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Solutions for nonlinear elliptic equations with variable growth and degenerate coercivity

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Abstract In this paper, based on the theory of variable exponent Sobolev space, we study a class of nonlinear elliptic equations with principal part having degenerate coercivity and obtain some existence and regularity results for the solutions.

Keywords Variable exponent Sobolev space · Nonlinear elliptic equation · Degenerate coercivity

Mathematics Subject Classification 35J60 · 35B45

1 Introduction

Recently, the following problem

$$\begin{cases} -\operatorname{div}(a(x,u)|\nabla u|^{p-2}\nabla u) = f(x) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases}$$

with $a(x,t)$ vanishing for t going toward infinity has been considered in [1,4–6,15]. This degeneracy implies that the classical methods for elliptic equations cannot be applied even if the datum f is regular. In order to get existence and regularity of solutions, in [1,4], and [6], the authors use an approximation procedure.

Motivated by their works, in this paper, we will study the following nonlinear elliptic problem

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$$\begin{cases} -\operatorname{div} a(x, u, \nabla u) = f(x) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain and $f \in L^{m(x)}(\Omega)$, for some Lebesgue measurable function $m(x) \geq m_- \geq 1$. We will use the variable exponent spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, the definitions of which will be given in Sect. 2. Under various hypotheses on the data f, we obtain some existence and regularity results for (P).

In order to study problem (P), throughout this paper, we assume that $p \in C(\bar{\Omega})$ satisfies

$$1 < p_- \leq p(x) \leq p_+ < N \tag{1.1}$$

and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following conditions:

(a1) For any $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$a(x, t, \xi)\xi \geq b(x, |t|)|\xi|^{p(x)} \quad \text{a.e. in } \Omega,$$

where $b(x, t) = \frac{c_0}{(1+t)^{\theta(p(x)-1)}}$, for some $c_0 > 0$ and $\theta \geq 0$.

(a2) For any $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$|a(x, t, \xi)| \leq c_1(g(x) + |t|^{p(x)-1} + |\xi|^{p(x)-1}) \quad \text{a.e. in } \Omega,$$

where $c_1 > 0$ and g is a nonnegative function in $L^{p'(x)}(\Omega)$ with $p'(x) = \frac{p(x)}{p(x)-1}$.

(a3) For any $t \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$,

$$(a(x, t, \xi) - a(x, t, \eta))(\xi - \eta) > 0 \quad \text{a.e. in } \Omega.$$

Remark 1.1 0 is not the solution of problem (P). In fact, if we take $\xi \in \mathbb{R}^N$ with $\xi \neq 0$ and $\lambda > 0$, using assumption (a1), we obtain

$$a(x, t, \lambda\xi)\xi \geq \lambda^{p(x)-1} b(x, |t|)|\xi|^{p(x)}$$

and

$$-a(x, t, -\lambda\xi)\xi \geq \lambda^{p(x)-1} b(x, |t|)|\xi|^{p(x)}.$$

Let $\lambda \to 0+$, we get $a(x, t, 0) = 0$.

As far as the existence and regularity results for (P) are concerned, there are two difficulties associated with this kind of problems.

Firstly, from hypothesis (a2), the operator $A(u) = -\operatorname{div} a(x, u, \nabla u)$ is well defined between $W_0^{1,p(x)}(\Omega)$ and its dual space $W^{-1,p'(x)}(\Omega)$. However, by assumption (a1), the operator A is in general not coercive. For example, take $a(x, t, \xi) = \frac{|\xi|^{p(x)-2}\xi}{(1+|t|)^{\theta(p(x)-1)}}$ and $u_n(x) = |x|^{\frac{n(p_+-N)}{(n+1)p_+}} - 1$, for $|x| \leq 1$. We obtain that $\|\nabla u_n\|_{p(x)}$ tends to infinity while $\frac{\langle A(u_n), u_n\rangle}{\|\nabla u_n\|_{p(x)}} \to 0$. So, classical methods used in order to prove the existence of a solution for (P) cannot be applied.

The second difficulty appears when we give a variable exponential growth condition (a2) for a. The operator A possesses more complicated nonlinearities; thus, some techniques used in the constant exponent case cannot be carried out for the variable exponent case.

In this paper, the results are achieved by using the approximation procedure and we consider a sequence of nondegenerate Dirichlet problems, which thus have solutions. We will obtain some a priori estimates on approximate solutions and pass to the limit to find a solution for (P).

The first result deals with data f having higher summability.

Theorem 1.1 *Assume that* $0 \le \theta \le \frac{p_- - 1}{p_+ - 1}$ *and* $f \in L^{m(x)}(\Omega)$ *with* $m_- > \frac{N}{p_-}$. *Then, there exists at least one nontrivial weak solution* $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ *for problem* (P), *in the sense that for any* $v \in W_0^{1,p(x)}(\Omega)$,

$$\int_\Omega a(x, u, \nabla u)\nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x. \tag{1.2}$$

The next result deals with data f, which give unbounded solutions in $W_0^{1,p(x)}(\Omega)$.

Theorem 1.2 *Assume that* $0 \le \theta < \frac{p_- - 1}{p_+ - 1}$ *and* $f \in L^{m(x)}(\Omega)$ *with* $m_- = \frac{N}{p_-}$. *Then, there exists at least one nontrivial weak solution* $u \in W_0^{1,p(x)}(\Omega)$ *in the sense of* (1.2). *Moreover, for any* $r > 0$, $u \in L^r(\Omega)$.

Theorem 1.3 *Assume that* $0 \le \theta < \frac{p_- - 1}{p_+ - 1}$ *and* $f \in L^{m(x)}(\Omega)$ *with*

$$\frac{(\theta p_+ - \theta + 1)N + (p_- - 1 - \theta p_+ + \theta)N\alpha}{(\theta p_+ - \theta + 1)p_- + (p_- - 1 - \theta p_+ + \theta)N\alpha} \le m_- < \frac{N}{p_-}, \tag{1.3}$$

where $\alpha \in (0, 1)$. *Then, there exists at least one nontrivial weak solution* $u \in W_0^{1,p(x)}(\Omega)$ *for problem* (P) *in the sense of* (1.2). *Moreover,* $|u|^s \in L^1(\Omega)$ *with*

$$s = \frac{p_- - 1 - \theta(p_+ - 1)}{N - m_- p_-} m_- N\alpha. \tag{1.4}$$

Remark 1.2 Under condition (1.3), we could verify that $s > 1$.

If we continue to decrease the summability of f, solutions we obtain in general do not belong to $W_0^{1,p(x)}(\Omega)$. In the following, we will introduce a different definition of solution, which also involves a different definition of gradient for a measurable function.

We start with the existence of weak gradient for every measurable function u such that for any $k > 0$, $T_k(u) \in W_0^{1,p(x)}(\Omega)$, where the truncation function T_k is defined by

$$T_k(t) = \max\{-k, \min\{k, t\}\},$$

for any $t \in \mathbb{R}$. And we recall the following result which appears in [23].

Proposition 1.1 *If* u *is a measurable function such that for any* $k > 0$, $T_k(u) \in W_0^{1,p(x)}(\Omega)$, *then there exists a unique measurable function* $v : \Omega \to \mathbb{R}^N$ *such that for any* $k > 0$,

$$v\chi_{\{x \in \Omega : |u| < k\}} = \nabla T_k(u) \quad a.e.\ x \in \Omega, \tag{1.5}$$

where χ_E *denotes the characteristic function of a measurable set* E. *Moreover, if* u *belongs to* $W_0^{1,1}(\Omega)$, *then* v *coincides with the standard distributional gradient of* u.

A function u such that $T_k(u) \in W_0^{1,p(x)}(\Omega)$, for any $k > 0$, does not necessarily belong to $W_0^{1,1}(\Omega)$. However, according to the above proposition, it is possible to define its weak gradient, still denoted by ∇u, as the unique function v which satisfies (1.5).

We will extend the notion of entropy solution (see [23]) to problem (P) as follows:

Definition 1.1 A measurable function u is an entropy solution to problem (P) if, for any $k > 0$, $T_k(u) \in W_0^{1,p(x)}(\Omega)$ and

$$\int_{\{x\in\Omega:|u-\phi|<k\}} a(x,u,\nabla u)(\nabla u - \nabla\phi)\,\mathrm{d}x \le \int_\Omega f T_k(u-\phi)\,\mathrm{d}x,$$

for any $\phi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$.

It is useful to extend the above definition of entropy solution to more general truncation functions than T_k. We introduce the class $\mathcal{T}$ of functions $T \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfying:

$$T(0) = 0 \quad \text{and} \quad T(-t) = -T(t),\ T'(t) \ge 0,\ \text{for any}\, t \in \mathbb{R},$$

$$T'(t) = 0,\ \text{for any}\ t\ \text{large enough} \quad \text{and} \quad T''(t) \le 0,\ \text{for any}\, t \ge 0.$$

Proposition 1.2 *The definition 1.1 is equivalent to the following statement that*

$$\int_\Omega a(x,u,\nabla u)\nabla T(u-\phi)\,\mathrm{d}x \le \int_\Omega f T(u-\phi)\,\mathrm{d}x,$$

for any $T \in \mathcal{T}$ and $\phi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$.

The proof of Proposition 1.2 is similar to Lemma 3.2 in [3], and we will omit it here.

Theorem 1.4 *Assume that $0 \le \theta < \frac{p_- -1}{p_+ -1}$ and $f \in L^{m(x)}(\Omega)$ with*

$$\max\left\{1, \frac{(\theta p_+ - \theta + 1)N + (p_- - 1 - \theta p_+ + \theta)N\alpha}{(\theta p_+ - \theta + 1)p_- + (p_- - 1 - \theta p_+ + \theta)p_- N\alpha}\right\} < m_- < \frac{(\theta p_+ - \theta + 1)N + (p_- - 1 - \theta p_+ + \theta)N\alpha}{(\theta p_+ - \theta + 1)p_- + (p_- - 1 - \theta p_+ + \theta)N\alpha}, \tag{1.6}$$

where $\alpha \in (0, 1)$. Then, there exists at least one nontrivial entropy solution $u \in W_0^{1,q(x)}(\Omega)$ for problem (P), where

$$q(x) = \frac{(p_- - 1 - \theta p_+ + \theta)m_- N\alpha p(x)}{(\theta p_+ - \theta + 1)(N - m_- p_-) + (p_- - 1 - \theta p_+ + \theta)N\alpha}. \tag{1.7}$$

Moreover, $|u|^s \in L^1(\Omega)$ with s as in (1.4).

Remark 1.3 According to the definition of $q(x)$, we could verify that $q(x) \ge 1$ is equivalent to

$$m_- \ge \frac{(\theta p_+ - \theta + 1)N + (p_- - 1 - \theta p_+ + \theta)N\alpha}{(\theta p_+ - \theta + 1)p_- + (p_- - 1 - \theta p_+ + \theta)p_- N\alpha}.$$

Theorem 1.5 *Assume that $0 \le \theta < \frac{p_- -1}{p_+ -1}$ and $f \in L^{m(x)}(\Omega)$ with*

$$1 \le m_- \le \max\left\{1, \frac{(\theta p_+ - \theta + 1)N + (p_- - 1 - \theta p_+ + \theta)N\alpha}{(\theta p_+ - \theta + 1)p_- + (p_- - 1 - \theta p_+ + \theta)p_- N\alpha}\right\}, \tag{1.8}$$

where $\alpha \in (0, 1)$. Let $q(x)$ be as in (1.7), then there exists at least one nontrivial entropy solution u for problem (P) such that $\int_\Omega |\nabla u|^{q(x)}\,\mathrm{d}x < \infty$. Moreover, $|u|^s \in L^1(\Omega)$ with s as in (1.4).

2 Variable exponent function spaces

In recent years, the differential equations and variational problems with nonstandard variable growth conditions have been greatly studied, see for example [7,12,14,18–21]. In the studies of this class of nonlinear problems, variable exponent spaces play an important role. Since they were thoroughly studied by Kováčik and Rákosník [16], variable exponent spaces have been used to model various phenomena. In [22], Růžička presented the mathematical theory for the application of variable exponent spaces in electro-rheological fluids. As another application, Chen et al. [8] suggested a model for image restoration based on a variable exponent Laplacian.

For the convenience of the reader, we recall some definitions and basic properties of variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a domain. For a deeper treatment on these spaces, we refer to [9].

Let $\mathbf{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to [1, \infty]$, we denote

$$\rho_{p(x)}(u) = \int\limits_{\Omega \setminus \Omega_\infty} |u|^{p(x)}\, \mathrm{d}x + \sup_{x \in \Omega_\infty} |u(x)|,$$

where $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$.

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of all functions u such that $\rho_{p(x)}(tu) < \infty$, for some $t > 0$. $L^{p(x)}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{p(x)} = \inf\{\lambda > 0 : \rho_{p(x)}(\lambda u) \leq 1\}.$$

For any $p \in \mathbf{P}(\Omega)$, we define the conjugate function $p'(x)$ as

$$p'(x) = \begin{cases} \infty, & x \in \Omega_1 = \{x \in \Omega : p(x) = 1\}, \\ 1, & x \in \Omega_\infty, \\ \frac{p(x)}{p(x)-1}, & x \in \Omega \setminus (\Omega_1 \cup \Omega_\infty). \end{cases}$$

Theorem 2.1 *For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$,*

$$\int\limits_{\Omega} |uv|\, \mathrm{d}x \leq 2\|u\|_{p(x)}\|v\|_{p'(x)}.$$

For any $p \in \mathbf{P}(\Omega)$, denote

$$p_+ = \sup_{x \in \Omega} p(x), \quad p_- = \inf_{x \in \Omega} p(x)$$

and we denote by $p_1 \ll p_2$ the fact that $\inf_{x \in \Omega}(p_2(x) - p_1(x)) > 0$.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is the class of all functions $u \in L^{p(x)}(\Omega)$ such that $|\nabla u| \in L^{p(x)}(\Omega)$ and it can be equipped with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}. \tag{2.1}$$

By $W_0^{1,p(x)}(\Omega)$, we denote the subspace of $W^{1,p(x)}(\Omega)$, which is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1). We know that if $\Omega \subset \mathbb{R}^N$ is a bounded domain and $p \in C(\bar{\Omega})$, $\|u\|_{1,\,p(x)}$, and $\|\nabla u\|_{p(x)}$ are equivalent norms on $W_0^{1,\,p(x)}(\Omega)$. Under condition $1 \leq p_- \leq p(x) \leq p_+ < \infty$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.

And we denote the dual space of $W_0^{1,p(x)}(\Omega)$ by $W^{-1,p'(x)}(\Omega)$.

Theorem 2.2 (Fan, [11]) *Let Ω be a bounded domain with the cone property. If $p \in C(\bar{\Omega})$ satisfying (1.1) and q is a measurable function defined on Ω with*

$$p(x) \leq q(x) \ll p^*(x) \triangleq \frac{Np(x)}{N-p(x)} \quad a.e.\, x \in \Omega,$$

then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Theorem 2.3 (Fan, [11]) *Let Ω be a bounded domain with the cone property. If p is Lipschitz continuous and satisfies (1.1), q is a measurable function defined on Ω with*

$$p(x) \leq q(x) \leq p^*(x) \quad a.e.\, x \in \Omega,$$

then there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

3 A priori estimates

In order to prove all the existence results for problem (P), in this part, we will give some a priori estimates for solutions of the following approximating problems:

$$\begin{cases} -\operatorname{div} a(x, T_n(u), \nabla u) = f_n(x) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{P_n}$$

where $n \in \mathbb{N}$ and $f_n \in L^\infty(\Omega)$.

For $u \in W_0^{1,p(x)}(\Omega)$, define the operator $A_n : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ by

$$\langle A_n(u), v\rangle = \int_\Omega a(x, T_n(u), \nabla u)\nabla v \,\mathrm{d}x$$

for any $v \in W_0^{1,p(x)}(\Omega)$.

First, using the classical theory of pseudo-monotone operators in reflexive Banach spaces (see [17]), we obtain the following existence result for problem (P_n).

Lemma 3.1 *For any $n \in \mathbb{N}$, there exists at least one weak solution $u_n \in W_0^{1,p(x)}(\Omega)$ for (P_n) in the sense that for any $v \in W_0^{1,p(x)}(\Omega)$,*

$$\int_\Omega a(x, T_n(u_n), \nabla u_n)\nabla v \,\mathrm{d}x = \int_\Omega f_n v \,\mathrm{d}x. \tag{3.1}$$

Proof (1) The operator A_n is coercive on $W_0^{1,p(x)}(\Omega)$. In fact, using assumption (a1) we obtain

$$\langle A_n(u), u\rangle \geq \int_\Omega \frac{c_0|\nabla u|^{p(x)}}{(1+|T_n(u)|)^{\theta(p(x)-1)}} \,\mathrm{d}x \geq \frac{c_0}{(1+n)^{\theta(p_+-1)}} \int_\Omega |\nabla u|^{p(x)} \,\mathrm{d}x.$$

which implies

$$\frac{\langle A_n(u), u\rangle}{\|\nabla u\|_{p(x)}} \geq \frac{c_0}{(1+n)^{\theta(p_+-1)}} \frac{\|\nabla u\|_{p(x)}^{p_-}}{\|\nabla u\|_{p(x)}} \to \infty,$$

as $\|\nabla u\|_{p(x)} \to \infty$.

(2) The operator A_n is pseudo-monotone on $W_0^{1,p(x)}(\Omega)$.

(i) A_n is bounded on $W_0^{1,p(x)}(\Omega)$. In fact, let $E \subset W_0^{1,p(x)}(\Omega)$ be bounded. For any $u \in E$, condition (a2) yields

$$\begin{aligned}\|A_n(u)\| &= \sup_{\|\nabla v\|_{p(x)}=1} |\langle A_n(u), v\rangle| = \sup_{\|\nabla v\|_{p(x)}=1} |\int_\Omega a(x, T_n(u), \nabla u)\nabla v \,\mathrm{d}x| \\ &\le \sup_{\|\nabla v\|_{p(x)}=1} \int_\Omega c_1(g(x) + |T_n(u)|^{p(x)-1} + |\nabla u|^{p(x)-1})|\nabla v| \,\mathrm{d}x \\ &\le C \sup_{\|\nabla v\|_{p(x)}=1} \|g + |T_n(u)|^{p(x)-1} + |\nabla u|^{p(x)-1}\|_{p'(x)}\|\nabla v\|_{p(x)} \le C.\end{aligned}$$

(ii) If $u_m \to u$ weakly in $W_0^{1,p(x)}(\Omega)$, as $m \to \infty$ and $\limsup_{m\to\infty}\langle A_n(u_m), u_m - v\rangle \le 0$, for any $v \in W_0^{1,p(x)}(\Omega)$, then

$$\liminf_{m\to\infty}\langle A_n(u_m), u_m - v\rangle \ge \langle A_n(u), u - v\rangle.$$

In fact, by Theorem 2.2, we get $u_m \to u$ in $L^{p(x)}(\Omega)$. Moreover, we assume that $u_m \to u$ a.e. in Ω.

In the following, we will verify that $A_n(u_m) \to A_n(u)$ weakly $*$ in $W^{-1,p'(x)}(\Omega)$, as $m \to \infty$. For any $m \in \mathbb{N}$, denote

$$h_m(x) = (a(x, T_n(u_m), \nabla u_m) - a(x, T_n(u), \nabla u))(\nabla u_m - \nabla u).$$

Then, it follows that

$$\langle A_n(u_m) - A_n(u), u_m - u\rangle = \int_\Omega h_m(x) \,\mathrm{d}x.$$

Note that $\limsup_{m\to\infty}\langle A_n(u_m), u_m - u\rangle \le 0$, we get

$$\limsup_{m\to\infty} \int_\Omega h_m(x) \,\mathrm{d}x \le 0. \tag{3.2}$$

Denote $h_m^+(x) = \max\{h_m(x), 0\}$, $h_m^-(x) = \max\{-h_m(x), 0\}$, then $h_m(x) = h_m^+(x) - h_m^-(x)$. By condition (a2), we obtain

$$\begin{aligned}h_m(x) &= a(x, T_n(u_m), \nabla u_m)\nabla u_m - a(x, T_n(u_m), \nabla u_m)\nabla u - a(x, T_n(u), \nabla u)(\nabla u_m - \nabla u) \\ &\ge \frac{c_0}{(1+n)^{\theta(p_+-1)}}|\nabla u_m|^{p(x)} - c_1(g(x) + n^{p(x)-1} + |\nabla u_m|^{p(x)-1})|\nabla u| \\ &\quad - c_1(g(x) + n^{p(x)-1} + |\nabla u|^{p(x)-1})(|\nabla u_m| + |\nabla u|).\end{aligned}$$

Take $0 < \varepsilon \le \frac{c_0}{2(1+n)^{\theta(p_+-1)}}$, by Young inequality, we get

$$\begin{aligned}h_m(x) &\ge \frac{c_0}{(1+n)^{\theta(p_+-1)}}|\nabla u_m|^{p(x)} - \varepsilon|\nabla u_m|^{p(x)} - C(1 + g(x)^{p'(x)} + |\nabla u|^{p(x)}) \\ &\ge \frac{c_0}{2(1+n)^{\theta(p_+-1)}}|\nabla u_m|^{p(x)} - C(1 + g(x)^{p'(x)} + |\nabla u|^{p(x)-1}).\end{aligned} \tag{3.3}$$

Therefore, $\{|\nabla u_m|^{p(x)}\}$ is bounded almost everywhere in the set $\{x \in \Omega : h_m(x) < 0\}$ and

$$h_m^-(x) \le C(1 + g(x)^{p'(x)} + |\nabla u|^{p(x)-1}). \tag{3.4}$$

Note that

$$h_m(x) = (a(x, T_n(u_m), \nabla u_m) - a(x, T_n(u_m), \nabla u))(\nabla u_m - \nabla u) \\ + (a(x, T_n(u_m), \nabla u) - a(x, T_n(u), \nabla u))(\nabla u_m - \nabla u),$$

condition (a3) implies that

$$h_m(x) \geq (a(x, T_n(u_m), \nabla u) - a(x, T_n(u), \nabla u))(\nabla u_m - \nabla u).$$

If $h_m(x) < 0$, $-h_m^-(x) = h_m(x)$, we obtain

$$0 \geq \liminf_{m\to\infty} -h_m^-(x) \geq \liminf_{m\to\infty} (a(x, T_n(u_m), \nabla u) - a(x, T_n(u), \nabla u))(\nabla u_m - \nabla u) = 0,$$

which implies $\limsup_{m\to\infty} h_m^-(x) = 0$. Thus,

$$\lim_{m\to\infty} h_m^-(x) = 0.$$

It follows from inequality (3.4) that $\{h_m^-\}$ is equi-integral in $L^1(\Omega)$. Thus, using Vitali theorem [25], we obtain

$$\int_\Omega h_m^- \, dx \to 0, \quad \text{as } m \to \infty.$$

Note that

$$\int_\Omega h_m^+ \, dx = \int_\Omega h_m \, dx + \int_\Omega h_m^- \, dx,$$

combining with (3.2), we derive

$$\int_\Omega h_m^+ \, dx \to 0, \quad \text{as } m \to \infty.$$

Passing to a subsequence, still denoted by $\{h_m^+\}$, we assume that $h_m^+(x) \to 0$ a.e. in Ω. Therefore,

$$\int_\Omega h_m \, dx \to 0 \quad \text{and} \quad h_m(x) \to 0 \text{ a.e. in } \Omega.$$

By inequality (3.3), $\{|\nabla u_m|^{p(x)}\}$ is equi-integral in $L^1(\Omega)$ and bounded a.e. in Ω. For almost everywhere $x \in \Omega$, up to a subsequence, we assume that $\nabla u_m(x) \to \xi$, as $m \to \infty$. By the definition of $h_m(x)$, we get

$$(a(x, T_n(u), \xi) - a(x, T_n(u), \nabla u))(\xi - \nabla u) = 0.$$

Then, it follows from condition (a3) that $\nabla u(x) = \xi$. Thus, $\nabla u_m(x) \to \nabla u(x)$ a.e. in Ω, as $m \to \infty$, which implies

$$a(x, T_n(u_m), \nabla u_m) \to a(x, T_n(u), \nabla u) \quad \text{a.e. in } \Omega.$$

For any $v \in W_0^{1,p(x)}(\Omega)$, condition (a2) implies

$$|a(x, T_n(u_m), \nabla u_m)\nabla v| \leq c_1(g(x) + n^{p(x)-1} + |\nabla u_m|^{p(x)-1})|\nabla v|,$$

thus $\{|a(x, T_n(u_m), \nabla u_m)\nabla v|\}$ is equi-integral in $L^1(\Omega)$. By Vitali theorem, we get

$$\langle A_n(u_m), v\rangle = \int_\Omega a(x, T_n(u_m), \nabla u_m)\nabla v \,\mathrm{d}x \to \int_\Omega a(x, T_n(u), \nabla u)\nabla v \,\mathrm{d}x,$$

as $m \to \infty$, i.e. $A_n(u_m) \to A_n(u)$ weakly-$*$ in $W^{-1,p'(x)}(\Omega)$.

Note that

$$\begin{aligned}&\langle A_n(u_m), u_m - v\rangle \\ &=\langle A_n(u_m) - A_n(u), u_m - u\rangle + \langle A_n(u_m), u - v\rangle + \langle A_n(u), u_m - u\rangle,\end{aligned} \tag{3.5}$$

we derive

$$\lim_{m\to\infty} \langle A_n(u_m), u_m - v\rangle = \langle A_n(u), u - v\rangle.$$

By (1) and (2), the operator A_n is surjective (see Theorem 2.7 in [17]). As $f_n \in L^\infty(\Omega) \subset W^{-1,p'(x)}(\Omega)$, we obtain the existence of weak solution for (P_n). □

In the following, we will prove some a priori estimates on the solutions of approximating problems (P_n), which are based on the rearrangement techniques.

Let $u : \Omega \to \mathbb{R}$ be a measurable function, for simplicity, we denote

$$\{u > t\} = \{x \in \Omega : u(x) > t\} \quad \text{and} \quad \{u \le t\} = \{x \in \Omega : u(x) \le t\}.$$

Next, we recall the definition of decreasing rearrangement of u. If we denote $|E|$ by the Lebesgue measure of a measurable set E, the distribution function $\mu_u(t)$ of u is defined by

$$\mu_u(t) = |\{|u| > t\}|,$$

for any $t \ge 0$. The decreasing rearrangement u^* of u is defined by

$$u^*(\sigma) = \inf\{t \in \mathbb{R} : \mu_u(t) \le \sigma\},$$

for $\sigma \in (0, |\Omega|)$.

We recall that for any $t \ge 0$, $\mu_u(t) = \mu_{u^*}(t)$. Then, for any monotone function ψ, it follows

$$\int_\Omega \psi(|u(x)|)\,\mathrm{d}x = \int_0^{|\Omega|} \psi(u^*(\sigma))\,\mathrm{d}\sigma,$$

and in particular, for any $r \in [1, \infty]$, $\|u\|_{L^r(\Omega)} = \|u^*\|_{L^r(0,|\Omega|)}$.

We say that a measurable function u belongs to the Marcinkiewicz space $M^r(\Omega)$ with $r > 0$, if there exists a constant C such that for any $t > 0$,

$$\mu_u(t)t^r \le C.$$

And the above condition is equivalent to the following statement: for any $\sigma \in (0, |\Omega|)$, $u^*(\sigma)\sigma^{\frac{1}{r}} \le C$, where C is a positive constant.

The norm on Marcinkiewicz space is defined by

$$\|u\|_{M^r(\Omega)} = \sup_{\sigma\in(0,|\Omega|)} u^*(\sigma)\sigma^{\frac{1}{r}}.$$

We also recall that if $u \in L^r(\Omega)$, then $u \in M^r(\Omega)$.

For solutions of (P_n), we obtain the following differential inequality:

Lemma 3.2 *For any $t \geq 0$, define*

$$B(t) = \int_0^t (1+s)^{-\frac{\theta(p_+-1)}{p_--1}}\,\mathrm{d}s.$$

Suppose u_n is a weak solution of (P_n), then for almost every $\sigma \in (0, |\Omega|)$,

$$-\frac{\mathrm{d}}{\mathrm{d}\sigma} B\left(u_n^*(\sigma)\right) \leq 2^{\frac{1}{p_--1}} \left(\left(N C_N^{\frac{1}{N}} \sigma^{\frac{1}{N'}}\right)^{-1} + c_0^{\frac{1}{1-p_-}} \left(N C_N^{\frac{1}{N}} \sigma^{\frac{1}{N'}}\right)^{-p'_-} \left(\int_0^\sigma f_n^*(\tau)\,\mathrm{d}\tau\right)^{\frac{p'_-}{p_-}} \right), \tag{3.6}$$

where C_N is the measure of unit ball in $\mathbb{R}^N$, $p'_- = \frac{p_-}{p_--1}$, and $N' = \frac{N}{N-1}$.

Proof For any $k > 0$, $t > 0$, choose $v = T_k(u_n - T_t(u_n))$ as a test function in (3.1), we obtain

$$\int_\Omega a(x, T_n(u_n), \nabla u_n)\nabla T_k(u_n - T_t(u_n))\,\mathrm{d}x = \int_\Omega f_n T_k(u_n - T_t(u_n))\,\mathrm{d}x,$$

which implies

$$\int_{\{t<|u_n|\leq t+k\}} a(x, T_n(u_n), \nabla u_n)\nabla u_n\,\mathrm{d}x \leq k \int_{|u_n|>t} |f_n|\,\mathrm{d}x.$$

By condition (a1), we get

$$\int_{\{t<|u_n|\leq t+k\}} \frac{c_0|\nabla u_n|^{p(x)}}{(1+|T_n(u_n)|)^{\theta(p(x)-1)}}\,\mathrm{d}x \leq k \int_{|u_n|>t} |f_n|\,\mathrm{d}x,$$

then

$$\frac{c_0}{(1+k+t)^{\theta(p_+-1)}} \int_{\{t<|u_n|\leq t+k\}} |\nabla u_n|^{p(x)}\,\mathrm{d}x \leq k \int_{|u_n|>t} |f_n|\,\mathrm{d}x. \tag{3.7}$$

As $1 < p_- \leq p(x)$, using Young inequality, we obtain

$$\int_{\{t<|u_n|\leq t+k\}} |\nabla u_n|^{p_-}\,\mathrm{d}x \leq \int_{\{t<|u_n|\leq t+k\}} |\nabla u_n|\,\mathrm{d}x + \int_{\{t<|u_n|\leq t+k\}} |\nabla u_n|^{p(x)}\,\mathrm{d}x,$$

moreover,

$$\left(\frac{\int_{\{t<|u_n|\leq t+k\}} |\nabla u_n|^{p_-}\,\mathrm{d}x}{\int_{\{t<|u_n|\leq t+k\}} |\nabla u_n|\,\mathrm{d}x}\right)^{\frac{1}{1-p_-}} \geq \left(\frac{\int_{\{t<|u_n|\leq t+k\}} |\nabla u_n|\,\mathrm{d}x + \int_{\{t<|u_n|\leq t+k\}} |\nabla u_n|^{p(x)}\,\mathrm{d}x}{\int_{\{t<|u_n|\leq t+k\}} |\nabla u_n|\,\mathrm{d}x}\right)^{\frac{1}{1-p_-}}.$$

Note that the function $s^{\frac{1}{1-p_-}}$ is convex for $s > 0$, Jensen's inequality for convex functions gives

$$\left(\frac{\int_{\{t<|u_n|\le t+k\}} |\nabla u_n|^{p_-}\,\mathrm{d}x}{\int_{\{t<|u_n|\le t+k\}} |\nabla u_n|\,\mathrm{d}x}\right)^{\frac{1}{1-p_-}} \le \frac{\mu_{u_n}(t)-\mu_{u_n}(t+k)}{\int_{\{t<|u_n|\le t+k\}} |\nabla u_n|\,\mathrm{d}x}.$$

Combining with (3.7), we obtain

$$\frac{\mu_{u_n}(t)-\mu_{u_n}(t+k)}{\int_{\{t<|u_n|\le t+k\}} |\nabla u_n|\,\mathrm{d}x} \ge \left(\frac{\int_{\{t<|u_n|\le t+k\}} |\nabla u_n|\,\mathrm{d}x + c_0^{-1}k(1+k+t)^{\theta(p_+-1)}\int_{|u_n|>t}|f_n|\,\mathrm{d}x}{\int_{\{t<|u_n|\le t+k\}} |\nabla u_n|\,\mathrm{d}x}\right)^{\frac{1}{1-p_-}}.$$

Let $k \to 0+$, we get

$$\frac{-\mu'_{u_n}(t)}{\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u_n|\le t}|\nabla u_n|\,\mathrm{d}x} \ge \left(\frac{\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u_n|\le t}|\nabla u_n|\,\mathrm{d}x + c_0^{-1}(1+t)^{\theta(p_+-1)}\int_{|u_n|>t}|f_n|\,\mathrm{d}x}{\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u_n|\le t}|\nabla u_n|\,\mathrm{d}x}\right)^{\frac{1}{1-p_-}},$$

thus

$$(-\mu'_{u_n}(t))^{1-p_-}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{|u_n|\le t}|\nabla u_n|\,\mathrm{d}x\right)^{p_-} \le \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{|u_n|\le t}|\nabla u_n|\,\mathrm{d}x + c_0^{-1}(1+t)^{\theta(p_+-1)}\int\limits_{|u_n|>t}|f_n|\,\mathrm{d}x.$$

Note that

$$NC_N^{\frac{1}{N}}\mu_{u_n}(t)^{\frac{1}{N'}} \le \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{|u_n|\le t}|\nabla u_n|\,\mathrm{d}x, \tag{3.8}$$

where C_N is the measure of unit ball in $\mathbb{R}^N$. Inequality (3.8) is an easy consequence of Fleming–Rishel formula [13] and of the isoperimetrie theorem. The proof is essentially the same as Lemma 2 in [24] and we will omit it here.

Using inequality (3.8), we obtain

$$\begin{aligned}
&(-\mu'_{u_n}(t))^{-1}\\
&\le\left(NC_N^{\frac{1}{N}}\mu_{u_n}(t)^{\frac{1}{N'}}\right)^{-1}\left(1+c_0^{-1}(1+t)^{\theta(p_+-1)}\left(NC_N^{\frac{1}{N}}\mu_{u_n}(t)^{\frac{1}{N'}}\right)^{-1}\int\limits_{|u_n|>t}|f_n|\,\mathrm{d}x\right)^{\frac{1}{p_--1}}\\
&\le 2^{\frac{1}{p_--1}}\left(\left(NC_N^{\frac{1}{N}}\mu_{u_n}(t)^{\frac{1}{N'}}\right)^{-1}+c_0^{\frac{1}{1-p_-}}(1+t)^{\frac{\theta(p_+-1)}{p_--1}}\left(NC_N^{\frac{1}{N}}\mu_{u_n}(t)^{\frac{1}{N'}}\right)^{-p'_-}\right.\\
&\left.\left(\int\limits_{|u_n|>t}|f_n|\,\mathrm{d}x\right)^{\frac{1}{p_--1}}\right)
\end{aligned}$$

$$\leq 2^{\frac{1}{p_--1}}(1+t)^{\frac{\theta(p_+-1)}{p_--1}}\left(\left(NC_N^{\frac{1}{N}}\mu_{u_n}(t)^{\frac{1}{N'}}\right)^{-1}\right.$$

$$\left. + c_0^{\frac{1}{1-p_-}}\left(NC_N^{\frac{1}{N}}\mu_{u_n}(t)^{\frac{1}{N'}}\right)^{-p'_-}\left(\int\limits_{|u_n|>t}|f_n|\,\mathrm{d}x\right)^{\frac{p'_-}{p_-}}\right).$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}B(u_n^*(\sigma)) = B'(u_n^*(\sigma))(\mu'_{u_n}(t))^{-1} = \left((1+t)^{\frac{\theta(p_+-1)}{p_--1}}\mu'_{u_n}(t)\right)^{-1},$$

therefore,

$$-\frac{\mathrm{d}}{\mathrm{d}\sigma}B(u_n^*(\sigma)) \leq 2^{\frac{1}{p_--1}}\left(\left(NC_N^{\frac{1}{N}}\mu_{u_n}(t)^{\frac{1}{N'}}\right)^{-1}\right.$$

$$\left. + c_0^{\frac{1}{1-p_-}}\left(NC_N^{\frac{1}{N}}\mu_{u_n}(t)^{\frac{1}{N'}}\right)^{-p'_-}\left(\int\limits_{|u_n|>t}|f_n|\,\mathrm{d}x\right)^{\frac{p'_-}{p_-}}\right)$$

$$= 2^{\frac{1}{p_--1}}\left(\left(NC_N^{\frac{1}{N}}\sigma^{\frac{1}{N'}}\right)^{-1} + c_0^{\frac{1}{1-p_-}}\left(NC_N^{\frac{1}{N}}\sigma^{\frac{1}{N'}}\right)^{-p'_-}\left(\int\limits_0^{\sigma}f_n^*(\tau)\,\mathrm{d}\tau\right)^{\frac{p'_-}{p_-}}\right).$$

We get the inequality (3.6). □

Theorem 3.1 *Assume that* $0 \leq \theta \leq \frac{p_--1}{p_+-1}$ *and* $m_- > \frac{N}{p_-}$. *Let* u_n *be a weak solution of* (P_n), *then there exists a constant* $C > 0$ *which depends on* $\Omega, \theta, m_-, N, p_+, p_-$, *and* $\|f_n\|_{m(x)}$ *such that*

$$\|u_n\|_\infty \leq C$$

and

$$\|\nabla u_n\|_{p(x)} \leq C.$$

Proof Note that $u_n^*(|\Omega|) = 0$. Integrating both sides of (3.6) between $\widetilde{\sigma}$ and $|\Omega|$, we obtain

$$B(u_n^*(\widetilde{\sigma}))$$

$$\leq 2^{\frac{1}{p_--1}}\left(C_N^{-\frac{1}{N}}|\Omega|^{\frac{1}{N}} + c_0^{\frac{1}{1-p_-}}\left(NC_N^{\frac{1}{N}}\right)^{-p'_-}\int\limits_{\widetilde{\sigma}}^{|\Omega|}\sigma^{-\frac{p'_-}{N'}}\left(\int\limits_0^{\sigma}f_n^*(\tau)\,\mathrm{d}\tau\right)^{\frac{p'_-}{p_-}}\mathrm{d}\sigma\right). \tag{3.9}$$

As $f_n \in L^\infty(\Omega)$, $f_n^* \in L^{m_-}(0,|\Omega|)$. By Hölder inequality, we get

$$\int\limits_0^{\sigma}f_n^*(\tau)\,\mathrm{d}\tau \leq \sigma^{\frac{1}{m'_-}}\|f_n^*\|_{m_-},$$

where $m'_- = \frac{m_-}{m_- - 1}$, thus

$$\int_{\widetilde{\sigma}}^{|\Omega|} \sigma^{-\frac{p'_-}{N'}} \left(\int_0^{\sigma} f_n^*(\tau)\,\mathrm{d}\tau \right)^{\frac{p'_-}{p_-}} \mathrm{d}\sigma \le \| f_n^* \|_{m_-}^{\frac{p'_-}{p_-}} \int_{\widetilde{\sigma}}^{|\Omega|} \sigma^{\frac{p'_-}{m'_- p_-} - \frac{p'_-}{N'}} \mathrm{d}\sigma$$

$$= \frac{m_- N(p_- - 1)}{m_- p_- - N} \left(|\Omega|^{\frac{m_- p_- - N}{m_- N(p_- - 1)}} - \widetilde{\sigma}^{\frac{m_- p_- - N}{m_- N(p_- - 1)}} \right) \| f_n \|_{m_-}^{\frac{p'_-}{p_-}}.$$

Using inequality (3.9), we derive

$$B(u_n^*(\widetilde{\sigma})) \le 2^{\frac{1}{p_- - 1}} \left(C_N^{-\frac{1}{N}} |\Omega|^{\frac{1}{N}} + c_0^{\frac{1}{1-p_-}} \left(N C_N^{\frac{1}{N}} \right)^{-p'_-} \frac{m_- N(p_- - 1)}{m_- p_- - N} |\Omega|^{\frac{m_- p_- - N}{m_- N(p_- - 1)}} \| f_n \|_{m_-}^{\frac{p'_-}{p_-}} \right).$$

Note that $B(u_n^*(0)) = B(\|u_n\|_\infty)$, we get

$$B(\|u_n\|_\infty) \le 2^{\frac{1}{p_- - 1}} \left(C_N^{-\frac{1}{N}} |\Omega|^{\frac{1}{N}} + c_0^{\frac{1}{1-p_-}} \left(N C_N^{\frac{1}{N}} \right)^{-p'_-} \frac{m_- N(p_- - 1)}{m_- p_- - N} |\Omega|^{\frac{m_- p_- - N}{m_- N(p_- - 1)}} \| f_n \|_{m_-}^{\frac{p'_-}{p_-}} \right).$$

Thus,

$$\|u_n\|_\infty \le B^{-1}(C). \tag{3.10}$$

As u_n is a weak solution for (P_n), we obtain

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \nabla u_n \,\mathrm{d}x = \int_\Omega f_n u_n \,\mathrm{d}x.$$

By condition (a1), we derive

$$c_0 \int_\Omega \frac{|\nabla u_n|^{p(x)}}{(1 + |T_n(u_n)|)^{\theta(p(x)-1)}} \le \int_\Omega |f_n u_n| \,\mathrm{d}x,$$

which implies

$$\int_\Omega |\nabla u_n|^{p(x)} \,\mathrm{d}x \le C(1 + \|u_n\|_\infty)^{\theta(p_+ - 1)} \|u_n\|_\infty \|f_n\|_{m_-} \,\mathrm{d}x \le C.$$

Now, we complete the proof. □

Theorem 3.2 *Assume that* $0 \le \theta < \frac{p_- - 1}{p_+ - 1}$ *and* $m_- = \frac{N}{p_-}$. *Let* u_n *be a weak solution of* (P_n), *then for any* $r > 0$, *there exists a constant* $C > 0$ *which depends on* $\Omega, \theta, m_-, N, p_+, p_-, r$ *and* $\|f_n\|_{m(x)}$ *such that*

$$\| |u_n|^r \|_1 \le C$$

and

$$\|\nabla u_n\|_{p(x)} \le C.$$

Proof As $f_n \in L^{m_-}(\Omega) \subset M^{m_-}(\Omega)$,

$$f_n^*(\tau)\tau^{\frac{1}{m_-}} \leq \tau^{\frac{1}{m_-}-1}\int_0^\tau f_n^*(\tau)\,\mathrm{d}\tau \leq \|f_n\|_{m_-}.$$

Take $r > 0$, inequality (3.9) implies

$$\int_0^{|\Omega|}\left(B(u_n^*(\widetilde{\sigma}))\right)^r d\widetilde{\sigma} \leq 2^{r+\frac{r}{p_--1}}\left(C_N^{-\frac{r}{N}}|\Omega|^{\frac{r}{N}+1}\right.$$
$$\left. + c_0^{\frac{r}{1-p_-}}\left(NC_N^{\frac{1}{N}}\right)^{-p'_-r}\int_0^{|\Omega|}\left(\int_{\widetilde{\sigma}}^{|\Omega|}\sigma^{-\frac{p'_-}{N'}}\left(\int_0^\sigma f_n^*(\tau)\,\mathrm{d}\tau\right)^{\frac{p'_-}{p_-}}\mathrm{d}\sigma\right)^r d\widetilde{\sigma}\right).$$

Note that $\overline{f}_n(\sigma) = \frac{1}{\sigma}\int_0^\sigma f_n^*(\tau)\,\mathrm{d}\tau$ is decreasing, it follows from Lemma 2.1 in [2] that

$$\int_0^{|\Omega|}\left(\int_{\widetilde{\sigma}}^{|\Omega|}\sigma^{-\frac{p'_-}{N'}}\left(\int_0^\sigma f_n^*(\tau)\,\mathrm{d}\tau\right)^{\frac{p'_-}{p_-}}\mathrm{d}\sigma\right)^r d\widetilde{\sigma} \leq C\int_0^{|\Omega|}(\overline{f}_n(\widetilde{\sigma}))^{\frac{p'_-r}{p_-}}\widetilde{\sigma}^{\frac{p'_-r}{p_-}+r-\frac{p'_-r}{N'}}d\widetilde{\sigma}$$
$$= C\int_0^{|\Omega|}\left(\int_0^{\widetilde{\sigma}} f_n^*(\tau)\,\mathrm{d}\tau\right)^{\frac{p'_-r}{p_-}}\widetilde{\sigma}^{\frac{p'_-r}{p_-}+r-\frac{p'_-r}{N'}-\frac{p'_-r}{p_-}}d\widetilde{\sigma} \leq C\int_0^{|\Omega|}f_n^*(\widetilde{\sigma})^{\frac{p'_-r}{p_-}}\widetilde{\sigma}^{\frac{p'_-r}{N}}d\widetilde{\sigma}.$$

Thus, we obtain

$$\int_0^{|\Omega|}(B(u_n^*(\widetilde{\sigma})))^r d\widetilde{\sigma}$$
$$\leq 2^{r+\frac{r}{p_--1}}\left(C_N^{-\frac{r}{N}}|\Omega|^{\frac{r}{N}+1} + c_0^{\frac{r}{1-p_-}}\left(NC_N^{\frac{1}{N}}\right)^{-p'_-r}C\int_0^{|\Omega|}f_n^*(\tau)^{\frac{p'_-r}{p_-}}\tau^{\frac{p'_-r}{N}}\,\mathrm{d}\tau\right)$$
$$\leq 2^{r+\frac{r}{p_--1}}\left(C_N^{-\frac{r}{N}}|\Omega|^{\frac{r}{N}+1} + c_0^{\frac{r}{1-p_-}}\left(NC_N^{\frac{1}{N}}\right)^{-p'_-r}C\|f_n\|_{m_-}^{\frac{p'_-r}{p_-}}\int_0^{|\Omega|}\tau^{\frac{p'_-r}{N}-\frac{p'_-r}{m_-p_-}}\,\mathrm{d}\tau\right). \tag{3.11}$$

In the following, we will verify that

$$B(u_n^*(\sigma)) = (B(|u_n|))^*(\sigma).$$

If $(B(|u_n|))^*(\sigma) = t$, we obtain $|\{x \in \Omega : B(|u_n(x)|) > t\}| = \sigma$, thus $|\{x \in \Omega : |u_n(x)| > B^{-1}(t)\}| = \sigma$. Moreover, we get $u_n^*(\sigma) = B^{-1}(t)$, i.e., $B(u_n^*(\sigma)) = t$.

Therefore, we obtain

$$\|(B(|u_n|))^r\|_1 = \|((B(|u_n|))^*)^r\|_1 = \|(B(u_n^*))^r\|_1 \leq C,$$

which implies

$$\left\| |u_n|^{\left(1-\frac{\theta(p_+-1)}{p_--1}\right)r} \right\|_1 \leq C. \tag{3.12}$$

As r is arbitrary, we obtain

$$\| |u_n|^r \|_1 \leq C.$$

Similarly to Lemma 3.2, we obtain

$$\frac{c_0}{k} \int\limits_{\{t<|u_n|\leq k+t\}} \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\theta(p(x)-1)}} \, \mathrm{d}x \leq \int\limits_{|u_n|>t} |f_n| \, \mathrm{d}x,$$

which follows

$$\frac{c_0}{k} \int\limits_{\{t<|u_n|\leq k+t\}} \frac{|\nabla u_n|^{p(x)}(1+|u_n|)^{-\theta(p_+-1)}}{(1+|u_n|)^{\theta(p(x)-1)-\theta(p_+-1)}} \, \mathrm{d}x \leq \int\limits_{|u_n|>t} |f_n| \, \mathrm{d}x.$$

Let $k \to 0+$, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{|u_n|\leq t} \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\theta(p(x)-p_+)}} \, \mathrm{d}x &\leq c_0^{-1}(1+t)^{\theta(p_+-1)} \int\limits_{|u_n|>t} |f_n| \, \mathrm{d}x \\ &= c_0^{-1}(1+t)^{\theta(p_+-1)} \int\limits_0^{\mu_{u_n(t)}} f_n^*(\sigma) \, \mathrm{d}\sigma, \end{aligned}$$

thus

$$\begin{aligned} &\int\limits_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{|u_n|\leq t} \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\theta(p(x)-p_+)}} \, \mathrm{d}x \, \mathrm{d}t \leq c_0^{-1} \int\limits_0^\infty (1+t)^{\theta(p_+-1)} \int\limits_0^{\mu_{u_n(t)}} f_n^*(\sigma) \, \mathrm{d}\sigma \, \mathrm{d}t \\ &= c_0^{-1} \int\limits_0^{|\Omega|} \int\limits_0^{u_n^*(\sigma)} (1+t)^{\theta(p_+-1)} f_n^*(\sigma) \, \mathrm{d}t \, \mathrm{d}\sigma \\ &= c_0^{-1} \int\limits_0^{|\Omega|} \frac{1}{\theta(p_+-1)+1} \left(\left(1+u_n^*(\sigma)\right)^{\theta(p_+-1)+1} - 1 \right) f_n^*(\sigma) \, \mathrm{d}\sigma \\ &\leq C \int\limits_0^{|\Omega|} u_n^*(\sigma)^{\theta(p_+-1)+1} f_n^*(\sigma) \, \mathrm{d}\sigma + C \int\limits_0^{|\Omega|} f_n^*(\sigma) \, \mathrm{d}\sigma \\ &\leq C \| f_n^* \|_{m_-} \left(\int\limits_0^{|\Omega|} u_n^*(\sigma)^{m'_-(\theta(p_+-1)+1)} \, \mathrm{d}\sigma \right)^{\frac{1}{m'_-}} + C \| f_n^* \|_{m_-}, \end{aligned}$$

which implies

$$\int_\Omega \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\theta(p(x)-p_+)}}\,\mathrm{d}x \le C,$$

then

$$\int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x \le \int_\Omega \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\theta(p(x)-p_+)}}\,\mathrm{d}x \le C.$$

Now, we get the result. □

Theorem 3.3 *Assume that* $0 \le \theta < \frac{p_- -1}{p_+ -1}$ *and* $1 \le m_- < \frac{N}{p_-}$. *Let* s *and* $q(x)$ *be as in* (1.4) *and* (1.7)*, respectively, and* u_n *be a weak solution of* (P_n)*, then there exists a constant* $C > 0$ *which depends on* $\Omega, \theta, m_-, N, p_+, p_-$*, and* $\|f_n\|_{m(x)}$ *such that*

$$\||u_n|^s\|_1 \le C.$$

Moreover, we obtain

(1) *if* m_- *satisfies* (1.3)*,* $\|\nabla u_n\|_{p(x)} \le C$;
(2) *if* m_- *satisfies* (1.6)*,* $\|\nabla u_n\|_{q(x)} \le C$;
(3) *if* m_- *satisfies* (1.8)*,* $\||\nabla u_n|^{q(x)}\|_1 \le C$ *and for any* $k > 0$*,* $\{T_k(u_n)\}$ *is bounded in* $W_0^{1,p(x)}(\Omega)$.

Proof Take $r = \frac{m_- N(p_- -1)}{N - m_- p_-}\alpha$, where $\alpha \in (0, 1)$. Inequality (3.11) implies

$$\|((B(|u_n|))^*)^r\|_1 \le C,$$

thus $\|(B(|u_n|))^r\|_1 \le C$. Moreover,

$$\||u_n|^s\|_1 \le C. \tag{3.13}$$

Therefore,

$$u_n^*(\sigma)^s\sigma \le \int_0^\sigma u_n^*(\tau)^s\,\mathrm{d}\tau \le \|(u_n^*)^s\|_1 = \||u_n|^s\|_1 \le C,$$

which implies

$$\|u_n\|_{M^s(\Omega)} \le C. \tag{3.14}$$

Similarly to Lemma 3.2, we obtain

$$\frac{c_0}{k}\int_{\{t<|u_n|\le k+t\}} \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\theta(p(x)-1)}}\,\mathrm{d}x \le \int_{|u_n|>t} |f_n|\,\mathrm{d}x,$$

which follows

$$\frac{c_0}{k}\int_{\{t<|u_n|\le k+t\}} \frac{|\nabla u_n|^{p(x)}(1+|u_n|)^{1-\frac{s}{m'_-}}}{(1+|u_n|)^{\theta(p(x)-1)+1-\frac{s}{m'_-}}}\,\mathrm{d}x \le \int_{|u_n|>t} |f_n|\,\mathrm{d}x.$$

Let $k \to 0+$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u_n|\le t}\frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\theta(p(x)-1)+1-\frac{s}{m'_-}}}\,\mathrm{d}x \le c_0^{-1}(1+t)^{\frac{s}{m'_-}-1}\int_{|u_n|>t}|f_n|\,\mathrm{d}x$$
$$= c_0^{-1}(1+t)^{\frac{s}{m'_-}-1}\int_0^{\mu_{u_n}(t)} f_n^*(\sigma)\,\mathrm{d}\sigma.$$

Similarly to Theorem 3.2, we obtain

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\int_{|u_n|\le t}\frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\theta(p(x)-1)+1-\frac{s}{m'_-}}}\,\mathrm{d}x\,\mathrm{d}t \le C\|f_n^*\|_{m_-}\left(\int_0^{|\Omega|}|u_n^*(\sigma)|^s\,\mathrm{d}\sigma\right)^{\frac{1}{m'_-}} + C\|f_n^*\|_{m_-},$$

which implies

$$\int_\Omega \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\theta(p_+-1)+1-\frac{s}{m'_-}}}\,\mathrm{d}x \le C.$$

(i) If $\frac{(\theta p_+-\theta+1)N+(p_--1-\theta p_++\theta)N\alpha}{(\theta p_+-\theta+1)p_-+(p_--1-\theta p_++\theta)N\alpha} \le m_- < \frac{N}{p_-}$, where $\alpha \in (0,1)$, we could derive

$$\theta(p_+-1)+1-\frac{s}{m'_-} \le 0,$$

then

$$\int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x \le \int_\Omega \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\theta(p_+-1)+1-\frac{s}{m'_1}}}\,\mathrm{d}x \le C.$$

(ii) If $\max\left\{1, \frac{(\theta p_+-\theta+1)N+(p_--1-\theta p_++\theta)N\alpha}{(\theta p_+-\theta+1)p_-+(p_--1-\theta p_++\theta)p_-N\alpha}\right\}$ $< m_- < \frac{(\theta p_+-\theta+1)N+(p_--1-\theta p_++\theta)N\alpha}{(\theta p_+-\theta+1)p_-+(p_--1-\theta p_++\theta)N\alpha}$, we obtain

$$\theta(p_+-1)+1-\frac{s}{m'_-} > 0$$

and

$$1 \le q(x) < p(x).$$

Thus, using Theorem 2.1 we obtain

$$\int_\Omega |\nabla u_n|^{q(x)}\,\mathrm{d}x$$
$$= \int_\Omega \frac{|\nabla u_n|^{q(x)}}{(1+|u_n|)^{\left(\theta(p_+-1)+1-\frac{s}{m'_-}\right)\frac{q(x)}{p(x)}}}(1+|u_n|)^{\left(\theta(p_+-1)+1-\frac{s}{m'_-}\right)\frac{q(x)}{p(x)}}\,\mathrm{d}x$$
$$\le C\left\|\frac{|\nabla u_n|^{q(x)}}{(1+|u_n|)^{\left(\theta(p_+-1)+1-\frac{s}{m'_-}\right)\frac{q(x)}{p(x)}}}\right\|_{\frac{p(x)}{q(x)}}\left\|(1+|u_n|)^{\left(\theta(p_+-1)+1-\frac{s}{m'_-}\right)\frac{q(x)}{p(x)}}\right\|_{\left(\frac{p(x)}{q(x)}\right)'}.$$

It follows from (1.4) and (1.7) that

$$s = \left(\theta(p_+ - 1) + 1 - \frac{s}{m'_-}\right) \frac{q(x)}{p(x) - q(x)},$$

which implies

$$\int_\Omega (1 + |u_n|)^{\left(\theta(p_+-1)+1-\frac{s}{m'_-}\right)\frac{q(x)}{p(x)}\left(\frac{p(x)}{q(x)}\right)'} \mathrm{d}x = \int_\Omega (1 + |u_n|)^s \, \mathrm{d}x \le C,$$

thus

$$\int_\Omega |\nabla u_n|^{q(x)} \, \mathrm{d}x \le C.$$

(iii) If $1 < m_- \le \max\left\{1, \frac{(\theta p_+ - \theta + 1)N + (p_- - 1 - \theta p_+ + \theta)N\alpha}{(\theta p_+ - \theta + 1)p_- + (p_- - 1 - \theta p_+ + \theta)p_- N\alpha}\right\}$. Similarly to the above discussion, we obtain

$$\int_\Omega |\nabla u_n|^{q(x)} \, \mathrm{d}x \le C.$$

In addition, we could verify

$$s < N' < m'_-.$$

Note that

$$\frac{c_0}{k} \int_{t<|u_n|\le k+t} \frac{|\nabla u_n|^{p(x)}}{(1 + |u_n|)^{\theta(p(x)-1)}} \mathrm{d}x \le \int_{|u_n|>t} |f_n| \, \mathrm{d}x.$$

Let $k \to 0+$, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{|u_n|\le t} |\nabla u_n|^{p(x)} \, \mathrm{d}x &\le c_0^{-1}(1+t)^{\theta(p_+-1)} \int_{|u_n|>t} |f_n| \, \mathrm{d}x \\ &= c_0^{-1}(1+t)^{\theta(p_+-1)} \int_0^{\mu_{u_n}(t)} f_n^*(\sigma) \, \mathrm{d}\sigma, \end{aligned}$$

then

$$\int_{|u_n|\le k} |\nabla u_n|^{p(x)} \, \mathrm{d}x \le c_0^{-1} \int_0^k (1+t)^{\theta(p_+-1)} \int_0^{\mu_{u_n}(t)} f_n^*(\sigma) \, d\sigma \, \mathrm{d}t.$$

Using Hölder inequality and (3.14), we obtain

$$\int_0^{\mu_{u_n}(t)} f_n^*(\sigma) \, \mathrm{d}\sigma \le (\mu_{u_n}(t))^{\frac{1}{m'_-}} \|f_n^*\|_{m_-} \le C t^{-\frac{s}{m'_-}},$$

thus

$$\int_{|u_n|\le k} |\nabla u_n|^{p(x)}\,\mathrm{d}x \le C\int_0^k (1+t)^{\theta(p_+-1)} t^{-\frac{s}{m'_-}}\,\mathrm{d}t$$
$$\le C(1+k)^{\theta(p_+-1)} k^{1-\frac{s}{m'_-}}$$
$$\le Ck^{\theta(p_+-1)-\frac{s}{m'_-}+1},$$

as k is large enough, i.e.,

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,\mathrm{d}x \le Ck^{\theta(p_+-1)-\frac{s}{m'_-}+1}. \tag{3.15}$$

(iv) If $m_- = 1$. It follows from (3.9) that

$$(B(|u_n|))^*(\widetilde{\sigma}) \le 2^{\frac{1}{p_- -1}} \times\left(C_N^{-\frac{1}{N}}|\Omega|^{\frac{1}{N}} + c_0^{\frac{1}{1-p_-}}\left(NC_N^{\frac{1}{N}}\right)^{-p'_-}\|f_n^*\|_1^{\frac{p'_-}{p_-}}\frac{N(p_- -1)}{N-p_-}\widetilde{\sigma}^{1-\frac{p'_-}{N'}}\right)$$
$$\le 2^{\frac{1}{p_- -1}}\widetilde{\sigma}^{1-\frac{p'_-}{N'}}\left(C_N^{-\frac{1}{N}}|\Omega|^{\frac{1}{N}+\frac{N-p_-}{N(p_- -1)}} + c_0^{\frac{1}{1-p_-}}\left(NC_N^{\frac{1}{N}}\right)^{-p'_-}\frac{N(p_- -1)}{N-p_-}\|f_n^*\|_1^{\frac{p'_-}{p_-}}\right),$$

thus $(B(|u_n|))^*(\widetilde{\sigma}) \le C\widetilde{\sigma}^{1-\frac{p'_-}{N'}}$, which implies

$$\|B(|u_n|)\|_{M^{\frac{N(p_- -1)}{N-p_-}}(\Omega)} \le C.$$

We obtain $\|u_n\|_{M^{\frac{N(p_- -1-\theta p_+ +\theta)}{N-p_-}}(\Omega)} \le C$, then

$$\||u_n|^s\|_1 \le C.$$

Similarly to the above discussion, we could obtain

$$\int_\Omega |\nabla u_n|^{q(x)}\,\mathrm{d}x \le C$$

and

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,\mathrm{d}x \le Ck^{\theta(p_+-1)+1}. \tag{3.16}$$

Now, we complete the proof. □

Remark 3.1 The definition of the function $B(t)$ in Lemma 3.2 implies the following:

(i) If $\theta > \frac{p_- -1}{p_+ -1}$, $B(t) \le \frac{1}{\frac{\theta(p_+-1)}{p_- -1}-1}$, for any $t \ge 0$;

(ii) If $\theta \le \frac{p_- -1}{p_+ -1}$, $\lim_{t\to\infty} B(t) = \infty$, i.e. $B(t)$ is unbounded.

In the proof of Theorems 3.1–3.3, we assume that $\theta \le \frac{p_- -1}{p_+ -1}$, thus, we could obtain (3.10), (3.12) and (3.13). We also observe that in Theorems 3.2 and 3.3, the case $\theta = \frac{p_- -1}{p_+ -1}$ is a limit case where it is not possible. In fact, from $\|(B(|u_n|))^r\|_1 \le C$ we can only get that $\{(\ln(1+|u_n|))^r\}$ is bounded in $L^1(\Omega)$.

4 Proof of the main results

In order to obtain the existence of solution for problem (P), firstly, we will prove the following result about the almost everywhere convergence of the gradients of the approximate solutions u_n, which allow us to pass to the limit in the approximate equations (P_n).

Lemma 4.1 *Let $f_n \in L^\infty(\Omega)$ be a sequence of functions which is strongly convergent to $f \in L^1(\Omega)$ and let u_n be the weak solution of (P_n) which converges to u almost everywhere in Ω. If*

(1) *for any $k > 0$, $T_k(u) \in W_0^{1,p(x)}(\Omega)$;*
(2) *there exists $r_1 > 0$ such that $\{|u_n|^{r_1}\}$ is bounded in $L^1(\Omega)$ and $|u|^{r_1} \in L^1(\Omega)$;*
(3) *there exists $r_2 > 0$ such that $\{|\nabla u_n|^{r_2}\}$ is bounded in $L^1(\Omega)$ and $|\nabla u|^{r_2} \in L^1(\Omega)$, then, up to a subsequence, ∇u_n converges to ∇u almost everywhere in Ω.*

Proof Take $\lambda \in (0,1)$. Denote

$$I_n = \int_\Omega \big((a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u))\nabla(u_n - u)\big)^\lambda \, dx,$$

using condition (a3), we obtain $I_n \ge 0$.

In the following, we will verify that $\lim_{n\to\infty} I_n = 0$. Take $k > 0$. By condition $(a2)$, the integral on $\{|u| \ge k\}$ gives

$$\begin{aligned} I^1_{n,k} &\triangleq \int_{\{|u|\ge k\}} \big((a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u))\nabla(u_n - u)\big)^\lambda \, dx \\ &\le C \int_{\{|u|\ge k\}} (g^{\lambda p'(x)} + |u_n|^{\lambda p(x)} + |\nabla u_n|^{\lambda p(x)} + |\nabla u|^{\lambda p(x)}) \, dx \\ &\le C \int_{\{|u|\ge k\}} (1 + g^{p'(x)} + |u_n|^{\lambda p_+} + |\nabla u_n|^{\lambda p_+} + |\nabla u|^{\lambda p_+}) \, dx. \end{aligned}$$

Take λ sufficiently small such that $\lambda p_+ < \min\{r_1, r_2\}$. By Hölder inequality, we obtain

$$\int_{\{|u|\ge k\}} |u_n|^{\lambda p_+} \, dx \le \left(\int_{|u|>k} |u_n|^{r_1} \right)^{\frac{\lambda p_+}{r_1}} |\{|u| \ge k\}|^{1-\frac{\lambda p_+}{r_1}}$$

and

$$\int_{\{|u|\ge k\}} |\nabla u_n|^{\lambda p_+} \, dx \le \left(\int_{|u|>k} |\nabla u_n|^{r_2} \right)^{\frac{\lambda p_+}{r_2}} |\{|u| \ge k\}|^{1-\frac{\lambda p_+}{r_2}}.$$

Note that $|u|^{r_1} \in L^1(\Omega)$, we obtain $u \in M^{r_1}(\Omega)$, i.e., for any $k > 1$,

$$|\{|u| \geq k\}| \leq |\{|u| > k-1\}| \leq \frac{C}{(k-1)^{r_1}}.$$

As $g^{p'(x)}$, $|\nabla u|^{\lambda p_+} \in L^1(\Omega)$ and $\{|u_n|^{r_1}\}$, $\{|\nabla u_n|^{r_2}\}$ are bounded in $L^1(\Omega)$, respectively, using the above inequalities and the absolute continuity of Lebesgue integral, we obtain

$$\lim_{k\to} \limsup_{n\to\infty} I^1_{n,k} = 0.$$

Denote

$$I^2_{n,k} = \int\limits_{\{|u|<k\}} \big((a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u))\nabla(u_n - u)\big)^{\lambda}\, \mathrm{d}x.$$

As $\nabla u = \nabla T_k(u)$ on the set $\{|u| < k\}$, we obtain

$$I^2_{n,k} \leq \int\limits_{\Omega} \big((a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla T_k(u)))\nabla(u_n - T_k(u))\big)^{\lambda}\, \mathrm{d}x.$$

Take $h > k+1$. We split the integral on the right side of the above inequality into two parts:

$$I^3_{n,k} = \int\limits_{\{|u_n - T_k(u)|\geq h\}} \big((a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla T_k(u)))\nabla(u_n - T_k(u))\big)^{\lambda}\, \mathrm{d}x,$$

$$I^4_{n,k} = \int\limits_{\{|u_n - T_k(u)|< h\}} \big((a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla T_k(u)))\nabla(u_n - T_k(u))\big)^{\lambda}\, \mathrm{d}x.$$

As $|u_n| \geq h - k$ on the set $\{|u_n - T_k(u)| \geq h\}$, we get

$$|\{|u_n - T_k(u)| \geq h\}| \leq |\{|u_n| \geq h-k\}| \leq \frac{C}{(h-k-1)^{r_1}}.$$

Similarly to the discussion of $I^1_{n,k}$, we obtain

$$\lim_{k\to\infty} \lim_{h\to\infty} \limsup_{n\to\infty} I^3_{n,k} = 0.$$

Note that $\nabla T_h(u_n - T_k(u)) = \nabla(u_n - T_k(u))$ on the set $\{|u_n - T_k(u)| < h\}$, by Hölder inequality, we derive

$$I^4_{n,k} \leq |\Omega|^{1-\lambda} \left(\int\limits_{\Omega} (a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla T_k(u)))\nabla T_h(u_n - T_k(u))\, \mathrm{d}x \right)^{\lambda}$$

Next, we will verify

$$\lim_{k\to\infty} \lim_{h\to\infty} \limsup_{n\to\infty} \int\limits_{\Omega} (a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla T_k(u)))\nabla T_h(u_n - T_k(u))\, \mathrm{d}x = 0,$$

which implies

$$\lim_{k\to\infty} \lim_{h\to\infty} \limsup_{n\to\infty} I^4_{n,k} = 0.$$

Denote

$$I_{n,k}^5 = \int_\Omega a(x, T_n(u_n), \nabla u_n)\nabla T_h(u_n - T_k(u))\,\mathrm{d}x$$

and

$$I_{n,k}^6 = \int_\Omega a(x, T_n(u_n), \nabla T_k(u))\nabla T_h(u_n - T_k(u))\,\mathrm{d}x.$$

As u_n is a weak solution for (P_n), we get

$$\int_\Omega a(x, T_n(u_n), \nabla u_n)\nabla T_h(u_n - T_k(u))\,\mathrm{d}x = \int_\Omega f_n T_h(u_n - T_k(u))\,\mathrm{d}x.$$

As $f_n \to f$ in $L^1(\Omega)$, we obtain $\lim_{n\to\infty} I_{n,k}^5 = \int_\Omega f T_h(u - T_k(u))\,\mathrm{d}x$.

For almost every $x \in \Omega$, there exists $k_0 > 0$ such that $|u(x)| < k_0$. If $k > k_0$, $T_k(u(x)) = u(x)$, thus

$$\lim_{k\to\infty}\lim_{h\to\infty}\lim_{n\to\infty} I_{n,k}^5 = 0.$$

Take n sufficiently large such that $n > h+k$. As $|u_n| < h+k$ on the set $\{|u_n - T_k(u)| < h\}$, we obtain

$$I_{n,k}^6 = \int_\Omega a(x, T_{h+k}(u_n), \nabla T_k(u))\nabla T_h(u_n - T_k(u))\,\mathrm{d}x.$$

By condition (a2), we obtain

$$|a(x, T_{h+k}(u_n), \nabla T_k(u))| \le c_1 \left(g(x) + (h+k)^{p(x)-1} + |\nabla T_k(u)|^{p(x)-1}\right).$$

Note that

$$a(x, T_{h+k}(u_n), \nabla T_k(u)) \to a(x, u, \nabla T_k(u)) \quad \text{a.e. in } \Omega,$$

using Lebesgue dominated convergence theorem, we derive

$$a(x, T_{h+k}(u_n), \nabla T_k(u)) \to a(x, u, \nabla T_k(u)) \quad \text{in } (L^{p'(x)}(\Omega))^N, \quad \text{as } n \to \infty. \tag{4.1}$$

As u_n is a weak solution for (P_n), choose $v = T_{k+h}(u_n)$ as a test function in (3.1), we obtain that $\{T_{k+h}(u_n)\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. It follows that

$$\int_\Omega |\nabla T_h(u_n - T_k(u))|^{p(x)}\,\mathrm{d}x \le C \int_{|u_n|\le k+h} (|\nabla u_n|^{p(x)} + |\nabla T_k(u)|^{p(x)})\,\mathrm{d}x \le C.$$

Thus, $\{T_h(u_n - T_k(u))\}$ is bounded in $W_0^{1,p(x)}(\Omega)$, passing to a subsequence, still denoted by $\{T_h(u_n - T_k(u))\}$, we assume that

$$T_h(u_n - T_k(u)) \to v \quad \text{weakly in } W_0^{1,p(x)}(\Omega), \quad \text{as } n \to \infty.$$

By Theorem 2.2, we get $T_h(u_n - T_k(u)) \to v$ in $L^{p(x)}(\Omega)$. Moreover, we assume $T_h(u_n - T_k(u)) \to v$ a.e. in Ω, as $n \to \infty$. Thus, $v = T_h(u - T_k(u))$ and

$$T_h(u_n - T_k(u)) \to T_h(u - T_k(u)) \quad \text{weakly in } W_0^{1,p(x)}(\Omega),$$

as $n \to \infty$. Using (4.1), we obtain

$$\lim_{n\to\infty} I_{n,k}^6 = \int_\Omega a(x, u, \nabla T_k(u))\nabla T_h(u - T_k(u))\,\mathrm{d}x,$$

thus $\lim_{k\to\infty}\lim_{h\to\infty}\lim_{n\to\infty} I_{n,k}^6 = 0$.

Combining with the above discussion, we get $\limsup_{n\to\infty} I_n = 0$, thus

$$\lim_{n\to\infty} I_n = 0,$$

which implies

$$((a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u))\nabla(u_n - u))^\lambda \to 0 \quad \text{in } L^1(\Omega).$$

Passing to a subsequence, we assume that

$$(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u))\nabla(u_n - u) \to 0 \quad \text{a.e. in } \Omega. \tag{4.2}$$

Next, we will verify that $\nabla u_n \to \nabla u$ a.e. in Ω. Firstly, we claim that for almost every $x \in \Omega$, the sequence $\{|\nabla u_n(x)|\}$ is bounded. In fact, on the contrary, there exists a subsequence, still denoted by $\{|\nabla u_n(x)|\}$, such that

$$|\nabla u_n(x) - \nabla u(x)| > 1, \qquad \frac{\nabla u_n(x) - \nabla u(x)}{|\nabla u_n(x) - \nabla u(x)|} \to \xi^* \neq 0,$$

as $n \to \infty$.

In the following, for the sake of simplicity, we omit the dependence of u_n and u on x. By condition (a3), we get

$$\left(a(x, T_n(u_n), \nabla u_n) - a\left(x, T_n(u_n), \nabla u + \frac{\nabla u_n - \nabla u}{|\nabla u_n - \nabla u|}\right)\right)\left(\nabla u_n - \nabla u - \frac{\nabla u_n - \nabla u}{|\nabla u_n - \nabla u|}\right) \geq 0,$$

which implies

$$\left(a(x, T_n(u_n), \nabla u_n) - a\left(x, T_n(u_n), \nabla u + \frac{\nabla u_n - \nabla u}{|\nabla u_n - \nabla u|}\right)\right)(\nabla u_n - \nabla u) \geq 0.$$

Note that

$$\begin{aligned}
0 \leq &\left(a\left(x, T_n(u_n), \nabla u + \frac{\nabla u_n - \nabla u}{|\nabla u_n - \nabla u|}\right) - a(x, T_n(u_n), \nabla u)\right)\frac{\nabla u_n - \nabla u}{|\nabla u_n - \nabla u|} \\
= &\left(a\left(x, T_n(u_n), \nabla u + \frac{\nabla u_n - \nabla u}{|\nabla u_n - \nabla u|}\right) - a(x, T_n(u_n), \nabla u_n)\right)\frac{\nabla u_n - \nabla u}{|\nabla u_n - \nabla u|} \\
&+ (a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u))\frac{\nabla u_n - \nabla u}{|\nabla u_n - \nabla u|} \\
\leq &(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u))\frac{\nabla u_n - \nabla u}{|\nabla u_n - \nabla u|} \\
\leq &(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u))(\nabla u_n - \nabla u),
\end{aligned}$$

using (4.2), we obtain

$$\left(a\left(x, T_n(u_n), \nabla u + \frac{\nabla u_n - \nabla u}{|\nabla u_n - \nabla u|}\right) - a(x, T_n(u_n), \nabla u)\right)\frac{\nabla u_n - \nabla u}{|\nabla u_n - \nabla u|} \to 0.$$

For almost every $x \in \Omega$, $u_n(x) \to u(x)$, which implies $|u_n(x)| \leq |u(x)| + 1 \leq n$ for n large enough, thus $T_n(u_n(x)) = u_n(x)$. We obtain

$$\left(a\left(x, u_n, \nabla u + \frac{\nabla u_n - \nabla u}{|\nabla u_n - \nabla u|}\right) - a(x, u_n, \nabla u)\right) \frac{\nabla u_n - \nabla u}{|\nabla u_n - \nabla u|} \to 0,$$

as $n \to \infty$, thus

$$(a(x, u, \nabla u + \xi^*) - a(x, u, \nabla u))\xi^* = 0,$$

this contradiction proves our claim. Therefore, there exists a subsequence, still denoted by $\{|\nabla u_n(x)|\}$ such that

$$\nabla u_n(x) \to \xi,$$

as $n \to \infty$. It follows from (4.2) that

$$(a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u))\nabla(u_n - u) \to 0,$$

which implies

$$(a(x, u, \xi) - a(x, u, \nabla u))(\xi - \nabla u) = 0.$$

By condition (a3), we obtain $\xi = \nabla u(x)$, thus

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega, \quad \text{as } n \to \infty.$$

Therefore, we get the result. □

In the following, we are going to prove Theorems 1.1–1.5.

As $C_0^\infty(\Omega)$ is dense in $(L^{m(x)}(\Omega), \|\cdot\|_{m(x)})$ (see Theorem 1.8 in [10]), there exists $\{f_n\} \subset C_0^\infty(\Omega)$ such that

$$f_n \to f \quad \text{in } L^{m(x)}(\Omega).$$

Then, there exists a weak solution u_n for (P_n) in the sense of (3.1).

Firstly, we will verify that $\{u_n\}$ satisfies the assumptions of Lemma 4.1.

For any $k > 0$, choose $v = T_k(u_n)$ as a test function in (3.1), we obtain

$$\int_\Omega a(x, T_n(u_n), \nabla u_n)\nabla T_k(u_n)\,\mathrm{d}x = \int_\Omega f_n T_k(u_n)\,\mathrm{d}x.$$

By condition (a1), it is easy to verify that $\{T_k(u_n)\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. We assume that $T_k(u_n) \to v$ weakly in $W_0^{1,p(x)}(\Omega)$, as $n \to \infty$. Using Theorem 2.2, we obtain $T_k(u_n) \to v$ in $L^{p(x)}(\Omega)$. Moreover, we assume that $T_k(u_n) \to v$ a.e. in Ω, as $n \to \infty$.

Let s be as in (1.4), it follows from Theorems 3.1–3.3 that $\{u_n\}$ is bounded in $M^s(\Omega)$. Then, there exists $C > 0$ such that for any $k > 0$,

$$|\{|u_n| > k\}| \leq \frac{C}{k^s},$$

we get

$$\lim_{k\to\infty} \limsup_{n\to\infty} |\{|u_n| > k\}| = 0.$$

Take $t > 0$, we obtain

$$\{|u_n - u_m| > t\} \subset \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|T_k(u_n) - T_k(u_m)| > t\},$$

thus

$$|\{|u_n - u_m| > t\}| \leq |\{|u_n| > k\}| + |\{|u_m| > k\}| + |\{|T_k(u_n) - T_k(u_m)| > t\}|.$$

Note that

$$\begin{aligned}
&|\{|T_k(u_n) - T_k(u_m)| > t\}| \\
&\quad \leq \int\limits_{\{|T_k(u_n)-T_k(u_m)|>t\}} \left|\frac{T_k(u_n) - T_k(u_m)}{t}\right|^{p(x)} \mathrm{d}x \\
&\quad \leq (t^{-p_+} + t^{-p_-}) \int\limits_{\{|T_k(u_n)-T_k(u_m)|>t\}} |T_k(u_n) - T_k(u_m)|^{p(x)} \mathrm{d}x
\end{aligned}$$

thus

$$\lim_{m,n\to\infty} |\{|T_k(u_n) - T_k(u_m)| > t\}| = 0,$$

which implies

$$\lim_{m,n\to\infty} |\{|u_n - u_m| > t\}| = 0,$$

i.e. $\{u_n\}$ is a Cauchy sequence in measure. Furthermore, there exists a measurable function u such that

$$u_n \to u \quad \text{in measure and a.e. in } \Omega,$$

as $n \to \infty$, then $v = T_k(u) \in W_0^{1,p(x)}(\Omega)$. Moreover,

$$T_k(u_n) \to T_k(u) \quad \text{weakly in } W_0^{1,p(x)}(\Omega),$$

as $n \to \infty$.

(1) There exists $r_1 > 0$ such that $\{|u_n|^{r_1}\}$ is bounded in $L^1(\Omega)$ and $|u|^{r_1} \in L^1(\Omega)$. In fact, it follows from Theorems 3.1–3.3 that

$$\int\limits_\Omega |u_n|^s \,\mathrm{d}x \leq C,$$

where s is as in (1.4). By Fatou Lemma, we obtain

$$\int\limits_\Omega |u|^s \,\mathrm{d}x \leq \liminf_{n\to\infty} \int\limits_\Omega |u_n|^s \,\mathrm{d}x \leq C.$$

Thus $|u|^s \in L^1(\Omega)$, furthermore, $u \in M^s(\Omega)$.

(2) There exists $r_2 > 0$ such that $\{|\nabla u_n|^{r_2}\}$ is bounded in $L^1(\Omega)$ and $|\nabla u|^{r_2} \in L^1(\Omega)$.

(i) If $m_- \geq \frac{(\theta p_+ - \theta + 1)N + (p_- - 1 - \theta p_+ + \theta)N\alpha}{(\theta p_+ - \theta + 1)p_- + (p_- - 1 - \theta p_+ + \theta)N\alpha}$, where $\alpha \in (0, 1)$.

It follows from Theorems 3.1, 3.2, and 3.3(1) that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. As $W_0^{1,p(x)}(\Omega)$ is reflexive, passing to a subsequence, we assume that $u_n \to \widetilde{u}$ weakly in $W_0^{1,p(x)}(\Omega)$, as $n \to \infty$. By Theorem 2.2, $u_n \to \widetilde{u}$ in $L^{p(x)}(\Omega)$. Furthermore, we assume that $u_n \to \widetilde{u}$ a.e. in Ω, thus $\widetilde{u} = u$.

Note that $\int_\Omega |\nabla v|^{p(x)}\,\mathrm{d}x$ is convex and continuously differentiable on $W_0^{1,p(x)}(\Omega)$, we obtain

$$\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x \le \liminf_{n\to} \int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x \le C.$$

(ii) If $\max\left\{1, \frac{(\theta p_+-\theta+1)N+(p_--1-\theta p_++\theta)N\alpha}{(\theta p_+-\theta+1)p_-+(p_--1-\theta p_++\theta)p_-N\alpha}\right\}$
$< m_- < \frac{(\theta p_+-\theta+1)N+(p_--1-\theta p_++\theta)N\alpha}{(\theta p_+-\theta+1)p_-+(p_--1-\theta p_++\theta)N\alpha}$.

It follows from Theorem 3.3(2) that $\{u_n\}$ is bounded in $W_0^{1,q(x)}(\Omega)$. Similarly to (i), up to a subsequence, we assume that $u_n \to u$ weakly in $W_0^{1,q(x)}(\Omega)$, then

$$\int_\Omega |\nabla u|^{q(x)}\,\mathrm{d}x \le \liminf_{n\to} \int_\Omega |\nabla u_n|^{q(x)}\,\mathrm{d}x \le C.$$

(iii) If $1 < m_- \le \max\left\{1, \frac{(\theta p_+-\theta+1)N+(p_--1-\theta p_++\theta)N\alpha}{(\theta p_+-\theta+1)p_-+(p_--1-\theta p_++\theta)p_-N\alpha}\right\}$.

Using (3.15), we obtain

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,\mathrm{d}x \le Ck^{\theta(p_+-1)-\frac{s}{m_-'}+1},$$

thus

$$\int_\Omega |\nabla T_k(u)|^{p(x)}\,\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega |\nabla T_k(u_n)|^{p(x)}\,\mathrm{d}x \le Ck^{\theta(p_+-1)-\frac{s}{m_-'}+1}.$$

For any $t > 1$, we have

$$\begin{aligned} |\{|\nabla u| > t\}| &\le |\{|\nabla u| > t, |u| \le k\}| + |\{|\nabla u| > t, |u| > k\}| \\ &\le \int_{\{|\nabla u|>t,|u|\le k\}} \left|\frac{\nabla u}{t}\right|^{p(x)} \mathrm{d}x + \frac{C}{k^s} \\ &\le C\left(k^{\theta(p_+-1)-\frac{s}{m_-'}+1} t^{-p_-} + k^{-s}\right). \end{aligned}$$

Take the minimization of $h(k) = k^{\theta(p_+-1)-\frac{s}{m_-'}+1} t^{-p_-} + k^{-s}$, we obtain

$$|\{|\nabla u| > t\}| \le Ct^{-\gamma},$$

where $\gamma = \frac{sp_-}{\theta(p_+-1)-\frac{s}{m_-'}+1+s}$, thus $|\nabla u| \in M^\gamma(\Omega)$. Furthermore, there exists $0 < \widetilde{\gamma} < \gamma$, such that

$$|\nabla u|^{\widetilde{\gamma}} \in L^1(\Omega).$$

(iv) If $m_- = 1$.

By (3.16), we get

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,\mathrm{d}x \le Ck^{\theta(p_+-1)+1},$$

similarly to (iii), there exists $\overline{\gamma} > 0$, such that

$$|\nabla u|^{\overline{\gamma}} \in L^1(\Omega).$$

Combining with the above discussion, it follows from Lemma 4.1 that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega, \quad \text{as } n \to \infty. \tag{3.3}$$

The Proof of Theorems 1.1–1.3 As u_n is a weak solution for (P_n) in the sense of (3.1), for any $v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, we derive

$$\int_\Omega a(x, T_n(u_n), \nabla u_n)\nabla v \, dx = \int_\Omega f_n v \, dx.$$

As $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$, using condition (a2), we get $\{a(x, T_n(u_n), \nabla u_n)\}$ is bounded in $(L^{p'(x)}(\Omega))^N$. Note that $a(x, T_n(u_n), \nabla u_n) \to a(x, u, \nabla u)$ a.e. in Ω, then $a(x, T_n(u_n), \nabla u_n) \to a(x, u, \nabla u)$ weakly in $(L^{p'(x)}(\Omega))^N$. Moreover, we obtain

$$\int_\Omega a(x, u, \nabla u)\nabla v \, dx = \int_\Omega f v \, dx,$$

i.e., u is a solution for (P) in the sense of (1.2). □

The Proof of Theorems 1.4–1.5 For any $T \in \mathcal{T}$ and $\phi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, choose $v = T(u_n - \phi)$ as a test function in (3.1), we obtain

$$\int_\Omega a(x, T_n(u_n), \nabla u_n)\nabla T(u_n - \phi) \, dx = \int_\Omega f_n T(u_n - \phi) \, dx.$$

Take $s_0 > 0$ such that $T'(s) = 0$, for any $s \geq s_0$. Denote $M = \|\phi\|_\infty + s_0$ and take $n > M$, we get

$$\begin{aligned}
&\int_\Omega a(x, T_n(u_n), \nabla u_n)\nabla T(u_n - \phi) \, dx \\
&\quad = \int_{\{|u_n - \phi| < s_0\}} a(x, u_n, \nabla u_n)\nabla(u_n - \phi) \cdot T'(u_n - \phi) \, dx \\
&\quad = \int_\Omega a(x, u_n, \nabla u_n)\nabla u_n \cdot T'(u_n - \phi) \, dx - \int_\Omega a(x, u_n, \nabla u_n)\nabla \phi \cdot T'(u_n - \phi) \, dx.
\end{aligned}$$

By Fatou Lemma,

$$\int_\Omega a(x, u, \nabla u)\nabla u \cdot T'(u - \phi) \, dx \leq \liminf_{n \to \infty} \int_\Omega a(x, u_n, \nabla u_n)\nabla u_n \cdot T'(u_n - \phi) \, dx.$$

Using condition (a2), we obtain

$$|a(x, u_n, \nabla u_n)T'(u_n - \phi)| \leq C(g(x) + M^{p(x)-1} + |\nabla T_M(u_n)|^{p(x)-1}),$$

thus $\{|a(x, u_n, \nabla u_n)T'(u_n - \phi)|\}$ is bounded in $L^{p'(x)}(\Omega)$. Note that $a(x, u_n, \nabla u_n)T'(u_n - \phi) \to a(x, u, \nabla u)T'(u - \phi)$ a.e. in Ω, we get

$$a(x, u_n, \nabla u_n)T'(u_n - \phi) \to a(x, u, \nabla u)T'(u - \phi) \quad \text{weakly in } (L^{p'(x)}(\Omega))^N,$$

thus

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla\phi\cdot T'(u_n-\phi)\,\mathrm{d}x \to \int_\Omega a(x,u,\nabla u)\nabla\phi\cdot T'(u-\phi)\,\mathrm{d}x.$$

It is easy to verify that

$$\int_\Omega f_n T(u_n-\phi)\,\mathrm{d}x \to \int_\Omega f T(u-\phi)\,\mathrm{d}x,$$

therefore,

$$\int_\Omega a(x,u,\nabla u)\nabla(u-\phi)T'(u-\phi)\,\mathrm{d}x \le \int_\Omega f T(u-\phi)\,\mathrm{d}x,$$

i.e., u is an entropy solution for (P_n).

Moreover, using (3.3) and Fatou Lemma, we obtain

$$\int_\Omega |\nabla u|^{q(x)}\,\mathrm{d}x \le \liminf_{n\to\infty}\int_\Omega |\nabla u_n|^{q(x)}\,\mathrm{d}x \le C.$$

Now, we get the result. □

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