} + \|u\|^p_{p,\partial\Omega}\right)^{1/p},$$

and to the norm

$$\|u\|_{\mathscr{W}^1_{p,p}(\Omega,\partial\Omega)} = \left(\|\nabla u\|^p_{p,\Omega} + \|u\|^p_{p,\partial\Omega}\right)^{1/p}.$$

It follows from Corollary 2.3 that if Ω has a finite measure, then the spaces $\mathscr{W}^1_{p,p}(\Omega, \partial\Omega)$ and $W^1_{p,p}(\Omega, \partial\Omega)$ coincide with equivalent norms. Moreover, by (2.2) and (2.3),

$$W^1_{p,p}(\Omega, \partial\Omega) \hookrightarrow L^{p^\star}(\Omega) \text{ (only continuous embedding)} \tag{2.4}$$

and

$$\mathscr{W}^1_{p,p}(\Omega, \partial\Omega) \hookrightarrow L^{p^\star}(\Omega) \text{ (only continuous embedding).} \tag{2.5}$$

Note that an example of a domain showing that the exponent $p^\star := pN/(N-1)$ in (2.5) cannot be improved without any regularity assumption on Ω is contained in [26, Example 2.11, p.123].

To conclude this section, we give the logarithmic Sobolev inequality associated with the Maz'ya space.

Lemma 2.4 *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set of finite measure with $N > 1$ and let $1 \le p < \infty$. Let $f \in \mathscr{W}^1_{p,p}(\Omega, \partial\Omega)$, $f \ge 0$ with $\|f\|_{p,\partial\Omega} = 1$. Then, for every $\varepsilon > 0$,*

$$\int_\Omega f^p \log(f)\, \mathrm{d}x \le \frac{N}{p}\left(-\log(\varepsilon) + \varepsilon C\|\nabla f\|^p_{p,\Omega} + \varepsilon C\right), \tag{2.6}$$

where $C = C(\Omega, N, p) > 0$ is the constant appearing in the Maz'ya inequality (2.3).

Proof Let $1 \le p < \infty$ and let $p^\star := Np/(N-1)$, $q = p/(N-1)$ so that $p + q = p^\star$. Let $f \in \mathscr{W}^1_{p,p}(\Omega, \partial\Omega)$, $f \ge 0$ with $\|f\|_{p,\partial\Omega} = 1$. Using the well-known Jensen's inequality, we get that

$$\begin{aligned}\int_\Omega f^p \log(f)\, \mathrm{d}x &\le \frac{1}{q}\log\left(\int_\Omega f^{p+q}\, \mathrm{d}x\right) = \frac{1}{q}\log\left(\int_\Omega f^{p^\star}\, \mathrm{d}x\right)\\ &\le \frac{1}{q}\log\|f\|^{p^\star}_{p^\star,\Omega} = \frac{N}{p}\log\|f\|^p_{p^\star,\Omega}.\end{aligned}$$

Since $\log\|f\|^p_{p^\star,\Omega} \le -\log(\varepsilon) + \varepsilon\|f\|^p_{p^\star,\Omega}$ for every $\varepsilon > 0$, it follows from the preceding estimate and (2.5) that for every $\varepsilon > 0$,

$$\begin{aligned}\int_\Omega f^p \log(f)\, \mathrm{d}x &\le \frac{N}{p}\left(-\log(\varepsilon) + \varepsilon\|f\|^p_{p^\star,\Omega}\right)\\ &\le \frac{N}{p}\left(-\log(\varepsilon) + \varepsilon C\|\nabla f\|^p_{p,\Omega} + \varepsilon C\|f\|^p_{p,\partial\Omega}\right)\\ &\le \frac{N}{p}\left(-\log(\varepsilon) + \varepsilon C\|\nabla f\|^p_{p,\Omega} + \varepsilon C\right)\end{aligned}$$

and the proof is finished. □

We note that in several papers, the logarithmic Sobolev inequality is due to Gross [21,22], but at our knowledge, it has been first obtained by Federbush [19].

3 The elliptic problem

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set with boundary $\partial\Omega$. For $1 < p < \infty$, we let $\mathscr{V}^{1,p}_\Theta(\Omega, \partial\Omega)$ be the completion of

$$W_{\sigma,\Theta} := \left\{u \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega}) : \int_{\partial\Omega} |u(x)|^p\, \mathrm{d}\sigma + \iint_{\partial\Omega\times\partial\Omega} \frac{|u(x) - u(y)|^p}{k_p(x,y)}\, \mathrm{d}\sigma_x\, \mathrm{d}\sigma_y < \infty\right\}$$

with respect to the norm

$$\begin{aligned}&\||u\||_{\mathscr{V}^{1,p}_\Theta(\Omega,\partial\Omega)}\\ &:= \left(\int_\Omega |\nabla u|^p\, \mathrm{d}x + \int_\Omega |u|^p\, \mathrm{d}x + \int_{\partial\Omega} |u|^p\, \mathrm{d}\sigma + \iint_{\partial\Omega\times\partial\Omega} \frac{|u(x) - u(y)|^p}{k_p(x,y)}\, \mathrm{d}\sigma_x\, \mathrm{d}\sigma_y\right)^{1/p}.\end{aligned}$$

It follows from the Maz'ya embedding (2.5) that if $|\Omega| < \infty$, then $\||\cdot\||_{\mathscr{V}_\Theta^{1,p}(\Omega,\partial\Omega)}$ is equivalent to the norm

$$\|u\|_{\mathscr{V}_\Theta^{1,p}(\Omega,\partial\Omega)} := \left(\int_\Omega |\nabla u|^p \, dx + \int_{\partial\Omega} |u|^p \, d\sigma + \iint_{\partial\Omega\times\partial\Omega} \frac{|u(x)-u(y)|^p}{k_p(x,y)} \, d\sigma_x \, d\sigma_y\right)^{1/p}.$$

It is clear that $\mathscr{V}_\Theta^{1,p}(\Omega,\partial\Omega)$ is a reflexive Banach space for every $p \in (1,\infty)$ and it is also continuously embedded into $W_{p,p}^1(\Omega,\partial\Omega)$. Let $(\mathscr{V}_\Theta^{1,p}(\Omega,\partial\Omega))^*$ denote its dual and let $\langle\cdot,\cdot\rangle_\Theta$ be the duality map. We consider the quasi-linear elliptic equation formally given by

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + a(x)|u|^{p-2}u = f & \text{in } \Omega \\ |\nabla u|^{p-2}\partial u/\partial\nu + b(x)|u|^{p-2}u + \Theta_p(u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where f is given in $(\mathscr{V}_\Theta^{1,p}(\Omega,\partial\Omega))^*$ and a, b are nonnegative measure functions that belong to $L^\infty(\Omega)$ and $L^\infty(\partial\Omega)$, respectively. We also assume that there is a constant $b_0 > 0$ such that

$$b(x) \geq b_0 > 0 \quad \text{for } \sigma - a.e.\ x \in \partial\Omega. \tag{3.2}$$

Definition 3.1 A function $u \in \mathscr{V}_\Theta^{1,p}(\Omega,\partial\Omega)$ is said to be a weak solution of (3.1) if for every $v \in \mathscr{V}_\Theta^{1,p}(\Omega,\partial\Omega)$,

$$\begin{aligned} \mathscr{A}(u,v) := &\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v \, dx + \int_\Omega a(x)|u|^{p-2}uv \, dx + \int_{\partial\Omega} b(x)|u|^{p-2}uv \, d\sigma \\ &+ \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^{p-2}}{k_p(x,y)}(u(x)-u(y))(v(x)-v(y)) \, d\sigma_x \, d\sigma_y = \langle f, v\rangle_\Theta. \end{aligned} \tag{3.3}$$

If $f \in L^q(\Omega)$ for some $q \in [1,\infty]$, then $\langle f, v\rangle_\Theta = \int_\Omega fv \, dx$. Using Brodwer results [17], it is straightforward to verify that for every $p \in (1,\infty)$ and for every $f \in (\mathscr{V}_\Theta^{1,p}(\Omega,\partial\Omega))^*$, Eq. (3.1) has a unique weak solution. In particular, if Ω is of finite measure, then for every $f \in L^q(\Omega)$ with $q \geq (p^\star)'$, Eq. (3.1) has a unique weak solution.

Before we state the main result of this section, we give the following lemma which is taken from [29, Lemma 3.13] and will be used in the proofs of the main results of this section.

Lemma 3.2 *Let $k_0 \geq 0$ and let $\Psi : [k_0,\infty) \to \mathbb{R}$ be a nonnegative, nonincreasing function such that there are positive constants c, α, and δ ($\delta > 1$) such that*

$$\Psi(h) \leq c(h-k)^{-\alpha}\Psi(k)^\delta \qquad \forall\, h > k \geq k_0.$$

Then $\Psi(k_0 + d) = 0$ with $d = c^{1/\alpha}\Psi(k_0)^{(\delta-1)/\alpha}2^{\delta(\delta-1)}$.

The following well-known inequalities will be also useful. For more details, we refer the reader to [5,6,16] and the references therein.

Lemma 3.3 *The following assertions hold true.*

(a) *Let $p \in (1,2]$ and $a, b \in \mathbb{R}^N$ with $a \neq b$. Then,*

$$\langle |a|^{p-2}a - |b|^{p-2}b, a-b\rangle \left[|a|^p + |b|^p\right]^{\frac{2-p}{p}} \geq (p-1)|a-b|^p. \tag{3.4}$$

(b) *Let $p \in (1,2]$ and $\varepsilon > 0$. Then, for every $a, b \in \mathbb{R}^N$ with $|a-b| \geq \varepsilon \cdot \min\{|a|,|b|\}$, we have*

$$\langle |a|^{p-2}a - |b|^{p-2}b, a-b\rangle \geq (p-1)(1+1/\varepsilon)^{p-2}|a-b|^p. \tag{3.5}$$

(c) *Let $p \in [2,\infty)$ and $a, b \in \mathbb{R}^N$. Then,*

$$\langle |a|^{p-2}a - |b|^{p-2}b, a-b\rangle \geq 2^{2-p} \cdot |a-b|^p. \tag{3.6}$$

(d) *Let $p \in [2,\infty)$ and $a, b \in \mathbb{R}^N$. The inequality (3.6) implies that*

$$2^{2-p} \cdot |a-b|^{p-1} \leq \left||a|^{p-2}a - |b|^{p-2}b\right|. \tag{3.7}$$

3.1 Case of open sets of finite measure

We have the following result.

Proposition 3.4 *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set of finite measure and let $f \in L^q(\Omega)$ with $q > N > 1$. Let $1 < p < \infty$, and let $u \in \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega)$ satisfy (3.3). Then, $u \in L^\infty(\Omega)$ and there is a constant $C > 0$ such that*

$$\|u\|_{\infty,\Omega}^{p-1} \leq C\|f\|_{q,\Omega}. \tag{3.8}$$

Proof Let $1 < p < \infty$ and let $u \in \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega)$ satisfy (3.3). Let $k \geq 0$ be a real number and set $u_k := (|u|-k)^+\mathrm{sgn}(u)$. Then, $u_k \in \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega)$. Let $A_k := \{x \in \overline{\Omega} : |u(x)| \geq k\}$. Since $u_k = 0$ on $\overline{\Omega} \setminus A_k$ and $\nabla u_k = \chi_{A_k} \cdot \nabla u$, we have that

$$\begin{aligned}
\mathscr{A}(u,u_k) &= \int_{A_k\cap\Omega} |\nabla u_k|^p \,\mathrm{d}x + \int_{A_k\cap\Omega} a(x)|u|^{p-2}uu_k \,\mathrm{d}x \\
&\quad + \int_{A_k\cap\partial\Omega} b(x)|u|^{p-2}uu_k \,\mathrm{d}\sigma + \iint_{A_k\cap\partial\Omega)\times(A_k\cap\partial\Omega)} \\
&\quad \times \frac{|u(x)-u(y)|^{p-2}}{k_p(x,y)}(u(x)-u(y))(u_k(x)-u_k(y)) \,\mathrm{d}\sigma_x \,\mathrm{d}\sigma_y \\
&= \mathscr{A}(u_k,u_k) + \int_{A_k\cap\Omega} a(x)(|u|^{p-2}uu_k - |u_k|^p) \,\mathrm{d}x \\
&\quad + \int_{A_k\cap\partial\Omega} b(x)(|u|^{p-2}uu_k - |u_k|^p) \,\mathrm{d}\sigma + \iint_{A_k\cap\partial\Omega)\times(A_k\cap\partial\Omega)} \\
&\quad \times \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u_k(x)-u_k(y)) - |u_k(x)-u_k(y)|^p}{k_p(x,y)} \,\mathrm{d}\sigma_x \,\mathrm{d}\sigma_y.
\end{aligned} \tag{3.9}$$

It is easy to check that $(|u|^{p-2}uu_k - |u_k|^p) \geq 0$ on A_k. Moreover, $|u(x)-u(y)|^{p-2}(u(x)-u(y))(u_k(x)-u_k(y)) - |u_k(x)-u_k(y)|^p \geq 0$ on $A_k \times A_k$. We have shown that for every $k \geq 0$,

$$\mathscr{A}(u_k,u_k) \leq \mathscr{A}(u,u_k) = \int_{A_k\cap\Omega} fu_k \,\mathrm{d}x.$$

Let $f \in L^q(\Omega)$ with $q > p^\star/(p^\star - p) = N$. Using the classical Hölder inequality, we have that for every $k \geq 0$,

$$\int_{A_k \cap \Omega} f u_k \, \mathrm{d}x \leq \|f\|_{q,\Omega} \|u_k\|_{p^\star,\Omega} \|\chi_{A_k}\|_s,$$

where $s \in [1, \infty]$ is such that $1/s + 1/q + 1/p^\star = 1$, that is, $1/s = (1 - 1/p^\star - 1/q) > (p-1)/p^\star$. Hence, there is a constant $C > 0$ such that for every $k \geq 0$,

$$C\|u_k\|^p_{\mathscr{W}^1_{p,p}(\Omega,\partial\Omega)} \leq \mathscr{A}(u_k, u_k) \leq \mathscr{A}(u, u_k) \leq \|f\|_{q,\Omega}\|u_k\|_{p^\star,\Omega}\|\chi_{A_k}\|_s.$$

This estimate, together with the embedding (2.5), show that there is a constant $C > 0$ such that for every $k \geq 0$,

$$\|u_k\|^{p-1}_{p^\star,\Omega} \leq C\|f\|_{q,\Omega}\|\chi_{A_k}\|_s.$$

Let $h > k$. Then, $A_h \subset A_k$ and on A_h the inequality $|u_k| \geq (h-k)$ holds. Therefore,

$$\|\chi_{A_h}\|^{p-1}_{p^\star} \leq C(h-k)^{-(p-1)}\|f\|_{q,\Omega}\|\chi_{A_k}\|_s. \tag{3.10}$$

Let $\delta := p^\star/s > p-1,\ \delta_0 := \delta/(p-1) > 1$. Then, $\|\chi_{A_k}\|_s = \|\chi_{A_k}\|^\delta_{p^\star}$ and using (3.10), we get that for $h > k \geq 0$,

$$\|\chi_{A_h}\|^{p-1}_{p^\star} \leq C(h-k)^{-(p-1)}\|f\|_{q,\Omega}\left[\|\chi_{A_k}\|^{p-1}_{p^\star}\right]^{\delta_0}. \tag{3.11}$$

Letting $\Psi(h) := \|\chi_{A_h}\|^{p-1}_{p^\star}$ in Lemma 3.2, on account of (3.11), we have that $\|\chi_{A_K}\|^{p-1}_{p^\star} = 0$ with the constant K given by $K := C^{1/(p-1)}\psi(0)^{\delta_0-1}2^{\delta_0(\delta_0-1)}\|f\||^{1/(p-1)}_{q,\Omega} = \tilde{C}\|f\||^{1/(p-1)}_{q,\Omega}$. Hence, $|u| \leq K$ a.e. on Ω. We have shown (3.8) and the proof is finished. □

Next, we consider the difference of weak solutions.

Theorem 3.5 *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set of finite measure with $N > 1$ and let $f_1, f_2 \in L^q(\Omega)$. Let $2(N-1)/N < p < \infty$ and let $u, v \in \mathscr{V}^{1,p}_\Theta(\Omega, \partial\Omega)$ be such that for every $\varphi \in \mathscr{V}^{1,p}_\Theta(\Omega, \partial\Omega)$, $\mathscr{A}(u, \varphi) = \int_\Omega f_1\varphi \, \mathrm{d}x$ and $\mathscr{A}(v, \varphi) = \int_\Omega f_2\varphi \, \mathrm{d}x$, so that,*

$$\mathscr{A}(u, \varphi) - \mathscr{A}(v, \varphi) = \int_\Omega (f_1 - f_2)\varphi \, \mathrm{d}x. \tag{3.12}$$

(a) *If $2 \leq p < \infty$ and $q > N$, then $u, v \in L^\infty(\Omega)$ and there is a constant $C > 0$ such that,*

$$\|u - v\|^{p-1}_{\infty,\Omega} \leq C\|f_1 - f_2\|_{q,\Omega}. \tag{3.13}$$

(b) *If $2(N-1)/N < p < 2$ and $q > Np/(Np - 2N + 2)$ then $u, v \in L^\infty(\Omega)$ and there is a constant $C > 0$ such that,*

$$\|u - v\|_{\infty,\Omega} \leq C\left[\sum_{j=1}^{2}\|f_j\|_{(p^\star)'}\right]^{\frac{2-p}{p-1}}\|f_1 - f_2\|_{q,\Omega}. \tag{3.14}$$

To prove the second part of the theorem, we need the following result.

Lemma 3.6 *Let* $A, B, C, \tau, \rho \in [0, +\infty)$, $p \in [1, +\infty)$ *and assume that* $A \le \tau\varepsilon^{-\rho} + \varepsilon^p C$ *for all* $\varepsilon \in (0, 1]$. *Then,*

$$A \le (\tau + 1)\left[B^{p/(p+\rho)}C^{\rho/(p+\rho)} + B\right]. \tag{3.15}$$

Proof Assume first that $0 < B \le C$ and let $\alpha := 1/(p + \rho)$ and $\varepsilon_1 := (B/C)^\alpha \in (0, 1]$. Then,

$$\begin{aligned} A &\le \tau\varepsilon_1^{-\rho}B + \varepsilon_1^p C = \tau(B/C)^{-\rho\alpha}B + (B/C)^{p\alpha}C = \tau B^{1-\rho\alpha}C^{\rho\alpha} + B^{p\alpha}C^{1-p\alpha} \\ &= (\tau + 1)B^{p\alpha}C^{\rho\alpha} = (\tau + 1)B^{p/(p+\rho)}C^{\rho/(p+\rho)}. \end{aligned}$$

If $B = 0$ then $0 \le A \le \varepsilon^p C$ for all $\varepsilon \in (0, 1]$. This shows that $A = 0$ and hence (3.15) holds. If $B > C$, let $\varepsilon_2 := 1$. Then, $A \le \tau\varepsilon_2^{-\rho}B + \varepsilon_2^p C\tau B + C \le (\tau + 1)B$ and (3.15) holds. □

Proof of Theorem 3.5: Let $2(N - 1)/N < p < \infty$, $f_1, f_2 \in L^q(\Omega)$ and let $u, v \in \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega)$ satisfy (3.12). Let $w := u - v \in \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega)$. For every real number $k \ge 0$, let $w_k := (|w| - k)^+ \operatorname{sgn}(w)$ and set $A(k) := \{x \in \overline{\Omega} : \ |w(x)| \ge k\}$. Then, $w_k \in \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega)$. Let $s \in [1, \infty]$ be such that $1/q + 1/s + 1/p^\star = 1$. Taking $\varphi = w_k$ as a test function in (3.12), we get that for every $k \ge 0$,

$$\mathscr{A}(u, w_k) - \mathscr{A}(v, w_k) \le \|f_1 - f_2\|_{q,\Omega}\|w_k\|_{p^\star,\Omega}\|\chi_{A(k)}\|_s. \tag{3.16}$$

Throughout the proof, we let $U(x, y) := u(x) - u(y)$, $V(x, y) = v(x) - v(y)$ and $W_k(x, y) := w_k(x) - w_k(y)$.

(a) Case $2 \le p < \infty$ **and** $q > N$: As $q > N$, it follows from Proposition 3.4 that $u, v \in L^\infty(\Omega)$. Note that

$$\begin{aligned} \mathscr{A}(u, w_k) - \mathscr{A}(v, w_k) &= \int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla w_k \, dx + 2^{2-p} \\ &\quad \times \int_{\partial\Omega} a(x)|w_k|^p \, dx + 2^{2-p}\int_{\partial\Omega} b(x)|w_k|^p \, d\sigma + 2^{2-p} \\ &\quad \times \iint_{\partial\Omega\times\partial\Omega} \frac{|w_k(x) - w_k(y)|^p}{k_p(x, y)} \, d\sigma_x \, d\sigma_y \\ &\quad + \int_\Omega a(x)(|u|^{p-2}u w_k - |v|^{p-2}v w_k - 2^{2-p}|w_k|^p) \, dx \\ &\quad + \int_{\partial\Omega} b(x)(|u|^{p-2}u w_k - |v|^{p-2}v w_k - 2^{2-p}|w_k|^p) \, d\sigma \\ &\quad + \iint_{\partial\Omega\times\partial\Omega} \frac{F(x, y, U, V, W_k)}{k_p(x, y)} \, d\sigma_x \, d\sigma_y, \end{aligned} \tag{3.17}$$

with

$$\begin{aligned} F(x, y, U, V, W_k) &:= [|U(x, y)|^{p-2}U(x, y) - |V(x, y)|^{p-2}V(x, y)] \\ &\times W_k(x, y) - 2^{2-p}|W_k(x, y)|^p. \end{aligned}$$

Since $w_k = 0$ on $\overline{\Omega} \setminus A(k)$ and $\nabla w_k = \chi_{A(k)} \cdot \nabla(u - v)$, it follows from (3.6) that for every $k \geq 0$,

$$\int_{\Omega} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla w_k \, dx \geq 2^{2-p} \int_{A_k} |\nabla w_k|^p \, dx. \tag{3.18}$$

Since $|u - v| \geq k$ on $A(k)$, it follows from (3.7) that on $A(k)$

$$||u|^{p-2}u - |v|^{p-2}v| \geq 2^{2-p}|u - v| \geq 2^{2-p}||u - v| - k|^{p-1} = 2^{2-p}|w_k|^{p-1}.$$

Multiplying this inequality by $|w_k|$, we get that

$$(|u|^{p-2}u - |v|^{p-2}v)w_k \geq 2^{2-p}|w_k|^p. \tag{3.19}$$

Similarly, we have that on $A(k) \times A(k)$,

$$\left[|U(x, y)|^{p-2}U(x, y) - |V(x, y)|^{p-2}V(x, y)\right][W_k(x, y)] \geq 2^{2-p}|W_k(x, y)|^p. \tag{3.20}$$

It follows from (3.18), (3.19), (3.20) and (3.16) that for every $k \geq 0$,

$$\begin{aligned} \min(1, b_0)2^{2-p}\|w_k\|^{p-1}_{\mathscr{W}^1_{p,p}(\Omega,\partial\Omega)} &\leq 2^{2-p}\mathscr{A}(w_k, w_k) \leq \mathscr{A}(u, w_k) - \mathscr{A}(v, w_k) \\ &\leq \|f_1 - f_2\|_{q,\Omega}\|\chi_{A(k)}\|_s. \end{aligned} \tag{3.21}$$

Using the embedding (2.5), we get that there exists a constant $C > 0$ such that for every $k \geq 0$,

$$\|w_k\|^{p-1}_{p^\star,\Omega} \leq C\|f_1 - f_2\||_{q,\Omega}\|\chi_{A(k)}\|_s. \tag{3.22}$$

Now proceeding exactly as at the end of the proof of Proposition 3.4 (after Eq. (3.10)), we get that $\|\chi_{A(K)}\|_{p^\star} = 0$ with the constant $K = \tilde{C}\|f_1 - f_2\|^{\frac{1}{p-1}}_{q,\Omega}$. Hence, $|w(x)| \leq K$ a.e. on Ω and we have shown (3.13).

(b) **Case** $2(N-1)/N < p < 2$ **and** $q > Np/(Np - 2N + 2)$**:** First, note that a simple calculation shows that $q > Np/(Np - 2N + 2) > N$ and hence, by Proposition 3.4, $u, v \in L^\infty(\Omega)$. Next, we claim that there is a constant $C_1 > 0$ such that for every $k \geq 0$,

$$\begin{aligned} \|\nabla w_k\|^p_{p,\Omega} \leq C_1 &\left[\int_{A(k)\cap\Omega} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla w_k \, dx\right]^{p/2} \\ &\cdot \left[\|\nabla u\|_{p,A(k)} + \|\nabla v\|_{p,A(k)}\right]^{p(1-p/2)}. \end{aligned} \tag{3.23}$$

For $\varepsilon \in (0,1]$, we let $B_\varepsilon := \{x \in \Omega : \ |\nabla u(x) - \nabla v(x)| \geq \varepsilon |\nabla u(x)|\}$. Using (3.5), we get that for every $k \geq 0$ and $\varepsilon > 0$,

$$\begin{aligned}
&\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla w_k \, \mathrm{d}x \\
&= \int_{A(k)\cap\Omega} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla (u-v) \, \mathrm{d}x \\
&\geq \int_{A(k)\cap B_\varepsilon} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla (u-v) \, \mathrm{d}x \\
&\geq (p-1)[1+1/\varepsilon]^{p-2} \int_{A(k)\cap B_\varepsilon} |\nabla u - \nabla v|^p \, \mathrm{d}x \\
&\geq (p-1)[1+1/\varepsilon]^{p-2} \left[\int_{A(k)\cap\Omega} |\nabla w_k|^p \, \mathrm{d}x - \int_{A(k)\setminus B_\varepsilon} |\nabla w_k|^p \, \mathrm{d}x \right] \\
&\geq (p-1)[1+1/\varepsilon]^{p-2} \left[\int_{A(k)\cap\Omega} |\nabla w_k|^p \, \mathrm{d}x - \varepsilon^p \int_{A(k)\cap\Omega} |\nabla w_k|^p \, \mathrm{d}x \right].
\end{aligned}$$

Using the fact that $[1+1/\varepsilon]^{2-p} \leq 2^{2-p}\varepsilon^{p-2}$ we get that

$$\|\nabla w_k\|_p^p \leq \frac{2^{2-p}}{p-1}\varepsilon^{p-2} \left[\int_{A(k)\cap\Omega} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla w_k \, \mathrm{d}x \right] + \varepsilon^p \|\nabla u\|_{p,A(k)}^p.$$

Applying Lemma 3.6 with

$$A := \|\nabla w_k\|_p^p, \ \tau := \frac{2^{2-p}}{p-1}, \ \rho = 2-p, \ C := \left[\|\nabla u\|_{p,A(k)} + \|\nabla v\|_{p,A(k)}\right]^p$$

and

$$B := \int_{A(k)} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla w_k \, \mathrm{d}x,$$

we get that

$$\|\nabla w_k\|_p^p \leq \frac{2^{2-p}+p-1}{p-1}\left[B^{p/2}C^{1-p/2} + B\right].$$

Using the estimate

$$B \leq \int_{A(k)\cap\Omega} |\nabla u|^{p-1}|\nabla u - \nabla v| \, \mathrm{d}x + \int_{A(k)\cap\Omega} |\nabla v|^{p-1}|\nabla u - \nabla v| \, \mathrm{d}x \leq 4C,$$

we get that there is a constant $C_1 > 0$ such that (3.23) holds. Similarly, we have that there is a constant $C_2 > 0$ such that for every $k \geq 0$,

$$\|w_k\|_{p,\partial\Omega}^p \leq C_2 \left[\int\limits_{A(k)\cap\partial\Omega} (|u|^{p-2}u - |v|^{p-2}v)w_k \, d\sigma \right]^{p/2} \cdot \left[\|u\|_{p,\partial\Omega} + \|v\|_{p,\partial\Omega}\right]^{p(1-p/2)}. \tag{3.24}$$

Combining (3.23) and (3.24) and using the fact that for every $k \geq 0$,

$$(|U(x, y|^{p-2}U(x, y) - |V(x, y)|^{p-2}V(x, y))W_k(x, y) \geq 0 \quad \text{on} \quad A(k) \times A(k),$$

we get that there is a constant $C > 0$ such that for every $k \geq 0$,

$$\|w_k\|^p_{\mathscr{W}^1_{p,p}(\Omega,\partial\Omega)} \leq C \cdot [\mathscr{A}(u, w_k - \mathscr{A}(v, w_k)]^{\frac{p}{2}} \cdot \left[\|u\|_{\mathscr{W}^1_{p,p}(\Omega,\partial\Omega)} + \|v\|_{\mathscr{W}^1_{p,p}(\Omega,\partial\Omega)}\right]^{p(1-\frac{p}{2})}. \tag{3.25}$$

Note that, since u satisfies (3.3), letting u as a test function, we get that there is a constant $C_3 > 0$ such that

$$\|u\|^p_{\mathscr{W}^1_{p,q}(\Omega,\partial\Omega)} \leq C_3 \|f_1\|_{(p^\star)',\Omega} \|u\|_{\mathscr{W}^1_{p,p}(\Omega,\partial\Omega)},$$

and this shows that

$$\|u\|^{p-1}_{\mathscr{W}^1_{p,p}(\Omega,\partial\Omega)} \leq C_3 \|f_1\|_{(p^\star)',\Omega}, \tag{3.26}$$

and similarly for v with f_1 replaced by f_2. Hence, there is a constant $C > 0$ such that

$$\|u\|_{\mathscr{W}^1_{p,p}(\Omega,\partial\Omega)} + \|v\|_{\mathscr{W}^1_{p,p}(\Omega,\partial\Omega)} \leq C \left[\sum_{j=1}^{2} \|f_j\|_{(p^\star)',\Omega} \right]^{1/(p-1)}. \tag{3.27}$$

It follows from (3.16), (3.25), (3.27), and the Sobolev embedding (2.5) that for every $k \geq 0$,

$$\|w_k\|^{\frac{p}{2}}_{p^\star,\Omega} \leq C \left[\sum_{j=1}^{2} \|f_j\|_{(p^\star)',\Omega} \right]^{p'(1-\frac{p}{2})} \cdot \|f_1 - f_2\|^{\frac{p}{2}}_{q,\Omega} \|\chi_{A(k)}\|_s^{\frac{p}{2}}. \tag{3.28}$$

Let $\delta := p^\star/s$. Since $q > Np/(Np - 2N + 2)$, then $\delta > 1$. Let $h > k$. Then, $A(h) \subset A(k)$ and $|w_k| \geq h - k$ on $A(k)$. Hence, $(h - k)^{p/2}\|\chi_{A(h)}\|^{p/2}_{p^\star} \leq \|w_k\|^{p/2}_{p^\star,\Omega}$. Letting $\Psi(h) = \|\chi_{A(h)}\|^{p/2}_{p^\star}$, it follows from (3.28) and the equality $\|\chi_{A(k)}\|_s = \|\chi_{A(k)}\|^\delta_{p^\star}$, that

$$\Psi(h) \leq C(h - k)^{-p/2} \left[\sum_{j=1}^{2} \|f_j\|_{(p^\star)'\Omega} \right]^{\frac{1}{p-1}} \cdot \|f_1 - f_2\|^{p/2}_{q,\Omega} \Psi(k)^\delta.$$

It follows from Lemma 3.2 that $\Psi(K) = 0$ with $K = C\left[\sum_{j=1}^{2} \|f_j\|_{(p^\star)',\Omega}\right]^{p'(\frac{2}{p}-1)}$ $\|f_1 - f_2\|_{q,\Omega}$. We have shown (3.14) and the proof is finished.

3.2 Case of general open sets (not necessarily of finite measure)

If $|\Omega| = \infty$, then we also assume that there exists a constant $a_0 > 0$ such that

$$a(x) \geq a_0 \text{ a.e. on } \Omega. \tag{3.29}$$

Proposition 3.7 *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set. Let $1 < p < \infty$, and let $u \in \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega)$ satisfy (3.3) where $f \in L^q(\Omega)$ with $q > N > 1$. Then, $u \in L^\infty(\Omega)$ and there is a constant $C = C(N, p, q, a_0, b_0) > 0$ such that*

$$\|u\|_{\infty,\Omega} \leq k_0 + \frac{C}{k_0^\alpha}\|u\|_{p^\star}^\alpha \|f\|_{q,\Omega}^{\frac{1}{p-1}}, \tag{3.30}$$

where $k_0 > 0$ is any fixed real number and $\alpha = \dfrac{p(q-N)}{q(N-1)(p-1)}$.

Proof Let $1 < p < \infty$ and let $u \in \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega)$ satisfy (3.3). Let $k_0 > 0$ be a fixed real number and set $u_k := (|u| - k)^+ \operatorname{sgn}(u)$ for $k \geq k_0$. Then, $u_k \in \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega)$. Let $A_k := \{x \in \overline{\Omega} : |u(x)| \geq k\}$. Since $k_0 \leq k \leq |u(x)|$ for every $x \in A_k$, we have that

$$|A_k| = \int\limits_{A_k \cap \Omega} dx = \frac{1}{k_0^p} \int\limits_{A_k \cap \Omega} k_0^p \, dx \leq \frac{1}{k_0^p} \int\limits_{A_k \cap \Omega} |u(x)|^p \, dx = \frac{1}{k_0^p}\|u\|_{p,\Omega}^p < \infty. \tag{3.31}$$

Hence, the set A_k has finite measure for every $k \geq k_0 > 0$. Now, proceeding exactly as in the proof of Proposition 3.4 (by using (3.29) and (2.4) if $|\Omega| = \infty$), we get that for every $h > k \geq k_0 > 0$,

$$\|\chi_{A_h}\|_{p^\star}^{p-1} \leq C(h-k)^{-(p-1)}\|f\|_{q,\Omega}\left[\|\chi_{A_k}\|_{p^\star}^{p-1}\right]^{\delta_0}, \tag{3.32}$$

where we recall that $\delta_0 := (p^\star q - q - p^\star)/q(p-1) > 1$. Letting $\Psi(h) := \|\chi_{A_h}\|_{p^\star}^{p-1}$ in Lemma 3.2, the estimate (3.32) shows that there is a constant K (independent of f) such that $\Psi(k_0 + K) := \|\chi_{A_{k_0+K}}\|_{p^\star}^{p-1} = 0$ with K given by $K := C\|\chi_{A_{k_0}}\|_{p^\star}^{\delta_0 - 1}\|f\||_{q,\Omega}^{1/(p-1)}$. Hence, $|u| \leq k_0 + K$ a.e. on Ω. Using (3.31), we get that

$$\|\chi_{A_{k_0}}\|_{p^\star}^{\delta_0-1} = \left(\int\limits_{A_{k_0}\cap\Omega} dx\right)^{\frac{\delta_0-1}{p^\star}} \leq \frac{1}{k_0^{\delta_0-1}}\|u\|_{p^\star,\Omega}^{\delta_0-1} = \frac{1}{k_0^\alpha}\|u\|_{p^\star,\Omega}^\alpha,$$

with $\alpha := \delta_0 - 1$. We have shown (3.30) and the proof is finished. □

We have the following result as a corollary of Proposition 3.7.

Corollary 3.8 *Let Ω, f, p, q, and u be as in Proposition 3.7. Then, there is a constant $C > 0$ such that*

$$\|u\|_{\infty,\Omega} \leq \|u\|_{p^\star} + C\|f\|_{q,\Omega}^{\frac{1}{p-1}}. \tag{3.33}$$

Proof Let $u \in \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega)$ satisfy (3.3) where $f \in L^q(\Omega)$ with $q > N > 1$. The inequality (3.33) is trivially satisfied if $u = 0$ a.e. on Ω. If $u \neq 0$ a.e. on Ω, taking $k_0 = \|u\|_{p^\star}$ in (3.30) we get (3.33). □

To conclude this section, we also consider the difference of weak solutions.

Proposition 3.9 *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set with $N > 1$ and let $f_1, f_2 \in L^q(\Omega)$. Let $2(N-1)/N < p < \infty$ and let $u, v \in \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega)$ be such that for every $\varphi \in \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega)$, $\mathscr{A}(u, \varphi) = \int_\Omega f_1\varphi \, \mathrm{d}x$ and $\mathscr{A}(v, \varphi) = \int_\Omega f_2\varphi \, \mathrm{d}x$, so that,*

$$\mathscr{A}(u, \varphi) - \mathscr{A}(v, \varphi) = \int_\Omega (f_1 - f_2)\varphi \, \mathrm{d}x. \tag{3.34}$$

(a) *If $2 \le p < \infty$ and $q > N$, then $u, v \in L^\infty(\Omega)$ and there is a constant $C > 0$ such that,*

$$\|u - v\|_{\infty,\Omega} \le k_0 + \frac{C}{k_0^{\beta_1}} \|u - v\|_{p^\star,\Omega}^{\beta_1} \|f_1 - f_2\|_{q,\Omega}^{\frac{1}{p-1}}, \tag{3.35}$$

where $k_0 > 0$ is any fixed real number and $\beta_1 := p(q-N)/q(N-1)(p-1)$.

(b) *If $2(N-1)/N < p < 2$ and $q > Np/(Np-2N+2)$ then $u, v \in L^\infty(\Omega)$ and there is a constant $C > 0$ such that,*

$$\|u - v\|_{\infty,\Omega} \le k_0 + \frac{C}{k_0^{\beta_2}} \|u - v\|_{p^\star,\Omega}^{\beta_2} \left[\sum_{j=1}^{2} \|f_j\|_{(p^\star)'}\right]^{p'(\frac{2}{p}-1)} \|f_1 - f_2\|_{q,\Omega}, \tag{3.36}$$

where $k_0 > 0$ is any fixed real number and $\beta_2 := (p^\star q - 2q - p^\star)/q$.

Proof Let $f_1, f_2 \in L^q(\Omega)$. Let $2(N-1)/N < p < \infty$ and let $u, v \in \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega)$ satisfy (3.34). Let $w := u - v \in \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega)$. Let $k_0 > 0$ be a fixed real number. For every real number $k \ge k_0$, let $w_k := (|w| - k)^+ \operatorname{sgn}(w)$ and set $A(k) := \{x \in \overline{\Omega} : |w(x)| \ge k\}$. Then, $w_k \in \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega)$.

(a) **Case** $2 \le p < \infty$ **and** $q > N$: It follows from Proposition 3.7 that $u, v \in L^\infty(\Omega)$. Let $\delta_0 := (p^\star q - q - p^\star)/q(p-1)$. Proceeding exactly as in the proof of Theorem 3.5 part (a) and noticing that $|A(k)| < \infty$ for every $k \ge k_0 > 0$, we get that there is a constant $C > 0$ such that $\Psi(h) := \|\chi_{A(h)}\|_{p^\star}^{p-1}$ satisfies the estimate

$$\Psi(h) \le C(h-k)^{-(p-1)} \|f_1 - f_2\|_{q,\Omega} \Psi(k)^{\delta_0}, \quad \forall\, h > k \ge k_0 > 0.$$

It follows from Lemma 3.2 that $\Psi(k_0 + K) = 0$ with $K = C\|\chi_{A(k_0)}\|_{p^\star}^{\delta_0 - 1} \|f_1 - f_2\|_{q,\Omega}^{\frac{1}{p-1}}$. Let $\beta_1 := \delta_0 - 1$. Since

$$\|\chi_{A(k_0)}\|_{p^\star}^{\delta_0 - 1} \le \frac{1}{k_0^{\beta_1}} \|u - v\|_{p^\star,\Omega}^{\beta_1},$$

we have shown (3.35) and the proof of part (a) is complete.

(b) **Case** $2(N-1)/N < p < 2$ **and** $q > Np/(Np-2N+2)$. Since $q > N$, it follows from Proposition 3.7 that $u, v \in L^\infty(\Omega)$. Let $\Psi(h) := \|A(h)\|_{p^\star}^{p/2}$ and let $\delta := (p^\star q - p - p^\star)/q > 1$. Proceeding as in the proof of Theorem 3.5 part (b), we get that

$$\Psi(h) \le C(h-k)^{-p/2} \left[\sum_{j=1}^{2} \|f_j\|_{(p^\star)'}\right]^{\frac{1}{p-1}} \|f_1 - f_2\|_{q,\Omega}^{p/2} \Psi(k)^{\delta}, \quad \forall\, h > k \ge k_0 > 0.$$

It follows from Lemma 3.2 that $\Psi(k_0 + K) = 0$ with $K = C\|\chi_{A(k_0)}\|_{p^\star}^{\delta-1}\left[\sum_{j=1}^{2}\|f_j\|_{(p^\star)'}\right]^{p'(\frac{2}{p}-1)}\|f_1 - f_2\|_{q,\Omega}$. Since

$$\|\chi_{A(k_0)}\|_{p^\star}^{\delta-1} \le \frac{1}{k_0^{\beta_2}}\|u - v\|_{p^\star,\Omega}^{\beta_2},$$

with $\beta_2 := \delta - 1 = (p^\star q - 2q - p^\star)/q$, we have shown (3.36) and the proof is finished. □

We have the following result as a corollary of Proposition 3.9.

Corollary 3.10 *Let Ω, f_1, f_2, p, q, and u, v be as in Proposition 3.9.*

(a) *If $2 \le p < \infty$ and $q > N$, then there is a constant $C > 0$ such that,*

$$\|u - v\|_{\infty,\Omega} \le \|u - v\|_{p^\star,\Omega} + C\|f_1 - f_2\|_{q,\Omega}^{\frac{1}{p-1}}.$$

(b) *If $2(N-1)/N < p < 2$ and $q > Np/(Np - 2N + 2)$, then there is a constant $C > 0$ such that,*

$$\|u - v\|_{\infty,\Omega} \le \|u - v\|_{p^\star,\Omega} + C\left[\sum_{j=1}^{2}\|f_j\|_{(p^\star)'}\right]^{p'(\frac{2}{p}-1)}\|f_1 - f_2\|_{q,\Omega}.$$

4 The relative *p*-capacity and admissible sets

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set with boundary $\partial\Omega$ and let $p \in [1, \infty)$. We let

$$\mathscr{W}^{1,p}(\Omega) := \overline{W^{1,p}(\Omega) \cap C_c(\overline{\Omega})}^{W^{1,p}(\Omega)}.$$

It is well known that $\mathscr{W}^{1,p}(\Omega)$ is a proper closed subspace of $W^{1,p}(\Omega)$, but they coincide if for example Ω is of class C (see [26, Theorem 1 p. 23]).

4.1 The relative capacity and a remark on the Maz'ya space

In this subsection, we introduce the relative capacity that plays an important role in the remainder of this article.

Definition 4.1 Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in [1, \infty)$. The *relative capacity* $\mathrm{Cap}_{p,\Omega}$ with respect to Ω is defined for sets $A \subset \overline{\Omega}$ by

$$\mathrm{Cap}_{p,\Omega}(A) := \inf\left\{\|u\|_{W^{1,p}(\Omega)}^{p} \;:\; \begin{array}{l} u \in \mathscr{W}^{1,p}(\Omega),\ \exists\, O \subset \mathbb{R}^N \text{ open}, \\ A \subset O \text{ and } u \ge 1 \text{ a.e. on } \Omega \cap O \end{array}\right\}.$$

- A set $P \subset \overline{\Omega}$ is called $\mathrm{Cap}_{p,\Omega}$-*polar* if $\mathrm{Cap}_{p,\Omega}(P) = 0$.
- We say that a property holds $\mathrm{Cap}_{p,\Omega}$-quasi everywhere (briefly q.e.) on a set $A \subset \overline{\Omega}$, if there exists a $\mathrm{Cap}_{p,\Omega}$-polar set P such that the property holds for all $x \in A \setminus P$.
- A function u is called $\mathrm{Cap}_{p,\Omega}$-quasi continuous on a set $A \subset \overline{\Omega}$ if for all $\varepsilon > 0$, there exists an open set O in the metric space $\overline{\Omega}$ such that $\mathrm{Cap}_{p,\Omega}(O) \le \varepsilon$ and u restricted to $A \setminus O$ is continuous.

The relative capacity $\mathrm{Cap}_{2,\Omega}$ has been introduced in [1] (see also [2,35]) to study the Laplace operator with linear Robin boundary conditions on arbitrary open subsets in $\mathbb{R}^N$. Biegert [4] has extended the definition of the relative capacity to every $p \in [1, \infty)$. Note that if $\Omega = \mathbb{R}^N$, then $\mathrm{Cap}_{p,\mathbb{R}^N} = \mathrm{Cap}_p$ is the classical Wiener capacity.

By [4], for every $u \in \mathscr{W}^{1,p}(\Omega)$, there exists a unique (up to a $\mathrm{Cap}_{p,\Omega}$-polar set) $\mathrm{Cap}_{p,\Omega}$-quasi continuous function $\tilde{u} : \overline{\Omega} \to \mathbb{R}$ such that $\tilde{u} = u$ a.e. on Ω. Moreover, if $u \in \mathscr{W}^{1,p}(\Omega)$ and $u_n \in \mathscr{W}^{1,p}(\Omega)$ is a sequence that converges to u in $\mathscr{W}^{1,p}(\Omega)$, then there is a subsequence of $\tilde{u}_n$ that converges to $\tilde{u}$ q.e. on $\overline{\Omega}$.

Remark 4.2 We have the following situation regarding the Maz'ya space. Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set with boundary $\partial\Omega$ and let

$$\Gamma_\infty := \{z \in \partial\Omega : \ \sigma(B(z,r) \cap \partial\Omega) = \infty \ \ \forall\, r > 0\}. \tag{4.1}$$

Then Γ_∞ is a relatively closed subset of $\partial\Omega$ and every function $u \in W_\sigma$ satisfies $u|_{\Gamma_\infty} = 0$, where we recall that

$$W_\sigma := \left\{ u \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega}) : \int\limits_{\partial\Omega} |u|^p \, d\sigma < \infty \right\}.$$

Since the closure of the set $\{u \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega}) : \ u|_{\Gamma_\infty} = 0\}$ in $W^{1,p}(\Omega)$ is the space $\{u \in \mathscr{W}^{1,p}(\Omega) : \ \tilde{u} = 0 \ \text{ q.e. on } \ \Gamma_\infty\}$, it follows that functions in $W^1_{p,p}(\Omega, \partial\Omega)$ are zero q.e. on Γ_∞. The complement of Γ_∞ denoted by

$$\Gamma_0 := \partial\Omega \setminus \Gamma_\infty = \{z \in \partial\Omega : \ \exists\, r > 0 : \ \sigma(B(z,r) \cap \partial\Omega) < \infty\} \tag{4.2}$$

is the relatively open subset of $\partial\Omega$ on which the measure σ is locally finite. Note that it may happen that $\Gamma_\infty = \partial\Omega$. In that case, $W^1_{p,p}(\Omega, \partial\Omega) = \mathscr{W}^1_{p,p}(\Omega, \partial\Omega) = W^{1,p}_0(\Omega) := \overline{\mathscr{D}(\Omega)}^{W^{1,p}(\Omega)}$. This is the case for the well-known 2-dimensional open set bounded by the von Kuch curve (also known as the snowflake) and is also the case for many domains with a fractal geometry.

Definition 4.3 Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set with boundary $\partial\Omega$. We say that a measurable subset Γ of $\partial\Omega$ is $\mathrm{Cap}_{p,\Omega}$*-admissible with respect to* σ, if $\mathrm{Cap}_{p,\Omega}(A) = 0$ implies $\sigma(A) = 0$ for every Borel set $A \subset \Gamma$.

The following result shows that the embedding (2.4) is not always injective. This result has been first proved in [35, Theorem 4.2.1] for $p = 2$. The case of general p is contained in [7, Theorem 2.11] where the authors have replaced the measure σ with a Radon measure μ on $\partial\Omega$. Since for "bad domains", σ is not always a Radon measure and for seek of completeness, we include the proof.

Theorem 4.4 *Let* $\Omega \subset \mathbb{R}^N$ *be an open set with boundary* $\partial\Omega$ *and let* $p \in [1, \infty)$ *be fixed. Then, the following assertions are equivalent.*

(i) *The operator* $R : \ W^1_{p,p}(\Omega, \partial\Omega) \to L^p(\Omega), u \mapsto u|_\Omega$ *is injective.*
(ii) *The set* Γ_0 *is* $\mathrm{Cap}_{p,\Omega}$*-admissible with respect to* σ.

Proof Let $p \in [1, \infty)$ be fixed.

(ii) $\Rightarrow$ (i): Assume that the set Γ_0 is $\mathrm{Cap}_{p,\Omega}$-admissible with respect to σ. We have to show that R is injective. Let $u \in W^1_{p,p}(\Omega, \partial\Omega)$ and suppose that $Ru = 0$. Then, there exists a sequence $u_n \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$ such that $u_n \to u$ in $W^1_{p,p}(\Omega, \partial\Omega)$, and therefore, $Ru_n \to Ru = 0$ in $\mathscr{W}^{1,p}(\Omega)$. By possibly passing to a subsequence, we have that u_n converges to zero q.e. on $\overline{\Omega}$. Since Γ_0 is $\mathrm{Cap}_{p,\Omega}$-admissible with respect to σ, it follows that u_n converges to zero σ-a.e. on Γ_0. As $u_n|_{\Gamma_0}$ converges to $u|_{\Gamma_0}$ in $L^p(\Gamma_0)$, the uniqueness of the limit implies that $u = 0$ σ-a.e. on Γ_0 and therefore $u = 0$ σ-a.e. on $\partial\Omega$ (since by definition, every $u \in W^1_{p,p}(\Omega, \partial\Omega)$ is such that $u = 0$ on $\Gamma_\infty := \partial\Omega \setminus \Gamma_0$).

(i) $\Rightarrow$ (ii): Assume that Γ_0 is not $\mathrm{Cap}_{p,\Omega}$-admissible with respect to σ. Then, there is a Borel set $K \subset \Gamma_0$ such that $\mathrm{Cap}_{p,\Omega}(K) = 0$ and $\sigma(K) > 0$. By the inner regularity of σ we may assume that K is compact. Since σ is locally finite on Γ_0, one also has that $\sigma(K) < \infty$. Since $\mathrm{Cap}_{p,\Omega}(K) = 0$, there exists a sequence $u_n \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$ such that $0 \le u_n \le 1$, $u_n = 1$ on K and $\|u_n\|_{W^{1,p}(\Omega)} \to 0$. For $k \in \mathbb{N}$ we let

$$O_k := \{x \in \mathbb{R}^N : \mathrm{dist}(x, K) < 1/k\}.$$

Then

$$K \subset O_{k+1} \subset O_k, \quad \bigcap_{k \ge 1} O_k = K \quad \text{and} \quad \sigma(O_k \cap \partial\Omega) \to \sigma(K).$$

Let $v_k \in \mathscr{D}(O_k)$ be such that $v_k = 1$ on K and $0 \le v_k \le 1$. It is clear that $v_k \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$ and $\|u_n v_k\|_{W^{1,p}(\Omega)} \to 0$ as $n \to \infty$. Note that $u_n v_k \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$, $0 \le u_n v_k \le 1$ and $u_n v_k = 1$ on K for all n, k. Now, let n_k be such that $\|w_k\|_{W^{1,p}(\Omega)} \le 2^{-k}$ where $w_k := u_{n_k} v_k$. Then, $w_k \to 0$ in $W^{1,p}(\Omega)$, $0 \le w_k \le 1$, $w_k = 1$ on K and $w_k \to 1_K$ everywhere on $\overline{\Omega}$. Since $w_k = 1$ on K, it follows that $\|w_k\|^p_{L^p(\Gamma_0)} \ge \sigma(K) > 0$. This shows that $1_K \in W^1_{p,p}(\Omega, \partial\Omega) \setminus \{0\}$ and $R1_K = 0$; hence, R is not injective. $\square$

4.2 **Admissible sets**

In this subsection, we give some examples of admissible and nonadmissible sets.

Definition 4.5 Let $p \in [1, \infty)$. We say that Ω has the $W^{1,p}$-extension property if for every $u \in W^{1,p}(\Omega)$ there exists $U \in W^{1,p}(\mathbb{R}^N)$ such that $U|_\Omega = u$ a.e.

In that case, by [24, Theorem 5], there exists a bounded linear extension operator $\mathscr{E}_p$ from $W^{1,p}(\Omega)$ into $W^{1,p}(\mathbb{R}^N)$. Moreover, the spaces $W^{1,p}(\Omega)$ and $\mathscr{W}^{1,p}(\Omega)$ coincide. In particular, one also obtains that for every $p \in (1, N)$, the space $W^{1,p}(\Omega)$ is continuously embedded into $L^{p_s}(\Omega)$ with $p_s = pN/(N-p)$.

Lemma 4.6 *Let $p \in [1, \infty)$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain which has the $W^{1,p}$-extension property and let $\mathscr{E}_p$ denote the bounded linear extension operator from $W^{1,p}(\Omega)$ into $W^{1,p}(\mathbb{R}^N)$. Then, for every $A \subset \partial\Omega$ one has,*

$$\frac{1}{\|\mathscr{E}_p\|^p} \mathrm{Cap}_p(A) \le \mathrm{Cap}_{p,\Omega}(A) \le \mathrm{Cap}_p(A). \tag{4.3}$$

Proof First, we claim that for every $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, there exists a function $U \in W^{1,p}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ such that $U|_\Omega = u$. Since Ω has the $W^{1,p}$-extension property, by [24, Theorem 2], it satisfies the measure density condition, that is, there exists a constant $c_\Omega > 0$ such that

$$|B(x,r) \cap \Omega| \ge c_\Omega r^N \quad \text{for all} \quad x \in \Omega \text{ and all } 0 < r \le 1.$$

Moreover, $\lambda_N(\partial\Omega) := |\partial\Omega| = 0$. For a measurable set $G \subset \mathbb{R}^N$, we let $M^{1,p}(G)$ be the Sobolev type space introduced by Hajłasz [23], which is the set of all functions $u \in L^p(G)$ with generalized gradient in $L^p(G)$. We recall that a measurable function g on Ω is called a generalized gradient of u if the inequality

$$|u(x) - u(y)| \leq |x - y|(g(x) + g(y))$$

holds a.e. on Ω, that is, there is a set $E \subset \Omega$ with $|E| = 0$ and the inequality holds for every $x, y \in \Omega \setminus E$. It follows from [31, Theorem 1.3] that $M^{1,p}(\mathbb{R}^N)|_{\overline{\Omega}} = M^{1,p}(\overline{\Omega})$ and there is a linear continuous extension operator $\mathscr{E}_p : M^{1,p}(\overline{\Omega}) \to M^{1,p}(\mathbb{R}^N)$. Using the fact that $M^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$ as sets with equivalent norms, we get that $M^{1,p}(\overline{\Omega}) = W^{1,p}(\Omega)$ as sets with equivalent norms; hence, the extension operator $\mathscr{E}_p$ constructed for $M^{1,p}(\overline{\Omega})$ is also a linear continuous extension operator from $W^{1,p}(\Omega)$ into $W^{1,p}(\mathbb{R}^N)$.

To verify that the extension operator $\mathscr{E}_p$ constructed by Shvartsman [31] maps $W^{1,p}(\Omega) \cap C(\overline{\Omega})$ into $W^{1,p}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, we describe shortly the construction of this explicit extension operator. By [31, Theorem 2.4], there exists a countable family of balls $W = W(\overline{\Omega})$ such that $\mathbb{R}^N \setminus \overline{\Omega} = \bigcup_{B \in W} B$, every ball $B = B(x_B, r_B) \in W$ satisfies $3r_B \leq \operatorname{dist}(B, \overline{\Omega}) \leq 25r_B$ and further every point of $\mathbb{R}^N \setminus \overline{\Omega}$ is covered by at most $C = C(N)$ balls from W. Let (ϕ_B) for $B \in W$ be a partition of unity associated with this Whitney covering W with properties $0 \leq \phi_B \leq 1$, $\operatorname{supp}(\phi_B) \subset B(x_B, (9/8)r_B)$, $\sum_{B \in W} \phi_B(x) = 1$ on $\mathbb{R}^N \setminus \overline{\Omega}$, and for all $x, y \in \mathbb{R}^N$, $|\phi_B(x) - \phi_B(y)| \leq C\operatorname{dist}(x, y)/r_B$ for some constant $C > 0$ independent of B. By [31, Theorem 2.6], there is a family of Borel sets $\{H_B : B \in W\}$ such that $H_B \subset B(x_B, \gamma_1 r_B) \cap \overline{\Omega}$, $\lambda_N(B) \leq \gamma_2 \lambda_N(H_B)$ for all $B \in W$ whenever $r_B \leq c_\Omega$, where γ_1 and γ_2 are positive constants. Now, by [31, Theorem 1.3 Equation (1.5)], a continuous extension operator $\mathscr{E}_p : M^{1,p}(\overline{\Omega}) \to M^{1,p}(\mathbb{R}^N)$ is given by

$$(\mathscr{E}_p u)(x) := \sum_{B \in W} u_{H_B} \phi_B(x) \quad \text{for } x \in \mathbb{R}^N \setminus \overline{\Omega}, \text{ where } u_{H_B} := \frac{1}{\lambda_N(H_B)} \int_{H_B} u \, \mathrm{d}x,$$

and $(\mathscr{E}_p u)(x) = u(x)$ if $x \in \overline{\Omega}$. Now, the claim follows from the construction of $\mathscr{E}_p$.

Next, let $A \subset \partial\Omega$ be an arbitrary set. By definition, $\operatorname{Cap}_{p,\Omega}(A) \leq \operatorname{Cap}_p(A)$. The proof of the first inequality in (4.3) follows as the case $p = 2$ included in [1, Proposition 1.4] (but in [1], the authors did not know that the previous claim holds for all extension domains). For seek of completeness, we include the proof for general p. Let $\varepsilon > 0$ be arbitrary.

Step 1: Assume that $A \subset \partial\Omega$ is a compact set and let $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that $u(x) \geq 1$ for all $x \in A$ and $\|u\|^p_{W^{1,p}(\Omega)} \leq \operatorname{Cap}_{p,\Omega}(A) + \varepsilon$. Let $U := \mathscr{E}_p u$. Then, $U \in W^{1,p}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and $U = u$ on Ω, and hence, $u = U$ on $\overline{\Omega}$ by continuity. Thus, $U(x) \geq 1$ on A and this implies that

$$\operatorname{Cap}_p(A) \leq \|U\|^p_{W^{1,p}(\mathbb{R}^N)} \leq \|\mathscr{E}_p\|^p \|u\|^p_{W^{1,p}(\Omega)} \leq \|\mathscr{E}_p\|^p (\operatorname{Cap}_{p,\Omega}(A) + \varepsilon).$$

Since ε was arbitrary, one obtains $\operatorname{Cap}_p(A) \leq \|\mathscr{E}_p\|^p \operatorname{Cap}_{p,\Omega}(A)$ for every compact set $A \subset \partial\Omega$.

Step 2: Assume that $A \subset \partial\Omega$ is a relatively open set. Then, there exists an open set $O \subset \mathbb{R}^N$ such that $A = O \cap \overline{\Omega}$. Since $\operatorname{Cap}_{p,\Omega}$ is a Choquet capacity (see [4]), we get

$$\begin{aligned}\operatorname{Cap}_p(A) &= \sup\{\operatorname{Cap}_p(K) : K \subset O \cap \overline{\Omega} \text{ compact}\} \\ &\leq \|\mathscr{E}_p\|^p \sup\{\operatorname{Cap}_{p,\Omega}(K) : K \subset O \cap \overline{\Omega} \text{ compact}\} \leq \|\mathscr{E}_p\|^p \operatorname{Cap}_{p,\Omega}(A).\end{aligned}$$

Step 3: Finally, if $A \subset \partial\Omega$ is arbitrary, then,

$$\mathrm{Cap}_p(A) = \inf\{\mathrm{Cap}_p(O) : A \subset O \subset \mathbb{R}^N \text{ open }\}.$$

Using the definition of $\mathrm{Cap}_{p,\Omega}$, we get that

$$\begin{aligned}\mathrm{Cap}_p(A) &= \inf\{\mathrm{Cap}_p(O) : A \subset O \subset \mathbb{R}^N \text{ open }\} \leq \|\mathscr{E}_p\|^p \inf\{\mathrm{Cap}_{p,\Omega}(O \cap \overline{\Omega}) \\ &: A \subset O \subset \mathbb{R}^N \text{ open }\} \leq \|\mathscr{E}_p\|^p \inf\{\mathrm{Cap}_{p,\Omega}(\mathscr{U}) : \\ &A \subset \mathscr{U} \subset \overline{\Omega}, \mathscr{U} \text{ relatively open }\} \leq \|\mathscr{E}_p\|^p \,\mathrm{Cap}_{p,\Omega}(A)\end{aligned}$$

and we have shown (4.3) for every $A \subset \partial\Omega$ and the proof is finished. □

Note that it is easy to verify that the inequalities in (4.3) hold for every $A \subset \Omega$. Now, we are ready to give a large class of admissible sets.

Lemma 4.7 *Let $p \in (1, N)$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain which has the $W^{1,p}$-extension property. Then, $\partial\Omega$ is $\mathrm{Cap}_{p,\Omega}$-admissible with respect to σ.*

Proof Let $p \in (1, N)$ and assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain which has the $W^{1,p}$-extension property and let $A \subset \partial\Omega$ be a Borel set such that $\mathrm{Cap}_{p,\Omega}(A) = 0$. Then, $\mathrm{Cap}_p(A) = 0$ by (4.3). Now, it follows from [18, Theorem 4.7.4, p.156] that $\mathscr{H}^s(A) = 0$ for any $s > N - p$, where $\mathscr{H}^s$ denotes the s-dimensional Hausdorff measure. Since $\sigma = \mathscr{H}^{N-1}|_{\partial\Omega}$ and $N - 1 > N - p$, we have that $\sigma(A) = 0$ and the proof is complete. □

Since bounded Lipschitz domains and the domain bounded by the von Kuch curve (for example) have the $W^{1,p}$-extension property, then their boundaries are $\mathrm{Cap}_{p,\Omega}$-admissible with respect to σ for every $p \in (1, N)$.

To conclude this section, we mention that all the examples of open sets, whose boundaries are not $\mathrm{Cap}_{2,\Omega}$-admissible with respect to σ, given in [1, Examples 1.5, 1.6, 4.2, 4.3], are also not $\mathrm{Cap}_{p,\Omega}$-admissible with respect to σ, for every $p \in [1, \infty)$.

5 The parabolic problem

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set of finite measure and with boundary $\partial\Omega$. In this section, given $p \in (1, \infty)$, we want to investigate the well-posedness of the first-order Cauchy problem formally given by

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \Delta_p u(t,x) & t > 0,\ x \in \Omega \\ |\nabla u(t,x)|^{p-2}\partial_\nu u(t,x) + b(x)|u(t,x)|^{p-2}u(t,x) & \\ \quad +\Theta_p(u(t,x)) = 0 & t > 0,\ \ x \in \partial\Omega \\ u(0,x) = u_0 & x \in \Omega, \end{cases} \tag{5.1}$$

where u_0 is a given function in $L^2(\Omega)$ and the coefficient $b \in L^\infty(\partial\Omega)$ and satisfies (3.2).

We define the functional $\Phi_\Theta : L^2(\Omega) \to [0, +\infty]$ by

$$\Phi_\Theta(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u|^p \,\mathrm{d}x + \dfrac{1}{p}\int_{\partial\Omega} b(x)|u|^p \,\mathrm{d}\sigma + \dfrac{1}{p}\iint_{\partial\Omega\times\partial\Omega} & \\ \quad \frac{|u(x)-u(y)|^p}{k_p(x,y)} \,\mathrm{d}\sigma_x \,\mathrm{d}\sigma_y, & \text{if } u \in D(\Phi_\Theta) \\ +\infty & \text{otherwise ,} \end{cases} \tag{5.2}$$

where $D(\Phi_\Theta) = \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega) \cap L^2(\Omega)$. As we have mentioned in Remark 4.2, if $u \in \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega) \subset \mathscr{W}_{p,p}^1(\Omega, \partial\Omega)$, one has

$$\int_{\partial\Omega} b(x)|u|^p \, \mathrm{d}\sigma = \int_{\Gamma_0} b(x)|u|^p \, \mathrm{d}\sigma$$

and

$$\iint_{\partial\Omega\times\partial\Omega} \frac{|u(x)-u(y)|^p}{k_p(x,y)} \, \mathrm{d}\sigma_x \, \mathrm{d}\sigma_y = \iint_{\Gamma_0\times\Gamma_0} \frac{|u(x)-u(y)|^p}{k_p(x,y)} \, \mathrm{d}\sigma_x \, \mathrm{d}\sigma_y$$

where Γ_0 denotes the relatively open subset of $\partial\Omega$ on which σ is locally finite and is given in (4.2).

5.1 **The nonlocal boundary conditions**

In this subsection, we justify the terminology *"nonlocal" boundary conditions*.

Definition 5.1 We say that a functional $\varphi : L^2(\Omega) \to (-\infty, +\infty]$ is *local*[1] if for every $u, v \in L^2(\Omega)$

$$|u| \wedge |v| = 0 \quad \Rightarrow \quad \varphi(u+v) = \varphi(u) + \varphi(v). \tag{5.3}$$

Here, $u \wedge v$ denotes the (pointwise) infimum of the functions u and v. A functional which is not local is said to be nonlocal.

Some properties of local functionals are given in [8] and the references therein.

Now, let Φ_Θ be the functional defined in (5.2). Let $u, v \in D(\Phi_\Theta) \cap C_c(\overline{\Omega})$ be such that $\mathrm{supp}[u] \cap \mathrm{supp}[v] = \emptyset$. It is clear that

$$\begin{aligned}
&\int_\Omega |\nabla(u+v)|^p \, \mathrm{d}x + \int_{\partial\Omega} b(x)|u+v|^p \, \mathrm{d}\sigma \\
&= \int_\Omega |\nabla u|^p \, \mathrm{d}x + \int_{\partial\Omega} b(x)|u|^p \, \mathrm{d}\sigma + \int_\Omega |\nabla v|^p \, \mathrm{d}x + \int_{\partial\Omega} b(x)|v|^p \, \mathrm{d}\sigma.
\end{aligned}$$

On the other hand, it is easy to see that there are functions $u, v \in D(\Phi_\Theta) \cap C_c(\overline{\Omega})$ with $\mathrm{supp}[u] \cap \mathrm{supp}[v] = \emptyset$ and

$$\begin{aligned}
|(u+v)(x) - (u+v)(y)| &= |u(x)-u(y)|_{\mathrm{supp}[u]\times\mathrm{supp}[u]} + |v(x)-v(y)|_{\mathrm{supp}[v]\times\mathrm{supp}[v]} \\
&\quad + |u(x)-v(y)|_{\mathrm{supp}[u]\times\mathrm{supp}[v]} + |v(x)-u(y)|_{\mathrm{supp}[v]\times\mathrm{supp}[u]} \\
&\neq |u(x)-u(y)|_{\mathrm{supp}[u]\times\mathrm{supp}[u]} + |v(x)-v(y)|_{\mathrm{supp}[v]\times\mathrm{supp}[v]}.
\end{aligned}$$

This shows that $\Phi_\Theta(u+v) \neq \Phi_\Theta(u) + \Phi_\Theta(v)$, and hence, the functional Φ_Θ is nonlocal. Since the nonlocality comes from the boundary conditions, more precisely from the operator Θ_p, we say that we have a nonlocal boundary condition. If Φ_Θ is lower semicontinuous (l.s.c.) its subgradient $\partial\Phi_\Theta$ is called a realization of the p-Laplace operator with the nonlocal Robin boundary conditions.

[1] In the literature, one can also find the term *additive*.

5.2 **The lower semicontinuity of the functional**

Let Φ_Θ be the functional defined in (5.2). We have the following result.

Theorem 5.2 *Let $\Omega \subset \mathbb{R}^N$ be an open set of finite measure and with boundary $\partial\Omega$ and let $p \in (1, \infty)$ be fixed. Then, the following assertions are equivalent.*

(i) *The functional Φ_Θ is l.s.c. on $L^2(\Omega)$.*
(ii) *The set Γ_0 is $\mathrm{Cap}_{p,\Omega}$-admissible with respect to σ.*

Proof Let $p \in (1, \infty)$ be fixed.

(ii) $\Rightarrow$ (i): Assume Γ_0 is $\mathrm{Cap}_{p,\Omega}$-admissible with respect to σ. We show that Φ_Θ is l.s.c. Let u_n be a sequence in $D(\Phi_\Theta) = \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega) \cap L^2(\Omega)$ that converges to $u \in L^2(\Omega)$. If $\liminf_{n\to\infty} \Phi_\Theta(u_n) = +\infty$, there is nothing to prove. Hence, we assume that $\liminf_{n\to\infty} \Phi_\Theta(u_n) < \infty$. Take any subsequence of u_n which we also denote by u_n, such that $\lim_{n\to\infty} \Phi_\Theta(u_n) = \text{const}$. Let the Banach space $\mathscr{W}_{p,p}^1(\Omega, \partial\Omega) \cap L^2(\Omega)$ be endowed with the norm

$$|||u||| := \left(\|\nabla u\|_{p,\Omega}^p + \|u\|_{p,\partial\Omega}^p\right)^{1/p} + \|u\|_{2,\Omega}.$$

Then u_n is a bounded sequence in the reflexive Banach space $\mathscr{W}_{p,p}^1(\Omega, \partial\Omega) \cap L^2(\Omega)$. Let v_n be a convex combination of u_n that converges strongly to v in $\mathscr{W}_{p,p}^1(\Omega, \partial\Omega) \cap L^2(\Omega)$ and hence converges strongly to v in $\mathscr{W}^{1,p}(\Omega)$. The uniqueness of the limit shows that $u = v$ a.e. on Ω. We have to show that $v|_{\partial\Omega} = u|_{\partial\Omega}$ σ-a.e. on $\partial\Omega$. Since $v = u = 0$ σ-a.e. on Γ_∞, it remains to show that $v|_{\Gamma_0} = u|_{\Gamma_0}$ σ-a.e. on Γ_0. Since v_n converges strongly to v on $\mathscr{W}^{1,p}(\Omega)$, by taking a subsequence if necessary, we may assume that v_n converges to v q.e. on Γ_0 and hence σ-a.e. (since by assumption Γ_0 is $\mathrm{Cap}_{p,\Omega}$-admissible with respect to σ). Now, since $v_n|_{\Gamma_0}$ converges to $v|_{\Gamma_0}$ in $L^p(\Gamma_0)$, the uniqueness of the limit shows that $v|_{\Gamma_0} = u|_{\Gamma_0}$ σ-a.e. on Γ_0. Using the convexity of Φ_Θ, we obtain that

$$\Phi_\Theta(u) = \liminf_{n\to\infty} \Phi_\Theta(v_n) \le \liminf_{n\to\infty} \Phi_\Theta(u_n),$$

and the proof of (i) is finished.

(i) $\Rightarrow$ (ii): Assume that Γ_0 is not $\mathrm{Cap}_{p,\Omega}$-admissible with respect to σ. Then, there is a Borel set $K \subset \Gamma_0$ such that $\mathrm{Cap}_{p,\Omega}(K) = 0$ and $\sigma(K) > 0$. By the inner regularity of the measure σ, we may assume that K is a compact set. Let w_k be the sequence constructed in the proof of Theorem 4.4 part (i) $\Rightarrow$ (ii). We recall that $w_k \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$, $w_k \to 0$ in $W^{1,p}(\Omega)$, $0 \le w_k \le 1$, $w_k = 1$ on K, and $w_k \to 1_K$ everywhere on $\overline{\Omega}$. Using this fact, one also has that $w_k \to 0$ in $L^2(\Omega)$. Without any restriction, we may assume that the sequence w_k is decreasing. Let $\tilde{w}_k := w_1 - w_k$. Since $w_k \to 1_K$ everywhere, it follows that $\tilde{w}_k \to w_1|_{\Gamma_0\setminus K}$ everywhere. Since $0 \le \tilde{w}_k \le w_1$, it follows that

$$\int_{\Gamma_0\setminus K} b|w_1|^p \, d\sigma \le \liminf_{k\to\infty} \int_{\Gamma_0} b|\tilde{w}_k|^p \, d\sigma = \liminf_{k\to\infty} \int_{\Gamma_0\setminus K} b|\tilde{w}_k|^p \, d\sigma \le \int_{\Gamma_0\setminus K} b|w_1|^p \, d\sigma.$$

Hence,

$$\int_{\Gamma_0\setminus K} b|w_1|^p \, d\sigma = \liminf_{k\to\infty} \int_{\Gamma_0} b|\tilde{w}_k|^p \, d\sigma. \tag{5.4}$$

Since $\tilde{w}_k \to w_1$ in $L^2(\Omega)$ and $\nabla\tilde{w}_k \to \nabla w_1$ in $L^p(\Omega)$ and $w_1 = 1$ on K, it follows from (5.4) and (3.2) that

$$\begin{aligned}
\Phi_\Theta(w_1) &= \frac{1}{p}\int_\Omega |\nabla w_1|^p\,\mathrm{d}x + \frac{1}{p}\int_{\Gamma_0} b|w_1|^p\,\mathrm{d}\sigma + \frac{1}{p}\iint_{\Gamma_0\times\Gamma_0} \frac{|w_1(x)-w_1(y)|^p}{k_p(x,y)}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y\\
&= \frac{1}{p}\int_\Omega |\nabla w_1|^p\,\mathrm{d}x + \frac{1}{p}\int_{\Gamma_0\setminus K} b|w_1|^p\,\mathrm{d}\sigma + \frac{1}{p}\int_K b|w_1|^p\,\mathrm{d}\sigma\\
&\quad + \frac{1}{p}\iint_{\Gamma_0\times\Gamma_0} \frac{|w_1(x)-w_1(y)|^p}{k_p(x,y)}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y\\
&\geq \frac{1}{p}\int_\Omega |\nabla w_1|^p\,\mathrm{d}x + \frac{1}{p}\int_{\Gamma_0\setminus K} b|w_1|^p\,\mathrm{d}\sigma + \frac{1}{p}b_0\sigma(K)\\
&\quad + \frac{1}{p}\iint_{\Gamma_0\times\Gamma_0} \frac{|w_1(x)-w_1(y)|^p}{k_p(x,y)}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y\\
&> \frac{1}{p}\int_\Omega |\nabla w_1|^p\,\mathrm{d}x + \frac{1}{p}\int_{\Gamma_0\setminus K} b|w_1|^p\,\mathrm{d}\sigma\\
&\quad + \frac{1}{p}\iint_{\Gamma_0\times\Gamma_0} \frac{|w_1(x)-w_1(y)|^p}{k_p(x,y)}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y = \liminf_{k\to\infty}\Phi_\Theta(\tilde{w}_k).
\end{aligned}$$

This shows thatΦ_Θ is not l.s.c. on $L^2(\Omega)$ and the proof is finished. □

Next, let $\Omega \subset \mathbb{R}^N$ be an open set of finite measure with boundary $\partial\Omega$. Let $p \in (1,\infty)$ and assume that Γ_0 is $\mathrm{Cap}_{p,\Omega}$-admissible with respect to σ. By Theorem 5.2, Φ_Θ is proper, convex, and l.s.c. Let $\partial\Phi_\Theta$ be its subgradient (which is trivially single value). Using the definition of the normal derivative given in (1.1) and the nonlocal operator in (1.3), it is straightforward to show that if $f \in L^2(\Omega)$ and $u \in D(\Phi_\Theta)$, then $f \in \partial\Phi_\Theta(u)$ if and only if u is a weak solution of (3.1). Moreover, $D(\partial\Phi_\Theta) \subset D(\Phi_\Theta) = \mathscr{V}_\Theta^{1,p}(\Omega,\partial\Omega) \cap L^2(\Omega)$ and for every $u \in D(\partial\Phi_\Theta)$ and $v \in D(\Phi_\Theta)$,

$$\begin{aligned}
\partial\Phi_\Theta(u)(v) &= \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,\mathrm{d}x + \int_{\partial\Omega} b(x)|u|^{p-2}uv\,\mathrm{d}\sigma\\
&\quad + \iint_{\partial\Omega\times\partial\Omega} \frac{|u(x)-u(y)|^{p-2}}{k_p(x,y)}(u(x)-u(y))(v(x)-v(y))\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y.
\end{aligned} \tag{5.5}$$

That is, $\partial\Phi_\Theta$ is the realization of the operator Δ_p with the nonlocal Robin boundary conditions

$$|\nabla u|^{p-2}\partial u/\partial\nu + b(x)|u|^{p-2}u + \Theta_p(u) = 0 \quad \text{weakly on } \partial\Omega.$$

As a corollary of Theorem 5.2, we have the following result.

Corollary 5.3 *Let $p \in (1,\infty)$ be fixed and let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set with finite measure and assume that Γ_0 is $\mathrm{Cap}_{p,\Omega}$-admissible with respect to σ. Then, the subgradient $-\partial\Phi_\Theta$ generates a strongly continuous semigroup $(S_\Theta(t))_{t\geq 0}$ of contractions on $L^2(\Omega)$.*

In particular, for every $u_0 \in L^2(\Omega)$, the orbit $u(\cdot) = S_\Theta(\cdot)u_0$ is the unique strong solution of the first-order Cauchy problem

$$\begin{cases} u \in C([0,\infty); L^2(\Omega)) \cap W^{1,\infty}_{loc}((0,\infty); L^2(\Omega)) \ \ \textit{and} \ \ u(t,\cdot) \in D(\partial\Phi_\Theta) \ \ \textit{a.e.} \\ \dfrac{\partial u(t,x)}{\partial t} + \partial\Phi_\Theta(u(t,x)) = 0, \quad t > 0, \ x \in \Omega; \quad u(0,x) = u_0(x). \end{cases} \tag{5.6}$$

Moreover, the semigroup $(S_\Theta(t))_{t\geq 0}$ is order preserving in the sense that $S_\Theta(t)v \leq S_\Theta(t)w$ for all $t \geq 0$ whenever $v, w \in L^2(\Omega)$ with $v \leq w$ and is nonexpansive on $L^\infty(\Omega)$ in the sense that $\|S_\Theta(t)v - S_\Theta(t)w\|_{\infty,\Omega} \leq \|v - w\|_{\infty,\Omega}$ for every $t \geq 0$ and $v, w \in L^\infty(\Omega) \cap L^2(\Omega)$.

Proof Let $p \in (1,\infty)$ be fixed and assume that Γ_0 is $\mathrm{Cap}_{p,\Omega}$-admissible with respect to σ. Since Φ_Θ is proper, convex, and l.s.c. (by Theorem 5.2) on the Hilbert space $L^2(\Omega)$, It follows from the well-known generation theorem by Minty [27,28] that $-\partial\Phi_\Theta$ generates a strongly continuous semigroup $(S_\Theta(t))_{t\geq 0}$ of contraction operators on $\overline{D(\Phi_\Theta)}^{L^2(\Omega)} = L^2(\Omega)$. By the theory of evolution equations governed by subgradient of l.s.c. functionals (see [30]), for every $u_0 \in L^2(\Omega)$, the orbit $u(\cdot) = S_\Theta(\cdot)u_0$ is the unique strong solution of the first-order Cauchy problem (5.6). Now, since $\Phi_\Theta(u \wedge v) + \Phi_\Theta(u \vee v) \leq \Phi_\Theta(u) + \Phi_\Theta(v)$ for every $u, v \in L^2(\Omega)$, it follows from [3, Théorème 2.1] (see also [11, Theorems 3.6 and 3.8]) that $S_\Theta(t)$ is order preserving. Finally, proceeding exactly as in the proof of [34, Theorem 3.4] by using [11, Theorem 3.6 and Corollary 3.9]), we can easily verify that $S_\Theta(t)$ is also nonexpansive on $L^\infty(\Omega)$. □

In the literature, *a semigroup that is order preserving and nonexpansive on $L^\infty(\Omega)$ is called submarkovian*. For more details, we refer the reader to the paper [11] and the references therein.

5.3 **The ultracontractivity**

Now, we formulate and prove the ultracontractivity property and the $(L^q - L^\infty)$-Hölder type continuity.

Theorem 5.4 *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set of finite measure with $N > 1$ and let $p \in [2,\infty)$. Assume that Γ_0 is $\mathrm{Cap}_{p,\Omega}$-admissible with respect to σ and let $(S_\Theta(t))_{t\geq 0}$ be the strongly continuous (nonlinear) semigroup on $L^2(\Omega)$ generated by $-\partial\Phi_\Theta$. Let $q \in [2,\infty]$ and let*

$$\beta := \frac{N-1}{N}\left[1 - \left(\frac{q}{q+p-2}\right)^N\right], \quad \delta := \frac{1}{p-2}\left[1 - \left(\frac{q}{q+p-2}\right)^N\right] \ \ \textit{and}$$

$$\gamma := \left(\frac{q}{q+p-2}\right)^N.$$

Then, there is a constant $C = C(N,p,q,\Omega) > 0$ such that for every $u_0, v_0 \in L^q(\Omega)$ and $t > 0$,

$$\|S_\Theta(t)u_0 - S_\Theta(t)v_0\|_\infty \leq C|\Omega|^\beta t^{-\delta}\|u_0 - v_0\|_q^\gamma. \tag{5.7}$$

Proof Let $p \in [2,\infty)$ be fixed. Let $u_0, v_0 \in L^\infty(\Omega)$, $u(s) := S_\Theta(s)u_0$ and $v(s) := S_\Theta(s)v_0$, where $s > 0$. Let $r \geq 2$ and consider the function $G_r : (0,\infty) \to [0,\infty)$ defined by $G_r(s) := \|u(s) - v(s)\|_r^r$. First, notice that G_r is well-defined because u and v are bounded in $\Omega \times (0,\infty)$ and because Ω has finite measure. We show that G_r is differentiable a.e. Since by Corollary 5.3, $(S_\Theta(t))_{t\geq 0}$ is the strongly continuous semigroup of contractions generated by the subdifferential $-\partial\Phi_\Theta$ of the proper, convex, and l.s.c. functional Φ_Θ, and $D(\partial\Phi_\Theta)$ is dense in $L^2(\Omega)$, it follows from a result of Brezis (see [30, Proposition IV.3.2 and Corollary IV.3.2] or Corollary 5.3 above), that, for every $u_0, v_0 \in L^\infty(\Omega) \subset L^2(\Omega) = \overline{D(\partial\Phi_\Theta)}$, the function $u(s)$ and $v(s)$ belong to $D(\partial\Phi_\Theta)$ for every $s > 0$, and u, v (as functions of s) are continuous from $[0,\infty) \to L^2(\Omega)$ and Lipschitz on $[a,\infty)$ for every $a > 0$. Hence, u, v (as functions of s) are differentiable a.e. Therefore, G_r is differentiable a.e. Throughout the remainder of the proof, we let $U(s) := u(s) - v(s)$, $\tilde{U}(s,x,y) = u(s,x) - u(s,y)$, $\tilde{V}(s,x,y) = v(s,x) - v(s,y)$ and in our notation, we sometime omit the dependence of u, v in x. Since $G_r(s) = \int_\Omega |U(s)|^r \, dx$, by Leibniz rule, and using the fact that $u(s)$, $v(s)$ are solutions of the Cauchy problem (5.6) with initial data u_0, v_0, we get that for a.e. $s > 0$,

$$\begin{aligned}
\frac{d}{ds}\|U(s)\|_r^r =& r\int_\Omega |U(s)|^{r-1}\operatorname{sgn}(U(s))U'(s)\, dx\\
=& -r\int_\Omega |U(s)|^{r-1}\operatorname{sgn}(U(s))\left[\partial\Phi_\Theta(u(s)) - \partial\Phi_\Theta(v(s))\right]\, dx\\
=& -r\int_\Omega |u(s)-v(s)|^{r-1}\operatorname{sgn}(u(s)-v(s))\partial\Phi_\Theta(u(s))\, dx\\
&+ r\int_\Omega |u(s)-v(s)|^{r-1}\operatorname{sgn}(u(s)-v(s))\partial\Phi_\Theta(v(s))\, dx.
\end{aligned}\tag{5.8}$$

Using (5.5), we get from (5.8) that

$$\begin{aligned}
\frac{d}{ds}\|U(s)\|_r^r =& -r(r-1)\int_\Omega |u(s)-v(s)|^{r-2}|\nabla u(s)|^{p-2}\nabla u(s)\nabla(u(s)-v(s))\, dx\\
& - r\int_{\partial\Omega} b(x)|u(s)|^{p-2}u(s)|u(s)-v(s)|^{r-1}\operatorname{sgn}(u(s)-v(s))\, d\sigma \; - r\iint_{\partial\Omega\times\partial\Omega}\frac{|\tilde{U}(x,y)|^{p-2}}{k_p(x,y)}\\
&\tilde{U}(x,y)(|U(s,x)|^{r-1}\operatorname{sgn}(U(s,x)) - |U(s,y)|^{r-1}\operatorname{sgn}(U(s,y)))\, d\sigma_x\, d\sigma_y\\
&+ r(r-1)\int_\Omega |u(s)-v(s)|^{r-2}|\nabla v(s)|^{p-2}\nabla v(s)\nabla(u(s)-v(s))\, dx\\
&+ r\int_{\partial\Omega} b(x)|v(s)|^{p-2}v(s)|u(s)-v(s)|^{r-1}\operatorname{sgn}(u(s)-v(s))\, d\sigma + r\iint_{\partial\Omega\times\partial\Omega}\frac{|\tilde{V}(s,x,y)|^{p-2}}{k_p(x,y)}\\
&\tilde{V}(s,x,y)(|U(s,x)|^{r-1}\operatorname{sgn}(U(s,x)) - |U(s,y)|^{r-1}\operatorname{sgn}(U(s,y)))\, d\sigma_x\, d\sigma_y\\
=& -r(r-1)\int_\Omega |u(s)-v(s)|^{r-2}\left[|\nabla u(s)|^{p-2}\nabla u(s) - |\nabla v(s)|^{p-2}\nabla v(s)\right]\nabla(u(s)\\
&- v(s))\, dx - r\int_{\partial\Omega} b(x)|u(s)-v(s)|^{r-2}\left[|u(s)|^{p-2}u(s) - |v(s)|^{p-2}v(s)\right](u(s)\\
&- v(s))\, d\sigma - r\iint_{\partial\Omega\times\partial\Omega}\frac{|\tilde{U}(x,y)|^{p-2}\tilde{U}(x,y) - |\tilde{V}(x,y|^{p-2}\tilde{V}(x,y)}{k_p(x,y)}(|U(x)|^{r-2}U(x)\\
&- |U(y)|^{r-2}U(y))\, d\sigma_x\, d\sigma_y.
\end{aligned}\tag{5.9}$$

Using (3.6) and (3.7), we get from (5.9), that there is a constant $C_1 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}s}\|U(s)\|_r^r \le -C_1 r(r-1)\int_\Omega |u(s)-v(s)|^{r-2}|\nabla(u(s)-v(s))|^p\,\mathrm{d}x$$
$$-C_1 r\int_{\partial\Omega} b(x)|u(s)-v(s)|^{r-2+p}\,\mathrm{d}\sigma - C_1 r\iint_{\partial\Omega\times\partial\Omega}\frac{|(u(x)-v(x))-(u(y)-v(y))|^{p-1}}{k_p(x,y)}$$
$$|(u(x)-v(x))-(u(y)-v(y))|^{r-1}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y$$
$$= -C_1 r(r-1)\int_\Omega |U(s)|^{r-2}|\nabla U(s)|^p\,\mathrm{d}x - C_1 r\int_{\partial\Omega} b(x)|U(s)|^{r-2+p}\,\mathrm{d}\sigma$$
$$-C_1 r\iint_{\partial\Omega\times\partial\Omega}\frac{|U(s,x)-U(s,y)|^{r-2+p}}{k_p(x,y)}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y. \tag{5.10}$$

Note that the integrals in the right hand side of (5.10) exist, since $u(s), v(s) \in \mathscr{V}_\Theta^{1,p}(\Omega,\partial\Omega)$ and $u(s), v(s)$ (as functions of x) are bounded.

Next, let $r : [0,\infty) \to [2,\infty)$ be an increasing differentiable function. Using the above argument, one has that the function $s \mapsto \|U(s)\|_{r(s)}^{r(s)}$ is differentiable a.e. Since

$$\frac{\mathrm{d}}{\mathrm{d}s}\|U(s)\|_{r(s)}^{r(s)} = r'(s)\frac{\partial}{\partial r}\|U(s)\|_r^r\Big|_{r=r(s)} + \frac{\partial}{\partial s}\|U(s)\|_r^r\Big|_{r=r(s)},$$

it follows from (5.10) that

$$\frac{\mathrm{d}}{\mathrm{d}s}\|U(s)\|_{r(s)}^{r(s)} \le r'(s)\int_\Omega |U(s)|^{r(s)}\log|U(s)|\,\mathrm{d}x - C_1 r(s)\int_{\partial\Omega} b(x)|U(s)|^{r(s)-2+p}\,\mathrm{d}\sigma$$
$$-C_1 r(s)\iint_{\partial\Omega\times\partial\Omega}\frac{|U(s,x)-U(s,y)|^{r(s)-2+p}}{k_p(x,y)}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y$$
$$-C_1 r(s)(r(s)-1)\int_\Omega |U(s)|^{r(s)-2}|\nabla U(s)|^p\,\mathrm{d}x. \tag{5.11}$$

Using (5.11), we obtain the following estimates:

$$\frac{\mathrm{d}}{\mathrm{d}s}\log\|U(s)\|_{r(s)} = -\frac{r'(s)}{r(s)}\log\|U(s)\|_{r(s)} + \frac{1}{r(s)}\frac{1}{\|U(s)\|_{r(s)}^{r(s)}}\frac{\mathrm{d}}{\mathrm{d}s}\|U(s)\|_{r(s)}^{r(s)}$$
$$\le -\frac{r'(s)}{r(s)}\log\|U(s)\|_{r(s)} - \frac{C_1(r(s)-1)}{\|U(s)\|_{r(s)}^{r(s)}}\int_\Omega |U(s)|^{r(s)-2}|\nabla U(s)|^p\,\mathrm{d}x$$
$$-\frac{C_1}{\|U(s)\|_{r(s)}^{r(s)}}\int_{\partial\Omega} b(x)|U(s)|^{r(s)-2+p}\,\mathrm{d}\sigma$$
$$-\frac{C_1}{\|U(s)\|_{r(s)}^{r(s)}}\iint_{\partial\Omega\times\partial\Omega}\frac{|U(s,x)-U(s,y)|^{r(s)-2+p}}{k_p(x,y)}\,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y$$
$$+\frac{r'(s)}{r(s)\|U(s)\|_{r(s)}^{r(s)}}\int_\Omega |U(s)|^{r(s)}\log|U(s)|\,\mathrm{d}x$$

$$
\begin{aligned}
&\le \frac{r'(s)}{r(s)} \int_\Omega \frac{|U(s)|^{r(s)}}{\|U(s)\|_{r(s)}^{r(s)}} \log\left(\frac{|U(s)|}{\|U(s)\|_{r(s)}}\right) \mathrm{d}x \\
&\quad - \frac{C_1(r(s)-1)}{\|U(s)\|_{r(s)}^{r(s)}} \int_\Omega |U(s)|^{r(s)-2} |\nabla U(s)|^p \, \mathrm{d}x \\
&\quad - \frac{C_1}{\|U(s)\|_{r(s)}^{r(s)}} \int_{\partial\Omega} b(x)|U(s)|^{r(s)-2+p} \, \mathrm{d}\sigma \\
&\quad - \frac{C_1}{\|U(s)\|_{r(s)}^{r(s)}} \iint_{\partial\Omega\times\partial\Omega} \frac{|U(s,x)-U(s,y)|^{r(s)-2+p}}{k_p(x,y)} \, \mathrm{d}\sigma_x \, \mathrm{d}\sigma_y \\
&\le \frac{r'(s)}{r(s)} \int_\Omega \frac{|U(s)|^{r(s)}}{\|U(s)\|_{r(s)}^{r(s)}} \log\left(\frac{|U(s)|}{\|U(s)\|_{r(s)}}\right) \mathrm{d}x \\
&\quad - \frac{C_1}{\|U(s)\|_{r(s)}^{r(s)}} \int_{\partial\Omega} b(x)|U(s)|^{r(s)-2+p} \, \mathrm{d}\sigma \\
&\quad - \frac{C_1}{\|U(s)\|_{r(s)}^{r(s)}} \iint_{\partial\Omega\times\partial\Omega} \frac{|U(x)-U(y)|^{r(s)-2+p}}{k_p(x,y)} \, \mathrm{d}\sigma_x \, \mathrm{d}\sigma_y \\
&\quad - \frac{C_1 p^p (r(s)-1)}{(r(s)-2+p)^p} \frac{1}{\|U(s)\|_{r(s)}^{r(s)}} \int_\Omega |\nabla |U(s)|^{\frac{r(s)-2+p}{p}}|^p \, \mathrm{d}x.
\end{aligned}
\tag{5.12}
$$

Let $F := \frac{|U(s)|^{\frac{r(s)-2+p}{p}}}{\|U(s)\|_{r(s)-2+p}^{\frac{r(s)-2+p}{p}}}$. It is clear that $F \in \mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega)$, $F \ge 0$ and $\|F\|_{p,\partial\Omega} = 1$. It follows from (3.2) and the logarithmic Sobolev inequality (2.6) (applied to $\mathscr{V}_\Theta^{1,p}(\Omega, \partial\Omega)$) that there is a constant $C_2 > 0$ such that for every $\varepsilon > 0$,

$$
\begin{aligned}
&- \int_\Omega |\nabla F|^p \, \mathrm{d}x - \int_{\partial\Omega} b(x)|F|^p \, \mathrm{d}\sigma - \iint_{\partial\Omega\times\partial\Omega} \frac{|F(x)-F(y)|^p}{k_p(x,y)} \, \mathrm{d}\sigma_x \, \mathrm{d}\sigma_y \\
&\le - \int_\Omega |\nabla F|^p \, \mathrm{d}x - \alpha - \iint_{\partial\Omega\times\partial\Omega} \frac{|F(x)-F(y)|^p}{k_p(x,y)} \, \mathrm{d}\sigma_x \, \mathrm{d}\sigma_y \\
&\le - \frac{1}{\|U(s)\|_{r(s)-2+p}^{r(s)-2+p}} \int_\Omega |\nabla |U(s)|^{\frac{r(s)-2+p}{p}}|^p \, \mathrm{d}x - \alpha - \iint_{\partial\Omega\times\partial\Omega} \frac{|F(x)-F(y)|^p}{k_p(x,y)} \, \mathrm{d}\sigma_x \, \mathrm{d}\sigma_y \\
&\le - \frac{p}{NC_2\varepsilon} \int_\Omega \frac{|U(s)|^{r(s)-2+p}}{\|U(s)\|_{r(s)-2+p}^{r(s)-2+p}} \log\left(\frac{|U(s)|^{\frac{r(s)-2+p}{p}}}{\|U(s)\|_{r(s)-2+p}^{\frac{r(s)-2+p}{p}}}\right) \mathrm{d}x - \frac{\log(\varepsilon)}{C_2\varepsilon}.
\end{aligned}
\tag{5.13}
$$

It follows from (5.13) and (5.12) that there is a constant $C_3 > 0$ such that for every $\varepsilon > 0$,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} \log \|U(s)\|_{r(s)} \leq \frac{r'(s)}{r(s)} \int_{\Omega} \frac{|U(s)|^{r(s)}}{\|U(s)\|_{r(s)}^{r(s)}} \log \left(\frac{|U(s)|}{\|U(s)\|_{r(s)}} \right) \mathrm{d}x \\
- \left(\frac{p}{r(s) - 2 + p} \right)^{p-1} \frac{p(r(s)-1)}{NC_3 \varepsilon} \frac{\|U(s)\|_{r(s)-2+p}^{r(s)-2+p}}{\|U(s)\|_{r(s)}^{r(s)}} \\
\times \int_{\Omega} \frac{|U|^{r(s)-2+p}}{\|U\|_{r(s)-2+p}^{r(s)-2+p}} \log \left(\frac{|U(s)|}{\|U(s)\|_{r(s)-2+p}} \right) \mathrm{d}x \\
- \left(\frac{p}{r(s) - 2 + p} \right)^{p} \frac{r(s)-1}{C_3} \frac{\|U(s)\|_{r(s)-2+p}^{r(s)-2+p}}{\|U(s)\|_{r(s)}^{r(s)}} \frac{\log(\varepsilon)}{\varepsilon}.
\end{aligned}
\tag{5.14}
$$

Let

$$
K(r(s), U(s)) := \int_{\Omega} \frac{|U(s)|^{r(s)}}{\|U(s)\|_{r(s)}^{r(s)}} \log \left(\frac{|U(s)|}{\|U(s)\|_{r(s)}} \right) \mathrm{d}x.
$$

By (5.14) we have that for every $\varepsilon > 0$,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} \log \|U(s)\|_{r(s)} \leq \frac{r'(s)}{r(s)} K(r(s), U(s)) - \left(\frac{p}{r(s) - 2 + p} \right)^{p-1} \\
\frac{p(r(s)-1)}{NC_3 \varepsilon} \frac{\|U(s)\|_{r(s)-2+p}^{r(s)-2+p}}{\|U(s)\|_{r(s)}^{r(s)}} K(r(s) - 2 + p, U(s)) \\
- \left(\frac{p}{r(s) - 2 + p} \right)^{p} \frac{r(s)-1}{C_3} \frac{\|U(s)\|_{r(s)-2+p}^{r(s)-2+p}}{\|U(s)\|_{r(s)}^{r(s)}} \frac{\log(\varepsilon)}{\varepsilon}.
\end{aligned}
\tag{5.15}
$$

Let

$$
\varepsilon := \frac{pr(s)(r(s)-1)}{NC_3 r'(s)} \left(\frac{p}{r(s) - 2 + p} \right)^{p-1} \frac{\|U(s)\|_{r(s)-2+p}^{r(s)-2+p}}{\|U(s)\|_{r(s)}^{r(s)}}.
$$

Then it follows from (5.15) that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} \log \|U(s)\|_{r(s)} \leq & \frac{r'(s)}{r(s)} \left[K(r(s), U(s)) - K(r(s) - 2 + p, U(s)) \right] \\
& - \frac{Nr'(s)}{(r(s) - 2 + p) r(s)} \\
& \log \left[\left(\frac{p}{r(s) - 2 + p} \right)^{p-1} \frac{pr(s)(r(s)-1)}{NC_3 r'(s)} \frac{\|U(s)\|_{r(s)-2+p}^{r(s)-2+p}}{\|U(s)\|_{r(s)}^{r(s)}} \right].
\end{aligned}
\tag{5.16}
$$

Since for every V, the mapping $q \mapsto \log \|V\|_q^q$ is convex and for a.e. q, $\frac{\mathrm{d}}{\mathrm{d}q} \log \|V\|_q^q = K(q, V) + \log \|V\|_q$, then the mapping $q \mapsto \frac{\mathrm{d}}{\mathrm{d}q} \log \|V\|_q^q$ is nondecreasing. Therefore, for every $q_2 \geq q_1 \geq 1$,

$$
K(q_1, V) - K(q_2, V) \leq \log \frac{\|V\|_{q_2}}{\|V\|_{q_1}}.
\tag{5.17}
$$

Applying (5.17) with $q_1 = r(s)$ and $q_2 = r(s) - 2 + p$, we get from (5.16) the following estimates:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log\|U(s)\|_{r(s)} \leq &\frac{r'(s)}{r(s)}\log\left(\frac{\|U(s)\|_{r(s)-2+p}}{\|U(s)\|_{r(s)}}\right) - \frac{Nr'(s)}{(r(s)-2+p)r(s)}\log\\ &\left[\frac{\|U(s)\|_{r(s)-2+p}^{r(s)-2+p}}{\|U(s)\|_{r(s)}^{r(s)}}\right] - \frac{Nr'(s)}{(r(s)-2+p)r(s)}\log\\ &\left[\left(\frac{p}{r(s)-2+p}\right)^{p-1}\frac{pr(s)(r(s)-1)}{NC_3r'(s)}\right].\end{aligned} \tag{5.18}$$

It follows from (5.18) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log\|U(s)\|_{r(s)} \leq &\frac{r'(s)}{r(s)}\left[\log\|U(s)\|_{r(s)-2+p} - \log\|U(s)\|_{r(s)}\right]\\ &- \frac{Nr'(s)}{r(s)}\log\|U(s)\|_{r(s)-2+p} + \frac{Nr'(s)}{r(s)-2+p}\log\|U(s)\|_{r(s)}\\ &- \frac{Nr'(s)}{(r(s)-2+p)r(s)}\log\left[\left(\frac{p}{r(s)-2+p}\right)^{p-1}\frac{pr(s)(r(s)-1)}{NC_3r'(s)}\right].\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\log\|U(s)\|_{r(s)} \leq\\ &\frac{r'(s)}{r(s)}\left[(1-N)\log\|U(s)\|_{r(s)-2+p} + \left(\frac{Nr(s)}{r(s)-2+p} - 1\right)\log\|U(s)\|_{r(s)}\right]\\ &- \frac{Nr'(s)}{(r(s)-2+p)r(s)}\log\left[\left(\frac{p}{r(s)-2+p}\right)^{p-1}\frac{pr(s)(r(s)-1)}{NC_3r'(s)}\right].\end{aligned} \tag{5.19}$$

Since $|\Omega| < \infty$, it follows from the classical Hölder inequality that,

$$\|U(s)\|_{r(s)} \leq |\Omega|^{\frac{p-2}{r(s)(r(s)-2+p)}}\|U(s)\|_{r(s)-2+p},$$

and this implies that

$$\log\|U(s)\|_{r(s)} \leq \frac{p-2}{r(s)(r(s)-2+p)}\log|\Omega| + \log\|U(s)\|_{r(s)-2+p}.$$

Since $(1-N) < 0$, it follows from the preceding estimate that

$$(1-N)\log\|U(s)\|_{r(s)-2+p} \leq (1-N)\log\|U(s)\|_{r(s)} + \frac{(N-1)(p-2)}{r(s)(r(s)-2+p)}\log|\Omega|.$$

Using this, we get from (5.19) the following estimates:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log\|U(s)\|_{r(s)} \leq &-\frac{N(p-2)r'(s)}{(r(s)-2+p)r(s)}\log\|U\|_{r(s)} - \frac{(1-N)(p-2)r'(s)}{r(s)(r(s)-2+p)}\log|\Omega|\\ &- \frac{Nr'(s)}{(r(s)-2+p)r(s)}\log\left[\left(\frac{p}{r(s)-2+p}\right)^{p-1}\frac{pr(s)(r(s)-1)}{NC_3r'(s)}\right].\end{aligned} \tag{5.20}$$

We set

$$A(s) := \frac{N(p-2)r'(s)}{(r(s)-2+p)r(s)}, \quad Y(s) := \log \|U(s)\|_{r(s)},$$

and

$$B(s) := \frac{Nr'(s)}{(r(s)-2+p)r(s)} \log\left[\left(\frac{p}{r(s)-2+p}\right)^{p-1} \frac{pr(s)(r(s)-1)}{NC_3 r'(s)}\right] + \frac{(1-N)(p-2)r'(s)}{r(s)(r(s)-2+p)} \log|\Omega|.$$

It follows from (5.20) that $Y(s)$ satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}s}Y(s) + A(s)Y(s) + B(s) \le 0, \quad \text{for all } s > 0.$$

Hence,

$$Y(s) \le X(s) := \exp\left[-\int_0^s A(\tau)\,\mathrm{d}\tau\right]\left[Y(0) - \int_0^s B(\tau)\exp\left(\int_0^\tau A(z)\,\mathrm{d}z\right)\mathrm{d}\tau\right]. \tag{5.21}$$

Let $r(s) := \frac{qt}{t-s}$ with $q \ge 2$ and $0 \le s < t$. A simple calculation gives

$$A(s) = \frac{N(p-2)}{(q-2+p)t + (2-p)s}.$$

Hence,

$$\int_0^s A(\tau)\,\mathrm{d}\tau = -N \log \frac{(q-2+p)t + (2-p)s}{(q-2+p)t},$$

and

$$\lim_{s\to t^-} \exp\left[-\int_0^s A(\tau)\,\mathrm{d}\tau\right] = \left(\frac{q}{q-2+p}\right)^N.$$

A similar simple calculation gives

$$\begin{aligned}\lim_{s\to t^-}&\left[Y(0) - \int_0^s B(\tau)\exp\left(\int_0^\tau A(z)\,\mathrm{d}z\right)\mathrm{d}\tau\right]\\ &= Y(0) - \frac{1}{p-2}\left[\left(\frac{q+p-2}{q}\right)^N - 1\right]\log(t)\\ &\quad + \frac{N-1}{N}\left[\left(\frac{q+p-2}{q}\right)^N - 1\right]\log|\Omega| + C_1,\end{aligned}$$

where C_1 is a constant depending on p, q, N and can be computed explicitly. Therefore,

$$\lim_{s\to t^-} X(s) = \left(\frac{q}{q-2+p}\right)^N Y(0) - \frac{1}{p-2}\left[1-\left(\frac{q}{q-2+p}\right)^N\right]\log(t) + \frac{N-1}{N}\left[\left(1-\frac{q}{q+p-2}\right)^N\right]\log|\Omega| + C_1\left(\frac{q}{q-2+p}\right)^N.$$

Using the submarkovian property (see Corollary 5.3) and (5.21), we get that for all $0 < s < t$,

$$\|U(t)\|_{r(s)} = \|u(t)-v(t)\|_{r(s)} \le \|u(s)-v(s)\|_{r(s)} = e^{Y(s)} \le e^{X(s)}. \tag{5.22}$$

Since $\lim_{s\to t} r(s) = \infty$ and

$$Y(0) := \log\|u(0)-v(0)\|_{r(0)} = \log\|u_0-v_0\|_q,$$

taking the limit as $s \to t^-$ of the inequality in (5.22), we get that

$$\|u(t)-v(t)\|_\infty = \lim_{s\to t^-}\|u(t)-v(t)\|_{r(s)} \le \lim_{s\to t^-} e^{X(s)}.$$

Calculating and using the fact that $r(0) = q$, we get that there is a constant $C > 0$ such that for every $t > 0$,

$$\|u(t)-v(t)\|_\infty \le C|\Omega|^\beta t^{-\delta}\|u_0-v_0\|_q^\gamma, \tag{5.23}$$

where

$$\beta := \frac{N-1}{N}\left[\left(1-\frac{q+p-2}{q}\right)^N\right], \quad \delta := \frac{1}{p-2}\left[1-\left(\frac{q}{q-2+p}\right)^N\right]$$
$$\text{and } \gamma := \left(\frac{q}{q-2+p}\right)^N.$$

Finally, to remove the requirement that $u_0, v_0 \in L^\infty(\Omega)$, let $u_0, v_0 \in L^q(\Omega)$ and $u_{n,0}, v_{n,0} \in L^\infty(\Omega)$ be sequences which converge, respectively, to u_0 and v_0 in $L^q(\Omega)$. Let $u_n(t) := S_\Theta(t)u_{n,0}$, $u(t) := S_\Theta(t)u_0$, $v_n(t) := S_\Theta(t)u_{n,0}$ and $v(t) := S_\Theta(t)v_0$. Using (5.23) with first $v_{n,0} = 0$ and then $u_{n,0} = 0$, we obtain that for every $t > 0$, $u_n(t)$ and $v_n(t)$ converge, respectively, to $u(t)$ and $v(t)$ in $L^\infty(\Omega)$. Therefore, for every $t > 0$, the sequence $(u_n(t) - v_n(t))$ converges in $L^\infty(\Omega)$. By uniqueness of the limit, $\lim_{n\to\infty}(u_n(t) - v_n(t)) = u(t) - v(t)$. Hence, for every $u_0, v_0 \in L^q(\Omega)$ and $t > 0$, we have

$$\|u(t)-v(t)\|_\infty = \|S_\Theta(t)u_0 - S_\Theta(t)v_0\|_\infty \le C|\Omega|^\beta t^{-\delta}\|u_0-v_0\|_q^\gamma.$$

We have shown (5.7) and the proof is finished. □

Note that a simple calculation shows that

$$\lim_{p\to 2}\beta = 0, \quad \lim_{p\to 2}\delta = \frac{N}{q} \quad \text{and} \quad \lim_{p\to 2}\gamma = 1$$

so that if $p = 2$ (that is the linear case), the estimate (5.7) reads

$$\|S_\Theta(t)u_0\|_\infty \le Ct^{-N/q}\|u_0\|_q.$$

This last estimate has been obtained in [1, Theorem 5.1] for the linear ($p = 2$) local Robin boundary conditions. The case of the Laplace operator with nonlinear local Robin type boundary conditions is contained in [6], where the authors have assumed that Ω has the $W^{1,2}$-extension property.

The proof of Theorem 5.4 giving here follows the lines of the proof of the abstract ultracontractivity result in the linear case contained in [15, Section 2.2], but the main ideas are similar to the ones contained in the works by Cipriani and Grillo [9,10] who have considered the p-Laplace operator (or more general quasi-linear operators) with Dirichlet boundary conditions.

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