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Further study of entire radial solutions of a biharmonic equation with exponential nonlinearity

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Abstract We study the structure of entire radial solutions of a biharmonic equation with exponential nonlinearity:

$$\Delta^2 u = \lambda e^u \quad \text{in } \mathbb{R}^N,\, N \geq 5 \tag{0.1}$$

with $\lambda = 8(N-2)(N-4)$. It is known from a recent interesting paper by Arioli et al. that (0.1) admits a singular solution $U_s(r) = \ln r^{-4}$. We show that for $5 \leq N \leq 12$, any regular entire radial solution u with $u(r) - \ln r^{-4} \to 0$ as $r \to \infty$ of (0.1) intersects with $U_s(r)$ infinitely many times. On the other hand, if $N \geq 13$, then $u(r) < U_s(r)$ for all $r > 0$, and the solutions are strictly ordered with respect to the initial value $a = u(0)$. Moreover, the asymptotic expansions of the entire radial solutions near ∞ are also obtained. Our main results give a positive answer to a conjecture in Arioli et al. (J Differ Equ 230:743–770, 2006) [see lines -11 to -9, p. 747 of Arioli et al. (J Differ Equ 230:743–770, 2006)].

Keywords Entire radial solutions · Biharmonic equations · Exponential nonlinearity

Mathematics Subject Classification Primary 35B45; Secondary 35J40

1 Introduction

We are interested in structure and asymptotic behaviors of entire radial solutions of the semilinear biharmonic equation

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$$\Delta^2 u = \lambda e^u \quad \text{in } \mathbb{R}^N, \quad N \geq 5, \qquad \lambda = 8(N-2)(N-4), \tag{1.1}$$

that is, in solutions $u = u(r)$, which exist for all $r = |x| > 0$.

It is known from [1] that (1.1) admits a singular radial solution:

$$U_s(r) = \ln r^{-4}.$$

For all $a, b \in \mathbb{R}$ we denote by $u_{a,b}$ the unique solution of the initial value problem

$$\begin{cases} \Delta^2 u(r) = \lambda e^{u(r)}, \quad \text{for } r \in [0, \infty) \\ u(0) = a, \quad \Delta u(0) = b, \quad u'(0) = (\Delta u)'(0) = 0. \end{cases} \tag{1.2}$$

It is known from (ii) of Theorem 2 of [1] that for any $a \in \mathbb{R}$, there is a unique $b_0 = b_0(a) < 0$ such that the unique solution $u_{a,b_0} \in C^4(0, \infty)$ of (1.2) satisfies

$$\lim_{r \to \infty} [u_{a,b_0}(r) - \ln r^{-4}] = 0. \tag{1.3}$$

This implies that for any $a > -\infty$, there is a unique radial solution $u_a(r) \in C^4([0, \infty))$ of (1.1) such that

$$\lim_{r \to \infty} [u_a(r) - \ln r^{-4}] = 0. \tag{1.4}$$

In this paper, we will use the main idea as in [9] to characterize the structure of $\{u_a\}_{a > -\infty}$, which gives a positive answer to a conjecture in [1]. Meanwhile, the asymptotic expansions of $\{u_a\}_{a > -\infty}$ near $r = \infty$ are also obtained. The main result of this paper is the following theorem.

Theorem 1.1 *Let $N \geq 5$. Then, for any $a > -\infty$, the Eq.* (1.1) *admits a unique solution $u = u(r)$ such that $u(0) = a$, $u'(r) < 0$, $\Delta u(r) < 0$ for $r \in (0, \infty)$ and $u(r) - \ln r^{-4} \to 0$ as $r \to \infty$. Moreover, if $5 \leq N \leq 12$, then $u(r) - \ln r^{-4}$ changes sign infinitely many times. If $N \geq 13$, then $u(r) < \ln r^{-4}$ and $\Delta u(r) > \Delta(\ln r^{-4})$ for all $r > 0$, and the solutions are strictly ordered with respect to the initial value $a = u(0)$. Namely, if $u_1(r)$ and $u_2(r)$ are two radial solutions of* (1.1) *with $u_1(0) < u_2(0)$, then $u_1(r) < u_2(r)$ and $\Delta u_1(r) > \Delta u_2(r)$ for $r > 0$.*

The existence and asymptotic behavior of solutions to the fourth-order Eq. (1.1) have been studied in the so-called conformal dimension $N = 4$ (see [5,11,13]) and in “supercritical dimension” $N \geq 5$ (see [1]). More recently, the stability properties of the solutions of (1.1) were determined in [12]. The authors in [3] classified solutions of (1.1) according to their stability outside compact sets of $\mathbb{R}^N$, complementing again the results in [12] in the conformal dimension $N = 4$ and showed different behaviors in “low dimensions” $5 \leq N \leq 12$ and in “high dimensions” $N \geq 13$. They obtained that for the first case, there exist both unstable solutions and solutions that are stable outside compact sets, and for the second case, any radially symmetric solution to (1.1) is fully stable. Meanwhile, the radial solutions of the Dirichlet and Navier boundary value problems of the equation $\Delta^2 u = \lambda e^u$ in the ball are widely studied, see [2,4,6,7].

2 Some preliminaries

We first show the following lemma.

Lemma 2.1 *For any $a \in \mathbb{R}$, (1.1) admits a unique radial solution $u_a(r)$ such that $u_a(0) = a$, $u_a'(0) = 0$, $u_a(r) - \ln r^{-4} \to 0$ as $r \to \infty$. Moreover,*

$$\Delta u_a(r) < 0, \quad u_a'(r) < 0 \quad \text{for } r \in (0, \infty).$$

Proof Existence and uniqueness of $u_a(r)$ of (1.1) satisfying $u_a(0) = a$ and $u_a(r) - \ln r^{-4} \to 0$ as $r \to \infty$ are known from (ii) of Theorem 2 of [1]. Moreover, it is also known from Theorem 2 of [1] that $(\Delta u_a)(0) < 0$.

To show that $u_a'(r) < 0$ and $\Delta u_a(r) < 0$ for $r \in (0, \infty)$, we only need to show that the second claim holds. Indeed, if $\Delta u_a(r) < 0$ for all $r > 0$, we see that $(r^{N-1}u_a'(r))' < 0$ for all $r > 0$ and thus $u_a'(r) < 0$ for all $r > 0$. Suppose that the second claim does not hold, we see that there is $r_0 \in (0, \infty)$ such that $\Delta u_a(r) < 0$ for $r \in (0, r_0)$, $\Delta u_a(r_0) = 0$ and $\Delta u_a(r) > 0$ for $r \in (r_0, \infty)$ since we know from the equation of u_a that $(\Delta u_a)'(r) > 0$ for all $r > 0$. This implies that there exist $\theta > 0$ and $R_0 > 3r_0$ such that $\Delta u_a(r) \geq \theta > 0$ for $r > R_0$. Therefore,

$$(r^{N-1}u_a'(r))' \geq \theta r^{N-1} \quad \forall r > R_0 \tag{2.1}$$

and thus

$$u_a'(r) \geq \left(\frac{R_0}{r}\right)^{N-1} u_a'(R_0) + \frac{\theta}{N} r \left(1 - \left(\frac{R_0}{r}\right)^N\right) \quad \forall r > R_0. \tag{2.2}$$

This contradicts the fact that $u_a(r) \to -\infty$ as $r \to \infty$ since (2.2) implies that $u_a(r) \to +\infty$ as $r \to \infty$. Hence, the second claim holds, and the proof is complete. □

In the following, we shall assume that u_a is the unique entire radial solution of (1.1) with $u_a(0) = a$. If there is no confusion, we drop the index a.

In radial coordinates $r = |x|$, Eq. (1.1) reads

$$\begin{aligned} &u^{(4)}(r) + \frac{2(N-1)}{r}u'''(r) + \frac{(N-1)(N-3)}{r^2}u''(r) - \frac{(N-1)(N-3)}{r^3}u'(r) \\ &= \lambda e^{u(r)}, \quad r \in [0, \infty). \end{aligned} \tag{2.3}$$

Using the transformation

$$w(s) := u(e^s) + 4s, \quad s = \ln r \ (r > 0), \tag{2.4}$$

we see that the Eq. (2.3) can be rewritten as

$$w^{(4)}(s) + K_3 w'''(s) + K_2 w''(s) + K_1 w'(s) - \lambda w(s) = \lambda(e^{w(s)} - w(s) - 1) \tag{2.5}$$

where

$$K_3 = 2(N-4), \quad K_2 = N^2 - 10N + 20, \quad K_1 = -2(N-2)(N-4).$$

The singular solution $r \mapsto \ln r^{-4}$ of the differential equation in (2.3) corresponds to the trivial solution $w(s) \equiv 0$ of (2.5). The characteristic polynomial (linearized at $w \equiv 0$) is

$$\nu \mapsto \nu^4 + 2(N-4)\nu^3 + (N^2 - 10N + 20)\nu^2 - 2(N-2)(N-4)\nu - 8(N-2)(N-4)$$

and, the eigenvalues are given by

$$\nu_1 = \frac{N_1 + \sqrt{N_2 + 4\sqrt{N_3}}}{2}, \quad \nu_2 = \frac{N_1 - \sqrt{N_2 + 4\sqrt{N_3}}}{2},$$
$$\nu_3 = \frac{N_1 + \sqrt{N_2 - 4\sqrt{N_3}}}{2}, \quad \nu_4 = \frac{N_1 - \sqrt{N_2 - 4\sqrt{N_3}}}{2},$$

where

$$N_1 = -(N-4), \quad N_2 = N^2 - 4N + 8, \quad N_3 = (9N-34)(N-2).$$

The following proposition is known from [2].

Proposition 2.2 (i) *For any* $N \geq 5$, *we have* $\nu_1, \nu_2 \in \mathbb{R}$ *and* $\nu_2 < 2 - N < 0 < \nu_1$.
(ii) *For any* $5 \leq N \leq 12$, *we have* $\nu_3, \nu_4 \notin \mathbb{R}$ *and* $Re\nu_3 = Re\nu_4 = -\frac{N-4}{2} < 0$.
(iii) *For any* $N \geq 13$, *we have* $\nu_4 < \nu_3 < 0$ *and*

$$\nu_2 < 4 - N < \nu_4 < \frac{(4-N)}{2} < \nu_3 < 0 < \nu_1, \quad \nu_3 + \nu_4 = 4 - N.$$

3 The case of $5 \leq N \leq 12$

In this section, we prove that for $5 \leq N \leq 12$, $u(r) - U_s(r)$ must have infinitely many intersections (and hence prove the first part of Theorem 1.1).

It is known from Proposition 2.2 that $\nu_3, \nu_4 \notin \mathbb{R}$ provided that $5 \leq N \leq 12$. Let $\nu_3 = \tau + i\kappa$. Then, $\tau = -\frac{N-4}{2} < 0$.

Set $\phi(r) = u(r) - U_s(r)$. The following theorem gives the asymptotic behavior of $\phi(r)$ near ∞, which is of independent interest.

Theorem 3.1 *Assume that* u *is the unique radial solution of* (1.1) *with* $5 \leq N \leq 12$, *which satisfies* $u(r) - \ln r^{-4} \to 0$ *as* $r \to \infty$. *Then, there exist constants* C, C_1, κ_1 *with* $C \neq 0$ *such that for* r *large:*

$$\phi(r) = Cr^{\tau} \sin(\kappa \ln r + \kappa_1) + C_1 r^{\nu_2} + O(r^{2\tau}). \tag{3.1}$$

Proof Note that (3.1) implies that ϕ admits infinitely many zeroes in $(0, \infty)$. Using the transformation

$$w(s) := u(e^s) + 4s \ (= \phi(r)), \quad s = \ln r \ (r > 0), \tag{3.2}$$

we see that $w(s)$ satisfies $w(s) \to 0$ as $s \to \infty$ and

$$w^{(4)}(s) + K_3 w'''(s) + K_2 w''(s) + K_1 w'(s) - \lambda w(s) = \lambda(e^{w(s)} - w(s) - 1), \quad s > 1. \tag{3.3}$$

We now write (3.3) as

$$(\partial_s - \nu_3)(\partial_s - \nu_4)(\partial_s - \nu_2)(\partial_s - \nu_1) w(s) = g(w(s)) \tag{3.4}$$

where $g(w(s)) = \lambda(\mathrm{e}^{w(s)} - w(s) - 1) = O(w^2(s))$. We claim that for any $S \gg 1$, there exist some constants A_i and B_i $(i = 1, 2, 3, 4)$ such that

$$\begin{aligned} w(s) = {} & A_1\mathrm{e}^{\tau s}\cos\kappa s + A_2\mathrm{e}^{\tau s}\sin\kappa s + A_3\mathrm{e}^{\nu_2 s} + A_4\mathrm{e}^{\nu_1 s} \\ & + B_1\int_S^s \mathrm{e}^{\tau(s-t)}\sin\kappa(s-t)g(w(t))\mathrm{d}t + B_2\int_S^s \mathrm{e}^{\tau(s-t)}\cos\kappa(s-t)g(w(t))\mathrm{d}t \\ & + B_3\int_S^s \mathrm{e}^{\nu_2(s-t)}g(w(t))\mathrm{d}t + B_4\int_S^s \mathrm{e}^{\nu_1(s-t)}g(w(t))\mathrm{d}t. \end{aligned}$$

Moreover, each A_i depends on S and ν_i $(i = 1, 2, 3, 4)$, whereas each B_i depends only on ν_i $(i = 1, 2, 3, 4)$. In fact, it follows from (3.4) and the ODE theory of second order (see [10]) that

$$\begin{aligned} (\partial_s - \nu_2)(\partial_s - \nu_1)w(s) = {} & \tilde{A}_1\mathrm{e}^{\tau s}\cos\kappa s + \tilde{A}_2\mathrm{e}^{\tau s}\sin\kappa s \\ & + \frac{1}{\kappa}\int_S^s \mathrm{e}^{\tau(s-t)}\sin\kappa(s-t)g(w(t))\mathrm{d}t, \end{aligned} \tag{3.5}$$

where $\tilde{A}_1$ and $\tilde{A}_2$ are constants depending on S, ν_3 and ν_4. Multiplying both the sides of (3.5) by $\mathrm{e}^{-\nu_2 s}$ and integrating it on (S, s), we obtain that

$$\begin{aligned} (\partial_s - \nu_1)w(s) = {} & \tilde{A}_3\mathrm{e}^{\nu_2 s} + \int_S^s \mathrm{e}^{\nu_2(s-t)}\left[\tilde{A}_1\mathrm{e}^{\tau t}\cos\kappa t + \tilde{A}_2\mathrm{e}^{\tau t}\sin\kappa t\right]\mathrm{d}t \\ & + \frac{1}{\kappa}\int_S^s \mathrm{e}^{\nu_2(s-t)}\int_S^t \mathrm{e}^{\tau(t-\xi)}\sin\kappa(t-\xi)g(w(\xi))d\xi\,\mathrm{d}t. \end{aligned}$$

We now switch the order of integrations to see that

$$\begin{aligned} (\partial_s - \nu_1)w(s) = {} & \hat{A}_1\mathrm{e}^{\tau s}\cos\kappa s + \hat{A}_2\mathrm{e}^{\tau s}\sin\kappa s + \hat{A}_3\mathrm{e}^{\nu_2 s} \\ & + \tilde{B}_1\int_S^s \mathrm{e}^{\tau(s-t)}\sin\kappa(s-t)g(w(t))\mathrm{d}t \\ & + \tilde{B}_2\int_S^s \mathrm{e}^{\tau(s-t)}\cos\kappa(s-t)g(w(t))\mathrm{d}t \\ & + \tilde{B}_3\int_S^s \mathrm{e}^{\nu_2(s-t)}g(w(t))\mathrm{d}t, \end{aligned}$$

where $\hat{A}_1$, $\hat{A}_2$ and $\hat{A}_3$ depend on S, ν_i $(i = 2, 3, 4)$, $\tilde{B}_i$ $(i = 1, 2, 3)$ depend only on ν_i $(i = 2, 3, 4)$. Repeating the same argument once again, we obtain our claim. We can also write $w(s)$ as

$$
\begin{aligned}
w(s) = {} & A_1 e^{\tau s}\cos\kappa s + A_2 e^{\tau s}\sin\kappa s + A_3 e^{\nu_2 s} + M_4 e^{\nu_1 s} \\
& + B_1 \int_S^s e^{\tau(s-t)} \sin\kappa(s-t) g(w(t))\mathrm{d}t + B_2 \int_S^s e^{\tau(s-t)} \cos\kappa(s-t) g(w(t))\mathrm{d}t \\
& + B_3 \int_S^s e^{\nu_2(s-t)} g(w(t))\mathrm{d}t - B_4 \int_s^\infty e^{\nu_1(s-t)} g(w(t))\mathrm{d}t
\end{aligned}
$$

by using the fact that $\int_S^s = \int_S^\infty - \int_s^\infty$. Since $w(s) \to 0$ as $s \to \infty$, we have $M_4 = 0$ (note $\nu_1 > 0$). Setting

$$w_1(s) = A_1 e^{\tau s}\cos\kappa s + A_2 e^{\tau s}\sin\kappa s + A_3 e^{\nu_2 s}$$

and

$$
\begin{aligned}
w_2(s) = {} & B_1 \int_S^s e^{\tau(s-t)} \sin\kappa(s-t) g(w(t))\mathrm{d}t + B_2 \int_S^s e^{\tau(s-t)} \cos\kappa(s-t) g(w(t))\mathrm{d}t \\
& + B_3 \int_S^s e^{\nu_2(s-t)} g(w(t))\mathrm{d}t - B_4 \int_s^\infty e^{\nu_1(s-t)} g(w(t))\mathrm{d}t,
\end{aligned}
$$

we see from the fact $g(w(t)) = O(w^2(t))$ that

$$|w_2(s)| \le C\left[\tilde{w}_1(s) + \tilde{w}_2(s)\right], \tag{3.6}$$

where $C > 0$ is independent of S and

$$
\begin{aligned}
\tilde{w}_1(s) &= \max\left\{ \int_S^s e^{\tau(s-t)}|w_1(t)|^2\mathrm{d}t, \int_S^s e^{\nu_2(s-t)}|w_1(t)|^2\mathrm{d}t, \int_s^\infty e^{\nu_1(s-t)}|w_1(t)|^2\mathrm{d}t \right\}, \\
\tilde{w}_2(s) &= \max\left\{ \int_S^s e^{\tau(s-t)}|w_2(t)|^2\mathrm{d}t, \int_S^s e^{\nu_2(s-t)}|w_2(t)|^2\mathrm{d}t, \int_s^\infty e^{\nu_1(s-t)}|w_2(t)|^2\mathrm{d}t \right\}.
\end{aligned}
$$

We now show

$$|w_2(s)| = o(e^{\tau s}). \tag{3.7}$$

There are three cases to be considered:

(i) $|w_2(s)| \le C\left[\tilde{w}_1(s) + \int_S^s e^{\tau(s-t)}|w_2(t)|^2\mathrm{d}t\right]$,
(ii) $|w_2(s)| \le C\left[\tilde{w}_1(s) + \int_S^s e^{\nu_2(s-t)}|w_2(t)|^2\mathrm{d}t\right]$,
(iii) $|w_2(s)| \le C\left[\tilde{w}_1(s) + \int_s^\infty e^{\nu_1(s-t)}|w_2(t)|^2\mathrm{d}t\right]$.

We only consider (i) and (iii). Case (ii) can be discussed similarly.

For case (i), we have

$$|w_2(s)| \le C\left[\tilde{w}_1(s) + \int_S^s e^{\tau(s-t)}|w_2(t)|^2\mathrm{d}t\right]. \tag{3.8}$$

Thus,

$$|w_2(s)| \le C\left[\tilde{w}_1(s) + \max_{s\ge S}|w_2(s)|\int\limits_S^s e^{\tau(s-t)}|w_2(t)|dt\right]. \tag{3.9}$$

Let $m(s) = \int_S^s e^{-\tau t}|w_2(t)|$. Then, it is seen from (3.9) that

$$m'(s) \le C\tilde{w}_1(s)e^{-\tau s} + C\max_{s\ge S}|w_2(s)|\, m(s). \tag{3.10}$$

For any $\epsilon > 0$ sufficiently small, we can choose S sufficiently large such that $0 < d_S := C\max_{s\ge S}|w_2(s)| < \epsilon$. It follows from (3.10) that

$$m(s) \le Ce^{d_S s}\int\limits_S^s \tilde{w}_1(t)e^{-\tau t}e^{-d_S t}dt. \tag{3.11}$$

Substituting $m(s)$ in (3.11) into (3.9), we see that

$$|w_2(s)| \le C\tilde{w}_1(s) + Cd_S e^{(\tau+d_S)s}\int\limits_S^s \tilde{w}_1(t)e^{-\tau t}e^{-d_S t}dt. \tag{3.12}$$

Note that $\tau + d_S < 0$ for S sufficiently large. We also know that if $\delta = \frac{\sqrt{N_2+4\sqrt{N_3}}}{2}$, then $\delta > 0$. Thus, $\nu_2 = \tau - \delta < \tau$. This implies $\tilde{w}_1(s) = o(e^{\tau s})$. We also know from (3.12) that $|w_2(s)| = o(e^{(\tau+d_S)s})$. Substituting this into (3.8), we eventually obtain that

$$|w_2(s)| = o(e^{\tau s}). \tag{3.13}$$

For case (iii), we have

$$|w_2(s)| \le C\left[\tilde{w}_1(s) + \int\limits_s^\infty e^{\nu_1(s-t)}|w_2(t)|^2dt\right]. \tag{3.14}$$

Thus,

$$|w_2(s)| \le C\tilde{w}_1(s) + C\max_{s\ge S}|w_2(s)|\int\limits_s^\infty e^{\nu_1(s-t)}|w_2(t)|dt. \tag{3.15}$$

Let $k(s) = \int_s^\infty e^{-\nu_1 t}|w_2(t)|$. Then, it is seen from (3.15) that

$$-k'(s) \le C\tilde{w}_1(s)e^{-\nu_1 s} + d_S k(s). \tag{3.16}$$

It follows from (3.16) that

$$k(s) \le Ce^{-d_S s}\int\limits_s^\infty \tilde{w}_1(t)e^{-\nu_1 t}e^{d_S t}dt. \tag{3.17}$$

Since $\tilde{w}_1(s) = o(e^{\tau s})$, we obtain from (3.17) that

$$k(s) = o(e^{(\tau-\nu_1)s}). \tag{3.18}$$

Substituting this into (3.15), we also have

$$|w_2(s)| = o(e^{\tau s}). \tag{3.19}$$

We now write $w(s)$ as

$$\begin{aligned} w(s) = {} & M_1 e^{\tau s}\cos\kappa s + M_2 e^{\tau s}\sin\kappa s + A_3 e^{\nu_2 s} \\ & - B_1 \int_s^\infty e^{\tau(s-t)}\sin\kappa(s-t) g(w(t))dt - B_2 \int_s^\infty e^{\tau(s-t)}\cos\kappa(s-t) g(w(t))dt \\ & + B_3 \int_S^s e^{\nu_2(s-t)} g(w(t))dt - B_4 \int_s^\infty e^{\nu_1(s-t)} g(w(t))dt. \end{aligned}$$

Then, it follows from the facts $g(w(s)) = O(w^2(s))$, $w_1(s) = O(e^{\tau s})$, $w_2(s) = o(e^{\tau s})$ and $\nu_2 < 2\tau$ that

$$w(s) = M_1 e^{\tau s}\cos(\kappa s) + M_2 e^{\tau s}\sin(\kappa s) + A_3 e^{\nu_2 s} + O(e^{2\tau s}). \tag{3.20}$$

We now claim that $M_1^2 + M_2^2 \neq 0$.

Suppose that $M_1 = M_2 = 0$. Then, it follows from (3.20) that $w(s) = O(e^{2\tau s})$ since $\nu_2 < 2\tau$. Substituting this into the expression of $w(s)$ and using the fact $4\tau < \nu_2$, we obtain that

$$w(s) = \tilde{A}_3 e^{\nu_2 s} + o(e^{\nu_2 s}), \tag{3.21}$$

where $\tilde{A}_3$ is a constant depending on S. We now show that $\tilde{A}_3 \neq 0$. On the contrary, $w(s) = o(e^{\nu_2 s})$. Substituting this into

$$\begin{aligned} w(s) = {} & M_3 e^{\nu_2 s} \\ & - B_1 \int_s^\infty e^{\tau(s-t)}\sin\kappa(s-t) g(w(t))dt - B_2 \int_s^\infty e^{\tau(s-t)}\cos\kappa(s-t) g(w(t))dt \\ & - B_3 \int_s^\infty e^{\nu_2(s-t)} g(w(t))dt - B_4 \int_s^\infty e^{\nu_1(s-t)} g(w(t))dt, \end{aligned}$$

we see that $M_3 = 0$ and $w(s) = o(e^{2\nu_2 s})$. Repeating this process, we eventually derive that $w(s) \equiv 0$. This is a contradiction. Moreover, $\tilde{A}_3 \neq 0$ implies $M_3 \neq 0$. Using the expression of $w(s)$, we obtain by direct calculations that for s sufficiently large

$$w'(s) = O(e^{\nu_2 s}), \quad w''(s) = O(e^{\nu_2 s}), \quad w'''(s) = O(e^{\nu_2 s}). \tag{3.22}$$

These also imply that

$$\begin{aligned} & \phi(r) \sim r^{2-N-\eta}, \quad \phi'(r) \sim r^{1-N-\eta}, \quad \Delta\phi(r) \sim r^{-N-\eta}, \\ & (\Delta\phi)'(r) \sim r^{-N-\eta-1} \quad \text{as } r \to \infty, \end{aligned} \tag{3.23}$$

where $\eta = -\nu_2 - (N-2) > 0$ by Proposition 2.2. Furthermore, $\phi(r)$ has no zeroes for r large. We show that this is impossible. In fact, it is easy to see that ϕ must change sign in $(0, \infty)$. Otherwise, we assume that $\phi(r) < 0$ for $r \geq 0$ (note that $u(r) < U_s(r)$ for r small). Then, using the behavior of ϕ near ∞ and integrating the equation $\Delta^2\phi = \lambda(e^{u(r)} - e^{U_s(r)})$ over $\mathbb{R}^N$, we see that

$$\int_0^\infty r^{N-1}(e^{u(r)} - e^{U_s(r)})dr = 0,$$

which contradicts $\phi = u - U_s < 0$. (Note that $r^{N-1}(\Delta U_s)'(r) \sim r^{N-4}$ for r near 0 and hence $\lim_{r\to 0^+} r^{N-1}(\Delta\phi)'(r) = 0$ since $N \geq 5$.)

Suppose that $\phi(r)$ has exactly k zeroes in $(0, +\infty)$ (recalling that ϕ has no zeroes when r is large) and $\phi(r) \sim r^{2-N-\eta}$ as $r \to \infty$, we easily see that $r^{N-1}\phi'(r)$ has at least k zeroes. On the other hand, since the function $\xi(r) := r^{N-1}\phi'(r)$ satisfies $\xi(0) = 0$ and $\xi(r) \to 0$ as $r \to \infty$, we see that $\xi'(r)$ has at least $k+1$ zeroes. Thus, $\Delta\phi(r) = \frac{1}{r^{N-1}}\xi'(r)$ has at least $k+1$ zeroes. A similar idea implies that $r^{N-1}(\Delta\phi)'(r)$ has at least k zeroes and $(r^{N-1}(\Delta\phi)'(r))'$ has at least $k+1$ zeroes. Therefore, $\Delta^2\phi = \frac{1}{r^{N-1}}(r^{N-1}(\Delta\phi)'(r))'$ has at least $k+1$ zeroes. This contradicts our assumption that ϕ has k zeroes, since $\Delta^2\phi = \lambda e^{\zeta}\phi$, where $\zeta(r) \in (\min\{u(r), U_s(r)\}, \max\{u(r), U_s(r)\})$ and $e^{\zeta(r)} > 0$ for all $r > 0$. This proves our claim.

Since $M_1^2 + M_2^2 \neq 0$, it follows from (3.3) that

$$\begin{aligned} w(s) = {} & M_1 e^{\tau s}\cos(\kappa s) + M_2 e^{\tau s}\sin(\kappa s) + \tilde{M}_3 e^{\nu_2 s} \\ & - B_1 \int_s^{\infty} e^{\tau(s-t)} \sin\kappa(s-t) g(w(t))dt - B_2 \int_s^{\infty} e^{\tau(s-t)} \cos\kappa(s-t) g(w(t))dt \\ & + B_3 \int_S^{s} e^{\nu_2(s-t)} g(w(t))dt - B_4 \int_s^{\infty} e^{\nu_1(s-t)} g(w(t))dt. \end{aligned} \tag{3.24}$$

Writing (3.20) as

$$w(s) = Ce^{\tau s}\sin(\kappa s + \kappa_1) + o(e^{\tau s}) \tag{3.25}$$

where $\tan\kappa_1 = M_2/M_1$ and $C = \sqrt{M_1^2 + M_2^2}$, and putting (3.25) back into (3.24), we obtain

$$w(s) = Ce^{\tau s}\sin(\kappa s + \kappa_1) + \hat{M}_3 e^{\nu_2 s} + O(e^{2\tau s}) \tag{3.26}$$

and hence

$$\phi(r) = Cr^{\tau}\sin(\kappa\ln r + \kappa_1) + \hat{M}_3 r^{\nu_2} + O(r^{2\tau}) \tag{3.27}$$

This implies (3.1) and completes the proof of this theorem. □

Corollary 3.2 *Let u_1 and u_2 be two different regular radial entire solutions of* (1.1) *satisfying $u_1(r) - \ln r^{-4} \to 0$ as $r \to \infty$ and $u_2(r) - \ln r^{-4} \to 0$ as $r \to \infty$. Then, the graph of u_1 intersects that of u_2 infinitely many times in* $(0, \infty)$.

Proof Define $w_i(s) = u_i(e^s) + 4s$ $(i = 1, 2)$ and $s = \ln r$, $w(s) = w_1(s) - w_2(s)$. We see that w satisfies the equation

$$w^{(4)}(s) + K_3 w'''(s) + K_2 w''(s) + K_1 w'(s) - \lambda w(s) = \lambda(e^{w_1(s)} - e^{w_2(s)} - w(s)), \quad s > 1. \tag{3.28}$$

It is clear that $w_i(s) \to 0$ as $s \to \infty$. Therefore,

$$\tilde{g}(w(s)) := e^{w_1(s)} - e^{w_2(s)} - w(s) = O(e^{\tau s})w(s)$$

Similar arguments to those in the proof of Theorem 3.1 imply that for r sufficiently large,

$$\varphi(r) := u_1(r) - u_2(r) = Q_1 r^{\tau}\cos(\kappa\ln r) + Q_2 r^{\tau}\sin(\kappa\ln r) + Q_3 r^{\nu_2} + O(r^{2\tau}) \tag{3.29}$$

with $Q_1^2 + Q_2^2 \neq 0$. This completes the proof of this corollary. □

4 The case of $N \geq 13$

In this section, we consider the case of $N \geq 13$. We first study the following linearized equation

$$\Delta^2\phi = 8(N-2)(N-4)\mathrm{e}^{u(r)}\phi, \quad \phi(r) \to 0 \ \text{ as } r \to \infty. \tag{4.1}$$

Lemma 4.1 *The solution $\phi(r)$ to (4.1) is given by*

$$\phi(r) = c(4 + ru'(r)) \tag{4.2}$$

for some $c \neq 0$.

Proof The proof can be done by three steps:

(i) We show that if $\phi(0) = 0$, then $\phi \equiv 0$.
(ii) We show that if $\phi(0) = 1$, then $\Delta\phi(0) < 0$.
(iii) We obtain (4.2).

(i) Suppose $\phi(0) = 0$ and $(\Delta\phi)(0) \neq 0$, we may assume that $(\Delta\phi)(0) > 0$. Since $\phi(0) = \phi'(0) = 0$, we may assume that $\phi(r) > 0$ for $r \in (0, R)$ and $\phi(R) = 0$. (R can be $+\infty$.) Then, in $(0, R)$, $(\Delta\phi)' > 0$, and hence $\Delta\phi(r) > 0$ for $r \in (0, R)$. This implies that $\phi'(r) > 0$ and $\phi(r) > 0$ for $r \in (0, R]$ and contradicts $\phi(R) = 0$. This implies that $\Delta\phi(0) = 0$. Since ϕ is the unique solution of the initial value problem

$$\begin{cases} \Delta^2\phi = \lambda \mathrm{e}^{u(r)}\phi & \text{for } r \in [0, \infty) \\ \phi(0) = \phi'(0) = \Delta\phi(0) = (\Delta\phi)'(0) = 0, \end{cases}$$

we then have $\phi \equiv 0$.
(ii) follows from the same arguments.
(iii) Let $\rho(r) = 4 + ru'(r)$. Then, $\rho(0) = 4$. Direct calculations imply that $\rho(r)$ is a solution of the equation in (4.1) (we can also obtain this claim by using the equation satisfied by $w'(s)$ with $w(s)$ being given in (3.2)). Moreover, we claim that

$$\rho(r) \to 0 \quad \text{as } r \to \infty. \tag{4.3}$$

To see this, we set $w(s)$ as in (3.2) and see that $w(s) \to 0$ and $w'(s) \to 0$ as $s \to \infty$. The second limit implies that our claim (4.3) holds and this limit can be obtained from the expression of $w(s)$ as in or similar to (3.24). (Note that we will obtain a similar expression of $w(s)$ for $N \geq 13$ later.) Let $\phi(r)$ be any nontrivial solution of (4.1) with $\phi(0) = \sigma \neq 0$. We see that $\tilde{\rho}(r) := \phi(r) - \frac{\sigma}{4}\rho(r)$ with $\tilde{\rho}(0) = 0$ is a solution of (4.1). It follows from (i) that $\tilde{\rho} \equiv 0$. Therefore, $\phi \equiv \frac{\sigma}{4}\rho$. This completes the proof of this lemma. □

Theorem 4.2 *For $N \geq 13$, we have $u(r) < U_s(r)$, $\Delta u(r) > \Delta U_s(r)$ for $r > 0$.*

Theorem 4.3 *For $N \geq 13$, the solution to (4.1) remains of constant sign; that is,*

$$4 + ru'(r) > 0, \quad \Delta(4 + ru'(r)) < 0. \tag{4.4}$$

Proof of Theorem 4.2 This theorem is just a slight improvement of Lemma 12 in [3]. Let $\phi(r) = U_s(r) - u(r)$. Then, ϕ satisfies

$$\Delta^2\phi = \lambda(\mathrm{e}^{U_s(r)} - \mathrm{e}^{U_s(r)-\phi(r)}) \leq \lambda \mathrm{e}^{U_s(r)}\phi(r), \quad \forall r > 0. \tag{4.5}$$

(Note that $1 - e^{-x} - x \le 0$ for all $x \in \mathbb{R}$. In fact, if we define $h(x) = 1 - e^{-x} - x$, we see that $h(0) = 0$ and $h'(x) > 0$ for $x < 0$; $h'(x) < 0$ for $x > 0$.) Now let $\psi(r) = r^{\nu_3}$. Then, by Proposition 2.2, $\nu_3 > \frac{4-N}{2}$. We have that

$$\Delta^2\psi = \lambda e^{U_s(r)}\psi. \tag{4.6}$$

Thus, for any $0 < R < 1$, $\int_{B_R(0)} r^{-4}|\phi|\psi \le C\int_{B_R(0)} r^{-4}|\ln r|r^{\frac{4-N}{2}} < +\infty$ since $N \ge 5$. This implies that the integral $\lambda e^{U_s(r)}\phi\psi$ is integrable. Multiplying (4.5) by ψ and (4.6) by ϕ and integrating over $B_r(0)$, we have the following inequality:

$$\int\limits_{\partial B_r(0)} [(\Delta\phi)'\psi - \Delta\phi\psi'] + \int\limits_{\partial B_r(0)} [\Delta\psi\phi' - (\Delta\psi)'\phi] \le 0. \tag{4.7}$$

Note that $\phi(r) > 0$ and $\Delta\phi(r) < 0$ for r small. If $\phi(r) > 0$ and $\Delta\phi(r) < 0$ for $r \in (0, \infty)$, we are done. Let us assume that there exist $r_1, r_2 \in (0, \infty]$ such that

$$\phi(r) > 0,\ r \in (0, r_1),\ \ \phi(r_1) = 0, \quad \Delta\phi(r) < 0, \quad r \in (0, r_2), \quad \Delta\phi(r_2) = 0. \tag{4.8}$$

Then, we have four cases here: (i) $r_1 = \infty$ and $r_2 = \infty$, (ii) $r_1, r_2 \in (0, \infty)$, (iii) $r_1 = \infty$ and $r_2 \in (0, \infty)$, (iv) $r_2 = \infty$ and $r_1 \in (0, \infty)$. If (i) occurs, then we are done. We only consider case (ii) in the following, the other two cases can be discussed similarly. (Note that case (iii) can be discussed as the case $r_1 > r_2$ in the proof of case (ii) and case (iv) can be discussed as the case $r_2 > r_1$ in the proof of case (ii).) We will derive contradictions from (4.7) and (4.8).

Let $I_1(r) = \int_{\partial B_r(0)}[(\Delta\phi)'\psi - \Delta\phi\psi']$ and $I_2(r) = \int_{\partial B_r(0)}[\Delta\psi\phi' - (\Delta\psi)'\phi]$. We first see $r_1 \ne r_2$. Otherwise, we take $r = r_1 = r_2$ and obtain that $I_1(r) > 0$, $I_2(r) > 0$. But these contradict with (4.7). (Note that $\Delta\psi < 0$. The fact $\phi'(r) < 0$ for $r \in (0, r_1]$ can be obtained from $\Delta\phi(r) < 0$ for $r \in (0, r_1)$. The fact $(\Delta\phi)'(r) > 0$ for $r \in (0, r_1]$ can be obtained from the equation of ϕ.)

We then see that $r_2 > r_1$. Otherwise, we have $r_2 < r_1$. In this case, we take $r = r_2$. Then, $I_1(r) > 0$. It remains to estimate $I_2(r_2)$.

To this end, we first show that $\Delta\phi > 0$ for $r \in (r_2, r_1)$. In fact, since $\Delta^2\phi = \lambda e^{\xi(r)}\phi > 0$ in $(0, r_1)$, where $\xi(r) \in (U_s(r) - \phi(r), U_s(r))$, we see that $(\Delta\phi)'(r) > 0$ for $r \in (0, r_1)$. (Note that if $k(r) = r^{N-1}(\Delta\phi)'(r)$, then $k(0) = 0$.) This implies that $\Delta\phi$ must be positive for $r > r_2$ and near r_2. Suppose that there exists $r_3 \le r_1$ such that $\Delta\phi(r_3) = 0$. Then, we have $\Delta\phi > 0$, $\Delta(\Delta\phi) > 0$ in (r_2, r_3). This is impossible, since $\Delta\phi$ must attain its maximum in (r_2, r_3) where $\Delta(\Delta\phi) \le 0$.

Now, we consider the function $\Psi(r) = r^{N-1}(\Delta\psi\phi' - (\Delta\psi)'\phi)$. Its derivative is given by

$$\begin{aligned}\Psi'(r) &= (r^{N-1}\phi'(r))'\Delta\psi(r) - (r^{N-1}(\Delta\psi)'(r))'\phi(r)\\ &= r^{1-N}[\Delta\phi(r)\Delta\psi(r) - \phi(r)\Delta^2\psi(r)] < 0 \quad \text{for } r \in (r_2, r_1).\end{aligned}$$

(Here, we have used the fact that $\Delta\psi < 0$.) So $\Psi(r_2) > \Psi(r_1) = r_1^{N-1}\Delta\psi(r_1)\phi'(r_1) \ge 0$. As a consequence, we have proved that $I_2(r_2) = r_2^{1-N}\int_{\partial B_{r_2}(0)}\Psi(r_2) > 0$. So again, we have $I_1(r_2) > 0$, $I_2(r_2) > 0$, and this gives a contradiction to the inequality (4.7).

Finally, we show that $r_2 > r_1$ is also impossible. If we take $r = r_1$ in (4.7), we see that $I_2(r_1) = \int_{\partial B_{r_1}(0)}[\Delta\psi\phi'] \ge 0$. It remains to estimate $I_1(r_1)$.

As before, we first show that $\phi(r) < 0$ for $r \in (r_1, r_2)$. In fact, since $\Delta\phi < 0$ in $(0, r_2)$, we see that ϕ must be negative for $r > r_1$ and near r_1. Suppose that there exists $r_3 \le r_2$ such that $\phi(r_3) = 0$. Then, we have $\Delta\phi < 0$, $\phi < 0$ in (r_1, r_3). This is impossible, since ϕ must attain its minimum in (r_3, r_2) where $\Delta\phi \geq 0$.

Now, we consider the function $\Phi(r) = r^{N-1}((\Delta\phi)'\psi - \Delta\phi\psi')$. Its derivative is given by

$$\begin{aligned}\Phi'(r) &= (r^{N-1}(\Delta\phi)'(r))'\psi(r) - (r^{N-1}\psi'(r))'\Delta\phi(r)\\ &= r^{1-N}[\Delta^2\phi(r)\psi(r) - \Delta\phi(r)\Delta\psi(r)] < 0 \quad \text{for } r \in (r_1, r_2).\end{aligned}$$

So, $\Phi(r_1) > \Phi(r_2) = r_2^{N-1}(\Delta\phi)'(r_2)\psi(r_2) \geq 0$. So, $I_1(r_1) = r_1^{1-N}\int_{\partial B_{r_1}(0)}\Phi(r_1) > 0$. So again, we have $I_1(r_1) > 0$, $I_2(r_1) \geq 0$ and a contradiction to (4.7). These contradictions imply that $\phi(r) > 0$, $\Delta\phi(r) < 0$ for $r \in (0, \infty)$ and this completes the proof. □

Proof of Theorem 4.3 Let $\tilde\phi(r)$ be a solution of (4.1). By Lemma 4.1, we may assume that $\tilde\phi(0) = 1$, $\Delta\tilde\phi(0) < 0$. We will show that $\tilde\phi(r) > 0$ and $\Delta\tilde\phi(r) < 0$ for $r \geq 0$.

Let $\tilde\psi(r) = r^{\nu_4}$. Then,

$$\Delta^2\tilde\psi = \lambda e^{U_s(r)}\tilde\psi. \tag{4.9}$$

By Proposition 2.2, we see that $\nu_4 > 4 - N$. This implies that $\int_{B_r(0)} r^{-4}|\tilde\phi|\tilde\psi < +\infty$. Multiplying (4.1) by $\tilde\psi$ and (4.9) by $\tilde\phi$ and integrating over $B_r(0)$, we obtain

$$\begin{aligned}0 &= \int_{B_r(0)}\lambda(e^{U_s} - e^u)\tilde\phi\tilde\psi + \int_{\partial B_r(0)}[(\Delta\tilde\phi)'\tilde\psi - \Delta\tilde\phi\tilde\psi'] + \int_{\partial B_r(0)}[\Delta\tilde\psi\tilde\phi' - (\Delta\tilde\psi)'\tilde\phi] \quad (4.10)\\ &= \tilde I_1(r) + \tilde I_2(r) + \tilde I_3(r)\end{aligned}$$

where $\tilde I_i(r)$ are defined in the last equality.

Let us assume that there exist $r_1, r_2 \in (0, \infty]$ such that

$$\tilde\phi(r) > 0, \quad r \in (0, r_1), \quad \tilde\phi(r_1) = 0, \tag{4.11}$$

and

$$\Delta\tilde\phi(r) < 0, \quad r \in (0, r_2), \quad \Delta\tilde\phi(r_2) = 0. \tag{4.12}$$

Then, we also have the four cases as those in the proof of Theorem 4.2. As in the proof of Theorem 4.2, we only need to consider the case $r_1, r_2 \in (0, \infty)$. By Theorem 4.2, we see that $\tilde I_1(r) > 0$ for $r \in (0, r_1]$. We first see that $r_1 \neq r_2$. On the contrary, we choose $r = r_1 = r_2$ and see that $\tilde I_2(r) \geq 0$, $\tilde I_3(r) \geq 0$. The identity (4.10) gives a contradiction.

We now show that $r_2 > r_1$. On the contrary, we see that $r_1 > r_2$. In this case, we take $r = r_2$ in (4.10). Then, $\tilde I_1(r_2) > 0$, $\tilde I_2(r_2) = \int_{\partial B_{r_2}(0)}(\Delta\tilde\phi)'\tilde\psi \geq 0$. It remains to estimate $\tilde I_3(r_2)$. By arguments similar to those in the proof of Theorem 4.2, we can see that $\tilde I_3(r_2) \geq 0$. So again, we have $\tilde I_1(r_2) > 0$, $\tilde I_2(r_2) \geq 0$, $\tilde I_3(r_2) \geq 0$, and these give a contradiction to the identity (4.10).

Finally, we show that $r_2 > r_1$ is also impossible. In this case, we take $r = r_1$ in (4.10). Then, $\tilde I_1(r_1) \geq 0$ by Theorem 4.2, $\tilde I_3(r_1) = \int_{\partial B_{r_1}(0)}\Delta\tilde\psi\tilde\phi \geq 0$. It remains to estimate $\tilde I_2(r_1)$. By arguments similar to those in the proof of Theorem 4.2, we obtain that $\tilde I_2(r_1) \geq 0$. So again, we have $\tilde I_1(r_1) > 0$, $\tilde I_2(r_1) \geq 0$, $\tilde I_3(r_1) \geq 0$, and these give a contradiction to the identity (4.10). These contradictions imply that $\tilde\phi$ and $\Delta\tilde\phi$ remain of constant sign, and the proof is complete. □

Let

$$f(x) = \frac{x - 4 + \sqrt{x^2 - 4x + 8 - 4\sqrt{(9x-34)(x-2)}}}{x - 4 - \sqrt{x^2 - 4x + 8 - 4\sqrt{(9x-34)(x-2)}}}.$$

Direct calculations show that $f(x)$ is an increasing function for $x \geq 13$. Moreover, $2 > f(13) > 1$. Note that $f(N) = \frac{-\nu_4}{-\nu_3}$. Let $\ell(k) := f^{-1}(x)$. We see that $\ell(k)$ is also an increasing function for $k \geq f(13) > 1$.

Theorem 4.4 *Assume that $N \geq 13$. Then, the set of solutions $\{u_a(r)\}$ to (1.1) is well ordered. That is, if $a > b$, then $u_a(r) > u_b(r)$ for all $r > 0$. Moreover, we also have the following asymptotic expansion for u:*

(i) *For any positive integer $k \geq 1$, if $N = \ell(k)$, then near ∞,*

$$u(r) = \ln r^{-4} + M_1 r^{\nu_3} + M_2 r^{2\nu_3} + \cdots + M_k r^{k\nu_3} + M_k r^{k\nu_3} \ln r + Q_1 r^{\nu_4} + o(r^{\nu_4}). \tag{4.13}$$

(ii) *For any positive integer $k \geq 1$, if $\ell(k) < N < \ell(k+1)$, then near ∞,*

$$u(r) = \ln r^{-4} + M_1 r^{\nu_3} + M_2 r^{2\nu_3} + \cdots + M_k r^{k\nu_3} + T_1 r^{\nu_4} + o(r^{\nu_4}). \tag{4.14}$$

where $M_1 \neq 0$ and the coefficients $M_2, M_3, \ldots, M_k$ are uniquely determined once, M_1 is determined.

Proof Since $u_a(r) = a + u_0(e^{\frac{a}{4}} r)$, if we define $\phi(r) := \frac{\partial u_a(r)}{\partial a}$, then

$$\phi(r) = \frac{1}{4}(4 + \rho(u_0)_\rho), \quad (\rho = e^{a/4} r).$$

It is clear that $\phi(0) = 1$. We see that $\hat{\phi}(\rho) := \frac{1}{4}(4 + \rho(u_0)_\rho)$ satisfies the equation

$$\Delta_\rho^2 \hat{\phi} = \lambda e^{u_0(\rho)} \hat{\phi}(\rho).$$

Then, Theorem 4.3 implies that

$$\hat{\phi}(\rho) > 0, \quad \Delta_\rho \hat{\phi}(\rho) < 0 \quad \text{for } \rho \geq 0.$$

This implies that $\phi(r) > 0$ and $\Delta_r \phi(r) < 0$ for $r \geq 0$. That is, if $a > b$, then $u_a(r) > u_b(r)$ and $\Delta u_a(r) < \Delta u_b(r)$ for all $r > 0$.

To see that expansions of u near ∞, we only show the second case. The first case can be done similarly. The arguments we use here are similar to those in the proof of Theorem 2.5 of [8].

As in the proof of Theorem 3.1, we see that for s sufficiently large, in the leading order,

$$w(s) = M_1 e^{\nu_3 s} + T_1 e^{\nu_4 s} + \hat{T} e^{\nu_2 s} + O(e^{\max\{2\nu_3, \nu_4\} s}). \tag{4.15}$$

Note that $\nu_2 < \nu_4 < \nu_3 < 0$ and that if $\ell(k) < N < \ell(k+1)$, then $-\nu_4 > k(-\nu_3)$.

$$\begin{aligned} w(s) = {} & \tilde{M}_1 e^{\nu_3 s} + \tilde{T} e^{\nu_4 s} + \hat{T} e^{\nu_2 s} \\ & + \tau_1 \int_S^s e^{\nu_3(s-t)} g(w(t)) dt + \tau_2 \int_S^s e^{\nu_4(s-t)} g(w(t)) dt \\ & + \tau_3 \int_S^s e^{\nu_2(s-t)} g(w(t)) dt + \tau_4 \int_s^\infty e^{\nu_1(s-t)} g(w(t)) dt. \end{aligned} \tag{4.16}$$

For each positive integer $M \geq 2$, $g(\omega)$ admits the following expansion

$$g(\omega) = d_2 \omega^2 + d_3 \omega^3 + \cdots + d_M \omega^M + O(\omega^{M+1}) \tag{4.17}$$

near $\tau = 0$, where $d_i = 1/(i!)$. Substituting (4.15) and (4.17) into (4.16) and iterating this process, after $(k-1)$ steps we arrive at

$$w(s) = M_1 e^{\nu_3 s} + M_2 e^{2\nu_3 s} + \cdots M_k e^{k\nu_3 s} + O(e^{\nu_4 s}) \tag{4.18}$$

near $s = \infty$. (We use arguments similar to those in the proof of Theorem 2.5 of [8] here. The relation between M_1 and each M_i with $i \in \{2, 3, \ldots, k\}$ is also known from the proof of Theorem 2.5 of [8].) Repeating this process once more, we obtain

$$w(s) = M_1 e^{\nu_3 s} + M_2 e^{2\nu_3 s} + \cdots M_k e^{k\nu_3 s} + T_1 e^{\nu_4 s} + O(e^{(k+1)\nu_3 s}) \tag{4.19}$$

near $s = \infty$. This implies (4.14).

If $M_1 = 0$, then $M_2 = \cdots = M_k = 0$ and

$$u(r) = \ln r^{-4} + O(r^{\nu_4}), \tag{4.20}$$

which implies that $\phi(r) = O(r^{\nu_4})$ for r near ∞, where $\phi(r) = U_s(r) - u(r)$. Since ϕ satisfies the equation

$$\Delta^2 \phi(r) = \lambda(e^{U_s(r)} - e^{u(r)}) = \lambda e^{\xi(r)} \phi(r)$$

where $\xi(r) \in (u(r), U_s(r))$, as in the proof of Theorem 4.3, we have that

$$\int_0^\infty (e^{U_s(r)} - e^{\xi(r)}) \phi(r) r^{\nu_4} r^{N-1} dr = 0 \tag{4.21}$$

where the integral is finite because $\nu_4 > 4 - N$ and $2\nu_4 < 4 - N$. (Note that we can obtain an identity similar to (4.10) for $\tilde{\phi} = \phi$ and $\tilde{\psi}$, to derive (4.21) we only need to send r in this identity to ∞. The behaviors of $(\Delta\phi)'(r)$, $\phi'(r)$ for r near ∞ can be obtained by arguments similar to those in the proof of (3.23).) This is impossible since $\phi(r) > 0$ for $r \in (0, \infty)$ by Theorem 4.2 and $e^{U_s(r)} - e^{\xi(r)} > 0$. Therefore, $M_1 \neq 0$. □

Remark 4.5 If we do the same procedure as that in Theorem 4.4 between ν_4 and ν_2, we can obtain more exact expansions of $u(r)$ near ∞.

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