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Levels and sublevels of algebras obtained by the Cayley–Dickson process

Cristina Flaut

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Abstract In this paper, we generalize the concepts of level and sublevel of a composition algebra to algebras obtained by the Cayley–Dickson process and we will show that, in the case of level for algebras obtained by the Cayley–Dickson process, the situation is the same as for the integral domains, proving that for any positive integer n, there is an algebra A obtained by the Cayley–Dickson process with the norm form anisotropic over a suitable field, which has the level $n \in \mathbb{N} - \{0\}$.

Keywords Cayley–Dickson process · Division algebra · Level and sublevel of an algebra

Mathematics Subject Classification (2000) 17A35 · 17A20 · 17A75 · 17A45

0 Introduction

In this paper, we assume that K is a commutative field with char$K \neq 2$ and all quadratic forms are nondegenerate. For the basic terminology of quadratic and symmetric bilinear spaces, the reader is referred to [16].

Definition 1 Let K be a field. The *level* of the field K, denoted by $s(K)$, is the smallest natural number n such that -1 is a sum of n squares of K. If -1 is not a sum of squares of K, then $s(K) = \infty$.

Pfister, in [12], showed that if a field has a finite level, then this level is a power of 2 and any power of 2 could be realized as the level of a field. The level of division algebras is defined in the same manner as for fields. In [9], Lewis constructed quaternion division algebras of level 2^k and $2^k + 1$ for all $k \in \mathbb{N} - \{0\}$ and he asked if there exist quaternion

In memory of my parents.

C. Flaut (✉)
"Ovidius" University of Constanţa, Constanţa, România
e-mail: cflaut@univ-ovidius.ro; cristina_flaut@yahoo.com
URL: http://cristinaflaut.wikispaces.com

division algebras whose levels are not of this form. Using function field techniques, these values were recovered for the quaternions by Laghribi and Mammone in [8]. Using the same technique, in [13], Susanne Pumpl ün constructed octonion division algebras of level 2^k and 2^k+1 for all $k \in \mathbb{N}-\{0\}$. In [4], Hoffman showed that there are many other values, other than 2^k or 2^k+1, which could be realized as a level of quaternion division algebras. In fact, he proved that for each $k \in \mathbb{N}$, $k \geq 2$, there exist quaternion division algebras D with level $s(D)$ bounded by the values 2^k+2 and $2^{k+1}-1$ (i.e., $2^k+2 \leq s(D) \leq 2^{k+1}-1$). In [10], Theorem 3.6, O'Shea proved the existence of octonion division algebras of level 6 and 7. These values, 6 and 7, are still the only known exact values for the level of octonion division algebras, other than 2^k or 2^k+1, $k \in \mathbb{N}-\{0\}$. It is still not known which exact numbers can be realized as levels and sublevels of quaternion and octonion division algebras, but, for the integral domains, the level problem was solved in [2], when Dai et al. proved that any positive integer could be realized as the level of an integral domain.

In this paper, we construct a division algebra, obtained by the Cayley–Dickson process, of dimension 2^t and prescribed level and sublevel 2^k, k, $t \in \mathbb{N}-\{0\}$, an algebra of dimension 2^t, and prescribed level 2^k+1, $k \in \mathbb{N}-\{0\}$, $t \in \mathbb{N}$, $t \geq 2$, and we will show that, in the case of the level for the algebras obtained by the Cayley–Dickson process, the situation is the same as for the integral domains, proving that for any positive integer n, there is an algebra A obtained by the Cayley–Dickson process with the norm form anisotropic over a suitable field, which has the level $n \in \mathbb{N}-\{0\}$.

1 Preliminaries

A quadratic form $q : V \to K$ is called *anisotropic* if $q(x) = 0$ implies $x = 0$, otherwise q is called *isotropic*.

Let φ be a n-dimensional quadratic irreducible form over K, $n \in N, n > 1$, which is not isometric to the hyperbolic plane. We may consider φ as a homogeneous polynomial of degree 2,

$$\varphi(X) = \varphi(X_1, \ldots, X_n) = \sum a_{ij} X_i X_j, a_{ij} \in K^*.$$

The *function field of* φ, denoted by $K(\varphi)$, is the quotient field of the integral domain

$$K[X_1, \ldots, X_n] / (\varphi(X_1, \ldots, X_n)).$$

For $n \in \mathbb{N}-\{0\}$, an *n-fold Pfister form* over K is a quadratic form of the type

$$\langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle, a_1, \ldots, a_n \in K^*.$$

A Pfister form is denoted by $\langle\langle a_1, a_2, \ldots, a_n \rangle\rangle$. For $n \in N, n > 1$, a Pfister form φ can be written as

$$\langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle = \langle 1, a_1, a_2, \ldots, a_n, a_1 a_2, \ldots, a_1 a_2 a_3, \ldots, a_1 a_2 \ldots a_n \rangle.$$

If $\varphi = \langle 1 \rangle \perp \varphi'$, then φ' is called *the pure subform* of φ. A Pfister form is hyperbolic if and only if is isotropic. Therefore a Pfister form is isotropic if and only if its pure subform is isotropic. (See [16]).

For the field L, we define

$$L^\infty = L \cup \{\infty\},$$

where $x + \infty = x$, for $x \in K$, $x\infty = \infty$ for $x \in K^*$, $\infty\infty = \infty$, $\frac{1}{\infty} = 0$, $\frac{1}{0} = \infty$.

An *L-place* of the field K is a map $\lambda : K \to L^{\infty}$ with the properties:

$$\lambda(x+y) = \lambda(x) + \lambda(y), \lambda(xy) = \lambda(x)\lambda(y),$$

whenever the right sides are defined.

A subset P of K is called an *ordering* of K if

$$P + P \subset P, P \cdot P \subset P, -1 \notin P,$$
$$\{x \in K \ / \ x \text{ is a sum of squares in } K\} \subset P, P \cup -P = K, P \cap -P = 0.$$

A field K with an ordering is called an *ordered* field. For $x, y \in K$, K an ordered field, we define $x > y$ if $(x - y) \in P$.

A *quadratic semi-ordering* (or *q-ordering*) of a field K is a subset P with the following properties:

$$P + P \subset P, \quad K^2 \cdot P \subset P, \quad 1 \in P, \quad P \cup -P = K, \quad P \cap -P = 0.$$

Obviously, every ordering is a q-ordering [16]. Let P_0 be a q-preordering, that is,

$$P_0 + P_0 \subset P_0, \quad K^2 \cdot P_0 \subset P_0, \quad P_0 \cap -P_0 = 0.$$

Then, there is a q-ordering P such that $P_0 \subset P$ or $-P_0 \subset P$. ([16], p. 133)

If $\varphi \cong \langle a_1, \ldots, a_n \rangle$ is a quadratic form over a formally real field K and P is an ordering on K, *the signature* of φ at P is

$$\text{sgn}(\varphi) = |\{i \mid a_i >_P 0\}| - |\{\{i \mid a_i <_P 0\}\}| .$$

The quadratic form q is *indefinite* at ordering P if $\dim \varphi > |\text{sgn}\varphi|$.

The *Witt index* of a quadratic form φ, denoted by $i_W(\varphi)$, is the dimension of a maximal totally isotropic subform of φ. Indeed, if

$$\varphi \cong \varphi_{\text{an}} \perp \varphi_{\text{h}},$$

with φ_{an} anisotropic and φ_{h} hyperbolic, the Witt index of φ is $\frac{1}{2} \dim \varphi_{\text{h}}$. The *first Witt index* of a quadratic form φ is the Witt index of φ over its function field and is denoted by $i_1(\varphi)$. The *essential dimension* of φ is

$$\dim_{\text{es}}(\varphi) = \dim(\varphi) - i_1(\varphi) + 1.$$

The *sublevel* of the algebra A, denoted by $\underline{s}(A)$, is the least integer n such that 0 is a sum of $n+1$ nonzero squares of elements in A. The *level* of the algebra A, denoted by $s(A)$, is the least integer n such that -1 is a sum of n squares in A. If these numbers do not exist, then the level and sublevel are infinite. Obviously, $\underline{s}(A) \leq s(A)$.

Cassels–Pfister Theorem *Let $\varphi, \psi = \langle 1 \rangle \perp \psi'$ be two quadratic forms over a field K, $\text{char}K \neq 2$. If φ is anisotropic over K and $\varphi_{K(\psi)}$ is hyperbolic, then $\alpha\psi < \varphi$ for any scalar represented by φ. In particular,* $\dim \varphi \geq \dim \psi$. [8, p. 1823, Theorem 1.3.]

Springer's Theorem *Let φ_1, φ_2 be two quadratic forms over a field K and $K(X)$ be the rational function field over K. Then, the quadratic form $\varphi_1 \perp X\varphi_2$ is isotropic over $K(X)$ if and only if φ_1 or φ_2 is isotropic over K.* [8, p. 1823, Theorem 1.1.]

In the following, we briefly present the *Cayley–Dickson process* and the properties of the algebras obtained. For details about the Cayley–Dickson process, the reader is referred to [14] and [15].

Let A be a finite-dimensional unitary algebra over a field K with a *scalar involution*

$$\overline{} : A \to A, a \to \overline{a},$$

that is, a linear map satisfying the following relations:

$$\overline{ab} = \overline{b}\overline{a},\ \overline{\overline{a}} = a,$$

and

$$a + \overline{a}, \quad a\overline{a} \in K \cdot 1 \text{ for all } \quad a, b \in A.$$

The element $\overline{a}$ is called the *conjugate* of the element a, the linear form

$$t : A \to K,\ t(a) = a + \overline{a}$$

and the quadratic form

$$n : A \to K,\ n(a) = a\overline{a}$$

are called the *trace* and the *norm* of the element a. Hence, an algebra A with a scalar involution is quadratic.

Let $\gamma \in K$ be a fixed nonzero element. We define the following algebra multiplication on the vector space

$$A \oplus A : (a_1, a_2)(b_1, b_2) = \left(a_1 b_1 + \gamma \overline{b_2} a_2, a_2 \overline{b_1} + b_2 a_1\right).$$

We obtain an algebra structure over $A \oplus A$, denoted by (A, γ) and called the *algebra obtained from A by the Cayley–Dickson process*. We have $\dim(A, \gamma) = 2 \dim A$.

Let $x \in (A, \gamma)$, $x = (a_1, a_2)$. The map

$$\overline{} : (A, \gamma) \to (A, \gamma),\ x \to \bar{x} = (\overline{a}_1, -a_2),$$

is a scalar involution of the algebra (A, γ), extending the involution $\overline{}$, of the algebra A. Let

$$t(x) = t(a_1)$$

and

$$n(x) = n(a_1) - \gamma n(a_2)$$

be the *trace* and the *norm* of the element $x \in (A, \gamma)$, respectively.

If we take $A = K$ and apply this process t times, $t \geq 1$, we obtain an algebra over K,

$$A_t = \left(\frac{\alpha_1, \ldots, \alpha_t}{K}\right).$$

By induction, in this algebra, the set $\{1, f_2, \ldots, f_q\}, q = 2^t$ generates a basis with the properties:

$$f_i^2 = \alpha_i 1, \quad \alpha_i \in K, \quad \alpha_i \neq 0, \quad i = 2, \ldots, q$$

and

$$f_i f_j = -f_j f_i = \beta_{ij} f_k,\ \beta_{ij} \in K,\ \beta_{ij} \neq 0, i \neq j, \quad i, j = 2, \ldots, q,$$

β_{ij} and f_k being uniquely determined by f_i and f_j.

If

$$x \in A_t, x = x_1 1 + \sum_{i=2}^{q} x_i f_i,$$

the quadratic form $T_C : A_t \to K$,

$$T_C = \left\langle 1, \alpha_1, \alpha_2, -\alpha_1\alpha_2, \alpha_3, \ldots, (-1)^t \left(\prod_{i=1}^{t} \alpha_i\right)\right\rangle = \langle 1, \beta_2, \ldots, \beta_q\rangle$$

is called *the trace form*, and the quadratic form $T_P = T_C \mid_{(A_t)_0} : (A_t)_0 \to K$,

$$T_P = \left\langle \alpha_1, \alpha_2, -\alpha_1\alpha_2, \alpha_3, \ldots, (-1)^t \left(\prod_{i=1}^{t} \alpha_i\right)\right\rangle = \langle \beta_2, \ldots, \beta_q\rangle$$

is called *the pure trace form* of the algebra A_t. We remark that $T_C = \langle 1\rangle \perp T_P$ (the orthogonal sum of two quadratic forms) and $n = n_C = \langle 1\rangle \perp -T_P$, therefore

$$n_C = \left\langle 1, -\alpha_1, -\alpha_2, \alpha_1\alpha_2, \alpha_3, \ldots, (-1)^{t+1} \left(\prod_{i=1}^{t} \alpha_i\right)\right\rangle = \langle 1, -\beta_2, \ldots, -\beta_q\rangle.$$

Generally, algebras A_t of dimension 2^t obtained by the Cayley–Dickson process are not division algebras for all $t \geq 1$. But there are fields on which, if we apply the Cayley–Dickson process, the resulting algebras A_t are division algebras for all $t \geq 1$. For example: the power-series field $K\{X_1, X_2, \ldots, X_t\}$ or the rational function field $K(X_1, X_2, \ldots, X_t)$, where $X_1, X_2, \ldots, X_t$ are t algebraically independent indeterminates over the field K. This construction was given by Brown in [1], in which for every t, he built a division algebra A_t of dimension 2^t over the power-series field $K\{X_1, X_2, \ldots, X_t\}$.We will briefly demonstrate this construction using polynomial rings over K and their rational function field instead of power-series field over K (as it was done by Brown).

First of all, we remark that if an algebra A is finite-dimensional, then it is a division algebra if and only if A does not contain zero divisors (See [14]). For every t, we construct a division algebra A_t over a field F_t. Let $X_1, X_2, \ldots, X_t$ be t algebraically independent indeterminates over the field K and $F_t = K(X_1, X_2, \ldots, X_t)$ be the rational function field. For $i = 1, \ldots, t$, we construct the algebra A_i over the rational function field $K(X_1, X_2, \ldots, X_i)$, by setting $\alpha_j = X_j$ for $j = 1, 2, \ldots, i$. Let $A_0 = K$. By induction over i, assuming that A_{i-1} is a division algebra over the field $F_{i-1} = K(X_1, X_2, \ldots, X_{i-1})$, we may prove that the algebra A_i is a division algebra over the field $F_i = K(X_1, X_2, \ldots, X_i)$.

Let $A_{F_i}^{i-1} = F_i \otimes_{F_{i-1}} A_{i-1}$. For $\alpha_i = X_i$, we apply the Cayley–Dickson process to the algebra $A_{F_i}^{i-1}$. The resulting algebra, denoted by A_i, is an algebra over the field F_i with the dimension 2^i.

Let

$$x = a + bv_i,\ y = c + dv_i,$$

be nonzero elements in A_i such that $xy = 0$, where $v_i^2 = \alpha_i$. Since

$$xy = ac + X_i\bar{d}b + (b\bar{c} + da)v_i = 0,$$

we obtain

$$ac + X_i\bar{d}b = 0 \tag{1.1}$$

and

$$b\bar{c} + da = 0. \tag{1.2}$$

The elements $a, b, c, d \in A_{F_i}^{i-1}$ are nonzero elements. Indeed, we have:

(i) If $a=0$ and $b\neq 0$, then $c=d=0\Rightarrow y=0$, false;
(ii) If $b=0$ and $a\neq 0$, then $d=c=0\Rightarrow y=0$, false;
(iii) If $c=0$ and $d\neq 0$, then $a=b=0\Rightarrow x=0$, false;
(iv) If $d=0$ and $c\neq 0$, then $a=b=0\Rightarrow x=0$, false.

This implies that $b\neq 0,\ a\neq 0,\ d\neq 0,\ c\neq 0$. If $\{1, f_2, \ldots, f_{2^{i-1}}\}$ is a basis in A_{i-1}, then $a=\sum_{j=1}^{2^{i-1}} g_j(1\otimes f_j)=\sum_{j=1}^{2^{i-1}} g_j f_j,\ g_j\in F_i,\ g_j=\frac{g_j'}{g_j''},\ g_j', g_j''\in K[X_1,\ldots,X_i],\ g_j''\neq 0,\ j=1,2,\ldots,2^{i-1}$, where $K[X_1,\ldots,X_t]$ is the polynomial ring. Let a_2 be the less common multiple of $g_1'',\ldots,g_{2^{i-1}}''$, then we can write $a=\frac{a_1}{a_2}$, where $a_1\in A_{F_i}^{i-1}, a_1\neq 0$.

Analogously, $b=\dfrac{b_1}{b_2}, c=\dfrac{c_1}{c_2}, d=\dfrac{d_1}{d_2}, b_1,c_1,d_1\in A_{F_i}^{i-1}-\{0\}$ and $a_2,b_2,c_2,d_2\in K[X_1,\ldots,X_t]-\{0\}$.

If we replace in relations (1.1) and (1.2), we obtain

$$a_1c_1d_2b_2+X_i\bar{d}_1b_1a_2c_2=0 \tag{1.3}$$

and

$$b_1\bar{c}_1d_2a_2+d_1a_1b_2c_2=0. \tag{1.4}$$

If we denote $a_3=a_1b_2, b_3=b_1a_2, c_3=c_1d_2, d_3=d_1c_2, a_3,b_3,c_3,d_3\in A_{F_i}^{i-1}-\{0\}$, relations (1.3) and (1.4) become

$$a_3c_3+X_i\bar{d}_3b_3=0 \tag{1.5}$$

and

$$b_3\bar{c}_3+d_3a_3=0. \tag{1.6}$$

Since the algebra $A_{F_i}^{i-1}=F_i\otimes_{F_{i-1}}A_{i-1}$ is an algebra over F_{i-1} with basis $X^i\otimes f_j$, $i\in\mathbb{N}$, and $j=1,2,\ldots,2^{i-1}$, we can write a_3,b_3,c_3,d_3 as $a_3=\sum_{j\geq m}x_jX_i^j, b_3=\sum_{j\geq n}y_jX_i^j, c_3=\sum_{j\geq p}z_jX_i^j, d_3=\sum_{j\geq r}w_jX_i^j$, where $x_j,y_j,z_j,w_j\in A_{i-1}, x_m,y_n,z_p,w_r\neq 0$. Since A_{i-1} is a division algebra, it follows that $x_mz_p\neq 0, w_ry_n\neq 0, y_nz_p\neq 0, w_rx_m\neq 0$. Using relations (1.5) and (1.6), we obtain that $2m+p+r=2n+p+r+1$, which is false. Therefore, the algebra A_i is a division algebra over the field $F_i=K(X_1,X_2,\ldots,X_i)$ of dimension 2^i. (See [3]).

Proposition 1.1 [3] *Let A be an algebra obtained by the Cayley–Dickson process. With the above notations, we have:*

(i) *If $s(A)\leq n$, then -1 is represented by the quadratic form $n\times T_C$.*
(ii) *-1 is a sum of n squares of pure elements in A if and only if the quadratic form $n\times T_P$ represents -1.*
(iii) *For $n\in\mathbb{N}-\{0\}$, if the quadratic form $\langle 1\rangle\perp n\times T_P$ is isotropic over K, then $s(A)\leq n$.*

Proposition 1.2 [3] *Let A be an algebra obtained by the Cayley–Dickson process. The following statements are true:*

(a) *If $n\in\mathbb{N}-\{0\}$, such that $n=2^k-1$, for $k>1$, then $s(A)\leq n$ if and only if $\langle 1\rangle\perp n\times T_P$ is isotropic.*
(b) *If -1 is a square in K, then $\underline{s}(A)=s(A)=1$.*
(c) *If $-1\notin K^{*2}$, then $s(A)=1$ if and only if T_C is isotropic.*

Remark 1.3 (i) If an algebra A, obtained by the Cayley–Dickson process, is a division algebra, then its norm form, n_C^A, is anisotropic. But there are algebras A obtained by the Cayley–Dickson process with the norm form n_C^A anisotropic, which are not division algebras. For example, if $K=\mathbb{R}$ and $t=4$, the real sedenion algebra

$$\left(\frac{-1,-1,-1,-1}{\mathbb{R}}\right)$$

with the basis $\{1, f_1, \ldots, f_{15}\}$ has the norm form anisotropic and is not a division algebra. For example, $(f_3+f_{10})(f_6-f_{15})=0$.

(ii) Using Proposition 1.2, if the algebra A is an algebra obtained by the Cayley–Dickson process of dimension greater than 2 and if n_C^A is isotropic, then $s(A)=\underline{s}(A)=1$. Indeed, if -1 is a square in K, the statement follows from the above. If $-1\notin K^{*2}$, since $n_C=\langle 1\rangle\perp -T_P$ and n_C is a Pfister form, we obtain that $-T_P$ is isotropic, therefore T_C is isotropic, and, from the above proposition, it results that $s(A)=\underline{s}(A)=1$.

2 Levels and sublevels of algebras obtained by the Cayley–Dickson process

Let A be an algebra over a field K obtained by the Cayley–Dickson process of dimension $q=2^t$ and let T_C, T_P, n_C be its trace, pure trace, and norm forms, respectively.

Theorem 2.1 *Let A be an algebra of dimension* 2^t *obtained by the Cayley–Dickson process of finite level over a field K. Then,*

$$\underline{s}(A)\le s(A)\le \underline{s}(A)+1.$$

Proof Denoting $n=\underline{s}(A)$, we find the nonzero elements

$$u_i=x_{i1}+x_{i2}f_2+\cdots+x_{iq}f_q\in A,$$

with

$$u_i''=x_{i2}f_2+\cdots+x_{iq}f_q\in A$$

the pure part of u_i, where $x_{ij}\in K, i\in\{1,2,\ldots,n+1\}, j\in\{1,2,\ldots,q\}, q=2^t$, such that $0=u_1^2+\cdots+u_{n+1}^2$. We obtain

$$\sum_{i=1}^{n+1}\left(x_{i1}^2+\left(u_i''\right)^2+2x_{i1}^2u_i''\right)=0,$$

therefore

$$\sum_{i=1}^{n+1}x_{i1}^2+\sum_{i=1}^{n+1}(u_i'')^2=0$$

and

$$\sum_{i=1}^{n+1}x_{i1}u_i''=0.$$

Case 1. If $x_{i1} = 0,\ \forall i \in \{1, 2, \ldots, n+1\}$. It results that

$$\sum_{i=1}^{n+1} (u_i'')^2 = 0,$$

hence, it follows that $(n+1) \times T_P$ is isotropic; therefore, it contains $\langle 1, -1 \rangle$ as a subform. We obtain that -1 is represented by the form $(n+1) \times T_P$. Therefore, -1 is a sum of square of $(n+1)$ pure elements from A, hence $s(A) \leq n+1$.

Case 2. There are at least two elements $x_{i1} \neq 0$ such that

$$\sum_{i=1}^{n+1} x_{i1}^2 = 0.$$

Since the elements $(u_i'')^2 \in K$ for all $i \in \{1, 2, \ldots, n+1\}$, it results that $s(A) \leq s(K)$. But $\underline{s}(K) = s(K) \leq n$, hence $s(A) \leq n$.

Case 3. If

$$\sum_{i=1}^{n+1} x_{i1}^2 \neq 0,$$

we denote $d_i = \dfrac{x_{i1}}{D} \in K$, where

$$D = \sum_{i=1}^{n+1} x_{i1}^2.$$

It follows that

$$\sum_{i=1}^{n+1} d_i u_i = \frac{1}{A} \sum_{i=1}^{n+1} \left(x_{i1}^2 + x_{i1} u_i'' \right) = 1,$$

since $\sum\limits_{i=1}^{n+1} x_{i1} u_i'' = 0$. We obtain

$$\sum_{i=1}^{n+1} \left(\left(\frac{D^{-1}+1}{2} \right) u_i - d_i \right)^2 = \left(\frac{D^{-1}+1}{2} \right)^2 \sum_{i=1}^{n+1} {u_i}^2 - \left(D^{-1}+1 \right) \sum_{i=1}^{n+1} u_i d_i + \sum_{i=1}^{n+1} d_i^2$$
$$= -(D^{-1}+1) + D^{-1} = -1,$$

therefore $s(A) \leq n+1$.

□

If A is a division algebra of dimension ≤ 8, the above result is a consequence of the main Theorem from [5].

Theorem 2.2 *Let K be a field, X be an algebraically independent indeterminate over K, A be a finite-dimensional K-algebra with finite level n and the scalar involution $\overline{}$. Let $k(A)$ be the least number such that the form $k \times n_C^A$ is isotropic over K, where n_C^A is the norm form of the algebra A, and let $A_1 = K(X) \otimes_K A$ and $B = (A_1, X)$. Then:*

(i) *If A is a division algebra, then B is a division algebra.*
(ii) $s(B) = \min\{s(A),\ k(A)\}$.
(iii) *If* $k(A) > 1$, $\underline{s}(B) = \min\{\underline{s}(A), k(A) - 1\}$.

Proof (i) It results by straightforward calculations, using the same arguments as in Brown's construction at step i, described above.

(ii) We have $s(B) \leq s(A)$. Let $k = k(A)$. If $k \times n_C^A$ is isotropic, it results that $k \times n_C^{A_1}$ is isotropic and therefore universal and it represents $-X^{-1}$. Hence, there are elements $z_1, \ldots, z_k \in A_1$ such that

$$\sum_{i=1}^{k} n_C^{A_1}(z_i) = -X^{-1}.$$

Let $w_i \in B,\ w_i = z_i u, u \in B, u^2 = X$. Since $t(w_i) = 0$, it follows that

$$w_i^2 = -n_C^B(w_i) = X n_C^{A_1}(z_i)$$

and

$$\sum_{i=1}^{k} w_i^2 = \sum_{i=1}^{k} X n_C^{A_1}(z_i) = -1.$$

It results that $s(B) \leq k$, therefore $s(B) \leq \min\{s(A),\ k(A)\}$.

Conversely, assuming that $s(B) = n$, we have $-1 = y_1^2 + \cdots + y_n^2$, where $y_i \in B$, $y_i = a_{i1} + a_{i2}u,\ u^2 = X, a_{i1}, a_{i2} \in A_1$ and we obtain

$$y_i^2 = a_{i1}^2 + X\overline{a}_{i2}a_{i2} + (a_{i2}\overline{a}_{i1} + a_{i2}a_{i1})u,$$

for $i \in \{1, 2, \ldots, n-1\}$. It follows that

$$-1 = \sum_{i=1}^{n} a_{i1}^2 + X \sum_{i=1}^{n} \overline{a}_{i2}a_{i2},$$

where $\psi = 1 \otimes {}^{-}$ is involution in A_1, $\psi(x) = \bar{x}$. We remark that $\overline{a}_{i2}a_{i2} \in K(X)$, $i \in \{1, \ldots, n\}$. Let $\{1, f_2, \ldots, f_q\}$, $q = 2^t$, be a basis in A, therefore

$$a_{i1} = \sum_{j=1}^{m} \frac{p_{ji1}(X)}{q_{ji1}(X)}(1 \otimes f_j),$$

with $\dfrac{p_{ji1}(X)}{q_{ji1}(X)} \in K(X)$, and

$$a_{i2} = \sum_{j=1}^{m} \frac{r_{ji2}(X)}{w_{ji2}(X)}(1 \otimes f_j),$$

with

$\frac{r_{ji2}(X)}{w_{ji2}(X)} \in K(X), i \in \{1, 2, \ldots, n\},\ j \in \{1, 2, \ldots, m\}$. It results that

$$-1 = \sum_{i=1}^{n}\left(\sum_{j=1}^{m}\frac{p_{ji1}(X)}{q_{ji1}(X)}(1 \otimes f_j)\right)^2 + X\sum_{i=1}^{n}\left(\sum_{j=1}^{m}\frac{r_{ji2}(X)}{w_{ji2}(X)}(1 \otimes f_j)\right) \times \left(\sum_{j=1}^{m}\frac{r_{ji2}(X)}{w_{ji2}(X)}(1 \otimes \overline{f_j})\right).$$

After clearing denominators, we obtain

$$-v^2(X) = \sum_{i=1}^{n}\left(\sum_{j=1}^{m}p'_{ji1}(X)(1 \otimes f_j)\right)^2 + X\sum_{i=1}^{n}\left(\sum_{j=1}^{m}r'_{ji2}(X)(1 \otimes f_j)\right) \times \left(\sum_{j=1}^{m}r'_{ji2}(1 \otimes \overline{f_j})\right), \tag{2.1}$$

where

$$v(X) = lcm\{q_{ji1}(X),\ w_{ji2}(X)\}, \quad i \in \{1, 2, \ldots, n\},\ j \in \{1, 2, \ldots, m\}$$

and

$$p'_{ji1}(X) = v(X)p_{ji1}(X), r'_{ji2}(X) = v(X)r_{ji2}(X), \quad i \in \{1, \ldots, n\},\ j \in \{1, 2, \ldots, m\}.$$

Case 1. If $p'_{ji1}(X)$ are not divisible by X, for some i and j, taking residues modulo X in (2.1), denoted with two-sided arrow, we obtain

$$\overleftrightarrow{-v^2(X)} = \sum_{i=1}^{n}\overleftrightarrow{\left(\sum_{j=1}^{m}p'_{ji1}(X)(1 \otimes f_j)\right)^2}.$$

In this relation, if $v(X)$ is not divisible by X, it results that $s(A) \le n$. If $v(X)$ is divisible by X, we have $\underline{s}(A) \le n-1$ and, from Theorem 2.1, we obtain $s(A) \le n$.

Case 2. If $p'_{ji1}(X)$ are divisible by X, for all i and j, it results that $v(X)$ is divisible by X, then dividing relation (2.1) by X and taking residues modulo X, we obtain

$$\overleftrightarrow{0} = \sum_{i=1}^{n}\overleftrightarrow{\left(\sum_{j=1}^{m}r'_{ji2}(X)(1 \otimes f_j)\right)\left(\sum_{j=1}^{m}r'_{ji2}(1 \otimes \overline{f_j})\right)}.$$

It follows that the form $n \times n_C^A$ is isotropic, therefore $k(A) \le n$.

It results that $s(B) = \min\{s(A),\ k(A)\}$.

(iii) Since $\underline{s}(B) \le s(B) \le s(A)$, then $\underline{s}(B) \le \underline{s}(A)$. Let $k = k(A)$. We have that $k \times n_C^A$ is isotropic, therefore $k \times n_C^{A_1}$ is isotropic. Hence, there are the elements $z_1, \ldots, z_k \in A_1$ such that $\sum_{i=1}^{k} n_C^{A_1}(z_i) = 0$. Let $w_i \in B$, $w_i = z_i u, u \in$

$B, u^2 = X$. Since $t(w_i) = 0$, we obtain $w_i^2 = -n_C^B(w_i) = Xn_C^{A_1}(z_i)$ and $\sum_{i=1}^k w_i^2 = \sum_{i=1}^k Xn_C^{A_1}(z_i) = 0$. It results that $\underline{s}(B) \leq k-1$, therefore

$$\underline{s}(B) \leq \min\{\underline{s}(A), k(A)-1\}.$$

Conversely, assuming that $\underline{s}(B) = n$, there are $y_1, \ldots, y_{n+1} \in B$, nonzero elements, such that $0 = y_1^2 + \cdots + y_{n+1}^2$, $y_i = a_{i1} + a_{i2}u$, $u^2 = X, a_{i1}, a_{i2} \in A_1$. Using the same notations as in (ii), after straightforward calculations, we obtain

$$\sum_{i=1}^{n+1}\left(\sum_{j=1}^{m} p'_{ji1}(X)(1 \otimes f_j)\right)^2 + X\sum_{i=1}^{n+1}\left(\sum_{j=1}^{m} r'_{ji2}(X)(1 \otimes f_j)\right) \times \left(\sum_{j=1}^{m} r'_{ji2}(1 \otimes \overline{f_j})\right) = 0. \tag{2.2}$$

Case 1. If $p'_{ji1}(X)$ are not divisible by X, for some i and j, taking residues modulo X in relation (2.2), we obtain

$$\overleftrightarrow{0} = \sum_{i=1}^{n+1}\overleftrightarrow{\left(\sum_{j=1}^{m} p'_{ji1}(X)(1 \otimes f_j)\right)^2},$$

therefore $\underline{s}(A) \leq n$.

Case 2. If $p'_{ji1}(X)$ are divisible by X, for all i and j, then dividing relation (2.2) by X and taking residues modulo X, we obtain

$$\overleftrightarrow{0} = \sum_{i=1}^{n+1}\overleftrightarrow{\left(\sum_{j=1}^{m} r'_{ji2}(X)(1 \otimes f_j)\right)\left(\sum_{j=1}^{m} r'_{ji2}(1 \otimes \overline{f_j})\right)},$$

therefore $k(A) \leq n+1$. It results that $\underline{s}(B) = \min\{\underline{s}(A), k(A)-1\}$.

□

Since $\underline{s}(B) \leq s(B) \leq s(A)$, in the above Theorem, we remark that if $k(A) = 1$, then $\underline{s}(B) = s(B) = s(A) = 1$. Results analogous to those in Theorem 2.2 are obtained for composition algebras in [17] and [11].

Let A_t be a division algebra over the field $K = K_0(X_1, \ldots, X_t)$ obtained by the Cayley–Dickson process and Brown's construction of dimension $q = 2^t$, where K_0 is a formally real field, $X_1, \ldots, X_t$ are algebraically independent indeterminates over the field K_0, and T_C and T_P are its trace and pure trace forms. Let

$$\varphi_n = \langle 1\rangle \perp n \times T_P, \psi_m = \langle 1\rangle \perp m \times T_C, n \geq 1,$$
$$A_t(n) = A_t \otimes_K K(\langle 1\rangle \perp n \times T_P), n \in \mathbb{N} - \{0\}. \tag{2.3}$$

We denote $K_n = K(\langle 1\rangle \perp n \times T_P)$, and let $n_C^{A_t}$ be the norm form of the algebra A_t.

Proposition 2.3 (i) *The norm form $n_C^{A_t(n)}$ is anisotropic over K_n.*

(ii) *With the above notations, for $t \geq 2$, if $n = 2^k + 1$ then $2^k \times n_C^{A_t(n)}$ is anisotropic over $K_0(X_1, X_2, \ldots, X_t)(\varphi_{2^k+1})$.*

Proof (i) First, we consider $n > 1$. Since $n_C^{A_t(n)}$ is a Pfister form and a Pfister form is isotropic if and only if it is hyperbolic, if $n_C^{A_t(n)}$ is isotropic over K_n, then it is hyperbolic. Since A_t is a division algebra, it follows that $n_C^{A_t}$ is anisotropic. From Cassels–Pfister Theorem, for some $\alpha \in K^*$, we obtain that $\alpha\varphi_n$ is a subform of the norm form $n_C^{A_t(n)}$. Since $\dim \varphi_n = 1 + n(2^t - 1)$ and $\dim n_C^{A_t(n)} = 2^t$, therefore $\dim \varphi_n > \dim n_C^{A_t(n)}$, false.
If $n = 1$, using the Cassels–Pfister Theorem, for some $\alpha \in K^*$, it results that $\alpha\varphi_1$ is a subform of the norm form $n_C^{A_t(1)}$. Since $\dim \varphi_1 = \dim n_C^{A_t(1)} = 2^t$ and the forms φ_1 and $n_C^{A_t(1)}$ are not similar, we obtain a contradiction.

(ii) We denote

$$\alpha_k = (2^k + 1) \times \langle 1, -X_1 \rangle.$$

It results that $X_2\alpha_k$ is a subform of φ_{2^k+1}, then

$$K_0(X_1, X_2, \ldots, X_t)(\alpha_k) \simeq K_0(X_1, X_2, \ldots, X_t)(X_2\alpha_k).$$

If $2^k \times n_C$ is isotropic over $K_0(X_1, X_2, \ldots, X_t)(\varphi_{2^k+1})$ there is a map

$K_0(X_1, X_2, \ldots, X_t)$-place: $K_0(X_1, X_2, \ldots, X_t)(\varphi_{2^k+1}) \to K_0(X_1, X_2, \ldots, X_t)(\alpha_k)$,

and $2^k \times n_C$ is isotropic over $K_0(X_1, X_2, \ldots, X_t)(\alpha_k)$ from [7, Theorem 3.3.]. By repeatedly applying of Springer's Theorem, it results that the quadratic form $2^k \times \langle 1, -X_1 \rangle$ is isotropic over $K_0(X_1)(\alpha_k)$, in contradiction with Proposition 2.2 from [8]. □

Remark 2.4 (i) The algebra $A_t(n)$ has dimension 2^t and is not necessarily a division algebra, but, using Remark 1.3, this algebra is of level greater than 1.
(ii) From Proposition 1.1(i) and (iii), if ψ_m is anisotropic and φ_n is isotropic over K_n, then $s(A_t(n)) \in [m + 1, n]$.

Example 2.5 Using the same notations as those in Theorem 2.2, let F be a field of level 2^k. If $A = A_0 = F$, $K = F$, $A_1 = K(X_1) \otimes_K A_0$, since $k(A) \geq 2^k + 1$, we obtain the division $K(X_1)$-algebra B of dimension 2 and level and sublevel 2^k. Using the same Theorem, we can continue the induction steps. Assuming that $A = A_{t-1}$ is a division algebra of dimension 2^{t-1} and level 2^k over the field $K = F(X_1, \ldots, X_{t-1})$, then, from Springer's Theorem, it results that $k(A_{t-1}) \geq 2^k + 1$. If $A = A_{t-1}$, $A_1 = K(X_t) \otimes_K A_{t-1}$, and B is the $K(X_t)$-algebra obtained by application of the Cayley–Dickson process with $\alpha = X_t$ to the $K(X_t)$-algebra A_1, then B is a division algebra of dimension 2^t and level and sublevel 2^k. This is an example of a division algebra of level and sublevel 2^k and dimension 2^t, $t, k \in \mathbb{N} - \{0\}$.

Proposition 2.6 *$i_1(\langle 1 \rangle \perp n \times T_P) = 1$ for all $n \in \mathbb{N} - \{0\}$, where T_P is the pure trace form for the algebra A_t, $t \geq 2$.*

Proof Let P be an arbitrary ordering over K such that $\beta_2, \ldots, \beta_q <_P 0$. We remark that such an ordering always exists. Indeed, since φ_n is anisotropic over K (from Springer's Theorem), it follows that $P_0 = \{a \mid a = 0 \text{ or } a \text{ is represented by } \varphi_n\}$ is a $q-$preordering, therefore there is a q-ordering P containing P_0 or $-P_0$. We have

$$|\mathrm{sgn}\varphi_n| = |\mathrm{sgn}\,(\langle 1 \rangle \perp n \times T_P)| = \left(2^t - 1\right) n - 1 < (2^t - 1)n + 1 = \dim \varphi_n.$$

It results that φ_n is indefinite at P over K, then P extends to K_n, from [4], Lemma 2.5. Since φ_n is isotropic over K_n, we obtain that

$$\dim((\varphi_n)_{K_n})_{\text{an}} \leq (2^t - 1)n - 1.$$

Since

$$\dim((\varphi_n)_{K_n})_{\text{an}} \geq |\text{sgn}\varphi_n| = (2^t - 1)n - 1,$$

then

$$\dim((\varphi_n)_{K_n})_{\text{an}} = (2^t - 1)n - 1 = \dim \varphi_n - 2$$

and therefore $i_1(\varphi_n) = \frac{1}{2}2 = 1$. □

Theorem 2.7 *With the above notations, we have*

$$s(A_t(n)) \in [n - [\frac{n}{2^t}], n],$$

for $t \geq 2$.

Proof From Proposition 2.6, we have that

$$\dim \varphi_n - i_1(\varphi_n) = (2^t - 1)n + 1 - i_1(\varphi_n) = (2^t - 1)n.$$

For the quadratic form ψ_m, the relation

$$\dim \psi_m - i_1(\psi_m) = 2^t n + 1 - i_1(\psi_m)$$

holds. The forms φ_n and ψ_m are anisotropic over $K = K_0(X_1, \ldots, X_t)$, by Springer's Theorem. From [6], Theorem 4.1, if

$$\dim \psi_m - i_1(\psi_m) < \dim \varphi_n - i_1(\varphi_n) \tag{2.4}$$

it results that ψ_m is anisotropic over K_n. From Proposition 2.6, we have $i_1(\varphi_n) = 1$ for all $n \in \mathbb{N} - \{0\}$; therefore, since $i_1(\psi_m) \geq 1$, if $\dim \psi_m < \dim \varphi_n$, we obtain relation (2.4). By straightforward calculations in relation (2.4), we obtain

$$2^t m + 1 - i_1(\psi_m) < (2^t - 1)n$$

and we remark that $n - [\frac{n}{2^t}] - 1$ is the highest value of $m \in \mathbb{N}$ such that the relation $\dim \psi_m < \dim \varphi_n$ holds. Hence, ψ_m is anisotropic over K_n for $m = n - [\frac{n}{2^t}] - 1$. From Remark 2.4, it results $s(A_t(n)) \geq n - [\frac{n}{2^t}]$. □

Theorem 2.8 *With the above notations, we have*

$$\underline{s}(A_t(n)) \in [n - \left[\frac{n + 2^t - 1}{2^t}\right], n\Big],$$

where $n \in \mathbb{N} - \{0\}$, $t \geq 2$.

Proof Using Proposition 1.1(i), if the quadratic form $\phi_m = (m + 1) \times T_C$ is anisotropic, then $\underline{s}(A_t(n)) \geq m + 1$ and if φ_n is isotropic, then $\underline{s}(A_t(n)) \leq n$. Using the same arguments as in the proof of Theorem 2.7, if

$$2^t(m + 1) - i_1(\phi_m) < (2^t - 1)n, \tag{2.5}$$

we have ϕ_m is anisotropic over K_n; therefore

$$\underline{s}(A_t(n)) \in [m+1, n].$$

Since $i_1(\phi_m) \geq 1$, the highest value of m such that relation (2.5) holds is $n - [\frac{n+2^t-1}{2^t}] - 1$. Indeed, relation (2.5) implies

$$2^t(m+1) - 1 < (2^t - 1)n,$$

therefore

$$m < n\frac{2^t-1}{2^t} + \frac{1}{2^t} - 1 = n - \frac{n+2^t-1}{2^t}$$

and we obtain

$$m \leq n - \left[\frac{n+2^t-1}{2^t}\right] - 1.$$

□

Theorems 2.7 and 2.8. generalize Theorem 3.8 from [10].

Theorem 2.9 *With the above notation, for each $n \in \mathbb{N} - \{0\}$, there is an algebra $A_t(n)$ such that $s(A_t(n)) = n$ and $\underline{s}(A_t(n)) \in \{n-1, n\}$.*

Proof Let $n \in \mathbb{N} - \{0\}$ and m be the least positive integer such that $n \leq 2^m$. For $n = 2^m$, there are quaternion $(A_2(n))$ and octonion $(A_3(n))$ division algebras of level $n = 2^m$, (see [8] and [13]). We assume that $n < 2^m$. With the above notations, for $t = m$, let $A_t(n)$ be the algebra of dimension $q = 2^t$. From Theorem 2.7, this algebra is of level

$$s(A_t(n)) \in \left[n - \left[\frac{n}{2^t}\right], n\right]$$

and sublevel

$$\underline{s}(A_t(n)) \in \left[n - \left[\frac{n+2^t-1}{2^t}\right], n\right], \quad n \in \mathbb{N} - \{0\}.$$

Since $n < 2^t$, it results that $[\frac{n}{2^t}] = 0$ and $[\frac{n+2^t-1}{2^t}] = 1$, therefore $s(A_t(n)) = n$ and $\underline{s}(A_t(n)) \in \{n-1, n\}$. □

Remark 2.10 Theorem 2.9 gives a positive partial answer to the question whether any number $n \in \mathbb{N} - \{0\}$ can be realized as a level of composition algebras. The answer becomes positive if we replace "composition algebras" with "algebras obtained by the Cayley–Dickson process." Therefore, we can say that any number $n \in \mathbb{N} - \{0\}$ can be realized as a level of an algebra obtained by the Cayley–Dickson process with the norm form anisotropic over a suitable field.

Example 2.11 If $n \in \{6, 7\}$, for $t \geq 3$, from Theorems 2.7 and 2.8, it follows that the algebra $A_t(n)$ has level 6 and 7, respectively. This remark generalizes the results obtained by O'Shea in [10] for the octonion division algebras.

Theorem 2.12 *With the above notations, we have that $s(A_t(n)) = n$, for $n = 2^k + 1$.*

Proof First, we prove that the form

$$\varkappa_n = n \times \langle 1\rangle \perp (n-1) \times T_P^{A_t}$$

is anisotropic over K_n. If the form $\varkappa_n$ is isotropic over K_n, since the form

$$\varphi_n' = \langle 1\rangle \perp n \times T_P^{A_{t-1}}$$

is a subform of the form φ_n and the norm φ_n' is isotropic over its function field $K(\varphi_n')$, then φ_n is isotropic over $K(\varphi_n')$. From [7, Theorem 3.3.], we have that there is a K-place from K_n to $K(\varphi_n')$. Let

$$\varkappa_n' = n \times \langle 1\rangle \perp (n-1) \times T_P^{A_{t-1}}.$$

Then, over K, we can write

$$\varkappa_n = \varkappa_n' \perp X_t (n-1) n_C^{A_{t-1}}.$$

If $\varkappa_n$ is isotropic over K_n, then $\varkappa_n$ is isotropic over $K(\varphi_n')$. We obtain that $\varkappa_n'$ or $(n-1)n_C^{A_{t-1}}$ are isotropic over $K(\varphi_n')$. Using the induction steps and the same arguments as in [8], Proposition 2.3, for $A_{t-1} = A_2$, we have that $\varkappa_n'$ is anisotropic over $K(\varphi_n')$ and from Proposition 2.3., (ii), we obtain that $(n-1)n_C^{A_{t-1}}$ is anisotropic over $K(\varphi_n')$. Therefore, $\varkappa_n$ is anisotropic over K_n.

Now, from Remark 2.4(ii), we have $s(A_t(n)) \leq n$. If $s(A_t(n)) < n$, then the form $\varkappa_n$ is isotropic over K_n, false. □

The above result generalizes Theorem 3.1 from [13].

3 Conclusions

In this paper, we generalized the concepts of level and sublevel of a composition algebra to algebras obtained by the Cayley–Dickson process. The main result of this paper is obtained in Theorem 2.9, where we proved that for any positive integer n, there is an algebra A, obtained by the Cayley–Dickson process with the norm form anisotropic over a suitable field, which has level $n \in \mathbb{N} - \{0\}$. Since it is still unknown what exact numbers can be realized as levels and sublevels of quaternion and octonion division algebras, as further research, I intend to improve the bounds for the level and sublevel of division quaternion and octonion algebras and to provide some new examples of values for the level and sublevel of division quaternion algebras or of division octonion algebras. It remains unknown whether there exist quaternion division algebras of sublevel 5, or quaternion division algebras of level 6. The result obtained in Theorem 2.9 seems to indicate that one of the problems in finding a given value for the level of division quaternion and octonion algebras can be the dimension of these algebras and it is easier to work with algebras obtained by the Cayley–Dickson process with higher dimension. This remark allows us to consider this problem in the reverse sense: for any positive integer n, how can the existence of an octonion division algebra of level n influence the existence of a quaternion division algebra of level n? Or, more generally, for any positive integer n, how can the existence of an algebra obtained by the Cayley–Dickson process, of dimension $2^t, t \geq 4$ and level n, influence the existence of a quaternion or an octonion division algebra of level n?

The remarks above can constitute the starting point for further research.

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