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Bifurcation phenomena for nonlinear superdiffusive Neumann equations of logistic type

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Abstract We consider a nonlinear Neumann logistic equation driven by the p-Laplacian with a general Carathéodory superdiffusive reaction. We are looking for positive solutions of such problems. Using minimax methods from critical point theory together with suitable truncation techniques, we show that the equation exhibits a bifurcation phenomenon with respect to the parameter $\lambda > 0$. Namely, we show that there is a $\lambda_* > 0$ such that for $\lambda < \lambda_*$, the problem has no positive solution; for $\lambda = \lambda_*$, it has at least one positive solution; and for $\lambda > \lambda_*$, it has at least two positive solutions.

Keywords Superdiffusive reaction · Neumann problem · Truncations · Local minimizers · Upper and lower solutions · p-Laplacian

Mathematics Subject Classification (2000) 35J65

1 Introduction

Let $\Omega \subseteq I\!R^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the existence and multiplicity of positive solutions for the following nonlinear Neumann problem:

$$(P)_\lambda \quad \begin{cases} -\Delta_p u(z) + \beta(z)|u(z)|^{p-2}u(z) = \lambda u(z)^{q-1} - f(z, u(z)) \ \text{ in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \end{cases}$$

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with $1 < p < q < p^*$, $\beta \in L^\infty(\Omega)_+$, $\beta \neq 0$, $\lambda > 0$; here Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u(z) = \operatorname{div}(\|Du(z)\|^{p-2} Du(z)) \text{ , for all } u \in W^{1,p}(\Omega) \text{ and all } z \in \Omega,$$

$n(\cdot)$ denotes the outward unit normal on $\partial\Omega$ and $p^* > 1$ is the Sobolev critical exponent defined by

$$p^* = \begin{cases} \frac{Np}{N-p} \ , & p < N \\ +\infty \ , & N \leq p \end{cases} .$$

In $(P)_\lambda$, f is a Carathéodory function (i.e., for all $x \in I\!R$, $z \to f(z, x)$ is measurable and, for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous), which exhibits a $(p-1)$-superlinear growth near $+\infty$.

If $p = 2$ and $f(z, x) = x^{r-1}$ with $2 < r < 2^*$, then the resulting equation is known as the *logistic equation* and models various population dynamics phenomena and reaction-diffusion processes (see [12,16]). More recently, there have been papers dealing with logistic equations driven by the p-Laplacian (p-logistic equation, for short). All of them consider Dirichlet problems and have a reaction term (right hand side) of the form

$$g(z, x) = \lambda x^{q-1} - x^{r-1}, \quad \text{for all } z \in \Omega \text{ and all } x \in I\!R$$

where $q < r$.

There are three different types of p-logistic equations depending on the relation between q and p:

- $q < p < r$ (subdiffusive case)
- $q = p < r$ (equidiffusive case)
- $p < q < r$ (superdiffusive case)

with $r < p^*$.

The first two cases are essentially similar and for the Dirichlet problem with large $\lambda > 0$ produce a unique solution u with flat core (i.e., $\mathcal{U} = \{z \in \Omega : u(z) = 1\}$ is nonempty). In contrast, the superdiffusive case differs and exhibits bifurcation phenomena.

The subdiffusive Dirichlet logistic equation was studied in [24] (for $N = 1$, ordinary differential equations) and in [7] (for $N \geq 2$, partial differential equations).

The equidiffusive Dirichlet equation was investigated in [14] (for $N = 1$) and in [7,9] (for $N \geq 2$).

Finally, the superdiffusive Dirichlet logistic equation is examined in the works of Dong [5] and Takeuchi [22,23]. We should also mention the very recent work [4], where a general form of the Dirichlet p-Laplacian equation is considered. More precisely, the authors deal with the problem

$$-\Delta_p u(z) = \lambda f(z, u(z)) \text{ in } \Omega, \quad u_{|\partial\Omega} = 0.$$

Indeed, their hypotheses restrict their work to $(p-1)$-sublinear reactions. Indeed, hypotheses H_4 and $H_5(i)$ are compatible within the context of a $(p-1)$-sublinear nonlinearity. Moreover, hypothesis $H_5(i)$ is needed to establish the bifurcation phenomenon (see the proofs of Lemmata 3.1 and 3.2, where $H_5(i)$ is used in an essential way). So Theorem 1.1, which is the main result in [4], covers a $(p-1)$-sublinear reaction, a situation that precludes superdiffusive logistic equations.

Another related Dirichlet work worth mentioning is that of Rabinowitz [21], where a logistic-type semilinear (i.e., $p = 2$) Dirichlet equation was studied, with the parameter

$\lambda > 0$ multiplying the whole reaction term. Rabinowitz established certain bifurcation phenomena for the equation using variational and topological methods.

Finally, we also mention the Dirichlet works [6,10,11,18,19]. In [6,11], the authors consider a somehow dual situation to the logistic equation, by studying problems where in the reaction we have the combined effects of concave and convex terms, i.e., the reaction has the form

$$g(z, x) = \lambda x^{\tau-1} + x^{r-1}, \text{ for all } z \in \Omega \text{ and all } x \geq 0$$

with $1 < \tau < p < r < p^*$.

They establish bifurcation phenomena for small values of $\lambda > 0$. Their work also produces results relating C^1 and $W^{1,p}$ local minimizers of a C^1-functional. On the other hand, Guo [10] studied nonlinear eigenvalue problems driven by the p-Laplacian and imposed more restrictive hypotheses on the reaction (namely, $f(z, x) = f(x)$ belongs to $C^2(\mathbb{R})$). We should also mention that in [10], as well as in [11,22] and [23], it is assumed that $p > 2$. Motreanu et al. [17] also consider nonlinear eigenvalue problems with a Carathéodory reaction of arbitrary polynomial growth near $\pm\infty$ and $(p-1)$-linear near 0. They prove a multiplicity theorem (three nontrivial smooth solutions) for all small $\lambda > 0$. Finally, Motreanu et al. [18] consider p-Lapalacian parametric equations with a nonsmooth potential (hemivariational inequalities) and examine the near resonant (from above and below the principal eigenvalue), the resonant (with respect to the principal eigenvalue) and the nonresonant cases.

In the best of our knowledge, there are no works on the nonlinear Neumann logistic equation. Somewhat related are the works [3,19,26]. In [3] the authors deal with the equation

$$-\Delta_p u(z) = \lambda a(z)|u(z)|^{p-2}u(z) + h(z)|u(z)|^{p^*-2}u(z) \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

with $a \in L^\infty(\Omega)_+$, $h \in C(\Omega)$, $\lambda > 0$. They prove the existence of one or two (for $p \geq 2$) solutions, when $\lambda > 0$ is in a certain bounded interval. In [19], the authors extend the work of [18] to Neumann problems.

Finally, in [26], the authors assume that $p > N$ (low dimensional problems), essinf $\beta > 0$ and $f(z, x)$ satisfies certain technical restrictive conditions. They show that there is an open interval $I \subset [0, +\infty)$ such that for all $\lambda \in I$ the problem

$$-\Delta_p u(z) = \lambda f(z, u(z)) \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

has three solutions. Their approach is completely different from the aforementioned works and uses the KKM-principle.

In the next section, for the convenience of the reader, we briefly review the main mathematical tools that we will use in this work.

2 Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle\cdot, \cdot\rangle$, we denote the duality brackets for the pair (X^*, X).

Let $\varphi \in C^1(X)$. We say that φ satisfies the *"Palais-Smale condition"* (the *"PS-condition"*, for short) if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $\varphi'(x_n) \to 0$ in X^* as $n \to \infty$ has a strongly convergent subsequence.

The topological notion of *linking sets* is crucial in the minimax characterization of the critical values of a C^1-functional.

Definition 2.1 Let Y be a Hausdorff topological space and $E_0,\ E,\ D$ nonempty subsets of Y such that $E_0 \subseteq E$. We say that the pair $\{E_0, E\}$ is linking with D in Y if

(a) $E_0 \cap D = \emptyset$;
(b) for any $\gamma \in C(E, Y)$ such that $\gamma_{|E_0} = id_{|E_0}$, we have $\gamma(E) \cap D \neq \emptyset$.

Using this notion, we have the following general minimax principle concerning the critical values of a C^1-functional (see, for example, [8], p. 644).

Theorem 2.1 *If X is a Banach space, $\varphi \in C^1(X)$ and satisfies the PS-condition, E_0, E, and D are nonempty closed subsets of X such that the pair $\{E_0, E\}$ is linking with D in X, $\sup\limits_{E_0} \varphi < \inf\limits_{D} \varphi$ and $c = \inf\limits_{\gamma\in\Gamma} \sup\limits_{x\in E} \varphi(\gamma(x))$, where $\Gamma = \{\gamma \in C(E, X) : \gamma_{|E_0} = id_{|E_0}\}$, then $c \geq \inf\limits_{D} \varphi$ and c is a critical value of φ.*

By appropriate choices of the linking sets, from Theorem 2.1 we obtain, as corollaries, the mountain pass, the saddle point and the generalized mountain pass theorems. For future use, we recall the mountain pass theorem.

Theorem 2.2 *If X is a Banach space, $\varphi \in C^1(X)$ and satisfies the PS-condition, there exist $x_0, x_1 \in X$ and $r > 0$ such that $\|x_1 - x_0\| > r$ and $\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = r\} = \eta_r$ and $c = \inf\limits_{\gamma\in\Gamma} \max\limits_{0\leq t\leq 1} \varphi(\gamma(t))$, where $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = x_0,\ \gamma(1) = x_1\}$, then $c \geq \eta_r$ and c is a critical value of φ.*

In the analysis of problem $(P)_\lambda$, we will use the following "natural" spaces:

$$C_n^1(\bar{\Omega}) = \left\{u \in C^1(\bar{\Omega}) \ :\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right\}$$

and its completion

$$W_n^{1,p}(\Omega) = \overline{C_n^1(\bar{\Omega})}^{\|\cdot\|},$$

where $\|\cdot\|$ is the norm of the Sobolev space $W_n^{1,p}(\Omega)$, that is, $\|u\| = \|Du\|_p + \|u\|_p$ for all $u \in W_n^{1,p}(\Omega)$.

The Banach space $C_n^1(\bar{\Omega})$ is an ordered Banach space with positive cone

$$C_+ = \{u \in C_n^1(\bar{\Omega}) \ :\ u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ \ :\ u(z) > 0 \text{ for all } z \in \bar{\Omega}\}.$$

Considered the nonlinear map $A : W_n^{1,p}(\Omega) \to W_n^{1,p}(\Omega)^*$ defined by

$$\langle A(u), y\rangle = \int\limits_{\Omega} \|Du(z)\|^{p-2}(Du(z), Dy(z))_{I\!R^N}\, \mathrm{d}z, \text{ for all } u, y \in W_n^{1,p}(\Omega), \tag{1}$$

the following result is well-known (see, for example, [2]).

Proposition 2.3 *The map $A : W_n^{1,p}(\Omega) \to W_n^{1,p}(\Omega)^*$ defined by (1) is continuous, bounded and of type $(S)_+$, that is, if $u_n \rightharpoonup u$ in $W_n^{1,p}(\Omega)$ and $\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ in $W_n^{1,p}(\Omega)$.*

If X is a Banach space, it is well-known that a vector $u_0 \in X$ is a local X-minimizer for a function $\varphi : X \to I\!R$ if there exists $r_0 > 0$ such that $\varphi_0(u_0) \leq \varphi_0(u_0 + h)$ for all $h \in X$ with $\|h\|_X \leq r_0$.

Next, we recall a result relating local minimizers in $C^1_n(\bar{\Omega})$ and in $W^{1,p}_n(\Omega)$ proved in [19]. (As we already mentioned analogous results for the "Dirichlet" Sobolev spaces can be found in [6,11].)

So, let $f_0 : \Omega \times I\!R \to I\!R$ be a Carathéodory function such that

$$|f_0(z,x)| \leq a_0(z) + c_0|x|^{r-1}, \text{ for a.a. } z \in \Omega \text{ and all } x \in I\!R,$$

with $a_0 \in L^\infty(\Omega)_+$, $c_0 > 0$, $p < r < p^*$.

We set $F_0 : \Omega \times I\!R \to I\!R$, $F_0(z,x) = \int_0^x f_0(z,s)\,\mathrm{d}s$ and consider the functional $\varphi_0 : W^{1,p}_n(\Omega) \to I\!R$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega F_0(z,u)\,\mathrm{d}z \text{ , for all } u \in W^{1,p}_n(\Omega).$$

Evidently, $\varphi_0 \in C^1(W^{1,p}_n(\Omega))$.

Proposition 2.4 ([19], Proposition 2.5) *If $u_0 \in W^{1,p}_n(\Omega)$ $(1 < p < \infty)$ is a local $C^1_n(\bar{\Omega})$-minimizer of φ_0, then it is a local $W^{1,p}_n(\Omega)$-minimizer of φ_0.*

In the analysis of problem $(P)_\lambda$, we will also use the notions of upper and lower solutions, which we recall next.

Definition 2.2 A function $\bar{u} \in W^{1,p}(\Omega)$ with $\dfrac{\partial \bar{u}}{\partial n} = 0$ is said to be an *upper solution* for problem $(P)_\lambda$ if

$$\int_\Omega \beta|\bar{u}|^{p-2}\bar{u}\,h\,\mathrm{d}z + \int_\Omega \|D\bar{u}\|^{p-2}(D\bar{u}, Dh)_{I\!R^N}\,\mathrm{d}z$$
$$\geq \lambda \int_\Omega \bar{u}^{q-1}h\,\mathrm{d}z - \int_\Omega f(z,\bar{u})h\,\mathrm{d}z, \text{ for all } h \in W^{1,p}_n(\Omega)\text{ , } h \geq 0.$$

An upper solution is a *strict upper solution* for problem $(P)_\lambda$ if it is not a solution.

Definition 2.3 A function $\underline{u} \in W^{1,p}(\Omega)$ with $\dfrac{\partial \underline{u}}{\partial n} = 0$ is said to be a *lower solution* for problem $(P)_\lambda$ if

$$\int_\Omega \beta|\underline{u}|^{p-2}\underline{u}\,h\,\mathrm{d}z + \int_\Omega \|D\underline{u}\|^{p-2}(D\underline{u}, Dh)_{I\!R^N}\,\mathrm{d}z$$
$$\leq \lambda \int_\Omega \underline{u}^{q-1}h\,\mathrm{d}z - \int_\Omega f(z,\underline{u})h\,\mathrm{d}z, \text{ for all } h \in W^{1,p}_n(\Omega)\text{ , } h \geq 0.$$

A lower solution is a *strict lower solution* for problem $(P)_\lambda$ if it is not a solution.

In what follows, we use the notation $r^\pm = max\{\pm r, 0\}$, for all $r \in I\!R$. Also, by $\|\cdot\|$, we denote either the norm of the Sobolev space $W^{1,p}(\Omega)$ and the one of $I\!R^N$; it will always be clear from the context which one is in use. Finally, by $\|\cdot\|_s$, we denote the norm of $L^s(\Omega)$ or of $L^s(\Omega, I\!R^N)$, $1 \leq s \leq \infty$, and by $|\cdot|_N$ the Lebesgue measure on $I\!R^N$.

3 Bifurcation theorem

In this section, we show that problem $(P)_\lambda$, governed by a superdiffusive nonlinear Neumann p-logistic equation, exhibits a bifurcation phenomenon with respect to the parameter $\lambda > 0$.

The hypotheses on the nonlinear perturbation f are:

H: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ a.e. in Ω and

(i) there exist $a \in L^\infty(\Omega)_+$, $c > 0$, $p < r < p^*$ such that
$$|f(z,x)| \le a(z) + c|x|^{r-1}\text{ , for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R};$$
(ii) for a.a. $z \in \Omega$ and all $x \ge 0$, $f(z,x) \ge 0$ and there exist $M, \gamma > 0$ and $\theta > q$ such that
$$f(z,x) \ge \gamma x^{\theta-1},\ \text{ for a.a. } z \in \Omega \text{ and all } x \ge M;$$
(iii) $\lim_{x\to 0^+} \frac{f(z,x)}{x^{p-1}} = 0$ uniformly, for a.a. $z \in \Omega$;

(iv) for every $r > 0$ and every bounded interval $I \subseteq (0, +\infty)$, there exists $\eta = \eta(r, I) > 0$ such that, for a.a. $z \in \Omega$, $x \mapsto \lambda x^{q-1} - f(z,x) + \eta x^{\theta-1}$ is nondecreasing on $[0, r]$, for all $\lambda \in I$.

Remark 3.1 Since we are interested in positive solutions and hypotheses H(ii),(iii),(iv) concern only the positive semiaxis $[0, +\infty)$, by truncating things if necessary, we may (and will) assume that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \le 0$. Hypothesis H(iv) implies that near $+\infty$ $f(z,\cdot)$ is $(p-1)$-superlinear. Similarly, hypothesis H(iii) dictates a $(p-1)$-sublinear behavior near zero.

Remark 3.2 Note that the following functions satisfy hypotheses H:
$$f_1(z,x) = \begin{cases} 0 & ,\ x \le 0 \\ b(z)x^{r-1} & ,\ x > 0 \end{cases}$$
where $b \in L^\infty(\Omega)_+$, $\underset{\Omega}{essinf}\, b > 0$ and $q < r < p^*$;
$$f_2(z,x) = \begin{cases} 0 & ,\ x \le 0 \\ x^{\tau-1} & ,\ 0 < x < 1 \\ x^{\theta-1} - x^{p-1}\ln x & ,\ 1 < x \end{cases}$$
where $p < \tau$, $q < \theta < p^*$.

The function f_1 corresponds to the standard nonlinear logistic equation (superdiffusive case) with a z-dependent coefficient in the extinction term.

By a positive solution of $(P)_\lambda$, we mean a function $u \in \operatorname{int} C_+$ that solves $(P)_\lambda$. Let
$$S = \{\lambda > 0 : \text{ problem } (P)_\lambda \text{ has a positive solution}\}.$$
First, we will show that $S \neq \emptyset$. To this end, we will need the following simple Lemma.

Lemma 3.1 *If $\beta \in L^\infty(\Omega)_+$, $\beta \neq 0$, then there exists $\xi_0 > 0$ such that*
$$\psi_0(u) = \|Du\|_p^p + \int_\Omega \beta|u|^p\, dz \ge \xi_0\|u\|^p,\ \textit{for all } u \in W_n^{1,p}(\Omega).$$

Proof Note that $\psi_0 \geq 0$. Suppose that the Lemma is not true. Exploiting the p-homogeneity of ψ_0, we can find $\{u_n\}_{n\geq 1} \subseteq W_n^{1,p}(\Omega)$ with $\|u_n\| = 1\,,\ n \geq 1\,$, such that $\psi_0(u_n) \to 0^+$ as $n \to \infty$. By passing to a suitable subsequence if necessary, we may assume that

$$u_n \rightharpoonup u \text{ in } W_n^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega) \text{ as } n \to \infty \tag{2}$$

(recall that $W_n^{1,p}(\Omega)$ is embedded compactly in $L^p(\Omega)$). Then, since (see (2))

$$\|Du\|_p^p \leq \liminf_{n\to\infty} \|Du_n\|_p^p \text{ and } \int_\Omega \beta|u_n|^p\,dz \to \int_\Omega \beta|u|^p\,dz,$$

in the limit as $n \to \infty$, we obtain $\psi_0(u) \leq 0$; so we have

$$\|Du\|_p^p \leq -\int_\Omega \beta|u|^p\,dz \leq 0. \tag{3}$$

Therefore, the limit function u is constant, that is, $u \equiv \xi \in I\!R$. If $\xi = 0$, then $Du_n \to 0$ in $L^p(\Omega, I\!R^N)$, hence $u_n \to 0$ in $W_n^{1,p}(\Omega)$ as $n \to \infty$ (see (2)), a contradiction to the fact that $\|u_n\| = 1$ for all $n \geq 1$.

If $\xi \neq 0$, then from (3), we have $\|Du\|_p^p \leq -|\xi|^p \int_\Omega \beta\,dz < 0$, again a contradiction. This proves the Lemma. □

Proposition 3.2 *If hypotheses H hold and $\beta \in L^\infty(\Omega)_+$, $\beta \neq 0$, then $S \neq \emptyset$.*

Proof We consider the following auxiliary Neumann problem

$$(\tilde{P})_\lambda \quad \begin{cases} -\Delta_p u(z) + \beta(z)|u(z)|^{p-2}u(z) = \lambda|u(z)|^{q-2}u(z) \text{ in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \end{cases}$$

with $\lambda > 0\,,\ p < q < p^*$.

Claim: for every $\lambda > 0$, problem $(\tilde{P})_\lambda$ has a solution $\bar{u} \in \operatorname{int} C_+$.

Let $\psi_\lambda : W_n^{1,p}(\Omega) \to I\!R$ be the Euler functional for problem $(\tilde{P})_\lambda$ defined by

$$\psi_\lambda(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_\Omega \beta|u|^p\,dz - \frac{\lambda}{q}\|u^+\|_q^q, \text{ for all } u \in W_n^{1,p}(\Omega).$$

Evidently, $\psi_\lambda \in C^1\left(W_n^{1,p}(\Omega)\right)$. We show that ψ_λ satisfies the PS-condition. So, let $\{u_n\}_{n\geq 1} \subseteq W_n^{1,p}(\Omega)$ be a sequence such that

$$|\psi_\lambda(u_n)| \leq M_1, \text{ for some } M_1 > 0 \text{ and all } n \geq 1 \tag{4}$$

and

$$\psi_\lambda'(u_n) \to 0 \text{ in } W_n^{1,p}(\Omega)^* \text{ as } n \to \infty. \tag{5}$$

From (5), we have

$$\left|\langle A(u_n), h\rangle + \int_\Omega \beta|u_n|^{p-2}u_n h\,dz - \lambda\int_\Omega (u_n^+)^{q-1}h\,dz\right| \leq \varepsilon_n\|h\|, \text{ for all } h \in W_n^{1,p}(\Omega) \tag{6}$$

where $\varepsilon_n \to 0^+$ and A is defined in (1).

In (6), we choose $h = -u_n^- \in W_n^{1,p}(\Omega)$. Then,

$$\|Du_n^-\|_p^p + \int_\Omega \beta(u_n^-)^p \,\mathrm{d}z \le \varepsilon_n \|u_n^-\|$$

so, by using Lemma 3.1, we have

$$\xi_0 \|u_n^-\|^p \le \varepsilon_n \|u_n^-\|, \text{ for all } n \ge 1$$

and, since $p > 1$,

$$u_n^- \to 0 \text{ in } W_n^{1,p}(\Omega) \text{ as } n \to \infty. \tag{7}$$

From (4) and (7), we have

$$\frac{q}{p}\|Du_n^+\|_p^p + \frac{q}{p}\int_\Omega \beta(u_n^+)^p \,\mathrm{d}z - \lambda \int_\Omega (u_n^+)^q \,\mathrm{d}z \le M_2, \text{ for all } n \ge 1, \tag{8}$$

for some $M_2 > 0$.

On the other hand, if in (6) we choose $h = u_n^+ \in W_n^{1,p}(\Omega)$, we obtain

$$-\|Du_n^+\|_p^p - \int_\Omega \beta(u_n^+)^p \,\mathrm{d}z + \lambda \int_\Omega (u_n^+)^q \,\mathrm{d}z \le \varepsilon_n \|u_n^+\|, \text{ for all } n \ge 1. \tag{9}$$

We add (8) and (9) and have

$$\left(\frac{q}{p} - 1\right)\left[\|Du_n^+\|_p^p + \int_\Omega \beta(u_n^+)^p \,\mathrm{d}z\right] \le M_3(\|u_n^+\| + 1), \text{ for all } n \ge 1,$$

where $M_3 > 0$.

Then, again by Lemma 3.1, we get

$$\left(\frac{q}{p} - 1\right)\xi_0 \|u_n^+\|^p \le M_3(\|u_n^+\| + 1), \text{ for all } n \ge 1\,,$$

therefore, since $1 < p < q$, we have that

$$\{u_n^+\}_{n\ge 1} \subseteq W_n^{1,p}(\Omega) \text{ is bounded.} \tag{10}$$

From (7) and (10), it follows that $\{u_n\}_{n\ge 1} \subseteq W_n^{1,p}(\Omega)$ is bounded, and so we may assume that

$$u_n \rightharpoonup u \text{ in } W_n^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^q(\Omega) \text{ as } n \to \infty \text{ (recall } q < p^*\text{).} \tag{11}$$

In (6), we choose $h = u_n - u \in W_n^{1,p}(\Omega)$ and pass to the limit as $n \to \infty$. Using (11), we have

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0\,;$$

then (see Proposition 2.3)

$$u_n \to u \text{ in } W_n^{1,p}(\Omega).$$

This proves that the functional ψ_λ satisfies the PS-condition.

Also, we have (see Lemma 3.1)

$$\psi_\lambda(u) \geq \frac{\xi_0}{p}\|u\|^p - \frac{\lambda}{q}\|u\|^q, \text{ for all } u \in W_n^{1,p}(\Omega). \tag{12}$$

Since $q > p$, from (12), it follows that there exists $\rho \in (0, 1)$ sufficiently small so that

$$\inf\,[\psi_\lambda(u) \,:\, \|u\| = \rho] = \eta_\rho > 0 = \psi_\lambda(0). \tag{13}$$

Finally, for $\xi \in (0, +\infty)$, we have

$$\psi_\lambda(\xi) = \frac{\xi^p}{p}\|\beta\|_1 - \frac{\lambda \xi^q}{q}|\Omega|_N.$$

Since $q > p$, it follows that

$$\psi_\lambda(\xi) \to -\infty \text{ as } \xi \to +\infty. \tag{14}$$

Then (13), (14) and the fact proven earlier that ψ_λ satisfies the PS-condition permit the use of Theorem 2.2. So (see (13)) we obtain $\bar{u} \in W_n^{1,p}(\Omega)$ such that

$$\psi_\lambda(0) = 0 < \eta_\rho \leq \psi_\lambda(\bar{u}) \tag{15}$$

and

$$\psi_\lambda'(\bar{u}) = 0. \tag{16}$$

From (15), we infer that $\bar{u} \neq 0$. From (16), we have

$$A(\bar{u}) + \beta|\bar{u}|^{p-2}\bar{u} = \lambda(\bar{u}^+)^{q-1}. \tag{17}$$

On (17), we act with $-\bar{u}^- \in W_n^{1,p}(\Omega)$ and obtain

$$\|D\bar{u}^-\|_p^p + \int_\Omega \beta(\bar{u}^-)^p \,\mathrm{d}z \;=\; 0\,;$$

so, from Lemma 3.1, we get

$$\xi_0\|\bar{u}^-\|^p \leq 0$$

and then

$$\bar{u}^- = 0\,, \text{ i.e. } \bar{u} \geq 0\,,\ \bar{u} \neq 0.$$

Then, from (17), we have

$$A(\bar{u}) + \beta\bar{u}^{p-1} = \lambda\bar{u}^{q-1}$$

and so (see [20])

$$-\,\Delta_p\bar{u}(z) + \beta(z)\bar{u}(z)^{p-1} = \lambda\bar{u}(z)^{q-1} \text{ a.e. in } \Omega\,,\ \frac{\partial\bar{u}}{\partial n} = 0 \text{ on } \partial\Omega. \tag{18}$$

Nonlinear regularity theory (see, for example, [8]), implies that $\bar{u} \in C_+ \setminus \{0\}$. From (18), we have

$$\Delta_p\bar{u}(z) \leq \beta(z)\bar{u}(z)^{p-1} \leq \|\beta\|_\infty\bar{u}(z)^{p-1} \text{ a.e. in } \Omega$$

so (see [25])

$$\bar{u} \in \operatorname{int} C_+.$$

By virtue of hypothesis H(ii), we have $f(z, \bar{u}(z)) \geq 0$ a.e. in Ω and so

$$-\Delta_p \bar{u}(z) + \beta(z)\bar{u}(z)^{p-1} = \lambda \bar{u}(z)^{q-1} \geq \lambda \bar{u}(z)^{q-1} - f(z, \bar{u}(z)) \text{ a.e. in } \Omega \tag{19}$$

that is, $\bar{u} \in \operatorname{int} C_+$ is an upper solution for $(P)_\lambda$ (see Definition 2.2).

We consider the following truncation of the reaction term in $(P)_\lambda$:

$$g_\lambda(z, x) = \begin{cases} 0 & , \ x \leq 0 \\ \lambda x^{q-1} - f(z, x) & , \ 0 < x < \bar{u}(z) \\ \lambda \bar{u}(z)^{q-1} - f(z, \bar{u}(z)) & , \ \bar{u}(z) \leq x. \end{cases} \tag{20}$$

This is a Carathéodory function. We set $G_\lambda(z, x) = \int_0^x g_\lambda(z, s)\, \mathrm{d}s$ and consider the C^1-functional $\hat{\varphi}_\lambda : W_n^{1,p}(\Omega) \to I\!R$ defined by

$$\hat{\varphi}_\lambda(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_\Omega \beta|u|^p\, \mathrm{d}z - \int_\Omega G_\lambda(z, u)\, \mathrm{d}z, \text{ for all } u \in W_n^{1,p}(\Omega).$$

From (20) and Lemma 3.1, we see that there exist $c_0, c_1 > 0$ such that

$$\hat{\varphi}_\lambda(u) \geq \frac{\xi_0}{p}\|u\|^p - c_0\|u\| - c_1, \text{ for all } u \in W_n^{1,p}(\Omega)$$

and then $\hat{\varphi}_\lambda$ is coercive.

Also, exploiting the compact embedding of $W_n^{1,p}(\Omega)$ into $L^p(\Omega)$, we can easily check that $\hat{\varphi}_\lambda$ is sequentially weakly lower semicontinuous. Therefore, by the Weierstrass theorem, we can find $u_0 \in W_n^{1,p}(\Omega)$ such that

$$\hat{\varphi}_\lambda(u_0) = \inf\left[\hat{\varphi}_\lambda(u) \ : \ u \in W_n^{1,p}(\Omega)\right] = \hat{m}_\lambda. \tag{21}$$

By hypothesis H(iii), there exists $\delta(1) > 0$ such that for every $x \in]0, \delta(1)[$ we have $f(z, x) < x^{p-1}$.

Let us fix $\xi \in (0, \min_{\bar{\Omega}} \bar{u})$ (recall that $\bar{u} \in \operatorname{int} C_+$) and $\xi < \delta(1)$. Then, since $f(z, x) \geq 0$ for a.a. $z \in \Omega$ and all $x \geq 0$ (see H(ii)), we have

$$\begin{aligned} \hat{\varphi}_\lambda(\xi) &= \frac{\xi^p}{p}\|\beta\|_1 - \frac{\lambda \xi^q}{q}|\Omega|_N + \int_\Omega \int_0^\xi f(z, s)\mathrm{d}s\, \mathrm{d}z \\ &\leq \frac{\xi^p}{p}\|\beta\|_1 - \frac{\lambda \xi^q}{q}|\Omega|_N + \frac{\lambda \xi^p}{p}|\Omega|_N. \end{aligned}$$

We observe that $\xi = \xi(\lambda)$; anyway, we can assume ξ not depending on λ. Indeed, fix $\tilde{\lambda} > 0$; for every $\lambda > \tilde{\lambda}$ the minimal solution $\bar{u}_\lambda$ of problem $(\tilde{P})_\lambda$ is an upper solution for the problem $(\tilde{P})_{\tilde{\lambda}}$; therefore $\bar{u}_{\tilde{\lambda}} \in [0, \bar{u}_\lambda] \cap \operatorname{int} C_+$. So we have $\xi \in (0, \min_{\bar{\Omega}} \bar{u}_{\tilde{\lambda}}) \subset (0, \min_{\bar{\Omega}} \bar{u}_\lambda)$ for every $\lambda \geq \tilde{\lambda}$ and we assume $\xi < \min_{\bar{\Omega}} \bar{u}_{\tilde{\lambda}}$. Hence, we can choose $\lambda > 0$ such that $\frac{q}{p}\frac{\|\beta\|_1 + |\Omega|_N}{|\Omega|_N} < \lambda \xi^{q-p}$. Then,

$$\hat{\varphi}_\lambda(\xi) < 0$$

so, by (21), we have

$$\hat{\varphi}_\lambda(u_0) = \hat{m}_\lambda < 0 = \hat{\varphi}_\lambda(0)$$

and then

$$u_0 \neq 0. \tag{22}$$

Also for such a $\lambda > 0$, from (21) we have

$$A(u_0) + \beta|u_0|^{p-2}u_0 = N_\lambda(u_0), \tag{23}$$

where $N_\lambda(u)(\cdot) = g_\lambda(\cdot, u(\cdot))$, for all $u \in W_n^{1,p}(\Omega)$.

On (23), first we act with $-u_0^- \in W_n^{1,p}(\Omega)$ and obtain (see (20))

$$\|Du_0^-\|_p^p + \int_\Omega \beta(u_0^-)^p \, dz = 0\,;$$

hence, by Lemma 3.1, we deduce

$$\xi_0\|u_0^-\|^p \le 0\,;$$

therefore, by (22),

$$u_0^- = 0\,, \;\text{ i.e. } u_0 \ge 0\,,\; u_0 \ne 0.$$

Also on (23), we act with $(u_0 - \bar{u})^+ \in W_n^{1,p}(\Omega)$. Then, (see (20) and (19))

$$\begin{aligned}
&\left\langle A(u_0), (u_0 - \bar{u})^+\right\rangle + \int_\Omega \beta u_0^{p-1}(u_0 - \bar{u})^+ \, dz \\
&= \int_\Omega g_\lambda(z, u_0)(u_0 - \bar{u})^+ \, dz \\
&= \lambda \int_\Omega \bar{u}^{q-1}(u_0 - \bar{u})^+ \, dz - \int_\Omega f(z, \bar{u})(u_0 - \bar{u})^+ \, dz \\
&\le \left\langle A(\bar{u}), (u_0 - \bar{u})^+\right\rangle + \int_\Omega \beta\bar{u}^{p-1}(u_0 - \bar{u})^+ \, dz\,;
\end{aligned}$$

then

$$\begin{aligned}
0 \le &\int_{\{u_0>\bar{u}\}} \left(\|D\bar{u}\|^{p-2}D\bar{u} - \|Du_0\|^{p-2}Du_0, Du_0 - D\bar{u}\right)_{I\!R^N} \, dz \\
&+ \int_{\{u_0>\bar{u}\}} \beta(\bar{u}^{p-1} - u_0^{p-1})(u_0 - \bar{u}) \, dz,
\end{aligned} \tag{24}$$

where $\{u_0 > \bar{u}\} \equiv \{z \in \Omega \,:\, u_0(z) > \bar{u}(z)\}$ (for the sake of simplicity, in the sequel, we will use this kind of notation).

Recall that the map $\zeta \to \|\zeta\|^{p-2}\zeta$, for $\zeta \in I\!R^N$, is strictly monotone. Hence, from (24), we have $|\{u_0 > \bar{u}\}|_N = 0$ and so $u_0 \le \bar{u}$, i.e. $u_0 \in [0, \bar{u}] = \{u \in W_n^{1,p}(\Omega) \,:\, 0 \le u(z) \le \bar{u}(z)$ a.e. in $\Omega\}$. So, by using (20), (23) becomes

$$A(u_0) + \beta u_0^{p-1} = \lambda u_0^{q-1} - N(u_0),$$

where

$$N(u)(\cdot) = f(\cdot, u(\cdot)),\; \text{for all } u \in W_n^{1,p}(\Omega) \tag{25}$$

so (see [20])

$$-\Delta_p u_0(z) + \beta(z)u_0(z)^{p-1} = \lambda u_0(z)^{q-1} - f(z, u_0(z)) \;\text{ a.e. in } \Omega\,, \quad \frac{\partial u_0}{\partial n} = 0 \text{ on } \partial\Omega.$$

Therefore, by H(iv), there exists η_λ such that

$$\Delta_p u_0(z) \leq \beta(z) u_0(z)^{p-1} + \eta_\lambda u_0(z)^{\theta-1}, \text{ for a.a. } z \in \Omega.$$

Hence (recall $\theta > q > p$), we can write

$$\Delta_p u_0(z) \leq \left(\|\beta\|_\infty + \eta_\lambda \|\bar{u}\|_\infty^{\theta-p} \right) u_0(z)^{p-1}$$

so (see [25])

$$u_0 \in \operatorname{int} C_+.$$

Therefore, we see that at least for $\lambda > \frac{q}{p} \frac{\|\beta\|_1 + |\Omega|_N}{|\Omega|_N} \frac{1}{\xi^{q-p}}$, problem $(P)_\lambda$ has a solution $u_0 \in \operatorname{int} C_+$, hence $S \neq \emptyset$. □

Let $\lambda_* = \inf S$. Evidently $\lambda_* \geq 0$.

Proposition 3.3 *If hypotheses H hold and $\beta \in L^\infty(\Omega)_+$, $\beta \neq 0$, then $\lambda_* > 0$.*

Proof Suppose that $\lambda_* = 0$. Then, we can find $\{\lambda_n\}_{n\geq 1} \subseteq S$ decreasing, $\lambda_n > 0$ for all $n \geq 1$, and $u_n \in \operatorname{int} C_+$ for all $n \geq 1$, such that (see (25))

$$A(u_n) + \beta u_n^{p-1} = \lambda_n u_n^{q-1} - N(u_n), \text{ for all } n \geq 1 \tag{26}$$

so, by considering the constant M postulated by H(ii), we have

$$\|Du_n\|_p^p + \int_\Omega \beta u_n^p \, dz \leq \lambda_1 \|u_n\|_q^q - \int_{\{u_n \geq M\}} f(z, u_n) u_n \, dz - \int_{\{0 < u_n < M\}} f(z, u_n) u_n \, dz \, ;$$

therefore, put $C_n = \{z \in \Omega : u_n(z) \geq M\}$, by Lemma 3.1 and H(ii), we obtain that there exists $c_2 > 0$ such that

$$\xi_0 \|u_n\|^p + \gamma \|\chi_{C_n} u_n\|_\theta^\theta \leq \lambda_1 \|\chi_{C_n} u_n\|_q^q + c_2, \text{ for all } n \geq 1. \tag{27}$$

It is easy to see that $\{\chi_{C_n} u_n\}_{n\geq 1}$ is bounded in $L^\theta(\Omega)_+$. Recalling that $\theta > q$ (see H(ii)), we get that the sequence $\{\chi_{C_n} u_n\}_{n\geq 1}$ is bounded in $L^q(\Omega)_+$. Then, by (27), we deduce that

$$\{u_n\}_{n\geq 1} \text{ is bounded in } W_n^{1,p}(\Omega). \tag{28}$$

So, by passing to a suitable subsequence if necessary, we may assume that $u_n \rightharpoonup u$ in $W_n^{1,p}(\Omega)$.

Because of (28) and Theorem 1.2 of [15] (see also [13], Proposition 5), we can find $\mu \in (0, 1)$ and $M_4 > 0$ such that

$$u_n \in C_n^{1,\mu}(\bar{\Omega}) \text{ and } \|u_n\|_{C_n^{1,\mu}(\bar{\Omega})} \leq M_4, \text{ for all } n \geq 1. \tag{29}$$

Exploiting the compact embedding of $C_n^{1,\mu}(\bar{\Omega})$ into $C_n^1(\bar{\Omega})$ and recalling that $u_n \rightharpoonup u$ in $W_n^{1,p}(\Omega)$, we have

$$u_n \to u \text{ in } C_n^1(\bar{\Omega}) \text{ as } n \to \infty. \tag{30}$$

Suppose $u = 0$ and set $y_n = \dfrac{u_n}{\|u_n\|}$, $n \geq 1$. Then, $\|y_n\| = 1$ for all $n \geq 1$, so we may assume that

$$y_n \rightharpoonup y \text{ in } W_n^{1,p}(\Omega) \quad \text{and so} \quad y_n \to y \text{ in } L^q(\Omega) \text{ (recall } q < p^*). \tag{31}$$

From (26), we have

$$A(y_n) + \beta y_n^{p-1} = \lambda_n y_n^{p-1} u_n^{q-p} - \frac{N(u_n)}{\|u_n\|^{p-1}}. \tag{32}$$

From (30) and hypothesis H(iii), we can easily check that

$$\frac{N(u_n)}{\|u_n\|^{p-1}} \rightharpoonup 0 \text{ in } L^{p'}(\Omega) \text{ as } n \to \infty \ (1/p + 1/p' = 1). \tag{33}$$

Also, acting on (32) with $y_n - y \in W_n^{1,p}(\Omega)$, passing to the limit as $n \to \infty$ and using (31) and (33), we obtain

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0\,;$$

so, by Proposition 2.3, we deduce

$$y_n \to y \text{ in } W_n^{1,p}(\Omega) \text{ as } n \to \infty$$

hence

$$\|y\| = 1. \tag{34}$$

Passing to the limit as $n \to \infty$ in (32) and using (30), (31) and (34), we have

$$A(y) + \beta y^{p-1} = 0$$

then, by Lemma 3.1, we can write

$$\xi_0 \|y\|^p \le 0$$

so that $y = 0$, which contradicts (34).

Therefore, $u \neq 0$.

So, recalling that $u_n \in \operatorname{int} C_+$, $n \geq 1$, we have

$$u \in C_+ \setminus \{0\}. \tag{35}$$

From (26), if we pass to the limit as $n \to \infty$ and use the fact that $\lambda_n \to 0^+$, we obtain

$$A(u) + \beta u^{p-1} = -N(u)$$

so, by Lemma 3.1, (35) and H(ii), $\xi_0\|u\|^p \le 0$ and then $u = 0$, which contradicts (35). Therefore, we can conclude that $\lambda_* > 0$. □

Proposition 3.4 *If hypotheses H hold and $\beta \in L^\infty(\Omega)_+$, $\beta \neq 0$ and $\lambda > \lambda_*$, then problem $(P)_\lambda$ has at least two distinct positive solutions $u_0, \hat{u} \in \operatorname{int} C_+$.*

Proof Let $\tilde{\lambda} \in (\lambda_*, \lambda) \cap S$. Then, we can find $\tilde{u} \in \operatorname{int} C_+$ such that

$$A(\tilde{u}) + \beta \tilde{u}^{p-1} = \tilde{\lambda}\tilde{u}^{q-1} - N(\tilde{u}) < \lambda \tilde{u}^{q-1} - N(\tilde{u}) \ \text{ in } W_n^{1,p}(\Omega)^*$$

so $\tilde{u} \in \operatorname{int} C_+$ is a strict lower solution for problem $(P)_\lambda$ (see Definition 2.3).

Let $\sigma(\xi) = \lambda\xi^{q-1} - \gamma\xi^{\theta-1}$ for $\xi \geq 0$ and with $\gamma > 0$ and $\theta > q$ as postulated by hypothesis H(ii). Since $\theta > q$, we see that $\sigma(\xi) \to -\infty$ as $\xi \to +\infty$. So, for $\xi \geq M$ ($M > 0$ as in H(ii)) large, we have $\sigma(\xi) < 0$; hence

$$\lambda\xi^{q-1} - \gamma\xi^{\theta-1} < 0$$

so, by H(ii), we get

$$\lambda\xi^{q-1} - f(z,\xi) < 0, \text{ for a.a. } z \in \Omega.$$

Therefore, $\bar{u} \equiv \xi$ is a strict upper solution for problem $(P)_\lambda$.

By taking $\xi \geq M$ even bigger if necessary, we may also assume that $\bar{u} \equiv \xi > \|\tilde{u}\|_\infty$.

Consider $\varphi_\lambda : W_n^{1,p}(\Omega) \to I\!R$ the Euler functional for problem $(P)_\lambda$ defined by

$$\varphi_\lambda(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_\Omega \beta|u|^p\,\mathrm{d}z - \frac{\lambda}{q}\|u^+\|_q^q + \int_\Omega F(z,u)\,\mathrm{d}z, \text{ for all } u \in W_n^{1,p}(\Omega).$$

Evidently, $\varphi_\lambda \in C^1(W_n^{1,p}(\Omega))$, it is sequentially weakly lower semicontinuous and clearly coercive on the order interval $[\tilde{u},\bar{u}] = \{u \in W_n^{1,p}(\Omega) : \tilde{u}(z) \leq u(z) \leq \bar{u}(z) \text{ a.e. } in\ \Omega\}$. So, by the Weierstrass theorem, we can find $u_0 \in W_n^{1,p}(\Omega)$ such that

$$\varphi_\lambda(u_0) = \inf\{\varphi_\lambda(u) : u \in [\tilde{u},\bar{u}]\}. \tag{36}$$

For any $y \in [\tilde{u},\bar{u}]$, let $s(t) = \varphi_\lambda(ty + (1-t)u_0),\ t \in [0,1]$. From (36), we have $0 \leq s'(0)$, from which

$$0 \leq \langle A(u_0), y-u_0\rangle + \int_\Omega \beta u_0^{p-1}(y-u_0)\,\mathrm{d}z - \lambda\int_\Omega u_0^{q-1}(y-u_0)\,\mathrm{d}z$$
$$+ \int_\Omega f(z,u_0)(y-u_0)\,\mathrm{d}z. \tag{37}$$

Let $h \in W_n^{1,p}(\Omega)$ and $\delta > 0$. We define

$$y(z) = \begin{cases} \tilde{u}(z) & ,\ z \in \{u_0 + \delta h \leq \tilde{u}\} \\ u_0(z) + \delta h(z) & ,\ z \in \{\tilde{u} < u_0 + \delta h < \bar{u}\} \\ \bar{u}(z) & ,\ z \in \{\bar{u} \leq u_0 + \delta h\}. \end{cases}$$

Then, $y \in [\tilde{u},\bar{u}]$ and so it can be used as a test function in (37). We obtain

$$\begin{aligned} 0 \leq\ & \delta\int_\Omega \|Du_0\|^{p-2}(Du_0, Dh)_{I\!R^N}\,\mathrm{d}z + \delta\int_\Omega \beta u_0^{p-1}h\,\mathrm{d}z - \lambda\delta\int_\Omega u_0^{q-1}h\,\mathrm{d}z \\ & + \delta\int_\Omega f(z,u_0)h\,\mathrm{d}z + \int_{\{u_0+\delta h\leq\tilde{u}\}} \|D\tilde{u}\|^{p-2}(D\tilde{u}, D(\tilde{u}-u_0-\delta h))_{I\!R^N}\,\mathrm{d}z \\ & + \int_{\{u_0+\delta h\leq\tilde{u}\}} \beta\tilde{u}^{p-1}(\tilde{u}-u_0-\delta h)\,\mathrm{d}z - \lambda\int_{\{u_0+\delta h\leq\tilde{u}\}} \tilde{u}^{q-1}(\tilde{u}-u_0-\delta h)\,\mathrm{d}z \\ & + \int_{\{u_0+\delta h\leq\tilde{u}\}} f(z,\tilde{u})(\tilde{u}-u_0-\delta h)\,\mathrm{d}z - \int_{\{\bar{u}\leq u_0+\delta h\}} \beta\bar{u}^{p-1}(u_0+\delta h-\bar{u})\,\mathrm{d}z \\ & + \lambda\int_{\{\bar{u}\leq u_0+\delta h\}} \bar{u}^{q-1}(u_0+\delta h-\bar{u})\,\mathrm{d}z - \int_{\{\bar{u}\leq u_0+\delta h\}} f(z,\bar{u})(u_0+\delta h-\bar{u})\,\mathrm{d}z \\ & + \lambda\int_{\{u_0+\delta h\leq\tilde{u}\}} (\tilde{u}^{q-1} - u_0^{q-1})(\tilde{u}-u_0-\delta h)\,\mathrm{d}z \end{aligned}$$

$$
\begin{aligned}
&- \int_{\{u_0+\delta h \le \tilde{u}\}} (f(z,\tilde{u}) - f(z,u_0))(\tilde{u} - u_0 - \delta h)\,\mathrm{d}z \\
&- \lambda \int_{\{\bar{u} \le u_0+\delta h\}} (\bar{u}^{q-1} - u_0^{q-1})(u_0 + \delta h - \bar{u})\,\mathrm{d}z \\
&+ \int_{\{\bar{u} \le u_0+\delta h\}} (f(z,\bar{u}) - f(z,u_0))(u_0 + \delta h - \bar{u})\,\mathrm{d}z \\
&- \int_{\{u_0+\delta h \le \tilde{u}\}} (\|Du_0\|^{p-2}Du_0 - \|D\tilde{u}\|^{p-2}D\tilde{u}, Du_0 - D\tilde{u})_{\mathbb{R}^N}\,\mathrm{d}z \\
&- \int_{\{u_0+\delta h \le \tilde{u}\}} \beta(u_0^{p-1} - \tilde{u}^{p-1})(u_0 - \tilde{u})\,\mathrm{d}z \\
&- \delta \int_{\{u_0+\delta h \le \tilde{u}\}} (\|Du_0\|^{p-2}Du_0 - \|D\tilde{u}\|^{p-2}D\tilde{u}, Dh)_{\mathbb{R}^N}\,\mathrm{d}z \\
&- \delta \int_{\{u_0+\delta h \le \tilde{u}\}} \beta(u_0^{p-1} - \tilde{u}^{p-1})h\,\mathrm{d}z \; - \int_{\{\bar{u} \le u_0+\delta h\}} \|Du_0\|^p\,\mathrm{d}z \\
&+ \int_{\{\bar{u} \le u_0+\delta h\}} \beta(\bar{u}^{p-1} - u_0^{p-1})(u_0 - \bar{u})\,\mathrm{d}z \\
&- \delta \int_{\{\bar{u} \le u_0+\delta h\}} \|Du_0\|^{p-2}(Du_0, Dh)_{\mathbb{R}^N}\,\mathrm{d}z + \delta \int_{\{\bar{u} \le u_0+\delta h\}} \beta(\bar{u}^{p-1} - u_0^{p-1})h\,\mathrm{d}z.
\end{aligned}
\tag{38}
$$

Since $\tilde{u} \in \operatorname{int} C_+$ is a (strict) lower solution for problem $(P)_\lambda$, we have

$$
\begin{aligned}
&\int_{\{u_0+\delta h \le \tilde{u}\}} \|D\tilde{u}\|^{p-2}(D\tilde{u}, D(\tilde{u} - u_0 - \delta h))_{\mathbb{R}^N}\,\mathrm{d}z + \int_{\{u_0+\delta h \le \tilde{u}\}} \beta\tilde{u}^{p-1}(\tilde{u} - u_0 - \delta h)\,\mathrm{d}z \\
&\quad -\lambda \int_{\{u_0+\delta h \le \tilde{u}\}} \tilde{u}^{q-1}(\tilde{u} - u_0 - \delta h)\,\mathrm{d}z + \int_{\{u_0+\delta h \le \tilde{u}\}} f(z,\tilde{u})(\tilde{u} - u_0 - \delta h)\,\mathrm{d}z \;\le\; 0.
\end{aligned}
\tag{39}
$$

Similarly, since $\bar{u} = \xi \in \operatorname{int} C_+$ is a (strict) upper solution for problem $(P)_\lambda$, we have

$$
\begin{aligned}
&- \int_{\{\bar{u} \le u_0+\delta h\}} \beta\bar{u}^{p-1}(u_0 + \delta h - \bar{u})\,\mathrm{d}z + \lambda \int_{\{\bar{u} \le u_0+\delta h\}} \bar{u}^{q-1}(u_0 + \delta h - \bar{u})\,\mathrm{d}z \\
&\quad - \int_{\{\bar{u} \le u_0+\delta h\}} f(z,\bar{u})(u_0 + \delta h - \bar{u})\,\mathrm{d}z \le\; 0.
\end{aligned}
\tag{40}
$$

Recall that the map $\zeta \mapsto \|\zeta\|^{p-2}\zeta$, $\zeta \in I\!R^N$, is monotone. Hence,

$$\int_{\{u_0+\delta h\le\tilde{u}\}} (\|Du_0\|^{p-2}Du_0 - \|D\tilde{u}\|^{p-2}D\tilde{u}, Du_0 - D\tilde{u})_{I\!R^N}\,\mathrm{d}z + \int_{\{u_0+\delta h\le\tilde{u}\}} \beta(u_0^{p-1} - \tilde{u}^{p-1})(u_0 - \tilde{u})\,\mathrm{d}z \;\ge\; 0. \tag{41}$$

Since $u_0 \in [\tilde{u}, \bar{u}]$, we have

$$\int_{\{\bar{u}\le u_0+\delta h\}} \beta(\bar{u}^{p-1} - u_0^{p-1})(\bar{u} - u_0)\,\mathrm{d}z \;\ge\; 0\,; \tag{42}$$

moreover, also using hypothesis H(i), we get

$$\lambda \int_{\{u_0+\delta h\le\tilde{u}\}} (\tilde{u}^{q-1} - u_0^{q-1})(\tilde{u} - u_0 - \delta h)\,\mathrm{d}z - \int_{\{u_0+\delta h\le\tilde{u}\}} (f(z,\tilde{u}) - f(z,u_0))(\tilde{u} - u_0 - \delta h)\,\mathrm{d}z \;\le\; -c_3\,\delta \int_{\{u_0+\delta h\le\tilde{u}<u_0\}} h\,\mathrm{d}z \tag{43}$$

for some $c_3 > 0$ (recall that $h(z) \le 0$ a.e. on $\{u_0 + \delta h \le \tilde{u}\}$), and

$$\lambda \int_{\{\bar{u}\le u_0+\delta h\}} (\bar{u}^{q-1} - u_0^{q-1})(u_0 + \delta h - \bar{u})\,\mathrm{d}z - \int_{\{\bar{u}\le u_0+\delta h\}} (f(z,\bar{u}) - f(z,u_0))(u_0 + \delta h - \bar{u})\,\mathrm{d}z \le\; c_4\,\delta \int_{\{u_0<\bar{u}\le u_0+\delta h\}} h\,\mathrm{d}z \tag{44}$$

for some $c_4 > 0$ (recall that $h(z) \ge 0$ a.e. on $\{\bar{u} \le u_0 + \delta h\}$).

We return to (38), use (39)–(44) and divide with $\delta > 0$. Then,

$$\begin{aligned}
0 \le& \int_\Omega \|Du_0\|^{p-2}(Du_0, Dh)_{I\!R^N}\,\mathrm{d}z + \int_\Omega \beta u_0^{p-1}h\,\mathrm{d}z - \lambda\int_\Omega u_0^{q-1}h\,\mathrm{d}z + \int_\Omega f(z,u_0)h\,\mathrm{d}z\\
&- c_3 \int_{\{u_0+\delta h\le\tilde{u}<u_0\}} h\,\mathrm{d}z \;+\; c_4 \int_{\{u_0<\bar{u}\le u_0+\delta h\}} h\,\mathrm{d}z\\
&- \int_{\{u_0+\delta h\le\tilde{u}\}} (\|Du_0\|^{p-2}Du_0 - \|D\tilde{u}\|^{p-2}D\tilde{u}, Dh)_{I\!R^N}\,\mathrm{d}z\\
&- \int_{\{u_0+\delta h\le\tilde{u}\}} \beta(u_0^{p-1} - \tilde{u}^{p-1})h\,\mathrm{d}z\\
&- \int_{\{\bar{u}\le u_0+\delta h\}} \|Du_0\|^{p-2}(Du_0, Dh)_{I\!R^N}\,\mathrm{d}z + \int_{\{\bar{u}\le u_0+\delta h\}} \beta(\bar{u}^{p-1} - u_0^{p-1})h\,\mathrm{d}z.
\end{aligned} \tag{45}$$

Note that

$$|\{u_0 + \delta h \le \tilde{u} < u_0\}|_N \to 0 \text{ and } |\{u_0 < \bar{u} \le u_0 + \delta h\}|_N \to 0 \text{ as } \delta \to 0^+.$$

Also from Stampacchia's theorem (see, for example, [8] pp. 195–196), we know that

$$Du_0(z) = D\tilde{u}(z) \text{ a.e. on } \{u_0 = \tilde{u}\} \text{ and } Du_0(z) = 0 \text{ a.e. on } \{u_0 = \bar{u}\}$$

(recall that $\bar{u} \equiv \xi > 0$).

So, if in (45) we pass to the limit as $\delta \to 0^+$ and recall that $u_0 \in [\tilde{u}, \bar{u}]$, we obtain

$$0 \leq \langle A(u_0), h\rangle + \int_\Omega \beta u_0^{p-1} h \, dz - \lambda \int_\Omega u_0^{q-1} h \, dz + \int_\Omega f(z, u_0) h \, dz. \tag{46}$$

Since $h \in W_n^{1,p}(\Omega)$ is arbitrary, from (46), we infer that

$$A(u_0) + \beta u_0^{p-1} = \lambda u_0^{q-1} - N(u_0)$$

then (see [20])

$$-\Delta_p u_0(z) + \beta(z) u_0(z)^{p-1} = \lambda u_0(z)^{q-1} - f(z, u_0(z)) \text{ a.e. in } \Omega, \ \frac{\partial u_0}{\partial n} = 0 \text{ on } \partial\Omega.$$

Therefore, $u_0 \in \operatorname{int} C_+$ (nonlinear regularity theory) is a solution of $(P)_\lambda$.

Let $\mu \in \big(0, \min_{\bar{\Omega}} \tilde{u}\big)$ (recall that $\tilde{u} \in \operatorname{int} C_+$). Also, let $r = \|\bar{u}\|_\infty + 1$, $I = [\lambda_*, \lambda + 1]$ and $\eta = \eta(r, I) > 0$ is the constant postulated by hypothesis H(iv). For $\delta \in (0, \mu)$ we set $u_\delta = u_0 - \delta$. Evidently, $u_\delta \in \operatorname{int} C_+$. For a.a. $z \in \Omega$, we have

$$\begin{aligned} -\Delta_p u_\delta(z) + \beta(z) u_\delta(z)^{p-1} + \eta u_\delta(z)^{\theta-1} = &-\Delta_p u_0(z) + \beta(z) u_0(z)^{p-1} \\ &+ \eta u_0(z)^{\theta-1} - \rho(\delta) \end{aligned}$$

with $\rho(\delta) \to 0^+$ as $\delta \to 0^+$. Since $u_0 \in \operatorname{int} C_+$ is a solution of $(P)_\lambda$, the previous becomes

$$-\Delta_p u_\delta(z) + \beta(z) u_\delta(z)^{p-1} + \eta u_\delta(z)^{\theta-1} = \lambda u_0(z)^{q-1} - f(z, u_0(z)) + \eta u_0(z)^{\theta-1} - \rho(\delta);$$

now, by H(iv) and since $\tilde{u} \in \operatorname{int} C_+$ solves $(P)_{\tilde{\lambda}}$, we have

$$\begin{aligned} -\Delta_p u_\delta(z) + \beta(z) u_\delta(z)^{p-1} + \eta u_\delta(z)^{\theta-1} &\geq \lambda \tilde{u}(z)^{q-1} - f(z, \tilde{u}(z)) + \eta \tilde{u}(z)^{\theta-1} - \rho(\delta) \\ &\geq (\lambda - \tilde{\lambda}) \mu^{q-1} - \rho(\delta) + \tilde{\lambda} \tilde{u}(z)^{q-1} - f(z, \tilde{u}(z)) + \eta \tilde{u}(z)^{\theta-1} \\ &= (\lambda - \tilde{\lambda}) \mu^{q-1} - \rho(\delta) - \Delta_p \tilde{u}(z) + \beta(z) \tilde{u}(z)^{p-1} + \eta \tilde{u}(z)^{\theta-1} \end{aligned} \tag{47}$$

(recall $\tilde{\lambda} < \lambda$).

We choose $\hat{\delta} \in (0, 1)$ small such that for $\delta \in (0, \hat{\delta}]$, we have

$$\rho(\delta) \leq (\lambda - \tilde{\lambda}) \mu^{q-1}.$$

Using this in (47), for $\delta \in (0, \hat{\delta}]$, we obtain

$$A(u_\delta) + \beta u_\delta^{p-1} + \eta u_\delta^{\theta-1} \geq A(\tilde{u}) + \beta \tilde{u}^{p-1} + \eta \tilde{u}^{\theta-1} \text{ in } W_n^{1,p}(\Omega)^*. \tag{48}$$

Fixed $\delta \in (0, \hat{\delta}]$, acting on (48) with $(\tilde{u} - u_\delta)^+ \in W_n^{1,p}(\Omega)$, we obtain

$$\begin{aligned} &\int_{\{\tilde{u} > u_\delta\}} (\|Du_\delta\|^{p-2} Du_\delta - \|D\tilde{u}\|^{p-2} D\tilde{u}, D\tilde{u} - Du_\delta)_{\mathbb{R}^N} dz \\ &+ \int_{\{\tilde{u} > u_\delta\}} \beta (u_\delta^{p-1} - \tilde{u}^{p-1})(\tilde{u} - u_\delta) \, dz + \eta \int_{\{\tilde{u} > u_\delta\}} (u_\delta^{\theta-1} - \tilde{u}^{\theta-1})(\tilde{u} - u_\delta) \, dz \geq 0. \end{aligned} \tag{49}$$

The strict monotonicity of $\zeta \mapsto \|\zeta\|^{\sigma-2}\zeta$, $\zeta \in I\!R^N$, for all $\sigma > 1$ and (49) imply $|\{\tilde{u} > u_\delta\}|_N = 0$, from which

$$\tilde{u} \le u_\delta,\ i.e.\ u_0 - \tilde{u} \in \operatorname{int} C_+. \tag{50}$$

Next let $u_\delta = u_0 + \delta$, $\delta \in (0, r]$. Then, for a.a. $z \in \Omega$, we have

$$-\Delta_p u_\delta(z) + \beta(z)u_\delta(z)^{p-1} + \eta u_\delta(z)^{\theta-1} = -\Delta_p u_0(z) + \beta(z)u_0(z)^{p-1} + \eta u_0(z)^{\theta-1} + \hat{\rho}(\delta)$$

with $\hat{\rho}(\delta) \to 0^+$ as $\delta \to 0^+$. Since $u_0 \in \operatorname{int} C_+$ is a solution of $(P)_\lambda$ and by recalling that $\bar{u} \equiv \xi \ge M$, $u_0 \le \bar{u}$ and H(iv), we can deduce the following estimate

$$\begin{aligned}-\Delta_p u_\delta(z) + \beta(z)u_\delta(z)^{p-1} + \eta u_\delta(z)^{\theta-1} &= \lambda u_0(z)^{q-1} - f(z, u_0(z)) + \eta u_0(z)^{\theta-1} + \hat{\rho}(\delta)\\ &\le \lambda\xi^{q-1} - f(z,\xi) + \eta\xi^{\theta-1} + \hat{\rho}(\delta).\end{aligned}$$

Recall that $\lambda\xi^{q-1} - f(z,\xi) \le \lambda\xi^{q-1} - \gamma\xi^{\theta-1} < 0$ for a.a. $z \in \Omega$ (see the first part of the proof). Since $\hat{\rho}(\delta) \to 0^+$ as $\delta \to 0^+$, we can find $\hat{\delta}_0 > 0$ such that $\lambda\xi^{q-1} - \gamma\xi^{\theta-1} + \hat{\rho}(\delta) \le 0$ for all $\delta \in (0, \hat{\delta}_0]$, so

$$\lambda\xi^{q-1} - f(z,\xi) + \hat{\rho}(\delta) \le 0\ , \text{ for a.a. } z \in \Omega \text{ and all } \delta \in (0, \hat{\delta}_0].$$

Therefore,

$$-\Delta_p u_\delta(z) + \beta(z)u_\delta(z)^{p-1} + \eta u_\delta(z)^{\theta-1} \le \beta(z)\xi^{p-1} + \eta\xi^{\theta-1},\ \text{ for a.a. } z \in \Omega \text{ and all } \delta \in (0, \hat{\delta}_0],$$

hence

$$A(u_\delta) + \beta u_\delta^{p-1} + \eta u_\delta^{\theta-1} \le A(\xi) + \beta\xi^{p-1} + \eta\xi^{\theta-1} \text{ in } W_n^{1,p}(\Omega), \text{ for all } \delta \in (0, \hat{\delta}_0]. \tag{51}$$

Acting on (51) with $(u_\delta - \xi)^+ \in W_n^{1,p}(\Omega)$ as before, we obtain

$$u_\delta \le \xi,\ \text{ i.e. } \bar{u} - u_0 \in \operatorname{int} C_+. \tag{52}$$

From (50) and (52), it follows that u_0 is a local $C_n^1(\overline{\Omega})$-minimizer of φ_λ. Invoking Proposition 2.4 it follows that u_0 is a local $W_n^{1,p}(\Omega)$-minimizer of φ_λ.

By virtue of hypothesis H(ii), we have $F(z,x) \ge 0$ for a.a. $z \in \Omega$ and all $x \ge 0$. So, for all $u \in W_n^{1,p}(\Omega)$, we have that there exist $\xi_0 > 0$ and $c_5 > 0$ such that (see Lemma 3.1)

$$\varphi_\lambda(u) \ge \frac{\xi_0}{p}\|u\|^p - \frac{\lambda}{q}\|u\|_q^q \ge \frac{\xi_0}{p}\|u\|^p - \frac{\lambda}{q}c_5\|u\|^q.$$

Since $q > p$, we can find $\bar{\delta} \in (0,1)$ small such that

$$\varphi_\lambda(u) > 0 = \varphi_\lambda(0),\ \text{ for all } 0 < \|u\| \le \bar{\delta},$$

then $u = 0$ is a strict local minimizer of φ_λ.

Without any loss of generality, we may assume that

$$\varphi_\lambda(0) = 0 \le \varphi_\lambda(u_0) \tag{53}$$

(the reasoning is similar if $\varphi_\lambda(u_0) < 0 = \varphi_\lambda(0)$). We may assume that u_0 is an isolated critical point of φ_λ. Indeed, otherwise we have a whole sequence of distinct positive solutions of $(P)_\lambda$ and so we have done. Then, as in the proof of Proposition 29 of [1], we can find $\rho \in (0,1)$ small such that $\|u_0\| > \rho$ and

$$\varphi_\lambda(u_0) < \inf\,[\varphi_\lambda(u)\ :\ \|u - u_0\| = \rho] = \eta_\lambda^\rho. \tag{54}$$

Let $E_0 = \{0, u_0\}$, $E = [0, u_0]$ and $D = \partial B_\rho(u_0) = \{u \in W_n^{1,p}(\Omega) : \|u - u_0\| = \rho\}$. Clearly the pair $\{E_0, E\}$ is linking with D in $W_n^{1,p}(\Omega)$ (see Definition 2.1). Moreover, since $\theta > q$ (see H(ii)), we see that φ_λ is coercive. Therefore, it satisfies the PS-condition. Hence, we can apply Theorem 2.1 and obtain $\hat{u} \in W_n^{1,p}(\Omega)$ such that (see (53) and (54))

$$\varphi_\lambda(0) = 0 \leq \varphi_\lambda(u_0) < \eta_\lambda^\rho \leq \varphi_\lambda(\hat{u}) \tag{55}$$

and

$$\varphi_\lambda'(\hat{u}) = 0. \tag{56}$$

From (55), we see that $\hat{u} \notin \{0, u_0\}$ and from (56)

$$A(\hat{u}) + \beta|\hat{u}|^{p-2}\hat{u} = \lambda(\hat{u}^+)^{q-1} - N(\hat{u}). \tag{57}$$

As before acting on (57) with $-\hat{u}^- \in W_n^{1,p}(\Omega)$, we obtain $\hat{u} \geq 0$, $\hat{u} \neq 0$. So

$$A(\hat{u}) + \beta\hat{u}^{p-1} = \lambda(\hat{u})^{q-1} - N(\hat{u})$$

which implies

$$-\Delta_p\hat{u}(z) + \beta(z)\hat{u}(z)^{p-1} = \lambda\hat{u}(z)^{q-1} - f(z, \hat{u}(z)) \text{ a.e. in } \Omega,\ \frac{\partial\hat{u}}{\partial n} = 0 \text{ on } \partial\Omega,$$

therefore $\hat{u} \in C_+ \setminus \{0\}$ (nonlinear regularity) solves $(P)_\lambda$.

Moreover, as before using hypothesis H(iv) and the nonlinear maximum principle of [25], we conclude that $\hat{u} \in \operatorname{int} C_+$. □

Next, we examine the critical case $\lambda = \lambda_*$.

Proposition 3.5 *If hypotheses H hold and $\beta \in L^\infty(\Omega)_+$, $\beta \neq 0$, then problem $(P)_{\lambda_*}$ has at least one positive solution.*

Proof Let $\{\lambda_n\}_{n\geq 1} \subseteq (\lambda_*, +\infty)$ be a decreasing sequence such that $\lambda_n \to \lambda_*$ as $n \to \infty$. From the proof of Proposition 3.4, we know that for every λ_n, we can find a solution $u_n \in \operatorname{int} C_+$ of $(P)_{\lambda_n}$ such that $\{u_n\}_{n\geq 1} \subseteq W_n^{1,p}(\Omega)$ is bounded (in fact we have $u_n \leq \bar{u}$ where $\bar{u} \equiv \xi$ with $\xi \geq M$ large such that $\lambda_1\xi^{q-1} - \gamma\xi^{\theta-1} < 0$, recall $\theta > q$). Then, by virtue of Theorem 2 of [15] (see also [13], Proposition 5), we can find $\mu \in (0, 1)$ and $M_4 > 0$ such that

$$u_n \in C_n^{1,\mu}(\overline{\Omega}) \text{ and } \|u_n\|_{C_n^{1,\mu}(\overline{\Omega})} \leq M_4\,, \quad \text{for all } n \geq 1.$$

Exploiting the compact embedding of $C_n^{1,\mu}(\overline{\Omega})$ into $C_n^1(\overline{\Omega})$, we may assume that

$$u_n \to u_* \text{ in } C_n^1(\overline{\Omega}).$$

Evidently, $u_* \in C_+$. If $u_* = 0$, then introducing $y_n = \dfrac{u_n}{\|u_n\|}$, $n \geq 1$ and reasoning as in the proof of Proposition 3.3, via hypothesis H(iii), we reach a contradiction. So, $u_* \neq 0$ and

$$A(u_*) + \beta u_*^{p-1} = \lambda_* u_*^{q-1} - N(u_*),$$

which implies

$$-\Delta_p u_*(z) + \beta(z)u_*(z)^{p-1} = \lambda_* u_*(z)^{q-1} - f(z, u_*(z)) \text{ a.e. in } \Omega\,,\ \frac{\partial u_*}{\partial n} = 0 \text{ on } \partial\Omega.$$

As before, via H(iv) and [25] we have $u_* \in \operatorname{int} C_+$ and of course solves $(P)_{\lambda_*}$. □

Summarizing we have the following bifurcation-type result for our superdiffusive nonlinear p-logistic equation.

Theorem 3.1 *If hypotheses H hold and $\beta \in L^\infty(\Omega)_+$, $\beta \neq 0$, then there exists $\lambda_* > 0$ such that*

(a) *for $\lambda \in (0, \lambda_*)$ problem $(P)_\lambda$ has no positive solution;*
(b) *for $\lambda = \lambda_*$ problem $(P)_\lambda$ has at least one positive solution;*
(c) *for $\lambda > \lambda_*$ problem $(P)_\lambda$ has at least two positive solutions.*

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