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Fourier decomposition of properly elliptic boundary value problems in a half-plane

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Abstract For a general class of second-order elliptic boundary value problems in the lower half-plane, we show that the existence and uniqueness of solutions in L^p Sobolev spaces is reduced to the invertibility of the ordinary differential operators obtained by Fourier decomposition. This terminology refers to the partial Fourier series expansion in the case of horizontally periodic solutions and to the partial Fourier transform otherwise. The problem is straightforward when $p = 2$ and, in the periodic case, the same question on a strip with finite width can also be quickly settled by indirect arguments irrespective of $p \in (1, \infty)$. However, in the half-plane, the infinite depth raises serious difficulties when $p \neq 2$. These difficulties are overcome by writing the problem as a first-order system and using existing abstract results about operator valued Fourier multipliers. In that approach, the randomized boundedness of the resolvent becomes the central issue.

Keywords Elliptic boundary value problem · Half-plane · Sobolev space · Fourier series · Fourier transform · r-boundedness

Mathematics Subject Classification (2000) 35B10 · 35J25 · 42A45 · 43A50

1 Introduction

The goal of this paper is to show that for a general class of second-order linear elliptic boundary value problems on the lower half-plane $\mathbb{R} \times \mathbb{R}_-$ with generic variable (x, y), the existence and uniqueness of solutions in L^p Sobolev spaces, $p \in (1, \infty)$, can be reduced to the same question for the collection of ODEs obtained by Fourier decomposition (Fourier series for

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solutions periodic in x, Fourier transform otherwise). While mostly routine when $p = 2$, the problem is delicate when $p \neq 2$. For expository purposes, the main focus will be on the 2π-periodic case.

Consider the homogeneous boundary value problem

$$\begin{cases} \mathfrak{P}u := u_{xx} + 2b(y)u_{xy} + c(y)u_{yy} + \alpha(y)u_x + \beta(y)u_y + \gamma(y)u = g, \\ u_y + \theta u = 0 \text{ on } \mathbb{R} \times \{0\}, \end{cases} \tag{1.1}$$

That the coefficient of u_{xx} is 1 is just a convenient normalization, which can always be achieved after factoring the coefficient of u_{xx}. The other coefficients are complex valued and $\theta \in \mathbb{C}$.

If g and u in (1.1) are 2π-periodic in x and if $g \in L^p((0, 2\pi) \times \mathbb{R}_-)$ for some $p \in (1, \infty)$, the Fourier series approach consists in expanding $g(x, y) = \sum_{k \in \mathbb{Z}} \mathrm{e}^{-ikx} g_k(y)$ with $g_k \in L^p_- := L^p(\mathbb{R}_-)$ and looking for solutions $u(x, y) = \sum_{k \in \mathbb{Z}} \mathrm{e}^{-ikx} u_k(y)$. Then, formally at least, the problem amounts to solving $P_k u_k = g_k$ where

$$P_k w := c(y)w'' + (\beta(y) - 2ikb(y))w' + (\gamma(y) - k^2 - ik\alpha(y))w, \tag{1.2}$$

with boundary condition $u_k'(0) + \theta u_k(0) = 0$. Here and throughout the paper, the "prime" refers to y-differentiation.

For consistency, the solutions u should be sought in the space

$$\begin{aligned} W^{2,p}_{(\theta),per}((0, 2\pi) \times \mathbb{R}_-) := \big\{ & u \in W^{2,p}((0, 2\pi) \times \mathbb{R}_-) : u_y + \theta u = 0 \\ & \text{on } (0, 2\pi) \times \{0\}, \quad u(0, \cdot) = u(2\pi, \cdot), \quad u_x(0, \cdot) = u_x(2\pi, \cdot) \big\} \end{aligned} \tag{1.3}$$

and the solutions u_k of $P_k u_k = g_k$ in the space

$$W^{2,p}_{(\theta)-} := \{ w \in W^{2,p}_- : w'(0) + \theta w(0) = 0 \}, \tag{1.4}$$

where $W^{m,p}_- := W^{m,p}(\mathbb{R}_-)$ for every $m \in \mathbb{N}$. All these function spaces consist of complex-valued functions.

The periodic boundary conditions incorporated in (1.3) ensure that the extension of $u \in W^{2,p}_{(\theta),per}((0, 2\pi) \times \mathbb{R}_-)$ by periodicity is in $\bigcap_{n \in \mathbb{N}} W^{2,p}((-n, n) \times \mathbb{R}_-)$ and that the 2π-periodic solutions u of (1.1) in the space $\bigcap_{n \in \mathbb{N}} W^{2,p}((-n, n) \times \mathbb{R}_-)$ are exactly the solutions of $\mathfrak{P}u = g$ in $W^{2,p}_{(\theta),per}((0, 2\pi) \times \mathbb{R}_-)$, extended by periodicity.

The question is whether the unique solvability of $\mathfrak{P}u = g$ in $W^{2,p}_{(\theta),per}((0, 2\pi) \times \mathbb{R}_-)$ is equivalent to the unique solvability of $P_k u_k = g_k$ in $W^{2,p}_{(\theta)-}$ for all $k \in \mathbb{Z}$. This is quite reminiscent of the question raised in the elementary treatment of evolution problems by separation of variables, but the functional setting raises the difficulty to a different level and it seems that it can only be resolved with the help of fairly recent concepts and developments.

When $p = 2$, basic Hilbert space theory shows that the answer is positive provided that the norm of P_k^{-1} in $\mathcal{L}(L^2_-, W^{2,2}_-)$ is uniformly bounded by a constant independent of $k \in \mathbb{Z}$. The existence of such a constant is not necessarily a trivial technical matter—it is not known without assumptions about the behavior of the coefficients of $\mathfrak{P}$ at infinity—but it is conceptually straightforward.

The real challenge arises when $p \neq 2$, for then the convergence of $\sum_{k \in \mathbb{Z}} \mathrm{e}^{-ikx} u_k(y)$ in $W^{2,p}((0, 2\pi) \times \mathbb{R}_-)$ *must* depend upon more than the uniform boundedness of the norm of P_k^{-1}. This is true even when the coefficients are constant. Indeed, already in the much simpler scalar case (when the u_k are just complex numbers), the Fourier series of an L^p function is generally not unconditionally convergent in L^p when $p \neq 2$, that is, the control of the size of the Fourier coefficients alone does not suffice to assess L^p convergence.

The exact condition that should be required of P_k^{-1} will be identified later on. For the time being, we sketch a simple indirect procedure when the half-plane is replaced by a strip with *finite* width: The domain is then a rectangle and the classical *a priori* estimates and the compactness of the Sobolev embeddings show that $\mathfrak{P}$ is semi-Fredholm with finite dimensional null space (due to the periodicity in x of the solutions, the corners of the rectangle are not an obstacle with the estimates). That its index is 0 can next be seen by using homotopy invariance and a direct calculation in a simpler special case. In a different context, this procedure is described in the appendix of Simpson and Spector [25]. Irrespective of $p \in (1, \infty)$, it is easily seen that the null space of $\mathfrak{P}$ is trivial if and only if the null space of P_k is trivial for every k. Since P_k is also Fredholm of index 0 (ordinary differential operator on a bounded interval), this proves that the invertibility of $\mathfrak{P}$ is equivalent to the invertibility of all the P_k.

The above method does not work in a half-space because the domain remains unbounded after attention is confined to an interval of length 2π in the x direction, so that the embeddings are not compact. As a result, the *a priori* estimates do not imply the semi-Fredholmness of $\mathfrak{P}$, let alone its Fredholmness of index 0. The same statement is even true for the P_k. Thus, the unique solvability does not follow from the triviality of the null space and the argument breaks down.

We now state our main result (Theorem 1.1 below) and explain our approach. We shall say that the coefficients of $\mathfrak{P}$ are *asymptotically periodic* if there are periodic functions $b_\sharp, \ldots, \gamma_\sharp$ with the same period L such that $\lim_{y \to -\infty} |b(y) - b_\sharp(y)| = \cdots = \lim_{y \to -\infty} |\gamma(y) - \gamma_\sharp(y)| = 0$. If so and if the coefficients of $\mathfrak{P}$ are continuous on $\overline{\mathbb{R}}_-$, their uniform continuity is equivalent to the continuity of $b_\sharp, \ldots, \gamma_\sharp$.

Theorem 1.1 *Suppose that $\mathfrak{P}$ in (1.1) is uniformly and properly elliptic with bounded uniformly continuous and asymptotically periodic coefficients. Let $p \in (1, \infty)$ be given. Then, $\mathfrak{P}$ is an isomorphism of $W^{2,p}_{(\theta),per}((0, 2\pi) \times \mathbb{R}_-)$ onto $L^p((0, 2\pi) \times \mathbb{R}_-)$ if and only if P_k is an isomorphism of $W^{2,p}_{(\theta)-}$ onto L^p_- for every $k \in \mathbb{Z}$.*

The subsequent comments help clarify the exact nature of the hypotheses made in Theorem 1.1. First, since a bounded uniformly continuous function on $\mathbb{R}_-$ has a unique bounded uniformly continuous extension to $\overline{\mathbb{R}}_-$, the boundedness and uniform continuity of the coefficients hold in $\mathbb{R}_-$ and $\overline{\mathbb{R}}_-$ simultaneously. Next, the uniform ellipticity condition means that

$$|\xi_1^2 + 2b(y)\xi_1\xi_2 + c(y)\xi_2^2| \geq \nu(\xi_1^2 + \xi_2^2),$$

for some constant $\nu > 0$, every $y \in \mathbb{R}_-$ (or, equivalently, $y \in \overline{\mathbb{R}}_-$) and every $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. Note that the strong ellipticity of $\mathfrak{P}$ is not assumed (variational arguments are nowhere involved).

The usual formulation of the proper ellipticity of $\mathfrak{P}$ is that it is elliptic and that, for every $y \in \overline{\mathbb{R}}_-$ and every pair of linearly independent vectors $\xi = (\xi_1, \xi_2)$ and $\xi' = (\xi_1', \xi_2')$ in $\mathbb{R}^2$, the polynomial of the variable $\tau \in \mathbb{C}$

$$(\xi_1 + \tau\xi_1')^2 + 2b(y)(\xi_1 + \tau\xi_1')(\xi_2 + \tau\xi_2') + c(y)(\xi_2 + \tau\xi_2')^2,$$

has exactly one root with (necessarily strictly) positive imaginary part, so that the other root has strictly negative imaginary part.

Traditionally, proper ellipticity is only involved in connection with the boundary conditions, which turns out to justify this rather convoluted definition. However, it is readily checked that, assuming ellipticity, the above root condition is equivalent to the requirement that the polynomial of the variable $\lambda \in \mathbb{C}$

$$c(y)\lambda^2 + 2b(y)\lambda + 1,$$

has exactly one root with strictly positive (negative) imaginary part. Since $\overline{\mathbb{R}}_-$ is connected and ellipticity rules out real roots, it suffices to check this property at a single point $y \in \overline{\mathbb{R}}$. This much simpler root condition is crucial to the arguments of this paper. In other words, for once, proper ellipticity is not merely needed for the treatment of the boundary conditions.

Lastly, it should also be stressed that the periodicity (in y) of the limiting coefficients has nothing to do with the periodicity (in x) of the solutions and that it is merely a coincidental technical limitation. This will be explained further below.

A special case of Theorem 1.1 arises when the coefficients are real and asymptotically constant, that is, $b_\sharp, \ldots, \gamma_\sharp$ are real constant functions $b_{-\infty}, \ldots, \gamma_{-\infty}$. This is often relevant in problems on unbounded domains. If so, Theorem 1.1 takes a simpler form (Theorem 6.1) and an even simpler one when the coefficients are real and constant (Theorem 4.3). When the coefficients and θ are real, unique solvability for complex valued u and g is equivalent to unique solvability when they are real valued, so that Theorem 1.1 and its variants are applicable to both settings.

Our strategy will be to view $\mathfrak{P}u = g$ as an evolution problem, with x playing the role of the time variable. In this perspective, the new variable $v := u_x$ is introduced and $\mathfrak{P}u = g$ becomes the first-order system (details in Sect. 2)

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u \\ v \end{pmatrix} + \mathfrak{A}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}, \tag{1.5}$$

where

$$\mathfrak{A} := \begin{pmatrix} 0 & -I \\ P & Q \end{pmatrix} \tag{1.6}$$

and P and Q are the differential operators on $\mathbb{R}_-$ given by

$$P := c(y)\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \beta(y)\frac{\mathrm{d}}{\mathrm{d}y} + \gamma(y), \tag{1.7}$$

$$Q := 2b(y)\frac{\mathrm{d}}{\mathrm{d}y} + \alpha(y). \tag{1.8}$$

For abstract first-order operators, the following theorem of Arendt and Bu [1, Theorem 2.3] (rephrased) gives a necessary and sufficient condition for solvability in spaces of periodic functions. It has already been used in various applications (integro-differential equations, delayed equations, etc.) albeit with a more apparent evolutionary nature than the elliptic problem of this paper.

Theorem 1.2 *Let X be a (complex) UMD Banach space. If $p \in (1, \infty)$, set*

$$W^{1,p}_{per}(0, 2\pi; X) := \{w \in W^{1,p}(0, 2\pi; X) : w(0) = w(2\pi)\} \tag{1.9}$$

and let A be a closed unbounded linear operator on X with domain W, equipped with the graph norm. Then, the operator $\frac{d}{dx} + A$ is an isomorphism of $L^p(0, 2\pi; W) \cap W^{1,p}_{per}(0, 2\pi; X)$ onto $L^p(0, 2\pi; X)$ if and only if $(A - ikI)^{-1} \in \mathcal{L}(X)$ exists for every $k \in \mathbb{Z}$ and the sequence $(k(A - ikI)^{-1})_{k\in\mathbb{Z}}$ is r-bounded in $\mathcal{L}(X)$.

Several comments are in order. First, A need not generate a semigroup and indeed the hypotheses are unchanged if A is replaced by $-A$. This (correctly) suggests that Theorem 1.2

does not implicitly contain any preferred direction of evolution, in sharp contrast with what is well known for initial value problems.

Next, recall that X is a UMD Banach space if the Hilbert transform is a bounded operator on $L^q(\mathbb{R}, X)$ for some (and then, as it turns out, every) $q \in (1, \infty)$. This is important, but not an issue here since all the reflexive L^p spaces, along with their closed subspaces (and hence all the reflexive Sobolev spaces and their closed subspaces) are UMD [4,5,13].

In contrast, the r-boundedness, shorthand for "randomized boundedness" (or "Rademacher boundedness") will be our main focus. It is a concept of boundedness for sets of bounded linear operators on Banach spaces, more restrictive than norm-boundedness, except in the Hilbert space case. It is typically much more difficult to check than norm-boundedness, but a *necessary* condition in Theorem 1.2. For convenience, the main properties of r-boundedness are collected in a short appendix.

There are related results in the literature that directly address the periodic solutions of abstract second-order operators $\frac{\mathrm{d}^2}{\mathrm{d}x^2} + Q\frac{\mathrm{d}}{\mathrm{d}x} + P$. See [1] when $Q = 0$ or [18] when both P and Q are closed operators on the space X. In general, these results cannot be used with (1.1). For example, Q in (1.8) with (natural) domain $W_-^{1,p}$ is not a closed operator on $X = L_-^p$ if b vanishes at some point of $\overline{\mathbb{R}}_-$.

Theorem 1.1 will follow from Theorem 1.2 with the choice

$$X := W_-^{1,p} \times L_-^p, \quad W := W_{(\theta)-}^{2,p} \times W_-^{1,p}, \tag{1.10}$$

and $A = \mathfrak{A}$ in (1.6). This is explained in the next section. The remainder of the paper is devoted to the proof of Theorem 1.1 and some of its variants.

The case when the coefficients are constant is treated in Sect. 3 when $\theta = 0$ (Neumann boundary condition) and in Sect. 4 in general. In both these sections, we rely on explicit formulas for solutions and estimates from harmonic analysis, in the spirit of Denk et al. [11], where the r-boundedness of resolvents of elliptic systems is discussed. Since $\mathfrak{A}$ is not (at all) an elliptic system, the results of [11] are not applicable.

The next step is to prove Theorem 1.1 when the coefficients are periodic, which is done in Sect. 5, by using the estimates when the coefficients are constant together with a partition of unity on $\overline{\mathbb{R}}_-$. In addition to the r-boundedness issues, there are technical difficulties related to the partition of unity, because it is crucial that the cut-off functions have uniformly bounded derivatives. The periodicity of the coefficients makes it possible to obtain such cut-off functions from a partition of unity on the circle and this is in fact the only reason why their periodicity is assumed. As a result, Theorem 1.1 is clearly true in a much broader setting, but finding alternative conditions as easily verifiable as the periodicity of the coefficients seems to be a rather tricky exercise.

The final step of the proof of Theorem 1.1, when the coefficients are asymptotically periodic, is given in Sect. 6, based on the results when the coefficients are periodic and with another, more standard, partition of unity.

As we shall see, the unique solvability of $\mathfrak{P}u = g$ is equivalent to the existence of an inverse R_k (actually, R_k^θ to keep track of the θ-dependence) of P_k for every $k \in \mathbb{Z}$, together with the r-boundedness of the sequences $(k^2 R_k)_{k\in\mathbb{Z}}$ and $(kR_k)_{k\in\mathbb{Z}}$ in various function spaces. Thus, when $p \neq 2$, r-boundedness rather than mere norm-boundedness is the correct and *necessary* condition to verify. The two concepts coincide when (and only when) $p = 2$, which explains why norm estimates suffice in this case.

Theorem 1.1 remains true for a Dirichlet boundary condition. It also has a direct impact on the unique solvability property when the boundary conditions are not homogeneous

(Sect. 6.3). By a simple change of variable, this can be extended to the more general boundary operator $\nabla u \cdot N + \theta u$ when $N = (\mu, 1)$ is nontangential (Remark 6.3).

The nonperiodic problem (1.1) is discussed in Sect. 7. The question now is to relate the invertibility of $\mathfrak{P}$ on a suitable space $W^{2,p}_{(\theta)}(\mathbb{R} \times \mathbb{R}_-)$ that does not incorporate periodicity in x [see (7.1)] to the invertibility of P_ξ on $W^{2,p}_{(\theta)-}$ for every $\xi \in \mathbb{R}$, where P_ξ is obtained from (1.2) by replacing k by ξ. Naturally, P_ξ arises, at least formally, from taking the Fourier transform of (1.1) in the x direction. The arguments are almost exactly the same, provided that Theorem 1.2 is replaced by an appropriate and already known substitute. Because the proof of the otherwise simple Lemma 2.4 does not go through in that setting, a notable difference is that the criterion obtained is only sufficient (Theorem 7.2). However, this is arguably the more useful part in the applications.

Notation As is customary, if Ω is an open subset of $\mathbb{R}^n$, the norm of the Sobolev space $W^{s,p}(\Omega)$ ($s \in \mathbb{R}, 1 \leq p \leq \infty$) is denoted by $|| \cdot ||_{s,p,\Omega}$. In particular, for consistency, $|| \cdot ||_{0,p,\Omega}$ is the norm of $L^p(\Omega)$.

2 An equivalent first-order system

This section gives further details about the reformulation of the problem as a first-order system and discusses related technical issues. The hypotheses of Theorem 1.1 are retained.

We begin with the remark that $L^p(0, 2\pi; L^p_-)$ is isometrically isomorphic to $L^p((0, 2\pi) \times \mathbb{R}_-)$ in the natural way, that is, by identifying $u \in L^p(0, 2\pi; L^p_-)$ with the complex-valued function $u(x, y) := u(x)(y)$. A proof[1] can be found in Benedek and Panzone [3, pp. 318-319]. It follows from this identification and from the definitions of the derivatives of scalar- and vector-valued distributions [22] that if $u \in W^{1,p}((0, 2\pi), L^p_-)$, the function of $L^p((0, 2\pi) \times \mathbb{R}_-)$ corresponding to $\frac{\mathrm{d}u}{\mathrm{d}x} \in L^p(0, 2\pi; L^p_-)$ is just the partial derivative u_x. Likewise, if $u \in L^p(0, 2\pi; W^{1,p}_-)$, then u_y corresponds to the derivative $\frac{\mathrm{d}u}{\mathrm{d}y} \in L^p(0, 2\pi; L^p_-)$. The repetition of these remarks makes it possible to give vector-valued characterizations of Sobolev spaces over $(0, 2\pi) \times \mathbb{R}_-$. For example,

$$W^{2,p}((0, 2\pi) \times \mathbb{R}_-) = L^p(0, 2\pi; W^{2,p}_-) \cap W^{1,p}(0, 2\pi; W^{1,p}_-) \cap W^{2,p}(0, 2\pi; L^p_-).$$

The space $W^{2,p}_{(\theta),per}((0, 2\pi) \times \mathbb{R}_-)$ in (1.3) can also be characterized in terms of vector-valued spaces. First, $u \in W^{2,p}_{(\theta),per}((0, 2\pi) \times \mathbb{R}_-)$ if and only if[2] $u \in L^p(0, 2\pi; W^{2,p}_{(\theta)-}) \cap W^{1,p}_{per}(0, 2\pi; W^{1,p}_-)$ *and* $\frac{\mathrm{d}u}{\mathrm{d}x} \in W^{1,p}_{per}(0, 2\pi; L^p_-)$ [see (1.4) and (1.9)]. Since $\frac{\mathrm{d}u}{\mathrm{d}x} \in L^p(0, 2\pi; W^{1,p}_-)$ (because $u \in W^{1,p}_{per}((0, 2\pi; W^{1,p}_-))$, the latter condition is equivalent to $\frac{\mathrm{d}u}{\mathrm{d}x} \in L^p(0, 2\pi; W^{1,p}_-) \cap W^{1,p}_{per}(0, 2\pi; L^p_-)$. This yields

$$u \in W^{2,p}_{(\theta),per}((0, 2\pi) \times \mathbb{R}_-) \Leftrightarrow \begin{pmatrix} u \\ \frac{\mathrm{d}u}{\mathrm{d}x} \end{pmatrix} \in L^p(0, 2\pi; W^{2,p}_{(\theta)-} \times W^{1,p}_-) \cap W^{1,p}_{per}(0, 2\pi; W^{1,p}_- \times L^p_-). \quad (2.1)$$

[1] Although this is mostly Fubini's theorem, measurability issues are not entirely trivial.

[2] Minor technicalities are involved in checking the equivalence of the periodicity/boundary conditions. For brevity, the verification is left to the reader.

This makes it obvious that solving $\mathfrak{P}u = g$ for $u \in W^{2,p}_{(\theta),per}((0, 2\pi) \times \mathbb{R}_-)$ and $g \in L^p((0, 2\pi) \times \mathbb{R}_-)$ is equivalent to solving (1.5) for $(u, v) \in L^p(0, 2\pi; W^{2,p}_{(\theta)-} \times W^{1,p}_-) \cap W^{1,p}_{per}(0, 2\pi; W^{1,p}_- \times L^p_-)$. More is true:

Lemma 2.1 *If the operator $\frac{d}{dx} + \mathfrak{A}$ with $\mathfrak{A}$ from (1.6) is an isomorphism of $L^p(0, 2\pi; W^{2,p}_{(\theta)-} \times W^{1,p}_-) \cap W^{1,p}_{per}(0, 2\pi; W^{1,p}_- \times L^p_-)$ onto $L^p(0, 2\pi; W^{1,p}_- \times L^p_-)$, then $\mathfrak{P}$ is an isomorphism of $W^{2,p}_{(\theta),per}((0, 2\pi) \times \mathbb{R}_-)$ onto $L^p((0, 2\pi) \times \mathbb{R}_-)$.*

Proof By (2.1), $(u, v) \in \ker\left(\frac{d}{dx} + \mathfrak{A}\right)$ if and only if $v = \frac{\mathrm{d}u}{\mathrm{d}x}$, $u \in W^{2,p}_{(\theta),per}((0, 2\pi)\times\mathbb{R}_-)$ and $\mathfrak{P}u = 0$, so that $\frac{d}{dx} + \mathfrak{A}$ and $\mathfrak{P}$ are simultaneously one to one. Also, $\mathfrak{P}$ is onto $L^p((0, 2\pi)\times\mathbb{R}_-)$ as soon as $\frac{d}{dx} + \mathfrak{A}$ is surjective, for then a solution $u \in W^{2,p}_{(\theta),per}((0, 2\pi) \times \mathbb{R}_-)$ to $\mathfrak{P}u = g \in L^p((0, 2\pi) \times \mathbb{R}_-) = L^p(0, 2\pi; L^p_-)$ is obtained by solving (1.5) for (u, v). □

The converse of Lemma 2.1 is true, but not trivial, for the invertibility of $\mathfrak{P}$ does not imply that $\frac{d}{dx} + \mathfrak{A}$ is onto $L^p(0, 2\pi; W^{1,p}_- \times L^p_-)$ in any obvious way: Only that its range contains the dense subspace $\left(L^p(0, 2\pi; W^{1,p}_-) \cap W^{1,p}_{per}(0, 2\pi; L^p_-)\right) \times L^p(0, 2\pi; L^p_-)$. Actually, the converse of Lemma 2.1 follows from the necessity of the criterion of Theorem 1.1—proved in Lemma 2.4 below—because this criterion suffices to use Theorem 1.2 with $A = \mathfrak{A}$. However, most of the proof of Theorem 1.1 consists precisely in justifying the latter claim.

In Theorem 1.2, the operator A must be closed. When $A = \mathfrak{A}$, this issue is resolved in:

Lemma 2.2 (i) *The operator P in (1.7) is a closed operator on L^p_- with domain $W^{2,p}_{(\theta)-}$.*
(ii) *The operator $\mathfrak{A}$ in (1.6) is a closed operator on $W^{1,p}_- \times L^p_-$ with domain $W^{2,p}_{(\theta)-} \times W^{1,p}_-$.*

Proof First, (i) ⇒ (ii) by (1.6), (1.7) and (1.8). To prove (i), it suffices to find a constant $M > 0$ such that $||w||_{2,p,\mathbb{R}_-} \leq M(||Pw||_{0,p,\mathbb{R}_-} + ||w||_{0,p,\mathbb{R}_-})$ for every $w \in W^{2,p}_-$. Since $|c|$ is bounded away from 0 (by the uniform ellipticity of $\mathfrak{P}$ in Theorem 1.1), this follows at once from $w'' = \frac{1}{c}Pw + Lw$ where L is a first-order differential operator with bounded continuous coefficients, together with the well-known inequality (see [14, p. 26] for comments and references)

$$||w'||_{0,p,\mathbb{R}_-} \leq \varepsilon||w''||_{0,p,\mathbb{R}_-} + K_\varepsilon||w||_{0,p,\mathbb{R}_-}, \qquad \forall w \in W^{2,p}_-,$$

where $\varepsilon > 0$ is arbitrary and $K_\varepsilon > 0$ is a suitable constant. □

It follows from Theorem 1.2 and Lemmas 2.1 and 2.2 that $\mathfrak{P}$ is invertible if the resolvent $(\mathfrak{A} - ikI)^{-1} \in \mathcal{L}(W^{1,p}_- \times L^p_-)$ exists for every $k \in \mathbb{Z}$ and the sequence $(k(\mathfrak{A} - ikI)^{-1})_{k\in\mathbb{Z}}$ is r-bounded. Our next task will be to rephrase these two conditions. In this aim, we introduce the sequence of differential operators P_k defined by

$$P_k := P - ikQ - k^2, \tag{2.2}$$

which are just the operators P_k of (1.2). We shall view P_k as an unbounded operator on L^p_- with domain $W^{2,p}_{(\theta)-}$ and set

$$R^\theta_k := P_k^{-1} \in \mathcal{L}(L^p_-), \tag{2.3}$$

whenever P_k is invertible. By the argument of the proof of part (i) of Lemma 2.2, P_k is a closed operator on L^p_-. As a result, it is invertible if and only if it is a linear isomorphism of $W^{2,p}_{(\theta)-}$ onto L^p_- (since the graph norm on $W^{2,p}_{(\theta)-}$ is equivalent to the $W^{2,p}_-$ norm).

Lemma 2.3 *For every $k \in \mathbb{Z}$, the operator $\mathfrak{A} - ikI$ is invertible if and only if P_k is invertible. If so,*

$$(\mathfrak{A} - ikI)^{-1} = \begin{pmatrix} R_k^\theta(Q - ik) & R_k^\theta \\ -PR_k^\theta + ik[Q, R_k^\theta] & -ikR_k^\theta \end{pmatrix}, \tag{2.4}$$

where $[Q, R_k^\theta] := QR_k^\theta - R_k^\theta Q$. Furthermore, $(k(\mathfrak{A} - ikI)^{-1})_{k\in\mathbb{Z}}$ is r-bounded in $\mathcal{L}(W_-^{1,p} \times L_-^p)$ if and only if $(k^2 R_k^\theta)_{k\in\mathbb{Z}}$ is r-bounded in $\mathcal{L}(L_-^p)$ and in $\mathcal{L}(W_-^{1,p})$ and $(kR_k^\theta)_{k\in\mathbb{Z}}$ is r-bounded in $\mathcal{L}(L_-^p, W_-^{1,p})$ and in $\mathcal{L}(W_-^{1,p}, W_-^{2,p})$.

Proof That $\mathfrak{A} - ikI$ and P_k are simultaneously invertible is trivial and both (2.4) and the sufficiency of the condition for the r-boundedness of $k(\mathfrak{A} - ikI)^{-1}$ follow from a routine verification [$P_k R_k^\theta = I$ must be used in the form $k^2 R_k^\theta = PR_k^\theta - I - ikQR_k^\theta$ to verify (2.4)].

The necessity of the condition for r-boundedness can be seen as follows. First, if $k(\mathfrak{A} - ikI)^{-1}$ is r-bounded in $\mathcal{L}(W_-^{1,p} \times L_-^p)$, then its restrictions to $\{0\} \times L_-^p$ and $W_-^{1,p} \times \{0\}$ are r-bounded, which implies that kR_k^θ is r-bounded in $\mathcal{L}(L_-^p, W_-^{1,p})$, that $k^2 R_k^\theta$ is r-bounded in $\mathcal{L}(L_-^p)$ and that $kR_k^\theta(Q - ik)$ is r-bounded in $\mathcal{L}(W_-^{1,p})$. Since Q is a first-order differential operator and we just saw that kR_k^θ is r-bounded in $\mathcal{L}(L_-^p, W_-^{1,p})$, it follows that $k^2 R_k^\theta$ is r-bounded in $\mathcal{L}(W_-^{1,p})$.

It remains to show that kR_k^θ is r-bounded in $\mathcal{L}(W_-^{1,p}, W_-^{2,p})$. The r-boundedness of $k(\mathfrak{A} - ikI)^{-1}$ in $\mathcal{L}(W_-^{1,p} \times L_-^p)$ is equivalent to the r-boundedness of $(\mathfrak{A} - ikI)^{-1}$ in $\mathcal{L}(W_-^{1,p} \times L_-^p, W_{(\theta)-}^{2,p} \times W_-^{1,p})$, that is, in $\mathcal{L}(W_-^{1,p} \times L_-^p, W_-^{2,p} \times W_-^{1,p})$ since $W_{(\theta)-}^{2,p}$ is equipped with the $W_-^{2,p}$ norm. To see this, use $\mathfrak{A}(\mathfrak{A} - ikI)^{-1} = I + ik(\mathfrak{A} - ikI)^{-1}$ and notice that, since $\mathfrak{A}$ is closed, the graph norm and the product norm are equivalent on the domain $W_{(\theta)-}^{2,p} \times W_-^{1,p}$. Thus, the restrictions of $(\mathfrak{A} - ikI)^{-1}$ to $\{0\} \times L_-^p$ and to $W_-^{1,p} \times \{0\}$ are r-bounded in $\mathcal{L}(W_-^{1,p} \times L_-^p, W_-^{2,p} \times W_-^{1,p})$. The former implies that R_k^θ is r-bounded in $\mathcal{L}(L_-^p, W_-^{2,p})$ and the latter that $R_k^\theta(Q - ik)$ is r-bounded in $\mathcal{L}(W_-^{1,p}, W_-^{2,p})$. But then $R_k^\theta Q$ is r-bounded in $\mathcal{L}(W_-^{1,p}, W_-^{2,p})$, so that the same thing is true of kR_k^θ. □

Remark 2.1 It is obvious, but important for future purposes, that Lemma 2.3 is still true if $k \in \mathbb{Z}$ is replaced by $\{k \in \mathbb{Z} : |k| \geq \kappa\}$ where κ is any positive integer.

The necessity part of Theorem 1.1 is essentially trivial:

Lemma 2.4 *In Theorem 1.1, assume that $\mathfrak{P}$ is an isomorphism of $W_{(\theta),per}^{2,p}((0, 2\pi) \times \mathbb{R}_-)$ onto $L^p((0, 2\pi) \times \mathbb{R}_-)$. Then, P_k is an isomorphism of $W_{(\theta)-}^{2,p}$ onto L_-^p for every $k \in \mathbb{Z}$.*

Proof If $w \in \ker P_k$, then $\mathrm{e}^{-ikx} w(y)$ is in $\ker \mathfrak{P}$, so that $w = 0$. Next, if $f \in L_-^p$, then $\mathrm{e}^{-ikx} f(y)$ is in $L^p((0, 2\pi) \times \mathbb{R}_-)$, so that there is $u \in W_{(\theta),per}^{2,p}((0, 2\pi) \times \mathbb{R}_-)$ such that $\mathfrak{P}u = \mathrm{e}^{-ikx} f(y)$. Upon multiplying both sides by e^{ikx} and integrating, it appears that $w(y) := \frac{1}{2\pi} \int_0^{2\pi} u(x, y)\mathrm{e}^{ikx} dx$ is in $W_{(\theta)-}^{2,p}$ and that $P_k w = f$. □

Since the necessity was settled in Lemma 2.4, it follows from Lemmas 2.1 and 2.2 and Theorem 1.2 that Theorem 1.1 is proved if, assuming that P_k is invertible (i.e., R_k^θ exists) for every $k \in \mathbb{Z}$, so that $(\mathfrak{A} - ikI)^{-1}$ exists by Lemma 2.3, it can be shown that $(k(\mathfrak{A} - ikI)^{-1})_{k\in\mathbb{Z}}$ is r-bounded. In turn, since finite sets are r-bounded, this amounts to proving that $(k(\mathfrak{A} - ikI)^{-1})_{|k|\geq\kappa}$ is r-bounded if $\kappa \in \mathbb{N}$ is large enough.

When the coefficients are constant, this will be done by showing that $(k^2 R_k^\theta)_{|k|\geq\kappa}$ is r-bounded in $\mathcal{L}(L_-^p)$ and in $\mathcal{L}(W_-^{1,p})$ and that $(kR_k^\theta)_{|k|\geq\kappa}$ is r-bounded in $\mathcal{L}(L_-^p, W_-^{1,p})$ and in $\mathcal{L}(W_-^{1,p}, W_-^{2,p})$ (see Lemma 2.3 and Remark 2.1).

3 Constant coefficients: Neumann boundary condition

We now assume that the coefficients $b(y), c(y), \ldots$ etc., are constant, and that $\theta = 0$. Under these conditions, following the strategy outlined at the end of the previous section, we prove the r-boundedness of $k^2 R_k^0$ and kR_k^0 for $|k|$ large enough provided that $\mathfrak{P}$ is properly elliptic.

In the Introduction, we pointed out that the proper ellipticity of $\mathfrak{P}$ boils down to the assumption that $\mathfrak{P}$ is elliptic and that $c\lambda^2 + 2b\lambda + 1$ has exactly one root in the upper (lower) open half-plane. In particular, $c \neq 0$ and $c - b^2 \neq 0$. If $(c - b^2)^{\frac{1}{2}}$ denotes either square root of $c - b^2$, this means that the two numbers $c^{-1}(-b \pm i(c - b^2)^{\frac{1}{2}})$ have imaginary parts of opposite signs. In turn, this amounts to saying that $c^{-1}\left(\pm(c - b^2)^{\frac{1}{2}} + ib\right)$ have nonzero *real* parts with opposite signs.

The operator P_k in (2.2) is

$$P_k w := cw'' + (\beta - 2ikb)w' + (\gamma - k^2 - ik\alpha)w \tag{3.1}$$

and its characteristic polynomial $c\lambda^2 + (\beta - 2ikb)\lambda + (\gamma - k^2 - ik\alpha)$ has roots

$$\lambda_\pm(k) = \frac{-\beta + 2ikb \pm [4k^2(c - b^2) - 4c\gamma + \beta^2 + 4ik(c\alpha - b\beta)]^{\frac{1}{2}}}{2c}, \tag{3.2}$$

from which it follows that

$$\lim_{|k|\to\infty} k^{-1}\lambda_\pm(k) = c^{-1}\left(\pm(c - b^2)^{\frac{1}{2}} + ib\right) \tag{3.3}$$

and

$$\lim_{|k|\to\infty} |k|^{-1}(\lambda_+(k) - \lambda_-(k)) = 2c^{-1}(c - b^2)^{\frac{1}{2}} \neq 0. \tag{3.4}$$

From the above introductory remarks, the proper ellipticity implies that the two limits in the right-hand side of (3.3) have nonzero real parts with opposite signs. As a result, when $|k|$ is large enough, one among $\lambda_+(k)$ and $\lambda_-(k)$ has positive real part and the other has negative real part. In addition, which is which depends only upon the sign of k and the choice of $(c - b^2)^{\frac{1}{2}}$ (but not upon k). From now on, this choice is made so that $\lambda_+(k)$ ($\lambda_-(k)$) is the root with positive (negative) real part. This holds for $|k| \geq \kappa$ with $\kappa \in \mathbb{N} \cup \{0\}$ large enough. Unless stated otherwise, any future reference to $\lambda_\pm(k)$ comes with the understanding that $|k| \geq \kappa$.

Because $\mathrm{Re}\,\lambda_-(k) < 0$, the only solutions $w \in W_-^{2,p}$ of $P_k w = 0$ are the constant multiples of $e^{\lambda_+(k)y}$, so that P_k is one to one on $W_{(0)-}^{2,p}$. Its surjectivity is proved below.

The function $c^{-1}(\lambda_-(k) - \lambda_+(k))^{-1}E_k(y)$ with

$$E_k(y) := \begin{cases} e^{\lambda_+(k)y} & \text{if } y < 0, \\ e^{\lambda_-(k)y} & \text{if } y > 0, \end{cases} \tag{3.5}$$

is a fundamental solution of P_k. As a result, the Green's function $G_k(y, z)$ of P_k on $\mathbb{R}_-$ with homogeneous Neumann boundary condition is $G_k(y, z) = c^{-1}(\lambda_-(k) - \lambda_+(k))^{-1}\left(E_k(y - z) + \xi_z e^{\lambda_+(k)y}\right)$, where $\xi_z \in \mathbb{C}$ is chosen such that $\frac{\partial G_k}{\partial y}(0, z) = 0$. A simple calculation yields $\xi_z = -\lambda_+(k)^{-1}\lambda_-(k)\mathrm{e}^{-\lambda_-(k)z}$. As a result, if $f \in L^p_-$, then $R^0_k f \in W^{2,p}_{(0)-}$ is given by

$$(R^0_k f)(y) = \frac{1}{c(\lambda_-(k)-\lambda_+(k))}\left((E_k * \widetilde{f})(y) - \frac{\lambda_-(k)}{\lambda_+(k)}\int_{-\infty}^{0} \mathrm{e}^{\lambda_+(k)y-\lambda_-(k)z} f(z)\mathrm{d}z\right), \quad (3.6)$$

where $\widetilde{f}$ denotes the extension of f by 0.

3.1 r-boundedness of $k^2R^0_k$ in $\mathcal{L}(L^p_-)$

By (3.3), (3.4), and (3.6) and the Kahane contraction principle, it is enough to prove that the sequences $f \in L^p_- \mapsto kE_k * \widetilde{f} \in L^p_-$ and $f \in L^p_- \mapsto k\int_{-\infty}^0 \mathrm{e}^{\lambda_+(k)y-\lambda_-(k)z} f(z)\mathrm{d}z \in L^p_-$ are r-bounded for $|k| \geq \kappa$.

We begin with the first sequence. Since $f \in L^p_- \mapsto \widetilde{f} \in L^p(\mathbb{R})$ is continuous, it suffices to show that $(kE_k*)_{|k|\geq\kappa}$ is r-bounded on $L^p(\mathbb{R})$, that is, that there is a constant $C > 0$ independent of $n \in \mathbb{N}$ such that for every $|k_j| \geq \kappa$ and $g_j \in L^p(\mathbb{R})$, $1 \leq j \leq n$ (see the Appendix),

$$\left\|\left(\sum_{j=1}^n |k_jE_{k_j} * g_j|^2\right)^{\frac{1}{2}}\right\|_{0,p,\mathbb{R}} \leq C\left\|\left(\sum_{j=1}^n |g_j|^2\right)^{\frac{1}{2}}\right\|_{0,p,\mathbb{R}}. \quad (3.7)$$

Furthermore, since finite subsets are r-bounded, it actually suffices to prove (3.7) when $|k_j| \geq \kappa'$ with $\kappa' \geq \kappa$ arbitrarily large.

Choose $0 < \rho < \left|\mathrm{Re}\left(c^{-1}\left(\pm(c-b^2)^{\frac{1}{2}} + ib\right)\right)\right|$. By (3.3), if $|k| \geq \kappa'$ and κ' is large enough, then $\mathrm{Re}\,\lambda_-(k) < -\rho|k| < \rho|k| < \mathrm{Re}\,\lambda_+(k)$. As a result, $|kE_k(y)| \leq H_{|k|}(y)$ for every $y \in \mathbb{R}$, where $H(y) := \mathrm{e}^{-\rho|y|}$ and $H_t(y) := tH(ty)$ for $t > 0$. In particular, $|kE_k * g| \leq H_{|k|} * |g|$ and so $\sum_{j=1}^n |k_jE_{k_j} * g_j|^2 \leq \sum_{j=1}^n (H_{|k_j|} * |g_j|)^2$. Since H is even and decreasing for $y > 0$, it is known (Stein [24, p. 57]) that

$$\sup_{t>0} |(H_t * |g|)(y)| \leq ||H||_{0,1,\mathbb{R}} M_g(y), \quad (3.8)$$

where M_g is the Hardy–Littlewood maximal function of g. Thus,

$$\sum_{j=1}^n |k_jE_{k_j} * g_j|^2 \leq 4\rho^{-2}\sum_{j=1}^n |M_{g_j}|^2. \quad (3.9)$$

On the other hand, since the maximal operator is bounded in $L^p(\mathbb{R}, \ell^2)$ ([24, p. 51]), there is a constant $C > 0$ independent of n such that $||\left(\sum_{j=1}^n |M_{g_j}|^2\right)^{\frac{1}{2}}||_{0,p,\mathbb{R}} \leq C||\left(\sum_{j=1}^n |g_j|^2\right)^{\frac{1}{2}}||_{0,p,\mathbb{R}}$. Thus, (3.7) follows from (3.9).

It remains to prove that $f \in L^p_- \mapsto k\int_{-\infty}^0 \mathrm{e}^{\lambda_+(k)y-\lambda_-(k)z} f(z)\mathrm{d}z \in L^p_-$ is r-bounded for $|k| \geq \kappa$. This can be written as $T_k\widetilde{f}$ where T_k is the operator on $L^p(\mathbb{R})$ given by $T_kg(y) := \int_{-\infty}^{\infty} \chi_{\mathbb{R}_-}(y)ke^{\lambda_+(k)y-\lambda_-(k)z}\chi_{\mathbb{R}_-}(z)g(z)\mathrm{d}z$. It is useful to notice that if $y, z < 0$, then $\mathrm{Re}\,\lambda_+(k)y - \mathrm{Re}\,\lambda_-(k)z \leq \min\{\mathrm{Re}\,\lambda_+(k)(y-z), \mathrm{Re}\,\lambda_-(k)(y-z)\}$. Thus, if $y, z < 0$, then $\left|ke^{\lambda_+(k)y-\lambda_-(k)z}\right| \leq |kE_k(y-z)| \leq H_{|k|}(y-z)$. It follows that

$\left|\chi_{\mathbb{R}_-}(y) k e^{\lambda_+(k)y-\lambda_-(k)z} \chi_{\mathbb{R}_-}(z)\right| \leq H_{|k|}(y-z)$ for every $y, z \in \mathbb{R}$ and large enough $|k|$. This implies $|T_k g| \leq H_{|k|} * |g|$ for every $g \in L^p(\mathbb{R})$, so that the exact same procedure as above proves the r-boundedness of T_k.

3.2 r-boundedness of kR_k^0 in $\mathcal{L}(L_-^p, W_-^{1,p})$

From the definition of the $W_-^{1,p}$ norm, kR_k^0 is r-bounded in $\mathcal{L}(L_-^p, W_-^{1,p})$ (for $|k| \geq \kappa$) if and only if kR_k^0 is r-bounded in $\mathcal{L}(L_-^p)$ (which follows from 3.1) and $k\left(R_k^0\right)'$ is r-bounded in $\mathcal{L}(L_-^p)$. From (3.6),

$$\left(R_k^0 f\right)'(y) = \frac{1}{c(\lambda_-(k)-\lambda_+(k))}\left((E_k' * \widetilde{f})(y) - \lambda_-(k)\int_{-\infty}^{0} \mathrm{e}^{\lambda_+(k)y-\lambda_-(k)z} f(z)\mathrm{d}z\right).$$

By (3.3), (3.4) and (3.5), $|E_k'| \leq C|k|E_k$ where $C > 0$ is a constant independent of $k \in \mathbb{Z}$. Thus, by (3.3), (3.4) and the Kahane contraction principle, the r-boundedness of $k\left(R_k^0\right)'$ in $\mathcal{L}(L_-^p)$ is proved if the sequences $f \in L_-^p \mapsto kE_k * \widetilde{f} \in L_-^p$ and $f \in L_-^p \mapsto k\int_{-\infty}^0 \mathrm{e}^{\lambda_+(k)y-\lambda_-(k)z} f(z)\mathrm{d}z \in L_-^p$ are r-bounded. Both issues were already settled in 3.1.

3.3 r-boundedness of $k^2R_k^0$ in $\mathcal{L}(W_-^{1,p})$

Initially, the question is a bit more technical to formulate, so we return to first principles. We must find a constant $C > 0$ independent of n such that for $|k_j| \geq \kappa$ and $f_j \in W_-^{1,p}$, $1 \leq j \leq n$,

$$\left(\int_0^1 \left\|\sum_{j=1}^n r_j(t)k_j^2 R_k^0 f_j\right\|_{1,p,\mathbb{R}_-}^p \mathrm{d}t\right)^{\frac{1}{p}} \leq C\left(\int_0^1 \left\|\sum_{j=1}^n r_j(t) f_j\right\|_{1,p,\mathbb{R}_-}^p \mathrm{d}t\right)^{\frac{1}{p}},$$

where $(r_j)_{j\in\mathbb{N}}$ is the sequence of Rademacher functions. By definition of the $W_-^{1,p}$ norm, this amounts to

$$\left\|\sum_{j=1}^n r_j k_j^2 R_k^0 f_j\right\|_{L^p((0,1)\times\mathbb{R}_-)} + \left\|\sum_{j=1}^n r_j k_j^2 (R_k^0 f_j)'\right\|_{L^p((0,1)\times\mathbb{R}_-)} \leq C\left(\left\|\sum_{j=1}^n r_j f_j\right\|_{L^p((0,1)\times\mathbb{R}_-)} + \left\|\sum_{j=1}^n r_j f_j'\right\|_{L^p((0,1)\times\mathbb{R}_-)}\right).$$

Therefore, it suffices to find a constant $C > 0$ such that

$$\left\|\sum_{j=1}^n r_j k_j^2 R_k^0 f_j\right\|_{L^p((0,1)\times\mathbb{R}_-)} \leq C\left\|\sum_{j=1}^n r_j f_j\right\|_{L^p((0,1)\times\mathbb{R}_-)} \tag{3.10}$$

and that

$$\left\|\sum_{j=1}^n r_j k_j^2 (R_k^0 f_j)'\right\|_{L^p((0,1)\times\mathbb{R}_-)} \leq C\left\|\sum_{j=1}^n r_j f_j'\right\|_{L^p((0,1)\times\mathbb{R}_-)}. \tag{3.11}$$

The inequality (3.10) is just the r-boundedness of $k^2 R_k^0$ in $\mathcal{L}(L_-^p)$ proved in 3.1. A major simplification in the proof of (3.11) arises from the remark that if $f \in W_-^{1,p}$ and if $w \in W_{(0)-}^{2,p}$ solves $P_k w = f$ (so that $w = R_k^0 f$), then $w' = \left(R_k^0 f\right)' \in W_{0-}^{1,p} := W_0^{1,p}(\mathbb{R}_-)$ solves the homogeneous Dirichlet problem

$$\begin{cases} P_k w' = f', \\ w'(0) = 0, \end{cases}$$

because the coefficients of P_k are constant. Now, since $\operatorname{sign} \operatorname{Re} \lambda_\pm(k) = \pm 1$ if $|k| \geq \kappa$, no nontrivial linear combination of $\mathrm{e}^{\lambda_\pm(k)y}$ is in $W_{0-}^{1,p}$, whence P_k is one to one on $W_{0-}^{1,p}$. Therefore, P_k is an isomorphism from $W_-^{2,p} \cap W_{0-}^{1,p}$ to L_-^p, with inverse S_k explicitly given by

$$(S_k g)(y) = \frac{1}{c(\lambda_-(k) - \lambda_+(k))} \left((E_k * \widetilde{g})(y) - \int_{-\infty}^{0} \mathrm{e}^{\lambda_+(k)y - \lambda_-(k)z} g(z)\mathrm{d}z \right), \quad (3.12)$$

for every $g \in L_-^p$. This implies that if $v \in W_{0-}^{1,p}$ and $P_k v = g \in L_-^p$, then $v \in W_-^{2,p} \cap W_{0-}^{1,p}$ (elliptic regularity) and $v = S_k g$. In particular, from the above, $\left(R_k^0 f\right)' = S_k f'$ whenever $f \in W_-^{1,p}$ and so the desired inequality (3.11) may be rewritten as

$$\left\| \sum_{j=1}^{n} r_j k_j^2 S_{k_j} f_j' \right\|_{L^p((0,1)\times\mathbb{R}_-)} \leq C \left\| \sum_{j=1}^{n} r_j f_j' \right\|_{L^p((0,1)\times\mathbb{R}_-)}.$$

This is nothing but the r-boundedness of $(k^2 S_k)_{|k|\geq\kappa}$ in $\mathcal{L}(L_-^p)$, which can be proved by the exact same arguments as in 3.1, just relying on the representation (3.12) instead of (3.6).

Remark 3.1 For use in 3.4 below, note that the result of 3.2 is also true—with the same proof—when R_k^0 is replaced by S_k, that is, $(kS_k)_{|k|\geq\kappa}$ is r-bounded in $\mathcal{L}(L_-^p, W_-^{1,p})$.

3.4 r-boundedness of kR_k^0 in $\mathcal{L}(W_-^{1,p}, W_-^{2,p})$

With the notation of 3.3, the problem is now to find a constant $C > 0$ independent of n such that for $|k_j| \geq \kappa$ and $f_j \in W_-^{1,p}$, $1 \leq j \leq n$,

$$\begin{aligned} &\left\| \sum_{j=1}^{n} r_j k_j R_k^0 f_j \right\|_{L^p((0,1)\times\mathbb{R}_-)} + \left\| \sum_{j=1}^{n} r_j k_j (R_k^0 f_j)' \right\|_{L^p((0,1)\times\mathbb{R}_-)} \\ &\quad + \left\| \sum_{j=1}^{n} r_j k_j (R_k^0 f_j)'' \right\|_{L^p((0,1)\times\mathbb{R}_-)} \\ &\quad \leq C \left(\left\| \sum_{j=1}^{n} r_j f_j \right\|_{L^p((0,1)\times\mathbb{R}_-)} + \left\| \sum_{j=1}^{n} r_j f_j' \right\|_{L^p((0,1)\times\mathbb{R}_-)} \right). \end{aligned}$$

Since the r-boundedness of $k^2 R_k^0$ in $\mathcal{L}(W_-^{1,p})$ proved in 3.3 implies that of kR_k^0 in $\mathcal{L}(W_-^{1,p})$, the sum of the first two terms in the left-hand side above is majorized as required. The only remaining question is the existence of $C > 0$ such that

$$\left\| \sum_{j=1}^{n} r_j k_j (R_k^0 f_j)'' \right\|_{L^p((0,1)\times\mathbb{R}_-)} \leq C \left(\left\| \sum_{j=1}^{n} r_j f_j \right\|_{L^p((0,1)\times\mathbb{R}_-)} + \left\| \sum_{j=1}^{n} r_j f_j' \right\|_{L^p((0,1)\times\mathbb{R}_-)} \right). \quad (3.13)$$

In 3.3, we noticed that if $f \in W_-^{1,p}$, then $(R_k^0 f)' = S_k f'$ with S_k from (3.12). Thus, $(R_k^0 f_j)'' = \left(S_k f_j'\right)'$ and so

$$\left\| \sum_{j=1}^{n} r_j k_j (R_k^0 f_j)'' \right\|_{L^p((0,1)\times\mathbb{R}_-)} = \left\| \sum_{j=1}^{n} r_j k_j \left(S_{k_j} f_j'\right)' \right\|_{L^p((0,1)\times\mathbb{R}_-)} \leq C \left\| \sum_{j=1}^{n} r_j f_j' \right\|_{L^p((0,1)\times\,\mathbb{R}_-)}$$

since $(kS_k)_{|k|\geq\kappa}$ is r-bounded in $\mathcal{L}(L_-^p, W_-^{1,p})$ (Remark 3.1), which is actually stronger than (3.13).

3.5 Dirichlet problem

As noted above, 3.1 and 3.2 are still true when R_k^0 is replaced by S_k from (3.12) (homogeneous Dirichlet problem), but 3.3 and 3.4 *fail* for S_k. For example, it is not difficult to see that $k^2 S_k$ is not even bounded in $\mathcal{L}(W_-^{1,p})$, let alone r-bounded.

However, for the Dirichlet problem, 3.3 and 3.4 are not quite the right properties: The corresponding spaces X and W of Theorem 1.2 are $W_{0-}^{1,p} \times L_-^p$ and $(W^{2,p} \cap W_{0-}^{1,p}) \times W_-^{1,p}$, respectively. Thus, the analog of Lemma 2.3 requires $(k^2 S_k)_{k\in\mathbb{Z}}$ to be r-bounded in $\mathcal{L}(L_-^p)$ and in $\mathcal{L}(W_{0-}^{1,p}, W^{1,p})$ and $(kR_k^0)_{k\in\mathbb{Z}}$ to be r-bounded in $\mathcal{L}(L_-^p, W_-^{1,p})$ and in $\mathcal{L}(W_{0-}^{1,p}, W_-^{2,p})$. In other words, in 3.3 and 3.4, the source space $W_-^{1,p}$ should be replaced by the smaller $W_{0-}^{1,p}$. With this modification, the desired r-boundedness properties continue to hold. For example, if $g \in W_{0-}^{1,p}$ in (3.12), then $g(0) = 0$, so that $\widetilde{g}' = \widetilde{g'}$ (extensions by 0), while integration by parts yields

$$\int_{-\infty}^{0} e^{\lambda_+(k)y-\lambda_-(k)z} g(z)\mathrm{d}z = \frac{1}{\lambda_-(k)} \int_{-\infty}^{0} e^{\lambda_+(k)y-\lambda_-(k)z} g'(z)\mathrm{d}z.$$

Thus,

$$(S_k g)'(y) = \frac{1}{c(\lambda_-(k) - \lambda_+(k))} \left((E_k * \widetilde{g}')(y) - \frac{\lambda_+(k)}{\lambda_-(k)} \int_{-\infty}^{0} e^{\lambda_+(k)y-\lambda_-(k)z} g'(z)\mathrm{d}z \right)$$

and hence $k^2(S_k)'$ is r-bounded in $\mathcal{L}(W_{0-}^{1,p}, L_-^p)$ by the arguments of 3.1 while $k(S_k)'$ is r-bounded in $\mathcal{L}(W_{0-}^{1,p}, W_-^{1,p})$ by the arguments of 3.2. Since it is already known that $k^2 S_k$

and kS_k are r-bounded in $\mathcal{L}(L^p_-)$ and $\mathcal{L}(L^p_-, W^{1,p}_-)$, respectively, it follows that $k^2 S_k$ and kS_k are r-bounded in $\mathcal{L}(W^{1,p}_{0-}, W^{1,p}_-)$ and $\mathcal{L}(W^{1,p}_{0-}, W^{2,p}_-)$, respectively.

4 Constant coefficients: general case

In this section, $\theta \in \mathbb{C}$ is arbitrary but we continue to assume that the coefficients are constant and that $\mathfrak{P}$ is properly elliptic. As in the previous section, let $\kappa \in \mathbb{N} \cup \{0\}$ be such that $\operatorname{sign} \operatorname{Re} \lambda_\pm(k) = \pm 1$ for $|k| \geq \kappa$. If so, the operator P_k is one to one on $W^{2,p}_{(\theta)-}$ if and only if the function $w(y) = \mathrm{e}^{\lambda_+(k)y}$ does not satisfy the condition $w'(0) + \theta w(0) = 0$, that is, if and only if $\theta \neq -\lambda_+(k)$. Then, the inverse R^θ_k of P_k exists and

$$R^\theta_k f = \frac{1}{c(\lambda_-(k) - \lambda_+(k))} \left((E_k * \widetilde{f})(y) - \frac{\lambda_-(k) + \theta}{\lambda_+(k) + \theta} \int_{-\infty}^{0} \mathrm{e}^{\lambda_+(k)y - \lambda_-(k)z} f(z) \mathrm{d}z \right), \quad (4.1)$$

for every $f \in L^p_-$, where E_k is given by (3.5) and $\widetilde{f}$ is the extension of f by 0. This is a minor modification of the formula (3.6) when $\theta = 0$, and so the arguments of 3.1 and 3.2 can be repeated verbatim to show that $(k^2 R^\theta_k)_{|k| \geq \kappa}$ is r-bounded in $\mathcal{L}(L^p_-)$ and $(kR^\theta_k)_{|k| \geq \kappa}$ is r-bounded in $\mathcal{L}(L^p_-, W^{1,p}_-)$ whenever $\theta \neq -\lambda_+(k)$ for every $|k| \geq \kappa$. At any rate, since $\theta \neq -\lambda_+(k)$ if $|k|$ is large enough by (3.3), $k^2 R^\theta_k$ and kR^θ_k are always r-bounded in $\mathcal{L}(L^p_-)$ and in $\mathcal{L}(L^p_-, W^{1,p}_-)$, respectively, provided that $|k|$ is large enough. For that reason, it will be convenient to increase κ, by requiring not only that $\operatorname{sign} \operatorname{Re} \lambda_\pm(k) = \pm 1$, but also that $\theta \neq -\lambda_+(k)$ for $|k| \geq \kappa$. With this new definition of κ, $(k^2 R^\theta_k)_{|k| \geq \kappa}$ is r-bounded in $\mathcal{L}(L^p_-)$ and $(kR^\theta_k)_{|k| \geq \kappa}$ is r-bounded in $\mathcal{L}(L^p_-, W^{1,p}_-)$ and all the r-boundedness results involving R^0_k when $|k| \geq \kappa$ in the previous section are of course unchanged.

Unlike in Sect. 3, if $w \in W^{2,p}_{(\theta)-}$ and $P_k w = f$, then w' does not solve a simple boundary value problem when $\theta \neq 0$, so that the procedure of 3.3 and 3.4 cannot be repeated to obtain the r-boundedness of $(k^2 R^\theta_k)_{|k| \geq \kappa}$ in $\mathcal{L}(W^{1,p}_-)$ and that of $(kR^\theta_k)_{|k| \geq \kappa}$ in $\mathcal{L}(W^{1,p}_-, W^{2,p}_-)$. We use another argument.

Given $k \in \mathbb{Z}$ with $|k| \geq \kappa$ and $v \in W^{1,p}_-$, set

$$\Gamma_k v := \frac{v(0)}{\lambda_+(k)} \mathrm{e}^{\lambda_+(k)y}. \quad (4.2)$$

Clearly, $\Gamma_k \in \mathcal{L}(W^{1,p}_-, W^{2,p}_-) \subset \mathcal{L}(W^{1,p}_-)$ and $P_k \Gamma_k = 0$. Thus, if $f \in L^p_-$ and $v := R^\theta_k f$, then $w := v + \theta \Gamma_k v$ solves $P_k w = f$ and $w_y(0) = 0$. Since R^0_k exists for $|k| \geq \kappa$, this means

$$R^\theta_k = R^0_k - \theta \Gamma_k R^\theta_k \quad \text{if} \quad |k| \geq \kappa. \quad (4.3)$$

Lemma 4.1 *The sequence $(\Gamma_k)_{|k| \geq \kappa}$ is r-bounded in $\mathcal{L}(W^{1,p}_-)$ and the r-bound of $(\Gamma_k)_{|k| \geq \ell}$ in $\mathcal{L}(W^{1,p}_-)$ tends to 0 as $\ell \to \infty$.*

Proof Of course, the former property follows from the latter. Since Γ_k is the composite of the evaluation $v \in W^{1,p}_- \mapsto v(0) \in \mathbb{C}$ and of the mapping $\Lambda_k : \mathbb{C} \to W^{1,p}_-$ defined by $\Lambda_k(\zeta) := \frac{\zeta}{\lambda_+(k)} \mathrm{e}^{\lambda_+(k)y} \in W^{1,p}_-$, it suffices to prove that the r-bound of $(\Lambda_k)_{|k| \geq \ell}$ in $\mathcal{L}(\mathbb{C}, W^{1,p}_-)$ tends to 0 as $\ell \to \infty$. In turn, this amounts to the similar question for Λ_k and

$(\Lambda_k)'$ in $\mathcal{L}(\mathbb{C}, L^p_-)$. But since $(\Lambda_k)' = \lambda_+(k)\Lambda_k$, it follows from (3.3) and the Kahane contraction principle that it suffices to show that the r-bound of $(\lambda_+(k)\Lambda_k)_{|k|\geq \ell}$ in $\mathcal{L}(\mathbb{C}, L^p_-)$ tends to 0 as $\ell \to \infty$.

Given $n \in \mathbb{N}$, let $k_j \in \mathbb{Z}$ be such that $|k_j| \geq \ell$ and let $\zeta_j \in \mathbb{C}, 1 \leq j \leq n$. Then, $\sum_{j=1}^n |\zeta_j e^{\lambda_+(k_j)y}|^2 = \sum_{j=1}^n |\zeta_j|^2 e^{2\,\mathrm{Re}\,\lambda_+(k_j)y}$. Since $\lim_{|k|\to\infty} \mathrm{Re}\,\lambda_+(k) = \infty$ by (3.3), it follows that $\mathrm{Re}\,\lambda_+(k_j)$ above is larger than any prescribed $\lambda > 0$ if ℓ is large enough. As a result, $\sum_{j=1}^n |\zeta_j|^2 e^{2\,\mathrm{Re}\,\lambda_+(k_j)y} = e^{2\lambda y}\sum_{j=1}^n |\zeta_j|^2 e^{2(\mathrm{Re}\,\lambda_+(k_j)-\lambda)y} \leq e^{2\lambda y}\sum_{j=1}^n |\zeta_j|^2$ since $y < 0$. This yields

$$\left\| \left(\sum_{j=1}^n |\lambda_+(k_j)\Lambda_{k_j}(\zeta_j)|^2 \right)^{\frac{1}{2}} \right\|_{0,p,\mathbb{R}_-} \leq (p\lambda)^{-\frac{1}{p}} \left(\sum_{j=1}^n |\zeta_j|^2 \right)^{\frac{1}{2}}.$$

In other words, the r-bound of $(\lambda_+(k)\Lambda_k)_{|k|\geq\ell}$ is (proportional to) $(p\lambda)^{-\frac{1}{p}}$, so that it can be made arbitrarily small if λ (and hence ℓ) is large enough. □

The next lemma proves the remaining two properties needed to establish the validity of Theorem 1.1 when the coefficients are constant.

Lemma 4.2 *The sequences $(k^2 R_k^\theta)_{|k|\geq\kappa}$ and $(kR_k^\theta)_{|k|\geq\kappa}$ are r-bounded in $\mathcal{L}(W^{1,p}_-)$ and in $\mathcal{L}(W^{1,p}_-, W^{2,p}_-)$, respectively.*

Proof We already know that R_k^θ exists if $|k| \geq \kappa$, so that, as usual, the r-boundedness needs to be establish only for $|k| \geq \ell$ with ℓ arbitrarily large. By Lemma 4.1, there is $\ell \geq \kappa$ such that the r-bound M_ℓ of $(\Gamma_k)_{|k|\geq\ell}$ satisfies $|\theta|M_\ell \leq \frac{1}{2}$.

Let $C_\ell > 0$ denote the r-bound of $(k^2 R_k^0)_{|k|\geq\ell}$ in $\mathcal{L}(W^{1,p}_-)$, which is finite by 3.3. Let now F be any finite subset of $\{k \in \mathbb{Z} : |k| \geq \ell\}$. Since finite subsets are r-bounded, $(k^2 R_k^\theta)_{k\in F}$ is r-bounded in $\mathcal{L}(W^{1,p}_-)$. Call $C_F > 0$ its r-bound.

By (4.3) and the properties of r-bounds, $C_F \leq C_\ell + |\theta|M_\ell C_F \leq C_\ell + \frac{1}{2}C_F$, so that $C_F \leq 2C_\ell$. From the definition of r-boundedness, $\sup_F C_F \leq 2C_\ell$ is the (finite) r-bound of $(k^2 R_k^\theta)_{|k|\geq\ell}$. in $\mathcal{L}(W^{1,p}_-)$.

Next, the r-boundedness of $(kR_k^\theta)_{|k|\geq\kappa}$ in $\mathcal{L}(W^{1,p}_-, W^{2,p}_-)$ is equivalent to its r-boundedness in $\mathcal{L}(W^{1,p}_-)$ plus the r-boundedness of $\left(k\left(R_k^\theta\right)'\right)_{|k|\geq\kappa}$ in $\mathcal{L}(W^{1,p}_-)$. The former follows from the stronger result above that $(k^2 R_k^\theta)_{|k|\geq\kappa}$ is r-bounded in $\mathcal{L}(W^{1,p}_-)$. To prove the latter, note that $(\Gamma_k v)' = \lambda_+(k)\Gamma_k v = v(0)e^{\lambda_+(k)y}$ by (4.2), so that the differentiation of (4.3) yields

$$k\left(R_k^\theta\right)' = k\left(R_k^0\right)' - k^{-1}\lambda_+(k)\Gamma_k\left(k^2 R_k^\theta\right). \tag{4.4}$$

Since kR_k^0 is r-bounded in $\mathcal{L}(W^{1,p}_-, W^{2,p}_-)$ by 3.4, then $k\left(R_k^0\right)'$ is r-bounded in $\mathcal{L}(W^{1,p}_-)$. Also, since it was just shown above that $k^2 R_k^\theta$ is r-bounded in $\mathcal{L}(W^{1,p}_-)$ and since Γ_k is r-bounded in $\mathcal{L}(W^{1,p}_-)$ (Lemma 4.1), it follows from (3.3) and the Kahane contraction principle that $k^{-1}\lambda_+(k)\Gamma_k(k^2 R_k^\theta)$ is r-bounded in $\mathcal{L}(W^{1,p}_-)$. Thus, by (4.4), $k\left(R_k^\theta\right)'$ is indeed r-bounded in $\mathcal{L}(W^{1,p}_-)$. □

From the discussion at the end of Sect. 2, Lemma 4.2 completes the proof of Theorem 1.1 when the coefficients are constant. If they are also real (so that ellipticity and proper ellipticity coincide), there is a simpler equivalent statement.

Theorem 4.3 *If $\mathfrak{P}$ is elliptic with constant real coefficients, then $\mathfrak{P}$ is an isomorphism of $W^{2,p}_{(\theta),per}((0,2\pi)\times\mathbb{R}_-)$ onto $L^p((0,2\pi)\times\mathbb{R}_-)$ if and only if $\gamma<0$ and $\theta\neq-\lambda_+(k)$ for every $k\in\mathbb{Z}$, where $\lambda_+(k)$ is the unique root with positive real part of the characteristic polynomial of P_k.*

Proof Note that $c>0$ by ellipticity and that, if $\beta\neq 0$ and $\gamma>0$, it follows from (3.2) that the real parts of both roots $\lambda_\pm(0)$ have the same nonzero sign. This implies at once that either $\ker P_0\neq\{0\}$ (if $\beta<0$) or that $cw_*''-\beta w_*'+\gamma w_*=0$, with boundary condition $w_*'(0)-(\frac{\beta}{c}-\theta)w_*(0)=0$, has a nonzero exponentially decaying (as $y\to-\infty$) solution $w_*\in W^{1,\infty}_-$ (if $\beta>0$). Since $\int_{-\infty}^0 fw_*=0$ whenever $f\in P_0\left(W^{2,p}_{(\theta)-}\right)$ (use integration by parts), this shows that $\overline{w}_*\in L^p_-$ and that $\overline{w}_*\notin P_0\left(W^{2,p}_{(\theta)-}\right)$. Thus, P_0 is not invertible if $\beta\neq 0$ and $\gamma>0$, and then it is not invertible either when $\gamma\geq 0$ and $\beta\in\mathbb{R}$ is arbitrary. That $\theta\neq-\lambda_+(k)$ is also needed for P_k to be invertible was already noted at the beginning of this section. Thus, by Theorem 1.1, both $\gamma<0$ and $\theta\neq-\lambda_+(k)$ for every k are necessary for $\mathfrak{P}$ to be invertible.

Conversely, if $\gamma<0$, an examination of (3.2) reveals that $\lambda_\pm(k)$ have real parts of opposite signs for *every* $k\in\mathbb{Z}$. (To see this, observe that if $\zeta\in\mathbb{C}$, then $(\mathrm{Re}\,\zeta)^2\geq\mathrm{Re}\,\zeta^2$ and let ζ be the square root of $4k^2(c-b^2)-4c\gamma+\beta^2+4ik(c\alpha-b\beta)$ with nonnegative real part.) This means that $\kappa=0$ in Sect. 3. If, in addition, $\theta\neq-\lambda_+(k)$ for every $k\in\mathbb{Z}$, then obviously κ need not be increased to ensure the extra condition $\theta\neq-\lambda_+(k)$ for $|k|\geq\kappa$, so that once again $\kappa=0$. Since the existence of R_k^θ was obtained earlier when $|k|\geq\kappa$, it follows that P_k is invertible for every $k\in\mathbb{Z}$ and then the invertibility of $\mathfrak{P}$ follows from Theorem 1.1. □

The results of this section show that the sequence $(k(\mathfrak{A}-ikI)^{-1})_{|k|\geq\kappa}$ is defined and r-bounded in $\mathcal{L}(W^{1,p}_-\times L^p_-)$ when the coefficients are constant, provided that $\kappa\in\mathbb{N}$ is large enough. This will be instrumental in the next step, which consists in proving the same property when the coefficients are periodic.

5 Periodic coefficients

In this section, we retain the hypotheses of Theorem 1.1 and also assume that the coefficients $b(y),\ldots,\gamma(y)$ of $\mathfrak{P}$ are periodic. In particular, they are defined for every $y\in\mathbb{R}$. With no loss of generality, we confine attention to the case when the period is 2π. We shall prove that, once again, the sequence $(k(\mathfrak{A}-ikI)^{-1})_{|k|\geq\kappa}$ is defined and r-bounded in $\mathcal{L}(W^{1,p}_-\times L^p_-)$ provided that $\kappa\in\mathbb{N}$ is large enough.

In what follows, $\mathfrak{v}(y):=(b(y),\ldots,\gamma(y))\in\mathbb{C}^5$ denotes the vector of coefficients of $\mathfrak{P}$ in (1.1). For future use, note that $\mathfrak{v}(\mathbb{R})=\mathfrak{v}([0,2\pi])$ is compact.

With $y_0\in\mathbb{R}$ being fixed, call $\mathfrak{A}_{y_0}$ the operator (1.6) when the coefficients $\mathfrak{v}(y)$ of $\mathfrak{P}$ are replaced by the constant coefficients $\mathfrak{v}(y_0)$. From the ellipticity of $\mathfrak{P}$ and the results of the previous section, $\left(k(\mathfrak{A}_{y_0}-ikI)^{-1}\right)_{|k|\geq\kappa}$ is defined and r-bounded in $\mathcal{L}(W^{1,p}_-\times L^p_-)$ if κ is large enough.

The r-boundedness of resolvents is unaffected by small enough relatively bounded perturbations of the operator. Since the spectrum of $\mathfrak{A}_{y_0}$ is not the entire complex plane, such perturbations include operators $\mathfrak{A}_0\in\mathcal{L}(W^{2,p}_{(\theta)-}\times W^{1,p}_-, W^{1,p}_-\times L^p_-)$ such that $||\mathfrak{A}_0-\mathfrak{A}_{y_0}||<\varepsilon$, where $\varepsilon>0$ is small enough and the norm is that of $\mathcal{L}(W^{2,p}_{(\theta)-}\times W^{1,p}_-, W^{1,p}_-\times L^p_-)$. More precisely, from [21, Lemma 3.4] (with $A = \mathfrak{A}_{y_0}$ and $K = \mathfrak{A}_0 - \mathfrak{A}_{y_0}$ in that lemma), the sequence $\left(k(\mathfrak{A}_0 - ikI)^{-1}\right)_{|k|\geq\kappa}$ is r-bounded in $\mathcal{L}(W_-^{1,p} \times L_-^p)$ after increasing κ if necessary.

The condition $||\mathfrak{A}_0 - \mathfrak{A}_{y_0}|| < \varepsilon$ is satisfied when $\mathfrak{A}_0$ is an operator having the same structure as $\mathfrak{A}$ in (1.6), (1.7) and (1.8) and continuous coefficients $\mathfrak{v}_0(y) := (b_0(y), \ldots, \gamma_0(y))$ such that $|\mathfrak{v}_0(y) - \mathfrak{v}(y_0)| \leq \delta_0$ for every $y \in \mathbb{R}$, where $\delta_0 > 0$ is small enough. We now construct such an operator $\mathfrak{A}_0$, with a useful extra feature. With $B(\mathfrak{v}(y_0), \delta_0)$ denoting the open ball in $\mathbb{C}^5$ with center $\mathfrak{v}(y_0)$ and radius δ_0, set

$$U_0 := \mathfrak{v}^{-1}(B(\mathfrak{v}(y_0), \delta_0)) = \{y \in \mathbb{R} : \mathfrak{v}(y) \in B(\mathfrak{v}(y_0), \delta_0)\},$$

an open subset of $\mathbb{R}$. For $y \in \overline{U}_0$, define $\mathfrak{v}_0(y) := \mathfrak{v}(y)$. Since $\overline{U}_0$ is closed in $\mathbb{R}$ and $\mathfrak{v}_0$ is continuous on $\overline{U}_0$, the classical Dugundji extension theorem [10,12] ensures that $\mathfrak{v}_0$ can be continuously extended to the entire line, in such way that the values of $\mathfrak{v}_0$ remain in the convex hull of $\mathfrak{v}_0\left(\overline{U}_0\right)$. Since $\mathfrak{v}_0\left(\overline{U}_0\right) \subset \overline{B}(\mathfrak{v}(y_0), \delta_0)$ and balls are convex, $|\mathfrak{v}_0(y) - \mathfrak{v}(y_0)| \leq \delta_0$ for every $y \in \mathbb{R}$. As a result, $\left(k(\mathfrak{A}_0 - ikI)^{-1}\right)_{|k|\geq\kappa}$ is r-bounded if κ is large enough. In addition, $\mathfrak{A}_0$ and $\mathfrak{A}$ coincide on U_0.

Cover $\mathfrak{v}(\mathbb{R})$ by finitely many balls $B(\mathfrak{v}(y_j), \delta_j)$, $1 \leq j \leq N$, as above. This produces N open subsets $U_j := \mathfrak{v}^{-1}(B(\mathfrak{v}(y_j), \delta_j))$ and N operators $\mathfrak{A}_j$ of the form (1.6) such that $\left(k(\mathfrak{A}_j - ikI)^{-1}\right)_{|k|\geq\kappa}$ is r-bounded for κ large enough and that $\mathfrak{A}_j$ and $\mathfrak{A}$ coincide on U_j. Obviously, κ may be chosen independent of j.

The open subsets U_j have two elementary but crucial properties: By the periodicity of $\mathfrak{v}$, each U_j is invariant by 2π-translation and $\cup_{j=1}^N U_j = \mathbb{R}$. Since U_j is open, $O_j := \mathrm{e}^{iU_j} = \{(\cos y, \sin y) : y \in U_j\}$ is open in the unit circle $\mathbb{S}^1$. Furthermore, by the translation invariance of U_j,

$$U_j = \{y \in \mathbb{R} : (\cos y, \sin y) \in O_j\}. \tag{5.1}$$

Since $\cup_{j=1}^N U_j = \mathbb{R}$, then $\cup_{j=1}^N O_j = \mathbb{S}^1$ and so there is a smooth partition of unity (ψ_j) of $\mathbb{S}^1$ subordinate to this covering. Define $\varphi_j(y) := \psi_j(\cos y, \sin y)$. Then, $0 \leq \varphi_j \leq 1$, $\operatorname{Supp} \varphi_j \subset U_j$ by (5.1) and $\sum_{j=1}^N \varphi_j = 1$ on $\mathbb{R}$. In addition, all the derivatives of φ_j are uniformly bounded on $\mathbb{R}$. Also, since $\operatorname{Supp} \psi_j$ is a compact subset of O_j, there is a smooth function $\widetilde{\psi}_j$ on $\mathbb{S}^1$ such that $\operatorname{Supp} \widetilde{\psi}_j \subset O_j$ and $\widetilde{\psi}_j = 1$ on $\operatorname{Supp} \psi_j$. Clearly, it may (and will) also be assumed that $\widetilde{\psi}_j = 0$ on a neighborhood of $(1, 0)$ in $\mathbb{S}^1$ for every index j such that $(1, 0) \notin \operatorname{Supp} \psi_j$. Then, with $\widetilde{\varphi}_j(y) := \widetilde{\psi}_j(\cos y, \sin y)$, it follows that $\operatorname{Supp} \widetilde{\varphi}_j \subset U_j$ (once again, by (5.1)), that $\widetilde{\varphi}_j = 1$ on $\operatorname{Supp} \varphi_j$, and that all the derivatives of $\widetilde{\varphi}_j$ are uniformly bounded on $\mathbb{R}$. In addition, $\widetilde{\varphi}'_j(0) = 0$ for $1 \leq j \leq N$ since $0 \in \operatorname{Supp} \varphi_j$ if and only if $(1, 0) \in \operatorname{Supp} \psi_j$, so that $\widetilde{\varphi}'_j(0) = 0$ irrespective of whether $0 \in \operatorname{Supp} \varphi_j$ by the assumed properties of $\widetilde{\psi}_j$. (This uses the elementary remark that the support of a continuous function is the closure of its interior.)

We now proceed to proving that, after increasing κ if necessary, $\mathfrak{A} - ikI$ is one to one on $W_{(\theta)-}^{2,p} \times W_-^{1,p}$ and onto $W_-^{1,p} \times L_-^p$ if $|k| \geq \kappa$—so that $(\mathfrak{A} - ikI)^{-1}$ exists for every such k—and that the sequence $(k(\mathfrak{A} - ikI)^{-1})_{|k|\geq\kappa}$ is r-bounded in $\mathcal{L}(W_-^{1,p} \times L_-^p)$. From this point on, the arguments parallel those in [11], with appropriate modifications.

5.1 Surjectivity

Let $(f, g) \in W_-^{1,p} \times L_-^p$ be given. Assuming that $|k| \geq \kappa$, the existence of $(\mathfrak{A}_j - ikI)^{-1}$ yields a solution $(u_j, v_j) \in W_{(\theta)-}^{2,p} \times W_-^{1,p}$ of $(\mathfrak{A}_j - ikI)(u_j, v_j) = (\varphi_j f, \varphi_j g) \in W_-^{1,p} \times L_-^p$. Then, $\widetilde{\varphi}_j(\mathfrak{A}_j - ikI)(u_j, v_j) = (\varphi_j f, \varphi_j g)$ since $\widetilde{\varphi}_j = 1$ on $\operatorname{Supp} \varphi_j$. Moreover, $\widetilde{\varphi}_j(\mathfrak{A}_j - ikI) =$

$\widetilde{\varphi}_j(\mathfrak{A} - ikI)$ since $\mathfrak{A}$ and $\mathfrak{A}_j$ coincide on Supp $\widetilde{\varphi}_j \subset U_j$. Therefore, $\widetilde{\varphi}_j(\mathfrak{A} - ikI)(u_j, v_j) = (\varphi_j f, \varphi_j g)$.

Now, $\widetilde{\varphi}_j(\mathfrak{A} - ikI)(u_j, v_j) = (\mathfrak{A} - ikI)(\widetilde{\varphi}_j u_j, \widetilde{\varphi}_j v_j) + (0, \mathfrak{E}_j(u_j, v_j))$, where $\mathfrak{E}_j$ is a k-independent scalar differential operator of first order in u_j and zeroth order in v_j, with *bounded* continuous coefficients (because the first and second derivatives of $\widetilde{\varphi}_j$ are bounded). In particular, $\mathfrak{E}_j \in \mathcal{L}(W^{1,p}_- \times L^p_-, L^p_-)$.

Altogether, $(\mathfrak{A} - ikI)(\widetilde{\varphi}_j u_j, \widetilde{\varphi}_j v_j) = (\varphi_j f, \varphi_j g) - (0, \mathfrak{E}_j(u_j, v_j))$, so that, if $(u, v) := \sum_{j=1}^N (\widetilde{\varphi}_j u_j, \widetilde{\varphi}_j v_j)$, then $(u, v) \in W^{2,p}_{(\theta)-} \times W^{1,p}_-$ and

$$\begin{aligned}(\mathfrak{A} - ikI)(u, v) &= (f, g) - \sum_{j=1}^N (0, \mathfrak{E}_j(u_j, v_j)) \\ &= \left(I - \sum_{j=1}^N (0, \mathfrak{E}_j(\mathfrak{A}_j - ikI)^{-1}\varphi_j) \right)(f, g). \quad (5.2)\end{aligned}$$

Indeed, that $(u, v) \in W^{2,p}_- \times W^{1,p}_-$ follows once again from the fact that the first and second derivatives of $\widetilde{\varphi}_j$ are bounded. To see that $u = \sum_{j=1}^N \widetilde{\varphi}_j u_j \in W^{2,p}_{(\theta)-}$, that is, that $u'(0) + \theta u(0) = 0$, just use $u_j \in W^{2,p}_{(\theta)-}$ and $\widetilde{\varphi}'_j(0) = 0$ for $1 \leq j \leq N$.

Since $\mathfrak{E}_j$ maps into L^p_-, the operator $\sum_{j=1}^N (0, \mathfrak{E}_j(\mathfrak{A}_j - ikI)^{-1}\varphi_j)$ maps $W^{1,p}_- \times L^p_-$ into itself and its norm is majorized by $\sum_{j=1}^N ||\mathfrak{E}_j|| \, ||(\mathfrak{A}_j - ikI)^{-1}|| \, ||\varphi_j||$ where $||\mathfrak{E}_j||$ is the norm in $\mathcal{L}(W^{1,p}_- \times L^p_-, L^p_-)$, $||(\mathfrak{A}_j - ikI)^{-1}||$ the norm in $\mathcal{L}(W^{1,p}_- \times L^p_-)$ and $||\varphi_j||$ the norm of the multiplication by φ_j in $\mathcal{L}(W^{1,p}_- \times L^p_-)$ (well defined since the first derivatives of φ_j are bounded). Since $(k(\mathfrak{A} - ikI)^{-1})_{|k| \geq \kappa}$ is r-bounded in $\mathcal{L}(W^{1,p}_- \times L^p_-)$, it is bounded and so $\lim_{|k| \to \infty} ||(\mathfrak{A}_j - ikI)^{-1}|| = 0$. Therefore, for large $|k|$, the operator $\mathfrak{S}_k := I - \sum_{j=1}^N (0, \mathfrak{E}_j(\mathfrak{A}_j - ikI)^{-1}\varphi_j)$ is invertible on $W^{1,p}_- \times L^p_-$. Since $\mathfrak{A} - ikI$ and $\mathfrak{S}_k$ have the same range by (5.2), $\mathfrak{A} - ikI$ is onto $W^{1,p}_- \times L^p_-$ for large $|k|$.

5.2 Injectivity

Let $(u, v) \in W^{2,p}_{(\theta)-} \times W^{1,p}_-$ be such that $(\mathfrak{A} - ikI)(u, v) = 0$, so that $\varphi_j(\mathfrak{A}_j - ikI)(u, v) = 0$ since $\mathfrak{A}$ and $\mathfrak{A}_j$ coincide on U_j and Supp $\varphi_j \subset U_j$. Now, $\varphi_j(\mathfrak{A}_j - ikI)(u, v) = (\mathfrak{A}_j - ikI)(\varphi_j u, \varphi_j v) + (0, \mathfrak{F}_j(u, v))$ where $\mathfrak{F}_j$ is a k-independent scalar differential operator of first order in u and zeroth order in v with *bounded* continuous coefficients (because the first and second derivatives of φ_j are bounded). Thus, $(\mathfrak{A}_j - ikI)(\varphi_j u, \varphi_j v) = -(0, \mathfrak{F}_j(u, v))$. For $|k|$ large enough, this means that $(\varphi_j u, \varphi_j v) = -(\mathfrak{A}_j - ikI)^{-1}(0, \mathfrak{F}_j(u, v))$, whence

$$\begin{pmatrix} u \\ v \end{pmatrix} = -\sum_{j=1}^N (\mathfrak{A}_j - ikI)^{-1} \left(0 \times \mathfrak{F}_j\right) \begin{pmatrix} u \\ v \end{pmatrix}.$$

From the above, $||(u, v)||_{1,p,\mathbb{R}_-} \leq \left(\sum_{j=1}^N ||(\mathfrak{A}_j - ikI)^{-1}|| \, ||\mathfrak{F}_j|| \right) ||(u, v)||_{1,p,\mathbb{R}_-}$ where $||(\mathfrak{A}_j - ikI)^{-1}||$ is the norm in $\mathcal{L}(W^{1,p}_- \times L^p_-)$ and $||\mathfrak{F}_j||$ is the norm in $\mathcal{L}(W^{1,p}_- \times L^p_-, L^p_-)$. Since $(k(\mathfrak{A} - ikI)^{-1})_{|k| \geq \kappa}$ is $(r$-)bounded in $\mathcal{L}(W^{1,p}_- \times L^p_-)$, the constant $\sum_{j=1}^N ||(\mathfrak{A}_j - ikI)^{-1}|| \, ||\mathfrak{F}_j||$ is less than 1 if $|k| \geq \kappa$ after increasing κ if necessary, so that $(u, v) = (0, 0)$.

5.3 r-boundedness

Now that we know from 5.1 and 5.2 that $(\mathfrak{A} - ikI)^{-1}$ exists when $|k| \geq \kappa$, we may rewrite (5.2) as

$$\begin{pmatrix} u \\ v \end{pmatrix} = (\mathfrak{A} - ikI)^{-1}\mathfrak{S}_k \begin{pmatrix} f \\ g \end{pmatrix}, \tag{5.3}$$

where $\mathfrak{S}_k \in \mathcal{L}(W_-^{1,p} \times L_-^p)$ is invertible (see 5.1). In 5.1, (u, v) was obtained in the form $(u, v) = \sum_{j=1}^N (\widetilde{\varphi}_j u_j, \widetilde{\varphi}_j v_j)$ with $(\mathfrak{A}_j - ikI)(u_j, v_j) = (\varphi_j f, \varphi_j g)$. If $|k| \geq \kappa$, the latter also reads $(u_j, v_j) = (\mathfrak{A}_j - ikI)^{-1}(\varphi_j f, \varphi_j g)$, so that $(u, v) = \left(\sum_{j=1}^N \widetilde{\varphi}_j (\mathfrak{A}_j - ikI)^{-1}\varphi_j\right)(f, g)$. Since this holds for every $(f, g) \in W_-^{1,p} \times L_-^p$, (5.3) becomes $\left(\sum_{j=1}^N \widetilde{\varphi}_j (\mathfrak{A}_j - ikI)^{-1}\varphi_j\right) = (\mathfrak{A} - ikI)^{-1}\mathfrak{S}_k$, that is,

$$(\mathfrak{A} - ikI)^{-1} = \left(\sum_{j=1}^N \widetilde{\varphi}_j (\mathfrak{A}_j - ikI)^{-1}\varphi_j\right)\mathfrak{S}_k^{-1}.$$

For $|k|$ large enough, $k\sum_{j=1}^N \widetilde{\varphi}_j (\mathfrak{A}_j - ikI)^{-1}\varphi_j \in \mathcal{L}(W_-^{1,p} \times L_-^p)$ is r-bounded (finite sum of r-bounded sequences). Thus, the r-boundedness of $(k(\mathfrak{A} - ikI)^{-1})_{|k|\geq\kappa}$ in $\mathcal{L}(W_-^{1,p} \times L_-^p)$ for large κ will follow from the same property for $(\mathfrak{S}_k^{-1})_{|k|\geq\kappa}$.

Recall that $\mathfrak{S}_k = I - \sum_{j=1}^N (0, \mathfrak{E}_j(\mathfrak{A}_j - ikI)^{-1}\varphi_j)$. Since $k(\mathfrak{A}_j - ikI)^{-1}$ is r-bounded for large $|k|$ and $1 \leq j \leq N$, it follows from the Kahane contraction principle that the r-bound of $\sum_{j=1}^N (0, \mathfrak{E}_j(\mathfrak{A}_j - ikI)^{-1}\varphi_j)$ is $O\left(\kappa^{-1}\right)$ when $|k| \geq \kappa$ and κ is large. Now, if X is any Banach space and $\mathcal{T} \subset \mathcal{L}(X)$ is r-bounded with r-bound $r(\mathcal{T}) < 1$, then $(I - \mathcal{T})^{-1}$ is also r-bounded in $\mathcal{L}(X)$ ([27, Lemma 2.4]). Thus, as claimed, $(\mathfrak{S}_k^{-1})_{|k|\geq\kappa}$ is r-bounded if κ is large enough.

6 Proof, variants, and implications of Theorem 1.1

We now complete the proof of Theorem 1.1 when the coefficients are asymptotically periodic. We also discuss the special case of real asymptotically constant coefficients, the Dirichlet problem, and nonhomogeneous boundary conditions.

6.1 Proof of Theorem 1.1

From the discussion at the end of Sect. 2, it suffices to show that the sequence $(k(\mathfrak{A} - ikI)^{-1})_{|k|\geq\kappa}$ is defined and r-bounded in $\mathcal{L}(W_-^{1,p} \times L_-^p)$ provided that $\kappa \in \mathbb{N}$ is large enough. Call $\mathfrak{P}_\sharp$ the operator with periodic coefficients $b_\sharp(y), \ldots, \gamma_\sharp(y)$ such that $\lim_{y\to-\infty} |b(y) - b_\sharp(y)| = \cdots = \lim_{y\to-\infty} |\gamma(y) - \gamma_\sharp(y)| = 0$. As observed just before Theorem 1.1, the uniform continuity of the coefficients of $\mathfrak{P}$ implies the (uniform) continuity of the coefficients of $\mathfrak{P}_\sharp$. It is equally clear that it implies the (uniform) ellipticity of $\mathfrak{P}_\sharp$.

The proper ellipticity is also inherited by $\mathfrak{P}_\sharp$. Indeed, by periodicity and continuity, the ellipticity of $\mathfrak{P}_\sharp$ shows that the imaginary parts of the roots of $c_\sharp(y)\lambda^2 + 2b_\sharp(y)\lambda + 1$ are uniformly bounded away from 0 when $y \to -\infty$. Meanwhile, the roots of $c(y)\lambda^2 + 2b(y)\lambda + 1$ tend to those of $c_\sharp(y)\lambda^2 + 2b_\sharp(y)\lambda + 1$ as $y \to -\infty$. By the proper ellipticity of $\mathfrak{P}$, it follows (by contradiction) that the roots of $c_\sharp(y)\lambda^2 + 2b_\sharp(y)\lambda + 1$ have imaginary parts of opposite signs when $-y > 0$ is large enough, and hence for every $y \in \mathbb{R}_-$.

Therefore, the results of the previous section are applicable to $\mathfrak{P}_\sharp$: If $\mathfrak{A}_\sharp$ denotes the operator $\mathfrak{A}$ in (1.6) with $\mathfrak{P}$ replaced by $\mathfrak{P}_\sharp$, then $(k(\mathfrak{A}_\sharp - ikI)^{-1})_{|k|\geq\kappa}$ is defined and r-bounded if κ is large enough. By using once more the stability of r-boundedness under small perturbations, there is $\varepsilon > 0$ such that if $\mathfrak{B} \in \mathcal{L}(W^{2,p}_{(\theta)-} \times W^{1,p}_-, W^{1,p}_- \times L^p_-)$ and $||\mathfrak{B} - \mathfrak{A}_\sharp|| < \varepsilon$, then after increasing κ if necessary, the sequence $(k(\mathfrak{B} - ikI)^{-1})_{|k|\geq\kappa}$ is defined and r-bounded. In particular, this is true if $\mathfrak{B}$ has the same form (1.6) as $\mathfrak{A}_\sharp$ with continuous coefficients uniformly δ-close to the coefficients of $\mathfrak{A}_\sharp$ for some $\delta > 0$ depending only upon ε.

A convenient choice of $\mathfrak{B}$ above is as follows: Due to the "asymptotic periodicity" assumption, there is $\rho > 1$ such that the coefficients of $\mathfrak{A}$ are δ-close to those of $\mathfrak{A}_\sharp$ on the interval $(-\infty, -\rho + 1] \subset \mathbb{R}_-$. Pick a continuous function η on $\overline{\mathbb{R}}_-$ such that $\operatorname{Supp}\eta \subset (-\infty, -\rho + 1)$, $0 \leq \eta \leq 1$ and $\eta = 1$ on $(-\infty, -\rho]$. Then the coefficients of $\mathfrak{B} := \eta\mathfrak{A} + (1-\eta)\mathfrak{A}_\sharp$ are continuous and uniformly δ-close to the coefficients of $\mathfrak{A}_\sharp$ on $\overline{\mathbb{R}}_-$. Thus, $(k(\mathfrak{B} - ikI)^{-1})_{|k|\geq\kappa}$ is defined and r-bounded if κ is large enough.

On the other hand, let $y_0 \in \overline{\mathbb{R}}_-$ be given along with $\delta_0 > 0$. If $\mathfrak{A}_0$ of the form (1.6) has continuous coefficients that coincide with those of $\mathfrak{A}$ on some open interval $J_0 := (y_0 - \delta_0, y_0 + \delta_0) \cap \overline{\mathbb{R}}_-$ of $\overline{\mathbb{R}}_-$ and are constant outside this interval, then an obvious simplification of the argument at the beginning of Sect. 5 shows that, if δ_0 is small enough, $(k(\mathfrak{A}_0 - ikI)^{-1})_{|k|\geq\kappa}$ is defined and r-bounded for some κ. Cover the compact interval $[-\rho - 1, 0]$ with finitely many such open intervals, say J_j, $1 \leq j \leq N - 1$. This yields $N - 1$ operators $\mathfrak{A}_j$ such that $\mathfrak{A}_j = \mathfrak{A}$ on J_j and that $(k(\mathfrak{A}_j - ikI)^{-1})_{|k|\geq\kappa}$ is defined and r-bounded if κ is large enough. This remains true for $j = N$ by defining $J_N := (-\infty, -\rho)$ and $\mathfrak{A}_N := \mathfrak{B}$ with ρ and $\mathfrak{B}$ as above.

Since the (relatively) open intervals J_j, $1 \leq j \leq N$ cover $\overline{\mathbb{R}}_-$, there is a smooth partition of unity φ_j subordinate to this covering and there is a smooth function $\widetilde{\varphi}_j$ with $\operatorname{Supp}\widetilde{\varphi}_j \subset J_j$ and $\widetilde{\varphi}_j = 1$ on $\operatorname{Supp}\varphi_j$ (support relative $\overline{\mathbb{R}}_-$). Furthermore, it may be assumed that $\widetilde{\varphi}_j = 0$ on a neighborhood of 0 in $\overline{\mathbb{R}}_-$ for every index j such that $0 \notin \operatorname{Supp}\varphi_j$. This ensures that $\widetilde{\varphi}'_j(0) = 0$ for $1 \leq j \leq N$.

Necessarily, $\varphi_N = \widetilde{\varphi}_N = 1$ on an unbounded interval, so that all the derivatives of φ_j and $\widetilde{\varphi}_j$ are uniformly bounded on $\overline{\mathbb{R}}$. Thus, the arguments of 5.1, 5.2, and 5.3 can be repeated verbatim to find that $(k(\mathfrak{A} - ikI)^{-1})_{|k|\geq\kappa}$ is defined and r-bounded in $\mathcal{L}(W^{1,p}_- \times L^p_-)$ if κ is large enough. The proof of Theorem 1.1 is finally complete.

6.2 Asymptotically constant coefficients

When the coefficients are real and the "limiting" coefficients $b_\sharp, \ldots, \gamma_\sharp$ are not only periodic but constant, Theorem 1.1 can be simplified further:

Theorem 6.1 *Suppose that the coefficients of $\mathfrak{P}$ in (1.1) are real, continuous on $\overline{\mathbb{R}}_-$ and asymptotically constant with limits $b_{-\infty}, \ldots, \gamma_{-\infty}$ and that $\mathfrak{P}$ is uniformly elliptic. Let $p \in (1, \infty)$ be given. If $\gamma_{-\infty} < 0$, then, $\mathfrak{P}$ is an isomorphism of $W^{2,p}_{(\theta),per}((0, 2\pi) \times \mathbb{R}_-)$ onto $L^p((0, 2\pi) \times \mathbb{R}_-)$ if and only if P_k is one to one on $W^{2,p}_{(\theta)-}$ for every $k \in \mathbb{Z}$.*

If $\gamma_{-\infty} \geq 0$, it can be shown that P_0 is not Fredholm (by a variant of the proof of Lemma 6.2 below), so that $\gamma_{-\infty} < 0$ is necessary in Theorem 6.1. Since the hypotheses of Theorem 6.1 imply that the coefficients of $\mathfrak{P}$ are uniformly continuous and since proper ellipticity is not an extra assumption when the coefficients are real, it is a rephrasing of Theorem 1.1 due to the following lemma:

Lemma 6.2 *Under the hypotheses of Theorem 6.1, the operator P_k is Fredholm of index* 0 *from $W^{2,p}_{(\theta)-}$ to L^p_- for every $k \in \mathbb{Z}$.*

Proof Recall that the uniform ellipticity of $\mathfrak{P}$ implies the ellipticity of the limiting operator, whence $c_{-\infty} > 0$ and $b^2_{-\infty} - c_{-\infty} < 0$.

Extend the coefficients of P and Q by symmetry, that is, set $\widetilde{c}(y) = c(-y)$, $\widetilde{\beta}(y) = \beta(-y)$, etc., for $y \geq 0$. This yields a second-order elliptic differential operator $\widetilde{P}_k$ on the line with continuous coefficients having (equal) limits as $y \to \pm\infty$. The Fredholmness of $\widetilde{P}_k : W^{2,p}(\mathbb{R}) \to L^p(\mathbb{R})$ is known to be equivalent to the symbol

$$\widetilde{\sigma}_k(y,\tau) := -\widetilde{c}(y)\tau^2 + 2\widetilde{b}(y)k\tau + \widetilde{\gamma}(y) - k^2 + i(\widetilde{\beta}(y)\tau - \widetilde{\alpha}(y)k), \tag{6.1}$$

satisfying the inequality

$$|\widetilde{\sigma}_k(y,\tau)| \geq \varepsilon_k(\tau^2 + 1), \tag{6.2}$$

for some $\varepsilon_k > 0$ and all pairs $(y,\tau) \in \mathbb{R}^2$ such that $y^2 + \tau^2$ is large enough. (More details about this criterion and an alternate option are given after the proof.) This amounts to the existence of $\varepsilon_k, \rho_k > 0$ such that (6.2) holds (i) for every $y \in \mathbb{R}$ and $|\tau| \geq \rho_k$ and (ii) for every $\tau \in \mathbb{R}$ and $|y| \geq \rho_k$.

Naturally, it suffices to check (6.2) with $|\widetilde{\sigma}_k(y,\tau)|$ replaced by $-\operatorname{Re}\widetilde{\sigma}_k(y,\tau)$, that is,

$$\widetilde{c}(y)\tau^2 - 2\widetilde{b}(y)k\tau - \widetilde{\gamma}(y) + k^2 \geq \varepsilon_k(\tau^2 + 1),$$

for every $y \in \mathbb{R}$ and $|\tau| \geq \rho_k$ and for every $\tau \in \mathbb{R}$ and $|y| \geq \rho_k$. Case (i) follows from $\widetilde{c}(y) > 0$ being uniformly bounded away from 0 (since $\lim_{|y|\to\infty}\widetilde{c}(y) = c_{-\infty} > 0$) while the other coefficients are bounded. Case (ii) is due to the similar inequality when the coefficients are replaced by their limits as $|y| \to \infty$. Indeed, $c_{-\infty}\tau^2 - 2b_{-\infty}k\tau - \gamma_{-\infty} + k^2 \geq \varepsilon_k(\tau^2 + 1)$ for some $\varepsilon_k > 0$ and every $\tau \in \mathbb{R}$ by the ellipticity condition $b^2_{-\infty} - c_{-\infty} < 0$ *and* the hypothesis $\gamma_{-\infty} < 0$ (if $\gamma_{-\infty} \geq 0$, the property fails for $k = 0$).

This shows that $\widetilde{P}_k$ is Fredholm from $W^{2,p}(\mathbb{R})$ to $L^p(\mathbb{R})$. Thus, there is a finite dimensional subspace $\widetilde{G}$ of $L^p(\mathbb{R})$ such that, for every $\widetilde{f} \in L^p(\mathbb{R})$, there is $\widetilde{g} \in \widetilde{G}$ such that $\widetilde{P}_k\widetilde{w} = \widetilde{f} - \widetilde{g}$ has a solution $\widetilde{w} \in W^{2,p}(\mathbb{R})$. In particular, let $f \in L^p_-$ be given and let $\widetilde{f}$ be the extension of f by 0. With $\widetilde{g}$ and $\widetilde{w}$ as above, the restriction w of $\widetilde{w}$ to $\mathbb{R}_-$ is in $W^{2,p}_-$ and solves $P_k w = f - g$, where $g := \widetilde{g}_{|\mathbb{R}_-} \in L^p_-$. Of course, $w \notin W^{2,p}_{(\theta)-}$ in general. Choose $\varphi \in C_0^\infty(\overline{\mathbb{R}}_-)$ once and for all such that $\varphi(0) = 0$ and $\varphi'(0) = 1$ and set $v := w - (w'(0) + \theta w(0))\varphi$. Then, $v \in W^{2,p}_{(\theta)-}$ and $P_k v = f - g - (w'(0) + \theta w(0))P_k\varphi$.

By definition, g belongs to the space G of restrictions to $\mathbb{R}_-$ of functions of $\widetilde{G}$. Since $\widetilde{G}$ is finite dimensional, the same thing is true of G. Thus, $G + \mathbb{C}P_k\varphi \subset L^p_-$ is finite dimensional and the above shows that given any $f \in L^p_-$, there is $h \in G + \mathbb{C}P_k\varphi$ (namely, $h = g + (w'(0) + \theta w(0))P_k\varphi$) such that $P_k v = f - h$ has a solution in $W^{2,p}_{(\theta)-}$. This proves that the range of $P_k : W^{2,p}_{(\theta)-} \to L^p_-$ has finite codimension. Therefore, P_k is Fredholm since $\dim\ker P_k \leq 2$ by the existence and uniqueness for ODE initial value problems.

It remains to show that the index of P_k is 0. The arguments proving the Fredholmness above can be repeated verbatim when P_k is replaced by $tP_k + (1-t)P_{-\infty,k}$, where $t \in [0,1]$ and $P_{-\infty,k}$ is the limiting operator with constant coefficients

$$P_{-\infty,k}w := c_{-\infty}w'' + (\beta_{-\infty} - 2ikb_{-\infty})w' + (\gamma_{-\infty} - k^2 - ik\alpha_{-\infty})w.$$

Thus, $P_{-\infty,k}$ is Fredholm and, by homotopy, P_k and $P_{-\infty,k}$ have the same index. A further simplification can be introduced by noticing that the ellipticity condition $b^2_{-\infty} - c_{-\infty} < 0$

is unaffected by changing $b_{-\infty}$ into $tb_{-\infty}$ for $t \in [0, 1]$ and that $\alpha_{-\infty}$ and $\beta_{-\infty}$ can also be replaced by $t\alpha_{-\infty}$ and $t\beta_{-\infty}$, respectively. Thus, by another homotopy, it suffices to show that the operator $c_{-\infty}w'' + (\gamma_{-\infty} - k^2)w$ has index 0 from $W^{2,p}_{(\theta)-}$ to L^p_-. A third homotopy shows that it is not restrictive to assume $c_{-\infty} = 1$ and $\gamma_{-\infty} - k^2 = -1$. Therefore, everything boils down to the fact that $w'' - w$ has index 0 from $W^{2,p}_{(\theta)-}$ to L^p_-.

If $\theta \neq -1$, it is an isomorphism, for its null space is clearly $\{0\}$ and if $f \in L^p_-$, it is a special case of (4.1) that the solution $w \in W^{2,p}_{(\theta)-}$ of $w'' - w = f$ is given by $w(y) := -\frac{1}{2}\left((E * \widetilde{f})(y) - \frac{\theta-1}{(\theta+1)}\mathrm{e}^y \int_{-\infty}^0 \mathrm{e}^z f(z)\mathrm{d}z\right)$, where $\widetilde{f}$ denotes the extension of f by 0 and $E(y) := \mathrm{e}^{-|y|}$.

If $\theta = -1$, the null space of $w'' - w$ in $W^{2,p}_{(-1)-}$ is $\mathbb{C}e^y$ (one-dimensional) while $w'' - w = f$ with $w'(0) - w(0) = 0$ is solvable only if $\int_{-\infty}^0 \mathrm{e}^z f(z)\mathrm{d}z = 0$, and then $w = (E * \widetilde{f}) \in W^{2,p}_{(-1)-}$ is a solution. Thus, the range has codimension 1. □

The symbol criterion for the Fredholmness of $\widetilde{P}_k$ used at the beginning of the proof of Lemma 2.4 has a rather long history and various ramifications. The variant we used is due to Cordes [8,9] and Illner [15]. It is valid for elliptic equations on $\mathbb{R}^n$ when the coefficients are continuous with "vanishing oscillation" at infinity (in particular, when they have limits). Many other proofs limited to $p = 2$ exist under (much) more restrictive smoothness conditions about the coefficients.

The Fredholmness of $\widetilde{P}_k$ can also be obtained by more elementary ODE arguments: Rewrite $\widetilde{P}_k$ as a first-order system $\frac{\mathrm{d}}{\mathrm{d}y} + \widetilde{B}_k(y)$, where $\widetilde{B}_k(y)$ is a 2×2 matrix with continuous coefficients having limits when $|y| \to \infty$. The uniform ellipticity condition for $\mathfrak{P}$ and the assumption $\gamma_{-\infty} < 0$ imply that if $|y|$ is large enough, then $\widetilde{B}_k(y)$ has exactly one eigenvalue with positive (negative) real part and that both real parts are bounded away from 0 as $|y| \to \infty$. By Coppel [7, Proposition 1, p. 50], $\frac{\mathrm{d}}{\mathrm{d}y} + B_k(y)$ has an exponential dichotomy[3] on $\mathbb{R}_\pm$ and then a straightforward variant of Palmer's method [20] (where a different functional setting is used) shows that $\frac{\mathrm{d}}{\mathrm{d}y} + \widetilde{B}_k(y)$ is Fredholm from $W^{2,p}(\mathbb{R}) \times W^{1,p}(\mathbb{R})$ to $W^{1,p}(\mathbb{R}) \times L^p(\mathbb{R})$, which in turn readily implies the Fredholmness of $\widetilde{P}_k$.

Remark 6.1 If the coefficients of $\mathfrak{P}$ are asymptotically constant but not real, the method of proof of Lemma 6.2 can be adapted to check whether the P_k are Fredholm of index 0 and hence whether Theorem 6.1 is still valid. This cannot be summarized by a simple condition about the limiting coefficients, but $\gamma_{-\infty} \neq 0$ is always necessary for P_0 to be Fredholm (and hence for $\mathfrak{P}$ to be invertible).

6.3 Other boundary conditions

The method of proof of Theorem 1.1 also works in the case of a homogeneous Dirichlet condition, based on 3.5 when the coefficients are constant. Thus, Theorem 1.1 is still true in this case, upon replacing $W^{2,p}_{(\theta),per}((0, 2\pi) \times \mathbb{R}_-)$ and $W^{2,p}_{(\theta)-}$ by $W^{2,p}((0, 2\pi) \times \mathbb{R}_-) \cap W^{1,p}_0((0, 2\pi) \times \mathbb{R}_-)$ and $W^{2,p}_- \cap W^{1,p}_{0-}$, respectively.

Remark 6.2 Theorem 1.1 is also true in the whole plane. Since there are no boundary conditions, the treatment of constant coefficients is significantly shorter, but the basic ingredients

[3] The result is phrased in a rather ambiguous way, which could erroneously suggest that the uniform continuity of $\widetilde{B}_k$ suffices for an exponential dichotomy. The correct reading of Coppel's condition is essentially the "vanishing oscillation" of Cordes and Illner.

(i.e., (3.4), (3.8) and the boundedness of the maximal function in $L^p(\mathbb{R}, \ell^2)$) remain the same. Note that proper ellipticity is still essential, even though boundary conditions are not involved.

More generally, the invertibility criterion of Theorem 1.1 implies the unique solvability for 2π-periodic boundary data. Specifically, define $W_{per}^{1-\frac{1}{p},p}(0, 2\pi)$ to be the space of restrictions to $(0, 2\pi)$ of the periodic functions of $\cap_{n\in\mathbb{N}} W^{1-\frac{1}{p},p}(-n, n)$. Then, $W_{per}^{1-\frac{1}{p},p}(0, 2\pi) \subset W^{1-\frac{1}{p},p}(0, 2\pi)$ and, if $p \neq 2$,

$$W_{per}^{1-\frac{1}{p},p}(0, 2\pi) = \begin{cases} \{h \in W^{1-\frac{1}{p},p}(0, 2\pi) : h(0) = h(2\pi)\} \text{ if } p > 2, \\ W^{1-\frac{1}{p},p}(0, 2\pi) \text{ if } p \in (1, 2). \end{cases} \tag{6.3}$$

When $p \in (1, 2)$, this is due to the fact that the extension by 0 outside $(0, 2\pi)$ is continuous from $W^{s,p}(0, 2\pi)$ to $W^{s,p}(\mathbb{R})$ if $s \in [0, \frac{1}{p})$ [16]. The justification of (6.3) when $p > 2$ is easy.

Even though $C_0^\infty(0, 2\pi)$ is dense in $W^{\frac{1}{2},2}(0, 2\pi)$ (so that $h(0)$ or $h(2\pi)$ are not defined when $h \in W^{\frac{1}{2},2}(0, 2\pi)$), the extension by 0 does not map into $W^{\frac{1}{2},2}(\mathbb{R})$ [17]. As a result, oddly enough, when $p = 2$, the space $W_{per}^{\frac{1}{2},2}(0, 2\pi)$ does not afford the simpler characterization (6.3).

If $h \in W_{per}^{1-\frac{1}{p},p}(0, 2\pi)$ is extended by periodicity, the classical method to prove the surjectivity of the trace produces some $v \in \cap_{n\in\mathbb{N}} W^{2,p}((-n, n) \times \mathbb{R}_-)$, also 2π -periodic in x, such that $(v, v_y) = (0, h)$ on $\mathbb{R} \times \{0\}$. The 2π -periodicity of v in x can be verified on the formulas giving v in terms of h through an integral operator [19,26]. This operator is of convolution type and thus produces a periodic v when h is periodic.

If v is chosen as above, then $v_y + \theta v = h$, so that if $g \in L^p((0, 2\pi) \times \mathbb{R}_-)$ and $h \in W_{per}^{1-\frac{1}{p},p}(0, 2\pi)$, the question of solving

$$\begin{cases} u_{xx} + 2b(y)u_{xy} + c(y)u_{yy} + \alpha(y)u_x + \beta(y)u_y + \gamma(y)u = g, \\ u_y + \theta u = h, \end{cases} \tag{6.4}$$

for $u \in W^{2,p}((0, 2\pi) \times \mathbb{R}_-)$ such that $u(0, \cdot) = u(2\pi, \cdot)$ and $u_x(0, \cdot) = u_x(2\pi, \cdot)$, say $u \in W_{per}^{2,p}((0, 2\pi) \times \mathbb{R}_-)$ for short, amounts to solving the same equation for $u - v \in W^{2,p}_{(\theta),per}((0, 2\pi) \times \mathbb{R}_-)$ after replacing g by $g - \mathfrak{P}v$. This problem is uniquely solvable for every g if and only if the criterion of Theorem 1.1 is satisfied, so that the same thing is true of (6.4).

Remark 6.3 The more general nontangential boundary condition $\nabla u \cdot N + \theta u = 0$ where $N := (\mu, 1)$ is constant and $\mu \in \mathbb{R}$ can be reduced to $v_y + \theta v = 0$ by the change of variable $u(x, y) = v(x + \phi(y), y)$ where ϕ is a smooth real-valued function with compact support in $(-\infty, 0]$ such that $\phi(0) = 0$ and $\phi'(0) = -\mu$. This changes the coefficients $b(y), \ldots, \gamma(y)$, but does not affect their asymptotic periodicity or the uniform ellipticity condition, or proper ellipticity[4]. In fact, initially, the coefficient of v_{xx} is no longer 1, but $1 + 2b(y)\phi'(y) + c(y)\phi'(y)^2$. This coefficient is bounded and bounded away from 0 (by uniform ellipticity), so that it can be factored to return to the case when the coefficient of v_{xx} is 1, and all the previous results are applicable.

4 Since it need only hold at one point to hold at every point and it does hold outside the support of ϕ.

7 The nonperiodic problem

The boundary value problem (1.1) makes sense when $g \in L^p(\mathbb{R} \times \mathbb{R}_-)$ and the solutions u are sought in the space

$$W^{2,p}_{(\theta)}(\mathbb{R} \times \mathbb{R}_-) := \{u \in W^{2,p}(\mathbb{R} \times \mathbb{R}_-) : u_y + \theta u = 0 \text{ on } \mathbb{R} \times \{0\}\}. \tag{7.1}$$

This may also be rewritten in the first-order form (1.5), with the system now acting from $L^p(\mathbb{R}; W) \cap W^{1,p}(\mathbb{R}; X)$ to $L^p(\mathbb{R}; X)$, where X and W are still given by (1.10). Thus, the only difference is that $(0, 2\pi)$ is replaced by $\mathbb{R}$ and the periodicity is dropped. For such a problem, we can use the following variant of Theorem 1.2:

Theorem 7.1 *Let X be a (complex) UMD Banach space and let A be a closed unbounded linear operator on X with domain W, equipped with the graph norm. Then, the operator $\frac{d}{dx} + A$ is an isomorphism of $L^p(\mathbb{R}; W) \cap W^{1,p}(\mathbb{R}; X)$ onto $L^p(\mathbb{R}; X)$ if and only if $(A - i\xi I)^{-1} \in \mathcal{L}(X)$ exists for every $\xi \in \mathbb{R}$ and $(\xi(A - i\xi I)^{-1})_{\xi \in \mathbb{R}}$ is r-bounded in $\mathcal{L}(X)$.*

The sufficiency was proved independently by the author [21, Theorem 4.1] and Schweiker [23]. The necessity is due to Arendt and Duelli [2]. In [21], the r-boundedness of $(\xi(A - i\xi I)^{-1})_{\xi \in \mathbb{R}}$ is replaced by a seemingly weaker, but more technical, condition. The necessity makes this refinement immaterial.

The proofs given in Sects. 3–6 reveal that the r-boundedness of $(k(\mathfrak{A} - ikI)^{-1})_{|k| \geq \kappa}$ never relies on k being an integer. Thus, what was actually proved there is that $(\xi(\mathfrak{A} - i\xi I)^{-1})_{\xi \in \mathbb{R}, |\xi| \geq \kappa}$ is r-bounded if $\kappa \geq 0$ is large enough.

Of course, the set $\{\xi(\mathfrak{A} - i\xi I)^{-1} : |\xi| < \kappa\}$ is no longer finite but, if defined for every $\xi \in \mathbb{R}$, $(\xi(\mathfrak{A} - i\xi I)^{-1})_{\xi \in \mathbb{R}, |\xi| \leq \kappa}$ is still r-bounded due to the compactness of $[-\kappa, \kappa]$ and the analyticity of $\xi(\mathfrak{A} - i\xi I)^{-1}$ ([11, p. 31]). If so, $(\xi(\mathfrak{A} - i\xi I)^{-1})_{\xi \in \mathbb{R}}$ is r-bounded. Lemma 2.1 is still true when $(0, 2\pi)$ is replaced by $\mathbb{R}$ and the periodicity is dropped and so is Lemma 2.3 if $k \in \mathbb{Z}$ is replaced by $\xi \in \mathbb{R}$ and

$$P_\xi w := c(y)w'' + (\beta(y) - 2i\xi b(y))w' + (\gamma(y) - \xi^2 - i\xi\alpha(y))w.$$

Thus, the sufficiency part of Theorem 7.1 yields:

Theorem 7.2 *Suppose that $\mathfrak{P}$ in (1.1) is uniformly and properly elliptic, with bounded uniformly continuous and asymptotically periodic coefficients. Let $p \in (1, \infty)$ be given. If P_ξ is an isomorphism of $W^{2,p}_{(\theta)-}$ onto L^p_- for every $\xi \in \mathbb{R}$, then $\mathfrak{P}$ is an isomorphism of $W^{2,p}_{(\theta)}(\mathbb{R} \times \mathbb{R}_-)$ onto $L^p(\mathbb{R} \times \mathbb{R}_-)$.*

If (and only if) $p = 2$, Theorem 7.2 can be proved by partial Fourier transform arguments and norm estimates for P_ξ^{-1}. We do not know whether the converse of Theorem 7.2 is true, which does *not* follow from Theorem 7.1. In that regard, see the comments after Lemma 2.1 and note that there is no obvious analog of Lemma 2.4 here.

As in Theorem 6.1, if the coefficients are real and asymptotically constant with $\gamma_{-\infty} < 0$, the invertibility of P_ξ amounts to P_ξ being one to one on $W^{2,p}_{(\theta)-}$ (Lemma 6.2 does not depend on k being an integer). If the coefficients are real and constant and $\gamma < 0$, this boils down to $\theta \neq -\lambda_+(\xi)$ for every $\xi \in \mathbb{R}$, where $\lambda_+(\xi)$ is the unique root with positive real part of $c\lambda^2 + (\beta - 2i\xi b)\lambda + (\gamma - \xi^2 - i\xi\alpha)$ (see Theorem 4.3).

As in the periodic case, the condition of Theorem 7.2 yields an existence and uniqueness result when the boundary condition is not homogeneous and the expected variant of the theorem for a Dirichlet boundary condition is true as well.

In Theorem 7.2, the half-plane can be replaced either by the whole plane or by the horizontal strip $\mathbb{R} \times (0, 1)$ (say). In the case of the strip, the coefficients need only be continuous on $[0, 1]$. The simpler indirect approach for periodic problems on a strip, mentioned in the Introduction and based on the Fredholmness of the operator, is no longer an option. Actually, it is not hard to prove (by using a suitable sequence) that the Laplace operator with homogeneous Neumann condition on the boundary of the strip is not semi-Fredholm[5]. For this example, the corresponding operators P_ξ are simply $w'' - \xi^2 w$ with condition $w'(0) = w'(1) = 0$ and P_0 is not invertible.

Appendix: *r*-boundedness

This appendix gives a very brief summary of the basic properties of r-boundedness used in this paper. Further details and various complements can be found in [1,6,21,27], among others.

If X and Y are Banach spaces and $\mathcal{T} \subset \mathcal{L}(X, Y)$, then $\mathcal{T}$ is said to be r-bounded if there is $p \in [1, \infty)$ and a constant $C_p \geq 0$ such that, for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in X$, every $T_1, \ldots, T_n \in \mathcal{T}$,

$$\left(\int_0^1 \left\| \sum_{j=1}^n r_j(t) T_j x_j \right\|_Y^p \mathrm{d}t \right)^{\frac{1}{p}} \leq C_p \left(\int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|_X^p \mathrm{d}t \right)^{\frac{1}{p}},$$

where $r_j(t) := \operatorname{sign} \sin(2^j \pi t)$ is the sequence of Rademacher functions. (Some authors replace the Rademacher functions by more general random variables, but the definitions are equivalent.) Although the constant C_p depends upon p, the concept of r-boundedness does not. For that reason, the value of p has little genuine importance in practice, but some choices may be more convenient than others in the applications. If $\mathcal{T}$ is r-bounded, the smallest constant C_p above is called the r-bound $r(\mathcal{T})$ of $\mathcal{T}$. Since this r-bound (but not its finiteness) depends upon p, it is implicitly assumed that p is chosen once and for all when referring to r-bounds without specifying p.

It is trivial that r-boundedness implies norm-boundedness (just choose $n = 1$). The converse is true only in special cases, most notably when both X and Y are Hilbert. Other properties of r-boundedness are that if $\mathcal{S}$ and $\mathcal{T}$ are r-bounded, then, $\mathcal{S} \cup \mathcal{T}$ and $\mathcal{S} + \mathcal{T}$ are r-bounded with $r(\mathcal{S} \cup \mathcal{T}) \leq r(\mathcal{S}) + r(\mathcal{T})$ and $r(\mathcal{S} + \mathcal{T}) \leq r(\mathcal{S}) + r(\mathcal{T})$. It should be observed that, in contrast to norm-boundedness, it is *false* that $r(\mathcal{S} \cup \mathcal{T})$ is majorized by $\max\{r(\mathcal{S}), r(\mathcal{T})\}$. However, if $\mathcal{T}$ is r-bounded and $\mathcal{S} \subset \mathcal{T}$, then $\mathcal{S}$ is r-bounded and $r(\mathcal{S}) \leq r(\mathcal{T})$.

Every singleton $\mathcal{T} = \{T\}$ is r-bounded (and $r(\mathcal{T}) = ||T||$), hence every finite $\mathcal{T}$ is r-bounded. Also, if Z is another Banach space and $\mathcal{U} \subset \mathcal{L}(Y, Z)$ is r-bounded, then $\mathcal{U}\mathcal{T}$ is r-bounded and $r(\mathcal{U}\mathcal{T}) \leq r(\mathcal{U}) r(\mathcal{T})$.

The so-called "Kahane contraction principle" refers to the fact that, if $\mathcal{T}$ is r-bounded and $K \subset \mathbb{C}$ satisfies $\sup |K| \leq M < \infty$, then $K\mathcal{T}$ is r-bounded and $r(K\mathcal{T}) \leq 2Mr(\mathcal{T})$ $(Mr(\mathcal{T})$ if $K \subset \mathbb{R})$.

Lastly, there is an important and somewhat simpler equivalent definition of r-boundedness when $p_1, p_2 \in [1, \infty)$ and $X = L^{p_1}(\Omega_1)$, $Y = L^{p_2}(\Omega_2)$ where Ω_1 and Ω_2 are (say) open subsets of euclidian space. The subset $\mathcal{T} \subset \mathcal{L}(L^{p_1}(\Omega_1), L^{p_2}(\Omega_2))$ is r-bounded if and only if there is a constant $C_{p_1 p_2} \geq 0$ such that, for every $n \in \mathbb{N}$ and every $f_1, \ldots, f_n \in L^{p_1}(\Omega_1)$, every $T_1, \ldots, T_n \in \mathcal{T}$,

[5] It is an isomorphism if a homogeneous Dirichlet condition is used instead.

$$\left\| \left(\sum_{j=1}^{n} |T_j f_j|^2 \right)^{\frac{1}{2}} \right\|_{0,p_2,\Omega_2} \leq C_{p_1 p_2} \left\| \left(\sum_{j=1}^{n} |f_j|^2 \right)^{\frac{1}{2}} \right\|_{0,p_1,\Omega_1}.$$

In addition, the best constant $C_{p_1 p_2}$ is equivalent to the r-bound $r(\mathcal{T})$ (so that, if convenient, it may also be referred to as being the r-bound).

References

1. Arendt, W., Bu, S.: The operator-valued Marcinkiewicz multiplier theorem and maximal regularity. Math. Z. **240**, 311–343 (2002)
2. Arendt, W., Duelli, M.: Maximal L^p-regularity for parabolic and elliptic equations on the line. J. Evolut. Equs. **6**, 773–790 (2006)
3. Benedek, A., Panzone, R.: The spaces L^P, with mixed norm. Duke Math. J. **28**, 301–324 (1961)
4. Burkholder, D.L.: Martingale transforms. Ann. Math. Stat. **37**, 1494–1504 (1966)
5. Clément, P., de Pagter, B.: Some remarks on the Banach space valued Hilbert transform. Indag. Math. (N.S.) **2**, 453–460 (1991)
6. Clément, P., de Pagter, B., Sukochev, F.A., Witvliet, H.: Schauder decomposition and multiplier theorems. Studia Math. **138**, 135–163 (2000)
7. Coppel, W.A.: Dichotomies in Stability Theory. Lecture Notes in Mathematics, vol. 629. Springer, Berlin (1978)
8. Cordes, H.O.: Lecture notes on Banach algebra methods in partial differential equations. Lund 1970/71 (unpublished)
9. Cordes, H.O.: Beispiele von pseudo-differentialoperator-algebren. Appl. Anal. **2**, 115–129 (1972)
10. Deimling, K.: Nonlinear Functional Analysis. Springer, Berlin (1985)
11. Denk, R., Hieber, M., Prüss, J.: $\mathcal{R}$-Boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic Type. Mem. Am. Math. Soc. no 788, Providence (2003)
12. Dugundji, J.: Topology. Allyn & Bacon, Boston (1973)
13. Fernandez, D.L., Garcia, J.B.: Interpolation of Orlicz-valued function spaces and U.M.D. property. Stud. Math. **99**, 23–39 (1991)
14. Grisvard, P.: Elliptic Problems in Nonsmooth Domains. Pitman, Boston (1985)
15. Illner, R.: On algebras of pseudo differential operators in $L^p(\mathbb{R}^n)$. Commun. Partial Differ. Equs. **2**, 359–393 (1977)
16. Lions, J.-L., Magenes, E.: Problèmes aux limites non homogènes (V). Ann. Sc. Norm. Sup. Pisa **16**, 1–44 (1962)
17. Lions, J.-L., Magenes, E.: Problèmes aux limites non homogènes et applications, vol. 1. Dunod, Paris (1968)
18. Lizama, C., Poblete, V.: Maximal regularity for perturbed integral equations on periodic Lebesgue spaces. J. Math. Anal. Appl. **348**, 775–786 (2008)
19. Nečas, J.: Les méthodes directes en théorie des équations elliptiques. Masson, Paris (1967)
20. Palmer, K.J.: Exponential dichotomies and transversal homoclinic points. J. Differ. Equs. **55**, 225–256 (1984)
21. Rabier, P.J.: An isomorphism theorem for linear evolution problems on the line. J. Dyn. Differ. Equs. **15**, 779–806 (2003)
22. Schwartz, L.: Distributions à valeurs vectorielles I. Ann. Inst. Fourier **7**, 1–141 (1957)
23. Schweiker, S.: Asymptotics, regularity and well-posedness of first and second-order differential equations on the line. Doctoral Dissertation, Ulm, Germany (2000)
24. Stein, E.M.: Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals. Princeton University Press, Princeton (1993)
25. Simpson, H.C., Spector, S.J.: On bifurcation in finite elasticity: buckling of a rectangular rod. J. Elast. **92**, 277–326 (2008)
26. Uspenskii, S.V.: Imbedding theorems for the generalized classes W_p^r of Sobolev. Sibirsk. Math. J. **3**, 418–445 (1962)
27. Weis, L.: Operator-valued Fourier multiplier theorems and maximal L_p-regularity. Math. Ann. **319**, 735–758 (2001)