Annali di Matematica (2014) 193:115–131
DOI 10.1007/s10231-012-0269-7

Higher differentiability for the solutions of nonlinear elliptic systems with lower-order terms and $L^{1,\theta}$-data

G. R. Cirmi · S. Leonardi

Received: 13 January 2012 / Accepted: 10 March 2012 / Published online: 24 March 2012

Abstract In this paper we study the higher differentiability of the solutions to a class of nonlinear systems of elliptic partial differential equations with a lower-order term having natural growth with respect to the gradient and with data belonging to a suitable Morrey space.

Keywords Nonlinear elliptic systems · L^1-data · Differentiability

Mathematics Subject Classification 35J25 · 35D10

1 Introduction

In this paper we study the higher differentiability of the weak solutions of a class of nonlinear elliptic systems whose model is

$$\begin{cases} -\text{div}\,[(s^2+|Du|^2)^{\frac{p-2}{2}}Du]+u|Du|^p=f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ ($n \geq 3$) with sufficiently regular boundary, $p \in [2, n[$, $u : \Omega \to \mathbb{R}^N$ ($N \geq 1$) is the unknown vector, $s \geq 0$ is a constant and $f \in L^1(\Omega, \mathbb{R}^N)$ is a vector-valued function belonging to a suitable Morrey space.

The existence of a weak solution with finite energy (that is $u \in W_0^{1,p}(\Omega, \mathbb{R}^N)$) for systems whose prototype is (1) has been proved by Bensoussan and Boccardo in [2], assuming that the main part of the operator satisfies the so-called "Landes condition" (see [9]), which

G. R. Cirmi · S. Leonardi (✉)
Dipartimento di Matematica ed Informatica, Università degli Studi di Catania, Viale A. Doria 6, 95125 Catania, Italy
e-mail: leonardi@dmi.unict.it

G. R. Cirmi
e-mail: cirmi@dmi.unict.it

amounts to a sort of diagonal structure of the system, and that the lower-order term verifies a sign (or angle) condition (see below for the precise statements of the assumptions).

Recently, Mingione (see [10]) has investigated the differentiability properties of the distributional solutions of a nonlinear elliptic equation ($N = 1$) of the type

$$-\text{div}\,[(s^2 + |Du|^2)^{\frac{p-2}{2}} Du] = \mu$$

where s is a non-negative constant, $p > 1$ and μ is a signed Radon measure with finite total variation $|\mu|(\Omega) < +\infty$ enjoying the following density condition

$$|\mu|(B_R) \leq M R^{n-\theta}, \quad \text{for some } M > 0,\ \theta \in [0, n]$$

for any ball $B_R \subset \Omega$.

This differentiability result has been extended to the very weak solutions of non-diagonal linear elliptic systems ($N \geq 2$) without lower-order terms in [5].

It is the aim of the present paper to prove similar differentiability properties for the usual weak solutions to systems of nonlinear elliptic equations, under the Landes condition, with lower-order terms having natural (or critical) growth with respect to the gradient and satisfying a sign condition.

Namely, at first, we will establish that if f belongs to the Morrey space $L^{1,\theta}(\Omega, \mathbb{R}^N)$, with $\theta \in [0, n[$, then any weak solution of that problem (1) has the property that the following expression of the gradient

$$V(Du) = (s^2 + |Du|^2)^{\frac{p-2}{4}} Du$$

belongs to the Morrey space $L^{2,\theta}_{loc}(\Omega, \mathbb{R}^{nN})$ (see also [4]). This Morrey regularity property in turn will allow us to gain that Du belongs to the space $L^{p,\theta}_{loc}(\Omega, \mathbb{R}^{nN})$ and that it has fractional derivatives in $L^{p}_{loc}(\Omega, \mathbb{R}^{nN})$.

Our result turns out to be optimal for this class of systems. As a matter of fact, as shown in the Remark 7, the differentiability of a solution fails whether $\theta = n$, that is under the sole requirement that f is just in $L^1(\Omega, \mathbb{R}^N)$, while in the case of the operator without lower-order term a small amount of differentiability still holds (see [10]).

2 Notations and results

In $\mathbb{R}^n$ ($n \geq 3$), with generic point $x = (x_1, x_2, \dots, x_n)$, we shall denote by Ω a bounded open non-empty set with diameter d_Ω and $C^{0,1}$-boundary $\partial\Omega$.

For $R > 0$ and $x^0 \in \mathbb{R}^n$, we define

$$\begin{aligned}
B_R(x^0) &= \{x \in \mathbb{R}^n : |x - x^0| < R\}, \\
\Omega(x^0, R) &= \Omega \cap B_R(x^0), \\
Q_R(x^0) &= \{x \in \mathbb{R}^n : \sup_{1 \leq i \leq n} |x_i - x_i^0| < R\}, \\
d(x^0, \partial\Omega) &= \text{dist}(x^0, \partial\Omega).
\end{aligned}$$

We shall often use the short notation B_R and Q_R, instead of $B_R(x^0)$ and $Q_R(x^0)$, respectively, when no ambiguity will arise.

Moreover, if $u \in L^1(B, \mathbb{R}^N),\ N \geq 1$, and $0 < |B| < +\infty$ ([1]), we denote by

$$u_B := \frac{1}{|B|} \int_B u(x)\,\mathrm{d}x$$

Now, let us define the functional spaces we will use. We propose a modification of the usual definitions, essentially equivalent, to simplify the treatment in the following.

Definition 1 (*Morrey space*) Let $q \geq 1$ and $\theta \in [0, n]$. By $L^{q,\theta}(\Omega, \mathbb{R}^N)$, we denote the space of all vector functions $u \in L^q(\Omega, \mathbb{R}^N)$ such that

$$\|u\|_{L^{q,\theta}(\Omega)} = \sup_{B_R \subset \Omega, R \leq 1} \left\{ R^{\theta - n} \int_{B_R} |u(x)|^q \mathrm{d}x \right\}^{1/q}$$

is finite. $L^{q,\theta}(\Omega, \mathbb{R}^N)$ equipped with the above norm is a Banach space.

Now, we recall some basic facts about fractional-order Sobolev spaces.

Definition 2 (*Fractional Sobolev space*) Let $t \in]0, 1]$ and $q \geq 1$. $W^{t,q}(\Omega, \mathbb{R}^N)$ is the space of all vector functions $u \in L^q(\Omega, \mathbb{R}^N)$ such that

$$\|u\|_{W^{t,q}(\Omega \mathbb{R}^N)} = \|u\|_{L^q(\Omega, \mathbb{R}^N)} + [u]_{t,q,\Omega} < +\infty$$

where

$$[u]_{t,q,\Omega} = \begin{cases} \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^q}{|x - y|^{n+tq}} \mathrm{d}x\, \mathrm{d}y \right)^{\frac{1}{q}} & \text{if } t < 1 \\ \|Du\|_{L^q(\Omega)} & \text{if } t = 1. \end{cases}$$

Here, Du represents the gradient of the vector-valued function u; that is,

$$Du \equiv \left(\frac{\partial u^\nu}{\partial x_i} \right)_{\nu=1,\ldots,N;\, i=1,\ldots,n} \equiv (D_i\, u^\nu)_{\nu=1,\ldots,N;\, i=1,\ldots,n}.$$

The following result is the Sobolev's embedding theorem in the case of fractional space (see [1] and also Lemma 3 of [6]).

Theorem 1 (Fractional Sobolev embedding) *Let Ω be a domain of $\mathbb{R}^n$ with $C^{0,1}$ boundary, $q \geq 1$ and $t \in]0, 1]$ such that $tq < n$. Then,*

$$W^{t,q}(\Omega, \mathbb{R}^N) \subset L^{\frac{nq}{n-tq}}(\Omega, \mathbb{R}^N)$$

with continuous embedding.

Moreover, the following proposition extends the classical Poincaré's inequality to the case of fractional Sobolev space (see [10] and related references).

Proposition 1 (Fractional Poincaré Inequality) *Let $B_R,\ R > 0$, be a ball in $\mathbb{R}^n$ and $u \in W^{t,q}(B_R, \mathbb{R}^N),\ t \in]0, 1[,\ q \geq 1$. Then,*

[1] $|B|$ is the n-dimensional Lebesgue measure of set B.

$$\int_{B_R} |u(x) - u_{B_R}|^q dx \leq c(n) R^{tq} \int_{B_R}\int_{B_R} \frac{|u(x) - u(y)|^q}{|x - y|^{n+tq}} \mathrm{d}x\, \mathrm{d}y([2]).$$

Given a vector-valued function $\omega : \Omega \to \mathbb{R}^N$ and a real number h, for any $i = 1, \ldots, n$, we define the finite difference operator τ_{ih} as

$$\tau_{ih}(\omega)(x) = \omega(x + he_i) - \omega(x),$$

for $x \in \Omega$ such that $x + he_i \in \Omega$, where $\{e_i\}_{i=1,\ldots,n}$ denotes the canonical basis of $\mathbb{R}^n$.

Lemma 1 *Let $u \in L^q(\Omega, \mathbb{R}^N)$, $q \geq 1$. Assume that there exist $\bar{t} \in]0, 1]$, $S > 0$ and an open set $\bar{\Omega} \subset\subset \Omega$ such that*

$$\|\tau_{ih}(u)\|_{L^q(\bar{\Omega})} \leq S|h|^{\bar{t}}$$

for every $1 \leq i \leq n$ and every $h \in \mathbb{R}$ satisfying $0 < |h| \leq \min\{1, dist(\bar{\Omega}, \partial\Omega)\}$.
Then,

$$u \in W^{t,q}_{loc}(\bar{\Omega}, \mathbb{R}^N) \quad \text{for every} \quad t \in]0, \bar{t}[$$

and for every open set $A \subset\subset \bar{\Omega}$ there exists a constant c, independent of S and u, such that

$$\|u\|_{W^{t,q}(A)} \leq c\left[S + \|u\|_{L^q(A)}\right].$$

We denote by $A(x, \xi)$ a matrix-valued function whose entries are the functions

$$A_i^\nu : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$$

for $i = 1, \ldots, n$ and $\nu = 1, \ldots, N$. Each entry is a Carathéodory functions (i.e., continuous in $\xi \in \mathbb{R}^{nN}$ for a.e. $x \in \Omega$ and measurable in x for every ξ) satisfying the following conditions for a.e. $x \in \Omega$, for every non-negative real number s and for every $\xi, \eta \in \mathbb{R}^{nN}$ such that $\xi \neq \eta$ ([3]):

$$\exists \Lambda_1 > 0 : (A_i^\nu(x, \xi) - A_i^\nu(x, \eta))(\xi_i^\nu - \eta_i^\nu) \geq \Lambda_1 (s^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2, \quad (2)$$

$$\exists\, \Lambda_2 > 0 : |A(x, \xi)| \leq \Lambda_2 (s^2 + |\xi|^2)^{\frac{p-2}{2}} |\xi|, \quad p \in [2, n[, \quad (3)$$

$$A_i^\nu(x, 0) = 0, \quad (4)$$

$$A_i^\nu(x, \xi)\left[\xi_i^\nu |\gamma|^2 - \gamma^\nu \gamma^\mu \xi_i^\mu\right] \geq 0 \quad \forall \gamma \in \mathbb{R}^N. \quad (5)$$

Remark 1 Since $p \geq 2$, the assumption (2) implies the strong monotonicity assumption

$$\left(A_i^\nu(x, \xi) - A_i^\nu(x, \eta)\right)(\xi_i^\nu - \eta_i^\nu) \geq c(\Lambda_1, p)\, |\xi - \eta|^p. \quad (6)$$

For $s \geq 0$, we set

$$V(\xi) \equiv V_s(\xi) := (s^2 + |\xi|^2)^{\frac{p-2}{4}} \xi \quad \forall \xi \in \mathbb{R}^{nN} \quad (7)$$

The assumptions (2) and (4) imply the ellipticity condition

$$A_i^\nu(x, \xi)\xi_i^\nu \geq \Lambda_1 |V(\xi)|^2, \text{ a.e. } x \in \Omega, \ \forall \xi \in \mathbb{R}^{nN}. \quad (8)$$

[2] As a permanent convention, we will denote by $c(\cdot, \ldots, \cdot)$ a positive constant which depends on various parameters.

[3] We assume the use of Einstein's convention throughout the paper.

Moreover, from Lemma 2.1 of [7] and (2) (see also [10]), we have the following properties

$$c(n,p)^{-1}(s^2+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}} \le \frac{|V(\xi)-V(\eta)|^2}{|\xi-\eta|^2} \le c(n,p)(s^2+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}, \tag{9}$$

$$(A_i^\nu(x,\xi)-A_i^\nu(x,\eta))(\xi_i^\nu-\eta_i^\nu) \ge c(\Lambda_1,n,p)\,|V(\xi)-V(\eta)|^2. \tag{10}$$

The assumption (3) and Young's inequality yield

$$|A(x,\xi)| \le c(\Lambda_2,p)\,(s^2+|\xi|^2)^{\frac{p-1}{2}}. \tag{11}$$

Remark 2 The assumption (5) is the so-called Landes condition. Note that it is automatically implied by (8), whenever $N=1$.

For $\nu=1,\ldots,N$ let $g^\nu:\Omega\times\mathbb{R}^N\times R^{nN}\to\mathbb{R}$ be Carathéodory functions and denote by $g(x,u,\xi)$ the vector-valued function whose ν-th component is g^ν. For $g(x,u,\xi)$, we will assume the following conditions for a.e. $x\in\Omega$, for every $u\in\mathbb{R}^N$ and for every $\xi\in\mathbb{R}^{nN}$:

$$|g(x,u,\xi)| \le b(|u|)\,[d(x)+|\xi|^p], \tag{12}$$

and

$$|g(x,u,\xi)| \ge \sigma\,|V(\xi)|^2 \quad \forall u\in\mathbb{R}^N : |u|\ge 1. \tag{13}$$

where $b(\cdot)$ is a real valued, positive, increasing and continuous function, $d(x)$ is a non-negative function in $L^{1,\theta}(\Omega,\mathbb{R}^N)$, $\theta\in[p,n[$, s is a non-negative real number and σ is a positive real number.

Moreover, we assume the following angle condition

$$g^\nu(x,u,\xi)(u^\nu-\tau^\nu)\ge 0, \quad \forall\tau,u\in\mathbb{R}^N : |\tau|\le|u| \tag{14}$$

which amounts to a sign condition in the scalar case $N=1$.

We consider the following system

$$\begin{cases} u\in W_0^{1,p}(\Omega,\mathbb{R}^N), \quad g(x,u,Du)\in L^1(\Omega,\mathbb{R}^N)\\ -D_iA_i^\nu(x,Du)+g^\nu(x,u,Du)=f^\nu \end{cases} \tag{15}$$

where, for any $\nu=1,\ldots,N$, f^ν denotes the ν-th component of the vector

$$f\in L^{1,\theta}(\Omega,\mathbb{R}^N), \quad \theta\in[p,n[. \tag{16}$$

By a *weak solution* of the system of equations (15), we mean a vector-valued function $u\in W_0^{1,p}(\Omega,\mathbb{R}^N)$ such that

$$\begin{cases} g(x,u(x),Du(x))\in L^1(\Omega,\mathbb{R}^N)\\ \int_\Omega A_i^\nu(x,Du)\,D_iv^\nu\,\mathrm{d}x+\int_\Omega g^\nu(x,u,Du)\,v^\nu\,\mathrm{d}x=\int_\Omega f^\nu\,v^\nu\,\mathrm{d}x\\ \forall\,v\in W_0^{1,p}(\Omega,\mathbb{R}^N)\cap L^\infty(\Omega,\mathbb{R}^N). \end{cases} \tag{17}$$

In [2], the following result has been proved

Theorem 2 *Let assumptions* (2)*,* (3)*,* (4)*,* (5)*,* (12)*,* (13)*,* (14) *be satisfied and let* $f\in L^1(\Omega,\mathbb{R}^N)$*. Then, there exists a weak solution* $u\in W_0^{1,p}(\Omega,\mathbb{R}^N)$ *of the problem* (15)*.*

Here, we prove the following

Theorem 3 *Let assumptions* (2)*,* (3)*,* (4)*,* (5)*,* (12)*,* (13)*,* (14)*,* (16) *be satisfied and let* $u \in W_0^{1,p}(\Omega, \mathbb{R}^N)$ *be a weak solution of the problem* (15)*.*

Then,

$$V(Du) \in L^{2,\theta}_{loc}(\Omega, \mathbb{R}^{nN}), \quad Du \in L^{p,\theta}_{loc}(\Omega, \mathbb{R}^{nN}), \tag{18}$$

and for any $\Omega' \subset\subset \Omega$, there exist two positive constants c_1 and c_2, depending only on data, such that

$$\|V(Du)\|_{L^{2,\theta}(\Omega')} \le c_1, \tag{19}$$

and

$$\|Du\|_{L^{p,\theta}(\Omega')} \le c_2. \tag{20}$$

Remark 3 The particular scalar case (i.e., $N = 1$) has been studied in [3].

Remark 4 The previous Morrey regularity result holds as well if $\theta \in [0, p[$ assuming also $u \in L^\infty(\Omega, \mathbb{R}^N)$. In the case $N = 1$, the boundedness of u has been proved in [3].

To prove the differentiability of a weak solution u, we shall require the following Hölder continuity assumption on the map $x \to A(x, \xi)$:

$$\begin{cases} \text{there exist } L > 0 \text{ and } \eta \in]0,1] \text{ such that} \\ |A(x,\xi) - A(x_0,\xi)| \le L|x - x_0|^\eta (s^2 + |\xi|^2)^{\frac{p-1}{2}}, \quad \forall x, x_0 \in \Omega, \xi \in \mathbb{R}^{nN}. \end{cases} \tag{21}$$

Theorem 4 *Let the assumptions* (2)*,* (3)*,* (4)*,* (5)*,* (12)*,* (13)*,* (14)*,* (16)*,* (21) *be satisfied and let* $u \in W_0^{1,p}(\Omega, \mathbb{R}^N)$ *be a weak solution of the problem* (15)*. Set*

$$\delta = \min\left\{1, \frac{n-\theta}{2}\right\}. \tag{22}$$

Then,

$$V(Du) \in W^{t,2}_{loc}(\Omega, \mathbb{R}^{nN}), \quad Du \in W^{2t/p,p}_{loc}(\Omega, \mathbb{R}^{nN}) \tag{23}$$

for every $t \in [0, \eta\delta[$.

Moreover, for every couple of open subset $\Omega' \subset\subset \Omega'' \subset\subset \Omega$*, there exist two positive constants* c_1 *and* c_2*, independent on u, such that*

$$[V(Du)]^2_{W^{t,2}(\Omega')} \le c_1 \left[\int_{\Omega''} (s^p + |Du|^p)\,dx + \|V(Du)\|^2_{L^{2,\theta}(\Omega'')}\right] \tag{24}$$

and

$$[Du]^p_{W^{2t/p,p}(\Omega')} \le c_2 \left[\int_{\Omega''} (s^p + |Du|^p)\,dx + \|Du\|^p_{L^{p,\theta}(\Omega'')}\right]. \tag{25}$$

Remark 5 In the case $\theta \in [0, p[$, the differentiability result stated above holds for the bounded weak solutions of the problem (15).

Remark 6 As a consequence of the fractional Sobolev embedding Theorem 1, we gain a better integrability on Du. Namely,

$$Du \in L_{loc}^{\frac{pn}{n-2t}}(\Omega, \mathbb{R}^{nN}) \text{ for every } t \in [0, \eta\delta[$$

where δ is the number defined in (22).

Given a vector $u \in \mathbb{R}^N$ and a real number $k > 0$ let us denote by $T_k(u)$, the vector-valued function whose components are defined by

$$[T_k(u)]^\nu = \begin{cases} u^\nu & \text{if } |u| \le k \\ k\dfrac{u^\nu}{|u|} & \text{if } |u| > k \end{cases} \tag{26}$$

for $\nu = 1, \dots, N$. Obviously

$$|T_k(u)| \le k, \quad |T_k(u)| \le |u| \quad \forall u \in \mathbb{R}^N, \quad \forall k \in \mathbb{R}^+.$$

Moreover, if $v \in W_0^{1,p}(\Omega, \mathbb{R}^N)$, then $T_k(v) \in W_0^{1,p}(\Omega, \mathbb{R}^N)$, and for any $i = 1, \dots, n$ and $\nu = 1, \dots, N$, we have

$$D_i[T_k(v)]^\nu = \begin{cases} D_i v^\nu & \text{if } |v| \le k \\ \dfrac{k}{|v|}\left[D_i v^\nu - \dfrac{1}{|v|^2} v^\nu v^\mu D_i v^\mu\right] & \text{if } |v| > k \end{cases}$$

see [9].

Remark 7 The regularity result stated in Theorem 4 fails if $f \in L^1(\Omega, \mathbb{R}^N)$. Indeed, let $n \ge 3,\ \theta = n,\ N = 2$ (or N=1), $\Omega = B(0, 1/2)$ and, for a. e. $x \in B(0, 1/2)$, define

$$u(x) = \left(\int_{|x|}^{1/2} \frac{1}{\rho^{n/2}|log\rho|} \mathrm{d}\rho, \int_{|x|}^{1/2} \frac{1}{\rho^{n/2}|log\rho|} \mathrm{d}\rho \right).$$

We can easily prove that $u \in W_0^{1,2}(B(0, 1/2), \mathbb{R}^2)$ is a solution of the Dirichlet problem associated with the system

$$-\Delta u + T_1(u)|Du|^2 = f(x)$$

where

$$f(x) = \frac{1 - (n/2 - 1)log|x|}{|x|^{n/2+1}log^2|x|} + T_1(u(x))|Du(x)|^2.$$

Easy calculations show that

$$Du \notin L_{loc}^{2,\theta}(\Omega, \mathbb{R}^{2n}) \text{ for any } \theta \in]0, n[,$$

this implies that the vector-valued function f belongs to $L^1(\Omega, \mathbb{R}^2)$ but doesn't belong to $L^{1,\theta}(\Omega, \mathbb{R}^2)$ for any $\theta \in]0, n[$.

Moreover,

$$Du \notin W_{loc}^{t,2}(\Omega, \mathbb{R}^{2n}) \text{ for any } t \in]0, 1[$$

since otherwise, being $W_{loc}^{t,2}(\Omega, \mathbb{R}^{2n}) \subset L_{loc}^{\frac{2n}{n-2t}}(\Omega, \mathbb{R}^{2n})$, it would be $Du \in L_{loc}^{2,n-2t}(\Omega, \mathbb{R}^{2n})$.

3 Proofs of the main results

In this section, we prove our results and we recall, among others, some well-known results on the weak solutions to homogeneous elliptic systems.

The following Lemma, whose proof can be found in [8, Lemma 3.3] or in [10, Lemma 3.2], concerns the $W^{2,p}$-regularity of the weak solutions u of homogeneous elliptic systems with coefficients depending only on Du.

Namely, we denote by $A_0(\xi)$ a matrix-valued function whose entries are the continuous functions

$$A_{0i}^{\nu} : \mathbb{R}^{nN} \to \mathbb{R}$$

for $i = 1, \dots, n$ and $\nu = 1, \dots, N$. Each entry satisfies the following conditions for every non-negative real number s and for every $\xi, \eta \in \mathbb{R}^{nN}$ such that $\xi \neq \eta$:

$$(A_{0i}^{\nu}(\xi) - A_{0i}^{\nu}(\eta))(\xi_i^{\nu} - \eta_i^{\nu}) \geq \alpha_0 \, (s^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \, |\xi - \eta|^2, \tag{27}$$

$$|A_0(\xi)| \leq \beta_0 \, (s^2 + |\xi|^2)^{\frac{p-2}{2}} \, |\xi|, \tag{28}$$

$$A_{0i}^{\nu}(0) = 0. \tag{29}$$

with α_0 and β_0 positive constants.

Lemma 2 *Let $v_0 \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution to the system*

$$div \, A_0(Dv_0) = 0 \ \ in \ \Omega.$$

Then,

$$V(Dv_0) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^{nN})$$

and there exists a constant $c = c(n, N, p, \Lambda_1, \Lambda_2) > 0$ such that for every $z_0 \in \mathbb{R}^{nN}$ and every ball $B_R \subset\subset \Omega$ we have

$$\int_{B_{R/2}} |DV(Dv_0)|^2 \, dx \leq \frac{c}{R^2} \int_{B_R} |V(Dv_0) - V(z_0)|^2 \, dx. \tag{30}$$

We can continue proving Theorem 3.

Proof of Theorem 3 It is enough to prove that, for any $R \leq 1$ for which $B_R \subset \Omega$, the integral

$$\frac{1}{R^{n-\theta}} \int_{B_R} |V(Du)|^2 \, dx$$

is bounded.

Let $\psi \in C^{\infty}(\mathbb{R}^n)$ be the standard cut-off function of the ball B_{2R}, that is

$$\begin{array}{ll} 0 \leq \psi(x) \leq 1 & \text{if } \ x \in \overline{B_{2R}} \\ \psi(x) = 1 & \text{if } \ x \in B_R \\ \psi(x) = 0 & \text{if } \ x \in \mathbb{R}^n \setminus \overline{B_{2R}} \\ |D\psi(x)| \leq \frac{c}{R} & \text{if } \ x \in \overline{B_{2R}} \setminus B_R \ . \end{array}$$

Let us take as test function in (17) the function

$$v = \psi^p T_1(u)$$

where $T_1(u)$ is the vector-valued function defined by (26). We obtain

$$\int_\Omega \psi^p A_i^\nu(x, Du)\, D_i[T_1(u)]^\nu\, dx + p\int_\Omega \psi^{p-1} A_i^\nu(x, Du)\, [T_1(u)]^\nu\, D_i\psi\, dx$$
$$+\int_\Omega \psi^p g^\nu(x, u, Du)\, [T_1(u)]^\nu\, dx = \int_\Omega \psi^p f^\nu\, [T_1(u)]^\nu\, dx. \tag{31}$$

Note that

$$\begin{aligned} I_1 &\equiv \int_\Omega \psi^p A_i^\nu(x, Du)\, D_i[T_1(u)]^\nu\, dx \\ &= \int_{\Omega\cap\{|u|\le 1\}} \psi^p A_i^\nu(x, Du)\, D_i u^\nu\, dx \\ &\quad + \int_{\Omega\cap\{|u|>1\}} \psi^p A_i^\nu(x, Du)\frac{1}{|u|}\left[D_i u^\nu - \frac{1}{|u|^2} u^\nu u^\mu D_i u^\mu\right]. \end{aligned} \tag{32}$$

By virtue of (5), the last integral in (32) is non-negative, and thus, using the assumptions (8) and (6), we estimate I_1 from above as follows

$$I_1 \ge c(\Lambda_1, p) \int_{\Omega\cap\{|u|\le 1\}} \psi^p (|V(Du)|^2 + |Du|^p)\, dx. \tag{33}$$

On the other hand, exploiting (11) and Young's inequality, we obtain

$$\begin{aligned} I_2 &\equiv p\int_\Omega \psi^{p-1} A_i^\nu(x, Du)\, [T_1(u)]^\nu D_i\psi\, dx \\ &\ge -p\int_\Omega \psi^{p-1} |A^\nu(x, Du)|\, |D\psi||[T_1(u)]^\nu|\, dx \\ &\ge -\varepsilon \int_\Omega \psi^p\, (s^2 + |Du|^2)^{\frac{p}{2}}\, |T_1(u)|\, dx - c(\varepsilon, \Lambda_2, p)\int_\Omega |D\psi|^p\, |T_1(u)|\, dx \\ &\ge -\varepsilon \int_\Omega \psi^p\, (|V(Du)|^2 + |Du|^p) dx - \varepsilon\, s^p\, R^n - c(\varepsilon, \Lambda_2, p, n)\, R^{n-p}. \end{aligned} \tag{34}$$

Moreover, by (14) we deduce,

$$\begin{aligned} I_3 &\equiv \int_\Omega \psi^p g^\nu(x, u, Du)\, [T_1(u)]^\nu\, dx \\ &= \int_{\Omega\cap\{|u|\le 1\}} \psi^p g^\nu(x, u, Du)\, u^\nu\, dx + \int_{\Omega\cap\{|u|>1\}} \psi^p g^\nu(x, u, Du)\, \frac{u^\nu}{|u|}\, dx \\ &\ge \int_{\Omega\cap\{|u|>1\}} \psi^p g^\nu(x, u, Du)\, \frac{u^\nu}{|u|}\, dx. \end{aligned} \tag{35}$$

Note that the angle condition (14) implies

$$g^\nu(x,u,Du)\,\frac{u^\nu}{|u|} \geq |g(x,u,Du)| \ \text{ for } \quad a.e. x \in \Omega \quad \text{and} \quad \text{for } |u| > 1 \tag{36}$$

so that, plugging (13) and (36) in (35) and observing that

$$|V(\xi)|^2 \geq |\xi|^p, \tag{37}$$

we get

$$I_3 \geq \sigma/2 \int\limits_{\Omega\cap\{|u|>1\}} \psi^p(|V(Du)|^2 + |Du|^p)\,\mathrm{d}x. \tag{38}$$

Gathering together (31), (33), (34) and (38), we obtain

$$\begin{aligned}(c(\Lambda_1,p)-\varepsilon) &\int\limits_{\Omega\cap\{|u|\leq 1\}} \psi^p(|V(Du)|^2 + |Du|^p)\,\mathrm{d}x \\ +(\sigma/2-\varepsilon) &\int\limits_{\Omega\cap\{|u|>1\}} \psi^p(|V(Du)|^2 + |Du|^p)\,\mathrm{d}x \\ \leq R^{n-\theta}&\|f\|_{L^{1,\theta}(\Omega)} + [\varepsilon\, s^p\, R^p + c(\varepsilon,\Lambda_2,p,n)]\,R^{n-p}\end{aligned} \tag{39}$$

whence, choosing a suitable $\varepsilon > 0$ and taking into account that $\theta \geq p$, we have

$$\int\limits_{B_R} (|V(Du)|^2+|Du|^p)\,\mathrm{d}x \leq R^{n-\theta}\left[(c(\Lambda_1,\Lambda_2,\sigma,p,n,\theta)+s^p d_\Omega^p)d_\Omega^{\theta-p}+\|f\|_{L^{1,\theta}(\Omega)}\right] \tag{40}$$

which, by a covering argument, implies the assertions (19) and (20).

We are now in position to prove Theorem 4. □

Proof of Theorem 4 We shall follow the outline of Lemma 6.2 of [10]. Let $B \subset\subset \Omega$ be a ball of radius R and let $\hat{B}$ be the enlarged ball of radius $16R$. We shall denote by $Q_{inn}(B)$ and $Q_{out}(B)$ the largest and the smallest cubes, concentric to B and with sides parallel to the coordinate axes, contained in B and containing B, respectively. If we put

$$Q_{\mathrm{inn}} = Q_{\mathrm{inn}}(B), \quad Q_{\mathrm{out}} = Q_{\mathrm{out}}(B)$$

and

$$\hat{Q}_{\mathrm{inn}} = Q_{\mathrm{inn}}(\hat{B}), \quad \hat{Q}_{\mathrm{out}} = Q_{\mathrm{out}}(\hat{B}),$$

we have the following inclusions

$$Q_{\mathrm{inn}} \subset B \subset\subset 4B \subset\subset \hat{Q}_{\mathrm{inn}} \subset \hat{B} \subset \hat{Q}_{\mathrm{out}}. \tag{41}$$

Let Ω' and Ω'' be a couple of open subset such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ and $x^0 \in \Omega'$. For any $\beta \in]0,1[$ (that will be chosen later) we fix $h \in \mathbb{R}$ with $0 < |h| << \min\{1, d(\Omega', \partial\Omega'')\}$ such that, denoted with $B = B(x^0, |h|^\beta)$ the ball centered in x^0 and with radius $|h|^\beta$, the outer cube of B, $\hat{Q}_{out}$ is included in Ω''.

Step 1

Let $v \in W^{1,p}(\hat{B}, \mathbb{R}^N)$ be the unique weak solution to the problem

$$\begin{cases} \operatorname{div} \ A(x, Dv) = 0 \text{ in } \ \hat{B} \\ v = u \qquad\qquad\qquad \text{on } \ \partial \hat{B}\,, \end{cases} \tag{42}$$

and let $v_0 \in W^{1,p}(8B, \mathbb{R}^N)$ be the unique weak solution to the problem

$$\begin{cases} \operatorname{div} \ A(x^0, Dv_0) = 0 \text{ in } \ 8B \\ v_0 = v \qquad\qquad\qquad \text{on } \ \partial 8B. \end{cases} \tag{43}$$

Then, we have

$$\int_B |\tau_{ih}(V(Du))|^2 \, \mathrm{d}x \le c \left[\int_B |\tau_{ih}(V(Dv_0))|^2 \, \mathrm{d}x + \int_{\hat{B}} |V(Du) - (Dv)|^2 \, \mathrm{d}x \right. \\ \left. + \int_{8B} |V(Dv) - V(Dv_0)|^2 \, \mathrm{d}x \right]. \tag{44}$$

First of all we estimate

$$\int_{\hat{B}} |V(Du) - V(Dv)|^2 \, \mathrm{d}x.$$

Let us observe that the function $w = v - u \in W_0^{1,p}(\hat{B}, \mathbb{R}^N)$ is a weak solution to the system

$$\begin{cases} \operatorname{div} \ A(x, D(w+u)) = 0 \text{ in } \ \hat{B} \\ w = 0 \qquad\qquad\qquad \text{on } \ \partial \hat{B}. \end{cases} \tag{45}$$

For such a solution we have

$$\int_{\hat{B}} A(x, D(w+u))Dw \, \mathrm{d}x - \int_{\hat{B}} A(x, Du)Dw \, \mathrm{d}x = - \int_{\hat{B}} A(x, Du)Dw \, \mathrm{d}x$$

whence, by (10), (3) and (9), we deduce

$$\begin{aligned} \int_{\hat{B}} |V(Du) - V(Dv)|^2 \, \mathrm{d}x &= \int_{\hat{B}} |V(D(u+w)) - V(Du)|^2 \, \mathrm{d}x \\ &\le \int_{\hat{B}} |A(x, Du)||Dw| \, \mathrm{d}x \\ &\le \Lambda_2 \int_{\hat{B}} (s^2 + |Du|^2)^{\frac{p-2}{2}} |Du||Dw| \, \mathrm{d}x \\ &\le \Lambda_2 \int_{\hat{B}} (s^2 + |Dv|^2 + |Du|^2)^{\frac{p-2}{4}} \, |D(u-v)| \, (s^2 + |Du|^2)^{\frac{p-2}{4}} |Du| \, \mathrm{d}x \end{aligned}$$

$$\leq \varepsilon \int\limits_{\hat{B}} (s^2 + |Dv|^2 + |Du|^2)^{\frac{p-2}{2}} |D(u-v)|^2 \, dx + c(\varepsilon) \int\limits_{\hat{B}} (s^2 + |Du|^2)^{\frac{p-2}{2}} |Du|^2 \, dx$$

$$\leq \varepsilon \int\limits_{\hat{B}} |V(Du) - V(Dv)|^2 \, dx + c(\varepsilon, p) \int\limits_{\hat{B}} |V(Du)|^2 \, dx \tag{46}$$

with c positive constant independent of the radius of $\hat{B}$.

In turn, for a sufficiently small ε, inequality (46) yields

$$\int\limits_{\hat{B}} |V(Du) - V(Dv)|^2 \, dx \leq c \int\limits_{\hat{B}} |V(Du)|^2 \, dx. \tag{47}$$

Now, we estimate

$$\int\limits_{8B} |V(Dv) - V(Dv_0)|^2 \, dx.$$

Since v and v_0 are the weak solutions, respectively, to the Dirichlet problems (42) and (43), the following integral identities hold

$$\int\limits_{8B} A(x, Dv) D(v - v_0) \, dx = 0 \tag{48}$$

and

$$\int\limits_{8B} A(x^0, Dv_0) D(v - v_0) \, dx = 0. \tag{49}$$

By virtue of the strong monotonicity condition (10) and using (21), (48), (49), Young's inequality and (9) we deduce

$$\begin{aligned}
&\int\limits_{8B} |V(Dv) - V(Dv_0)|^2 \, dx \\
&\quad \leq \frac{1}{c(\Lambda_1, n, p)} \int\limits_{8B} [A(x^0, Dv) - A(x^0, Dv_0)] D(v - v_0) \, dx \\
&\quad = \frac{1}{c} \int\limits_{8B} [A(x^0, Dv) - A(x, Dv)] D(v - v_0) \, dx \\
&\quad \leq \frac{1}{c} \int\limits_{8B} |A(x^0, Dv) - A(x, Dv)| |D(v - v_0)| \, dx \\
&\quad \leq \frac{L}{c} \int\limits_{8B} |x - x^0|^\eta \, (s^2 + |Dv|^2)^{\frac{p-1}{2}} \, |D(v - v_0)| \, dx \\
&\quad \leq \frac{L}{c} \left[c(\epsilon) |h|^{2\beta\eta} \int\limits_{8B} (s^p + |Dv|^p) \, dx + \epsilon \int\limits_{8B} |V(Dv) - V(v_0)|^2 \, dx \right].
\end{aligned}$$

Choosing ϵ sufficiently small, it follows

$$\int\limits_{8B} |V(Dv) - V(Dv_0)|^2 \, dx \leq c |h|^{2\eta\beta} \int\limits_{8B} (s^p + |Dv|^p) \, dx$$

consequently, using again the inequalities (37) and (47), we deduce

$$\int_{8B} |V(Dv) - V(Dv_0)|^2 \, dx \le c|h|^{2\eta\beta} \int_{\hat{B}} (s^p + |Du|^p) \, dx. \tag{50}$$

Summing up (47) and (50), we have

$$\int_{8B} |V(Du) - V(Dv_0)|^2 \, dx \le c \left[|h|^{2\eta\beta} \int_{\hat{B}} (s^p + |Du|^p) \, dx + \int_{\hat{B}} |V(Du)|^2 \, dx \right]. \tag{51}$$

Now, we have to estimate the first integral in the right-hand side of (44). Using a well-known result about the translation operator, we have

$$\int_{B} |\tau_{ih}(V(Dv_0))|^2 \, dx \le |h|^2 \int_{4B} |DV(Dv_0)|^2 \, dx$$

and by virtue of the estimate (30), we obtain

$$\int_{B} |\tau_{ih}(V(Dv_0))|^2 \, dx \le c|h|^{2(1-\beta)} \int_{8B} |V(Dv_0) - V(z_0)|^2 \, dx, \tag{52}$$

for every $z_0 \in \mathbb{R}^{nN}$.

Step 2.

Let us assume, for a moment, that there exists some $\bar{t} \in [0, \eta\delta[$ such that

$$V(Du) \in W^{\bar{t},2}_{loc}(\Omega, \mathbb{R}^{nN}), \tag{53}$$

and that for every couple of open subset $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, there exists a positive constant $\bar{c}$ depending on $dist(\Omega', \partial\Omega'')$ such that

$$[V(Du)]^2_{W^{\bar{t},2}(\Omega')} \le \bar{c} \left[\int_{\Omega''} (s^p + |Du|^p) \, dx \; + \; \|V(Du)\|^2_{L^{2,\theta}(\Omega'')} \right]. \tag{54}$$

We claim that

$$V(Du) \in W^{t,2}_{loc}(\Omega, \mathbb{R}^{nN}) \quad \text{for every} \quad t \in [0, \gamma(\bar{t})[, \tag{55}$$

where $\gamma(t) = \frac{\eta\delta}{1-t+\eta\delta}$, and that for every couple of open subset $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ there exists a positive constant $c = c(n, \Lambda_1, \Lambda_2, \sigma, p, N, \beta, \eta, L, d(\Omega', \partial\Omega''))$ such that

$$[V(Du)]^2_{W^{t,2}(\Omega')} \le c \left[\int_{\Omega''} (s^p + |Du|^p) \, dx \; + \|V(Du)\|^2_{L^{2,\theta}(\Omega'')} \right]. \tag{56}$$

As a matter of the fact, if $\bar{t} = 0$, we choose $z_0 = 0$ in (52) and we have

$$\begin{aligned} \int_{B} |\tau_{ih}(V(Dv_0))|^2 \, dx &\le c|h|^{2(1-\beta)} \int_{8B} |V(Dv_0)|^2 \, dx \\ &\le c|h|^{2(1-\beta)} \left(\int_{8B} |V(Dv_0) - V(Du)|^2 \, dx + \int_{8B} |V(Du)|^2 \, dx \right). \end{aligned} \tag{57}$$

Using inequality (51), we deduce

$$\int_B |\tau_{ih}(V(Dv_0))|^2 \le c|h|^{2(1-\beta)} \left[|h|^{2\eta\beta} \int_{\hat{B}} (s^p + |Du|^p)\,\mathrm{d}x + \int_{\hat{B}} |V(Du)|^2\,\mathrm{d}x \right]. \tag{58}$$

In the case $\bar{t} \in]0, \eta\delta[$, we choose $z_0 = V^{-1}((V(Du))_{8B})$ in (52), and again exploiting (51), we deduce

$$\begin{aligned}
\int_B |\tau_{ih}(V(Dv_0))|^2\,\mathrm{d}x &\le c|h|^{2(1-\beta)} \int_{8B} |V(Dv_0) - (V(Du))_{8B}|^2\,\mathrm{d}x \\
&\le c|h|^{2(1-\beta)} \left(\int_{8B} |V(Dv_0) - V(Du)|^2\,\mathrm{d}x + \int_{8B} |V(Du) - (V(Du))_{8B}|^2\,\mathrm{d}x \right) \\
&\le c|h|^{2(1-\beta)} \left[|h|^{2\eta\beta} \int_{\hat{B}} (s^p + |Du|^p)\,\mathrm{d}x + \int_{\hat{B}} |V(Du)|^2\,\mathrm{d}x \right. \\
&\quad \left. + \int_{8B} |V(Du) - (V(Du))_{8B}|^2\,\mathrm{d}x \right].
\end{aligned}$$

We estimate the last integral in the right-hand side by fractional Poincaré inequality and we obtain

$$\begin{aligned}
\int_B |\tau_{ih}(Dv_0)|^2 \le c|h|^{2(1-\beta)} &\left[|h|^{2\eta\beta} \int_{\hat{B}} (s^p + |Du|^p)\,\mathrm{d}x + \int_{\hat{B}} |V(Du)|^2\,\mathrm{d}x \right. \\
&\left. + |h|^{2\beta\bar{t}} \, [V(Du)]^2_{W^{\bar{t},2}(\hat{B})} \right].
\end{aligned} \tag{59}$$

We observe that the inequalities (58) and (59) may be summarize as follows

$$\begin{aligned}
\int_B |\tau_{ih}(Dv_0)|^2 \le c|h|^{2(1-\beta)} &\left[|h|^{2\eta\beta} \int_{\hat{B}} (s^p + |Du|^p)\,\mathrm{d}x \right. \\
&\left. + \int_{\hat{B}} |V(Du)|^2\,\mathrm{d}x + \chi(\bar{t})|h|^{2\beta\bar{t}} \, [V(Du)]^2_{W^{\bar{t},2}(\hat{B})} \right],
\end{aligned}$$

where $\chi(t) = 0$ if $t = 0$ and $\chi(t) = 1$ if $t > 0$.

Moreover, since $\bar{t} < \eta$ and $|h| < 1$, from the previous estimate, we deduce

$$\int_B |\tau_{ih}(V(Dv_0))|^2 \le c|h|^{2(1-\beta)+2\beta\bar{t}} \lambda(\hat{B}) + \int_{\hat{B}} |V(Du)|^2\,\mathrm{d}x \tag{60}$$

where λ is the set function defined by

$$\lambda(A) = \int_A (s^p + |Du|^p) + \chi(\bar{t})\,[V(Du)]^2_{W^{\bar{t},2}(A)} \tag{61}$$

for any measurable set $A \subset \Omega$.
We remark that the function λ is countable super-additive.
Gathering together (44), (47), (50) and (60), we find

$$\int_B |\tau_{ih}(V(Du))|^2\,\mathrm{d}x \le c\left[\left(|h|^{2(1-\beta)+2\beta\bar{t}} + |h|^{2\beta\eta}\right)\lambda(\hat{B}) + \int_{\hat{B}} |V(Du)|^2\,\mathrm{d}x\right]. \tag{62}$$

Now we use a covering argument as in [10]. Firstly, we take a lattice of cubes with equal side length, comparable to $|h|^\beta$, sides parallel to the coordinate axes, and centered in Ω', and we consider them as the inner cubes of the balls $B_j = B(x_j, |h|^\beta)$, with $x_j \in \Omega'$. By the compactness property of Ω' and the Vitali's covering Theorem, we can find $\bar{J} = \bar{J}(h) \in \mathbb{N}$ such that

$$\left|\Omega' \setminus \bigcup_{i=1}^{\bar{J}} Q_{inn}(B_i)\right| = 0 \quad \text{and} \quad Q_{inn}(B_i) \cap Q_{inn}(B_j) = \emptyset \text{ if } i \neq j.$$

Then, using (62), we obtain

$$\begin{aligned}\int_{\Omega'} |\tau_{ih}(V(Du))|^2\,\mathrm{d}x &= \sum_{i=1}^{\bar{J}} \int_{Q_{inn}(B_i)} |\tau_{ih}(V(Du))|^2\,\mathrm{d}x \le \sum_{i=1}^{\bar{J}} \int_{B_i} |\tau_{ih}(V(Du))|^2\,\mathrm{d}x \\ &\le \sum_{i=1}^{\bar{J}} \left[\left(|h|^{2(1-\beta)+2\beta\bar{t}} + |h|^{2\beta\eta}\right)\lambda(\hat{B}_i) + \int_{\hat{B}_i} |V(Du)|^2\,\mathrm{d}x\right].\end{aligned}$$

We point out that

$$\sum_{i=1}^{\bar{J}} \lambda(\hat{B}_i) \le c(n) \sum_{i=1}^{\bar{J}} \lambda(Q_{inn}(B_i)) \le c(n)\lambda(\bigcup_{i=1}^{\bar{J}} Q_{inn}(B_i)) \le c(n)\lambda(\Omega'')$$

and moreover that

$$\begin{aligned}\textstyle\sum_{i=1}^{\bar{J}} \int_{\hat{B}_i} |V(Du)|^2\,\mathrm{d}x &\le c(n) \textstyle\sum_{i=1}^{\bar{J}} \int_{Q_{inn}(B_i)} |V(Du)|^2\,\mathrm{d}x \\ &\le c(n)\left[|h|^{\beta(n-\theta)} \|V(Du)\|^2_{L^{2,\theta}(\Omega'')}\right].\end{aligned}$$

At least we have (recall that $\eta \le 1$)

$$\begin{aligned}\int_{\Omega'} |\tau_{ih}(V(Du))|^2\,\mathrm{d}x \le c(n)&\left[\left(|h|^{2(1-\beta)+2\beta\bar{t}} + |h|^{2\beta\eta} + |h|^{\beta\eta(n-\theta)}\right)\right. \\ &\left.\cdot\left(\lambda(\Omega'') + \|V(Du)\|^2_{L^{2,\theta}(\Omega'')}\right)\right]\end{aligned}$$

for any $\beta \in]0,1[$, $h \in \mathbb{R}$, $|h| < 1$, fixed at the beginning of the proof and for every couple of open subset $\Omega' \subset\subset \Omega'' \subset\subset \Omega$.

Now, since $\delta = \min\{1, \frac{n-\theta}{2}\}$ and $|h| < 1$ the powers $|h|^{2\beta\eta}$ and $|h|^{\beta\eta(n-\theta)}$ are less than $|h|^{2\beta\eta\delta}$ and the previous inequality implies

$$\int_{\Omega'} |\tau_{ih}(V(Du))|^2 \, dx \le c(n) \Big[\Big(|h|^{2(1-\beta)+2\beta\bar{t}} + |h|^{2\beta\eta\delta} \Big) \cdot \cdot \Big(\lambda(\Omega'') + \|V(Du)\|^2_{L^{2,\theta}(\Omega'')} \Big) \Big].$$

Choosing $\beta = \frac{1}{1-\bar{t}+\eta\delta}$, we minimize the right-hand side of the latter inequality with respect to h and we obtain

$$\int_{\Omega'} |\tau_{ih}(V(Du))|^2 \, dx \le c|h|^{2\gamma(\bar{t})} \Big[\lambda(\Omega'') + \|V(Du)\|^2_{L^{2,\theta}(\Omega'')} \Big] \tag{63}$$

where $\gamma(t) = \dfrac{\eta\delta}{1 - t + \eta\delta}$.
In turn, since the sets Ω', Ω'' are arbitrary, the previous estimate, Lemma 1 and the assumption (54) imply

$$V(Du) \in W^{t,2}_{loc}(\Omega, \mathbb{R}^{nN}) \quad \text{for every} \quad t \in [0, \gamma(\bar{t})[,$$

and the estimate (56) holds.
Step 3.

We shall complete the proof via iteration reasoning as in the proof of Lemma 6.2 of [10]. Namely, let us introduce the two sequences $\{t_k\}$ and $\{s_k\}$ defined by setting

$$s_1 = \frac{\eta\delta}{4(1+\eta\delta)}, \quad s_{k+1} = \gamma(s_k)$$
$$t_1 = 2s_1, \quad t_{k+1} = \frac{\gamma(s_k) + \gamma(t_k)}{2}$$

and we note that

$$\{s_k\} \text{ is increasing}$$
$$\lim s_k = \eta\delta$$
$$s_k < t_k < \eta\delta \quad \forall\, k \in \mathbb{N}.$$

Using the result of the previous step, we prove, by induction, that

$$V(Du) \in W^{t_k,2}_{loc}(\Omega, \mathbb{R}^{nN}) \quad \text{for every } k \in \mathbb{N}$$

and for every couple of open subset $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, there exists a positive constant c, independent of k, such that

$$[V(Du)]^2_{W^{t_k,2}(\Omega')} \le c \left[\int_{\Omega''} (s^p + |Du|^p) \, dx + \|V(Du)\|^2_{L^{2,\theta}(\Omega'')} \right], \quad \text{for every } k \in \mathbb{N}$$

and these two facts will imply the assertion (23) concerning $V(Du)$) and the estimate (24). Now we can prove the result concerning Du.

Let it be $\Omega' \subset\subset \Omega$. Using (9), we obtain

$$\begin{aligned}
[Du]^p_{W^{2t/p,p}(\Omega')} &= \int\limits_{\Omega'}\int\limits_{\Omega'} \frac{|Du(x)-Du(y)|^p}{|x-y|^{n+2t}}\,\mathrm{d}x\mathrm{d}y \\
&\le c \int\limits_{\Omega'}\int\limits_{\Omega'} \frac{(|Du(x)|+|Du(y)|)^{p-2}\,|Du(x)-Du(y)|^2}{|x-y|^{n+2t}}\,\mathrm{d}x\mathrm{d}y \\
&\le c \int\limits_{\Omega'}\int\limits_{\Omega'} \frac{\left(s^2+|Du(x)|^2+|Du(y)|^2\right)^{\frac{p-2}{2}}\,|Du(x)-Du(y)|^2}{|x-y|^{n+2t}}\,\mathrm{d}x\mathrm{d}y \\
&\le c \int\limits_{\Omega'}\int\limits_{\Omega'} \frac{|V(Du(x))-V(Du(y))|^2}{|x-y|^{n+2t}}\,\mathrm{d}x\mathrm{d}y = c\,[V(Du)]^2_{W^{t,2}(\Omega')}.
\end{aligned}$$

At least, the estimate (25) follows from the above inequality, by virtue of estimate (24) and the fact that

$$|V(\xi)|^2 \le 2(s^p + |\xi|^p).$$

□

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