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Prefactorized subgroups in pairwise mutually permutable products

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Abstract We continue here our study of pairwise mutually and pairwise totally permutable products. We are looking for subgroups of the product in which the given factorization induces a factorization of the subgroup. In the case of soluble groups, it is shown that a prefactorized Carter subgroup and a prefactorized system normalizer exist. A less stringent property have $\mathcal{F}$-residual, $\mathcal{F}$-projector and $\mathcal{F}$-normalizer for any saturated formation $\mathcal{F}$ including the supersoluble groups.

Keywords Finite group · Permutability · Factorization · Saturated formation

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1 Introduction and preliminaries

All groups considered throughout this paper are finite.

A group G is said to be the product of its subgroups A and B if $G = AB$. Sometimes, such a group is also called factorized by A and B or simply factorized. A subgroup S of $G = AB$ is called prefactorized if $S = (S \cap A)(S \cap B)$. We say that S is factorized if whenever $s \in S$ and $s = ab$ with $a \in A$ and $b \in B$, then $a \in S$ (and $b \in S$). According to a result of Wielandt ([1, Lemma 1.1.1]), S is factorized if and only if S is prefactorized and $A \cap B \leq S$. In particular, every factorized subgroup of G is prefactorized, and every subgroup of G containing A or B is factorized.

If $G = G_1G_2 \dots G_k$ is the pairwise permutable product of the subgroups $G_1, G_2, \dots, G_k$, we say that a subgroup S of G is prefactorized with respect to the above factorization if $S = (S \cap G_1)(S \cap G_2) \dots (S \cap G_k)$.

A group $G = G_1G_2 \dots G_k$ is said to be the product of its pairwise mutually permutable subgroups $G_1, G_2, \dots, G_k$, if G_i and G_j are mutually permutable subgroups of G, that is, G_i permutes with every subgroup of G_j, and G_j permutes with every subgroup of G_i for all $i, j \in \{1, 2, \dots, k\}$. G is said to be the pairwise totally permutable product of $G_1, G_2, \dots, G_k$ if G_i and G_j are totally permutable subgroups of G, that is, every subgroup of G_i permutes with every subgroup of G_j for all $i \neq j$. These kinds of products have been studied extensively with a lot of properties and results available (see [2–10] and the papers cited therein).

We continue the investigation about certain subgroups of pairwise mutually and pairwise totally permutable products that began by the authors in [5] and [6]. The new results presented here are often related to saturated formations and to subgroups of the given product which are factorized or prefactorized.

In [5, Lemma 1(ii)], we proved that if $G = G_1G_2 \dots G_k$ is the pairwise mutually permutable product of $G_1, G_2, \dots, G_k$ and S is a subgroup of G, then $(S \cap G_1)(S \cap G_2) \dots (S \cap G_k)$ is a subgroup of G which is the pairwise mutually permutable product of its factors. Moreover, if S is a normal subgroup of G, then the aforementioned product is also a normal subgroup of G. We are concerned here with the case when this product coincides with the subgroup S itself, that is, when S is prefactorized with respect to $G = G_1G_2 \dots G_k$.

The following example shows where we should not try:

Example 1 Let $G = A \times B$ where $A = \langle u, v \rangle$ is a nonabelian group of order p^3 and $B = \langle x \rangle$ is a group of order p with p an odd prime. Then, A and $\langle u, vx \rangle$ are mutually permutable subgroups of G and $G = A\langle u, vx \rangle$. We have $Z(G) = \langle [u, v], x \rangle$ and $Z(G) \cap A = Z(G) \cap \langle u, vx \rangle = \langle [u, v] \rangle \neq Z(G)$.

We begin with a result that allows us to extend the concept of factorized subgroup in the case of pairwise mutually permutable products.

Lemma 1 *Let the group $G = G_1G_2 \dots G_k$ be the pairwise mutually permutable product of the subgroups $G_1, G_2, \dots, G_k$. For a subgroup S of G, the following conditions are equivalent.*

(i) If $a_{i_1}a_{i_2} \dots a_{i_k} \in S$, with $a_{i_j} \in G_{i_j}$, where $\{i_1, i_2, \dots, i_k\} = \{1, 2, \dots, k\}$, then a_{i_j} belongs to S for all $i_j \in \{1, 2, \dots, k\}$.

(ii) $S = (S \cap G_1)(S \cap G_2) \dots (S \cap G_k)$ and $G_i \cap \prod_{j \neq i} G_j \leq S$ for all $i = 1, 2, \dots, k$.

Proof (i) implies (ii) If $x = a_1a_2 \dots a_k$ is an element of S, with $a_i \in G_i$, then by the hypothesis a_i belongs to $(S \cap G_i)$ for all $i = 1, 2, \dots, k$. This shows that

$S = (S \cap G_1)(S \cap G_2) \dots (S \cap G_k)$. Moreover, if $x \in G_i \cap \prod_{j \neq i} G_j$ for some $i = 1, 2, \dots, k$, then $x^{-1} \in \prod_{j \neq i} G_j$ and $xx^{-1} = 1 \in S$. Thus, by (i), x lies in S.

(ii) implies (i) Let $x = a_{i_1} a_{i_2} \dots a_{i_k}$ be an element of S with $a_{i_j} \in G_{i_j}$ where $\{i_1, i_2, \dots, i_k\} = \{1, 2, \dots, k\}$. Since $S = (S \cap G_1)(S \cap G_2) \dots (S \cap G_k)$ and the factors are pairwise permutable, we have also that $x = b_{i_1} b_{i_2} \dots b_{i_k}$ with $b_{i_j} \in (S \cap G_{i_j})$. Therefore, $b_{i_1}^{-1} a_{i_1} = (b_{i_2} \dots b_{i_k})(a_{i_2} \dots a_{i_k})^{-1} \in G_{i_1} \cap G_{i_2} \dots G_{i_k} \leq S$ by (ii) and so $a_{i_1} \in S$. Now we deduce that $a_{i_2} \dots a_{i_k}$ belongs to S. Arguing as before, $a_{i_2} \dots a_{i_k} = c_{i_1} c_{i_2} \dots c_{i_k} = d_{i_2} d_{i_1} \dots d_{i_k}$ with $c_{i_j} \in (S \cap G_{i_j})$ and $d_{i_j} \in (S \cap G_{i_j})$. Therefore, $d_{i_2}^{-1} a_{i_2}$ belongs to $G_{i_2} \cap \prod_{j \neq i_2} G_{i_j}$, which is contained in S by (ii). Consequently, we obtain that $a_{i_2} \in S$ and similarly $a_{i_3}, \dots, a_{i_k}$ belong to S. □

A subgroup S of a pairwise mutually permutable product $G = G_1 G_2 \dots G_k$ is said to be factorized if it satisfies one of the equivalent conditions of Lemma 1. It is obvious that every subgroup of $G = G_1 G_2 \dots G_k$ that contains all factors but one is factorized.

The following lemma studies the behavior of factorized (prefactorized) subgroups in pairwise mutually permutable products. It is an extension of some known properties of these types of subgroups in the two factors case (see [1, Lemma 1.1.2]).

Lemma 2 *Let the group $G = G_1 G_2 \dots G_k$ be the pairwise mutually permutable product of the subgroups $G_1, G_2, \dots, G_k$. Then:*

(i) If S is prefactorized in G, and N is a normal subgroup of G, then SN/N is a prefactorized subgroup of $G/N = (G_1N/N)(G_2N/N) \dots (G_kN/N)$.
(ii) If N is a prefactorized normal subgroup of $G = G_1 G_2 \dots G_k$ and N is contained in S, then S/N is a prefactorized subgroup of $G/N = (G_1N/N)(G_2N/N) \dots (G_kN/N)$ if and only if S is a prefactorized subgroup of $G = G_1 G_2 \dots G_k$.
(iii) If N is a normal subgroup of G, a subgroup S/N of the factorized quotient group $G/N = (G_1N/N)(G_2N/N) \dots (G_kN/N)$ is factorized if and only if S is a factorized subgroup of $G = G_1 G_2 \dots G_k$.
(iv) If U is a factorized (prefactorized) subgroup of G and V is a factorized (prefactorized) subgroup of U, then V is a factorized (prefactorized) subgroup of G.
(v) If U and V are factorized (prefactorized) subgroups of G, then also $\langle U, V \rangle$ is factorized (prefactorized).
(vi) If U and V are factorized subgroups of G, then $U \cap V$ is a factorized subgroup of G.

Proof (i) Clearly, $SN/N = (S \cap G_1)N/N \cdot (S \cap G_2)N/N \dots \cdot (S \cap G_k)N/N \leq (SN/N \cap G_1N/N)(SN/N \cap G_2N/N) \dots (SN/N \cap G_kN/N)$ which is contained in SN/N. This shows that SN/N is prefactorized.

(ii) If S is prefactorized, it is clear by the preceding statement that S/N is prefactorized. Conversely, suppose that S/N is prefactorized. Then, using the fact that N is prefactorized and $N \leq S$, we have $S = (S \cap G_1N)(S \cap G_2N) \dots (S \cap G_kN) = (S \cap G_1)(S \cap G_2) \dots (S \cap G_k)N = (S \cap G_1)(S \cap G_2) \dots (S \cap G_k)(N \cap G_1)(N \cap G_2) \dots (N \cap G_k) = (S \cap G_1)(S \cap G_2) \dots (S \cap G_k) \leq S$, which shows that S is prefactorized.

(iii) Let S be a factorized subgroup of G containing N. If $xN = a_{i_1} a_{i_2} \dots a_{i_k} N$ is an element of S/N, with $x \in S$ and $a_{i_j} \in G_{i_j}$, then $x = a_{i_1} a_{i_2} \dots a_{i_k} y$, where y belongs to $N \leq S$. Hence, $a_{i_1} a_{i_2} \dots a_{i_k} = xy^{-1}$ belongs to S, and so a_{i_j} belongs to S. Therefore, S/N is a factorized subgroup of G/N.

Conversely, suppose that the subgroup S/N is factorized in G/N. Let $x =$

$a_{i_1}a_{i_2}\dots a_{i_k}$ be an element of S, with $a_{i_j} \in G_{i_j}$. Since $xN = a_{i_1}a_{i_2}\dots a_{i_k}N$, it follows that $a_{i_j}N$ belongs to S/N. Hence, a_{i_j} belongs to S, and so S is factorized.

(iv) Suppose that $U = (U \cap G_1)(U \cap G_2)\dots(U \cap G_k)$ and V is a prefactorized subgroup of U. Then $V = (V \cap (U \cap G_1))(V \cap (U \cap G_2))\dots(V \cap (U \cap G_k)) = (V \cap G_1)(V \cap G_2)\dots(V \cap G_k)$, that is, V is a prefactorized subgroup of G. Assume now that U is factorized in G and V is factorized in U. Then, V is prefactorized in G. Moreover, $G_i \cap \prod_{j\neq i} G_j \leq U$ and $(U \cap G_i) \cap \prod_{j\neq i}(U \cap G_j) \leq V$. Note that the fact that U is factorized in G yields $(U \cap \prod_{j\neq i} G_j) \leq \prod_{j\neq i}(U \cap G_j)$. Consequently, $G_i \cap \prod_{j\neq i} G_j \leq V$ and V is a factorized subgroup of G.

(v) If $U = (U \cap G_1)(U \cap G_2)\dots(U \cap G_k)$ and $V = (V \cap G_1)(V \cap G_2)\dots(V \cap G_k)$, then $\langle U, V\rangle = \langle (U \cap G_1)(U \cap G_2)\dots(U \cap G_k), (V \cap G_1)(V \cap G_2)\dots(V \cap G_k)\rangle \leq \langle ((\langle U, V\rangle \cap G_1), (\langle U, V\rangle \cap G_2), \dots, (\langle U, V\rangle \cap G_k)\rangle = (\langle U, V\rangle \cap G_1)\cdot(\langle U, V\rangle \cap G_2)\cdot\dots\cdot(\langle U, V\rangle \cap G_k) \leq \langle U, V\rangle$. Thus, $\langle U, V\rangle$ is a prefactorized subgroup of G. Moreover, in the case that U and V are factorized subgroups of G, we have that $G_i \cap \prod_{j\neq i} G_j \leq U$ and $G_i \cap \prod_{j\neq i} G_j \leq V$ for all $i = 1, 2, \dots, k$. Consequently, $G_i \cap \prod_{j\neq i} G_j \leq \langle U, V\rangle$ for all $i = 1, 2, \dots, k$ and $\langle U, V\rangle$ is a factorized subgroup of G.

(vi) If $a_{i_1}a_{i_2}\dots a_{i_k} \in U \cap V$, with $a_{i_j} \in G_{i_j}$, where $\{i_1, i_2, \dots, i_k\} = \{1, 2, \dots, k\}$, then using the fact that U and V are factorized, we have that a_{i_j} belongs to $U \cap V$ for all $i_j \in \{1, 2, \dots, k\}$. Therefore, applying Lemma 1(i), $U \cap V$ is a factorized subgroup of G. □

2 Normal prefactorized subgroups

We begin with some prominent conjugacy classes of subgroups for a positive statement.

Proposition 1 *(i) Let the group $G = G_1G_2\dots G_k$ be a product of pairwise mutually permutable subgroups. For each prime p dividing $|\ G\ |$, there exists a Sylow p-subgroup of G, P say, such that P is prefactorized, that is, $P = (P \cap G_1)(P \cap G_2)\dots(P \cap G_k)$. Moreover, $P \cap G_i$ is a Sylow p-subgroup of G_i for all $i = 1, 2, \dots, k$.*

(ii) Let the group $G = G_1G_2\dots G_k$ be a product of pairwise mutually permutable soluble subgroups. For each set of primes π dividing the order of G, there exists a Hall π-subgroup of G, H say, such that H is prefactorized, that is, $H = (H \cap G_1)(H \cap G_2)\dots(H \cap G_k)$. Moreover, $H \cap G_i$ is a Hall π-subgroup of G_i for all $i = 1, 2, \dots, k$.

Proof We prove (ii). Since all G_i are soluble, applying [5, Theorem 1] G is soluble. We proceed by induction on k. The result is clear if $k = 1$. Suppose that $k > 1$ and the result is true for all groups that are pairwise mutually permutable products of less than k factors. Consider the product $G_1G_2\dots G_{k-1}$. Then, there exists a Hall π-subgroup T of $G_1G_2\dots G_{k-1}$ such that $T = (T \cap G_1)(T \cap G_2)\dots(T \cap G_{k-1})$. Moreover $T \cap G_i$ is a Hall π-subgroup of G_i for all $i = 1, 2, \dots, k-1$. Since the product is pairwise mutually permutable, it follows that TG_k is a subgroup of G. Let H be a Hall π-subgroup of TG_k containing T. Then $H = H \cap TG_k = T(H \cap G_k)$. Also $T \cap G_i \leq H \cap G_i$ for all $i = 1, 2, \dots k-1$. But $T \cap G_i$ is a Hall π-subgroup of G_i for all $i = 1, 2, \dots k-1$. Therefore $T \cap G_i = H \cap G_i$ for all $i = 1, 2, \dots k-1$ and $H = (H \cap G_1)(H \cap G_2)\dots(H \cap G_k)$. Note that $|\ TG_k : H\ |=|\ G_k : H \cap G_k\ |$ is a π'-number. Consequently, $(H \cap G_k)$ is a Hall π-subgroup of G_k. Now an argument on the orders shows that H is a Hall π-subgroup of G. □

Using the same arguments as above, we obtain (i) for nonsoluble groups.

The following example shows that in a mutually permutable product (even if it is a totally permutable one), there is not necessarily a factorized Sylow subgroup for each prime p dividing the order of G.

Example 2 Let $X = \langle x, y \mid x^3 = y^2 = 1, x^y = x^{-1}\rangle$ be a group isomorphic to the symmetric group of degree 3 and $Y = \langle a, b \mid a^5 = b^2 = [a, b] = 1\rangle \simeq C_5 \times C_2$. Consider $G = X \times Y$, $A = \langle y\rangle \times \langle a\rangle \times \langle b\rangle$ and $B = \langle x\rangle \times \langle b\rangle$. Then $G = AB$ is the totally permutable product of A and B. Moreover $A \cap B = \langle b\rangle$ and $\langle x\rangle$ is a Sylow 3 subgroup of G which is clearly prefactorized but not factorized.

Remark Assume that a certain normal subgroup N of a pairwise mutually permutable product $G = G_1G_2 \dots G_k$ satisfies that N/M is prefactorized in $G/M = (G_1M/M)(G_2M/M)\dots(G_kM/M)$ for each minimal normal subgroup M of G contained in N. Then, either N is prefactorized in $G = G_1G_2\dots G_k$, or N is a minimal normal subgroup of G.

Assume that N is not prefactorized in G. Then $N \neq 1$. If N is not a minimal normal subgroup of G, there exists a minimal normal subgroup M of G with $M \leq N$. By hypothesis, N/M is prefactorized in G/M, that is, $N/M = ((G_1M/M) \cap (N/M))((G_2M/M) \cap (N/M))\dots((G_kM/M) \cap (N/M))$ and from here $N = (N \cap G_1)(N \cap G_2)\dots(N \cap G_k)M$. On the other hand, by [5, Lemma 1(ii)], $(N\cap G_1)(N\cap G_2)\dots(N\cap G_k)$ is a normal subgroup of G. If $(N\cap G_1)(N\cap G_2)\dots(N\cap G_k) \neq 1$, then it contains a minimal normal subgroup of G, R say, and this implies $N = (N \cap G_1)(N \cap G_2)\dots(N \cap G_k)$, that is, N is prefactorized, contrary to assumption. Therefore, we may assume that $(N\cap G_1)(N\cap G_2)\dots(N\cap G_k) = 1$. Thus, $N = M$ and N is a minimal normal subgroup of G.

We present now an example of a p-group in which the derived subgroup is not prefactorized.

Example 3 Consider the group $M = (\langle a\rangle \times \langle b\rangle \times \langle x, y\rangle)/N$ with $N = \langle a^2b^2x^2\rangle$, a and b elements of order 4 and $\langle x, y\rangle \simeq Q_8$, the quaternion group of order 8. Denote by u^* the class uN of M. We show that $M = \langle a^*\rangle\langle a^*x^*\rangle\langle b^*\rangle\langle b^*y^*\rangle$ is a pairwise mutually permutable product: Since two of the factors are normal subgroups, we have to check only one case, and we obtain $(a^*x^*)(b^*y^*) = a^*b^*x^*y^* = a^*b^*y^*x^*(a^*b^*)^2 = b^*y^*a^*x^*(b^*y^*)^2(a^*x^*)^2 = (b^*y^*)^{-1}(a^*x^*)^{-1}$ since $(a^*x^*)^2 = (b^*)^2$ and $(b^*y^*)^2 = (a^*)^2$. Note that all squares belong to the center of M, so given relation leads to all the other further relations to be checked, so if $\{e, f\} \subseteq \{1, -1\}$ we find $(a^*x^*)^e(b^*y^*)^f = (b^*y^*)^{-f}(a^*x^*)^{-e}$. Note the subgroup $M' = M'M^4$ is not prefactorized.

We will now exhibit some positive results.

Theorem 1 *Let the group $G = G_1G_2\dots G_k$ be the product of the pairwise mutually permutable subgroups $G_1, G_2, \dots, G_k$. Then*

(i) The subgroup of G which is generated by all Π-elements of G (where Π denotes a set of primes dividing the order of G) is prefactorized in G.
(ii) $G^p = \langle x^p \mid x \in G\rangle$ is prefactorized in G for each prime p.
(iii) If M and N are prefactorized normal subgroups of G and N has exponent p, p a prime, then also $[M, N]$ is prefactorized in G.

Proof (i) Denote by N the subgroup generated by all Π-elements of G, Π a set of primes dividing the order of G. Assume that N is not prefactorized and choose for G a couterexample of minimal order. Then N/M is prefactorized in $G/M =$

$(G_1M/M)(G_2M/M)\dots(G_kM/M)$ for each minimal normal subgroup M of G contained in N. By the above Remark, N is a minimal normal subgroup of G and so $G_i \cap N = 1$ for all $i = 1, 2, \dots, k$. Since G/N is a Π'-group, it follows that G_iN/N is also a Π'-group for all i. Therefore, G is a Π'-group. This implies that $N = 1$, contrary to assumption. Hence, N is prefactorized.

(ii) Let $N = G^p$. Assume that $N \neq (N \cap G_1)(N \cap G_2)\dots(N \cap G_k)$ and let G be of minimal order. Arguing as in (i), N is a minimal normal subgroup of G and so $N \cap G_i = 1$ for all $i = 1, 2, \dots, k$. In particular, G_i have exponent p for all i. Hence G is a p-group and $|N| = p$. Choose $x \in G_j$. Then $\langle x\rangle G_i = G_i\langle x\rangle$ and G_i is a subgroup of index at most p in the product $G_i\langle x\rangle$. Thus, G_i is normalized by all elements x of G_j and so by G.
Let t be an element of G. Then, $t = u_1u_2\dots u_k$ with $u_i \in G_i$, $i = 1, 2, \dots, k$. Suppose that $t^p \neq 1$ and choose t with least number of nontrivial factors. Let u_w be the first nontrivial factor. Then $t = u_ws$ and $s^p = 1$ by our choice of t. Since G_w is normal in G, it follows that every conjugate of u_w belong to G_w. We have $t^p = (u_ws)^ps^{-p} = u_w(su_ws^{-1})(s^2u_ws^{-2})\dots(s^{p-1}u_ws^{1-p}) \in G_w$. This means that $t^p \in N \cap G_w = 1$, against the choice of t. Therefore, G has exponent p and $N = G^p = 1$, the final contradiction.

(iii) Assume the result is not true and let G be a minimal counterexample. By Lemma 2(v), MN is a prefactorized subgroup of G. If we suppose that MN is a proper subgroup of G, then it is clear that MN satisfies the same hypotheses as G. By the choice of G, $[M, N]$ is prefactorized in MN. Lemma 2(iv) yields $[M, N]$ is prefactorized in G. Therefore, we may assume that $G = MN$. Then, $[M, N]$ is a normal subgroup of G, and by the remark, we may assume that $[M, N]$ is a minimal normal subgroup of G such that $[M, N] \cap G_i = 1$ for every $i = 1, 2, \dots, k$. Let $1 \neq x \in N \cap G_j$. Then x has order p since $N^p = 1$. In particular, it normalizes G_i since $|G_i\langle x\rangle : G_i| \in \{1, p\}$. This implies that $N \cap G_j$ normalizes $M \cap G_i$ for all $i = 1, 2, \dots, k$. On the other hand, by hypothesis we have $M = (M \cap G_1)(M \cap G_2)\dots(M \cap G_k)$ and $N = (N \cap G_1)(N \cap G_2)\dots(N \cap G_k)$. Hence $[M, N] = [(N \cap G_1)(N \cap G_2)\dots(N \cap G_k), (M \cap G_1)(M \cap G_2)\dots(M \cap G_k)]$. Applying [11, A; 7.4(f)], $[M, N] = [N \cap G_1, M][N \cap G_2, M]\dots[N \cap G_k, M]$. Let $z \in [N \cap G_j, M]$. By [11, A; 7.2], $z = x_1x_2\dots x_k$ with $x_i \in [N \cap G_j, M \cap G_i]^g$, $g \in G$. Therefore $x_i \in G_i^g \cap [N, M]$. Now $x_i^{g^{-1}} \in G_i \cap [N, M] = 1$ and $x_i = 1$. Consequently $z = 1$ and from here $[M, N] = 1$. This final contradiction proves the claim. □

Corollary 1 *(i) If $G = G_1G_2\dots G_k$ is a pairwise mutually permutable product of exponent p, then every term of the descending central series and every term of the derived series of G are prefactorized in G. Furthermore, for every prefactorized normal subgroup M of G, we have that $[M, G]$ and M' are prefactorized subgroups of G.*

(ii) If $G = G_1G_2\dots G_k$ is a pairwise mutually permutable product, then G^pG' is prefactorized in G.

(iii) If $G = G_1G_2\dots G_k$ is a pairwise mutually permutable product and m is a squarefree integer, then G^mG' is prefactorized in G.

Proof (i) is a direct consequence of part (iii) of Theorem 1. □

To prove (ii), notice that applying (i), $(G/G^p)' = G^pG'/G^p$ is prefactorized in G/G^p and G^p is prefactorized in G by Theorem 1(ii). The result now follows from Lemma 2(ii).

We proceed by induction on the number k of prime factors of m for statement (iii). The result is true if m is a prime by (ii). Assume that the result is true for all squarefree integers that are products of $k-1$ primes and choose a prime p dividing m. Then $m = pn$ and $H_n = G^n G'$ and $H_p = G^p G'$ are prefactorized by the induction hypothesis. Hence $(H_n)^p (H_n)'$ and $(H_p)^n (H_p)'$ are prefactorized in G. Now we have the following inclusions:

$$G^m (G')^p \subseteq (H_n)^p (H_n)' \subseteq G^m G'$$
$$G^m (G')^n \subseteq (H_p)^n (H_p)' \subseteq G^m G'$$

and therefore $G^m G' = ((H_n)^p (H_n)' (H_p)^n (H_p)')$ is prefactorized in G by Lemma 2(v).

Note that the hypothesis about m in the above corollary is essential (see Example 3).

The following statement gives an indication that there is quite a range of prefactorized normal subgroups.

Corollary 2 *Let the soluble group $G = G_1 G_2 \dots G_k$ be a pairwise mutually permutable product of the subgroups $G_1, G_2, \dots, G_k$. Then, there is a descending sequence of characteristic prefactorized subgroups $G = A_0 \supset A_1 \supset A_2 \supset \dots \supset A_m = 1$ such that every quotient A_s/A_{s+1} is an elementary abelian p-group.*

Proof Since $G \neq 1$ is soluble, we have $G' \neq G$. Choose a prime dividing $| G/G' |$. Then also $G^p G' \neq G$, and we may take $A_1 = G^p G'$ by the statement (ii) of the above result. The corollary now follows by induction on $| G |$. □

The following result shows that $\mathcal{F}$-residuals, for $\mathcal{F}$ a saturated formation of soluble groups, have a good behavior concerning pairwise mutually permutable products and prefactorizations.

Theorem 2 *Let the group $G = G_1 G_2 \dots G_k$ be the product of the pairwise mutually permutable subgroups $G_1, G_2, \dots, G_k$. If $\mathcal{F}$ is a saturated formation of soluble groups, then $G^{\mathcal{F}}$, the $\mathcal{F}$-residual of G, is prefactorized.*

Proof Denote by $N = G^{\mathcal{F}}$ the $\mathcal{F}$-residual of G. Assume that the result is not true and let G be a counterexample of minimal order. By the remark, we may assume that N is a minimal normal subgroup of G and $N \cap G_i = 1$ for all $i = 1, 2, \dots, k$. Since G/N is soluble, we that $G_i \simeq G_i N/N$ are soluble and so G is soluble by [5, Theorem 1]. Then, N is an elementary abelian p-group for some prime p. Let F denote the canonical local definition of $\mathcal{F}$, that is, the uniquely determined function defining $\mathcal{F}$ which is integrated and full, that is, $F(q) \subseteq \mathcal{F}$ and $F(q) = \mathcal{S}_q F(q)$ for all primes q (see [11, IV; 3.9]). Clearly, $G^{\mathcal{F}}$ is contained in $L = G^{F(p)}$, the $F(p)$-residual of G and L/N is p-nilpotent by [11, IV; 3.2(b)]. Denote by M/N the normal Hall p'-subgroup of L/N. By Proposition 1(ii), there exists a prefactorized Hall p'-subgroup H of G. Then $H = (H \cap G_1)(H \cap G_2) \dots (H \cap G_k)$ is a product of pairwise mutually permutable subgroups, $G_i H$ is a subgroup of G, and the Sylow p-subgroups of G_i are Sylow p-subgroups of $G_i H$. Now $HG_i \cap N$ is a normal p-subgroup of HG_i, and so it is contained in every Sylow p-subgroup of HG_i. In particular, it is contained in every Sylow p-subgroup of G_i. Consequently $HG_i \cap N = 1$. Now $M \cap H$ is a Hall p'-subgroup of M. Since M/N is a p'-group, it follows that $M = (M \cap H)N$. Then $M \cap HG_i = (M \cap H)N \cap HG_i = (M \cap H)(N \cap HG_i) = M \cap H$. This implies that $M \cap H$ is normalized by all G_i and therefore also by G. Therefore, $M = (M \cap H) \times N$ and $M \leq C_G(N)$. Since N is a minimal normal p-subgroup of G, we have that $G/C_G(N)$ does not possess nontrivial normal p-subgroups. Consequently $LC_G(N)/C_G(N) = 1$, that is, $L \leq C_G(N)$. We have that G/N is an $\mathcal{F}$-group and $G/C_G(N) \in F(p)$. Therefore applying [11, IV; 3.2], $G \in \mathcal{F}$ and $N = 1$. This is a contradiction. □

Unfortunately, $\mathcal{F}$-residuals, even for subgroup-closed saturated formations, are not necessarily factorized subgroups in the group as the following example shows:

Example 4 Consider the group G as in Example 2. Then, it is clear that the nilpotent residual of G, $G^{\mathcal{N}}$, is equal to $\langle x\rangle$, which is clearly prefactorized in G. However, $A \cap B = \langle b\rangle$ is not contained in $G^{\mathcal{N}}$.

3 Prefactorized projectors and normalizers

From now on, all groups considered will be finite and soluble.

In this section, we analyze the behavior of projectors and normalizers associated with saturated formations in pairwise totally and mutually permutable products.

Lemma 3 *Let the group $G = G_1G_2 \dots G_k$ be the pairwise totally permutable product of the subgroups $G_1, G_2, \dots, G_k$. If G is a primitive group, then either G is supersoluble or $\prod_{i\neq j} G_j = 1$ for some $i \in \{1, 2, \dots, k\}$.*

Proof Let N be the unique minimal normal subgroup of G. We know that N is abelian and $C_G(N) = N$. Applying [4, Lemma 1], the supersoluble residual $(G_i)^{\mathcal{U}}$ of G_i is a normal subgroup of G for all $i = 1, 2, \dots, k$. If $(G_i)^{\mathcal{U}} = 1$ for all $i \in \{1, 2, \dots, k\}$, then G is supersoluble by [4, Theorem 1]. Hence, we may assume without loss of generality that $(G_1)^{\mathcal{U}} \neq 1$. Then, N is contained in $(G_1)^{\mathcal{U}}$. Applying [3, Corollary], $G_2 \dots G_k$ centralizes $(G_1)^{\mathcal{U}}$. Therefore $G_2 \dots G_k \leq C_G(N) = N \leq G_1$. Consequently $G = G_1$. This implies that, if $i \neq 1$, all subgroups of G_i are permutable in G. Assume that $G_i \neq 1$ for some $i \in \{2, 3, \dots, k\}$ and consider a cyclic subgroup W of prime order of G_i. Then, if K is a complement of N in G, we have that WK is a subgroup of G and $W = N \cap WK$ is a normal subgroup of G. The minimality of N implies that $N = W$ and G is supersoluble, contrary to assumption. Consequently $G_i = 1$ for all $i \in \{2, 3, \dots, k\}$. □

Lemma 4 *Let the group $G = G_1G_2 \dots G_k$ be the pairwise totally permutable product of the subgroups $G_1, G_2, \dots, G_k$. If G is supersoluble and it is a primitive group with unique minimal normal subgroup N, then one of the following cases holds:*

- *(i) If $N \leq G_i$ for all $i \in \{1, 2, \dots, k\}$ then there exists $r \in \{1, 2, \dots, k\}$ such that $G = G_r \neq N$ and either $G_i = N$ for all $i \neq r$ or $G = G_1 = G_2 = \dots = G_k = N$.*
- *(ii) If $N \leq G_i$ for $i = 1, 2, \dots, r$, $r < k$, and $G_j \neq 1$, for some $j > r$, then $G_1 = G_2 = \dots = G_r = N$ and $G_{r+1}G_{r+2} \dots G_k \neq 1$ is a complement of N in G.*
- *(iii) If $N \leq G_i$ for $i = 1, 2, \dots, r$, $k > r \geq 2$ and $G_j = 1$ for all $j > r$, then there exists $1 \leq s \leq r$ such that $G = G_s \neq N$ and either $G_i = N$ for all $s \neq i \leq r$, or $G = G_1 = G_2 = \dots = G_r = N$.*
- *(iv) There exists $i \in \{1, 2, \dots, k\}$ such that $G = G_i$ and $G_j = 1$ for all $j \neq i$.*

Proof Let p be the prime dividing $|N|$. Since G is supersoluble, we have that N is of order p and $C_G(N) = N = \langle x\rangle$ is the Sylow p-subgroup of G. Let M denote a core-free complement of N in G. Then $M = \langle y\rangle$ is an abelian maximal subgroup of G with exponent dividing $p-1$. In particular, M is a Hall p'-subgroup of G. Therefore, N has to be contained in at least one of the factors G_i, $i = 1, 2, \dots, k$.

(i) Assume first that $N \leq G_i$ for all $i \in \{1, 2, \dots, k\}$. Consider one of the factors G_i and take G_j with $G_j \neq G_i$. We see that $G_i = N$ or $G_j = N$. It is clear that $G_i = \langle x\rangle\langle y^{\alpha_i}\rangle$ and in the same way $G_j = \langle x\rangle\langle y^{\alpha_j}\rangle$ with $\langle y^{\alpha_i}\rangle \neq \langle y^{\alpha_j}\rangle$. Suppose that $y^{\alpha_i} \neq 1 \neq y^{\alpha_j}$. Write $H = \langle xy^{\alpha_i} \rangle$. If p divides the order of H, then $N \leq H$. Since H is abelian, $H = N$ and so $y^{\alpha_i} \in N$ and $G_i = N$. This contradiction implies that $H = \langle xy^{\alpha_i} \rangle$ is a p'-group. Now the fact that G_i and G_j are totally permutable yields that $\langle xy^{\alpha_i}, y^{\alpha_j} \rangle = \langle xy^{\alpha_i} \rangle \langle y^{\alpha_j} \rangle$ is a p'-group and therefore it is abelian. Consequently $[xy^{\alpha_i}, y^{\alpha_j}] = 1$. Moreover $[y^{\alpha_i}, y^{\alpha_j}] = 1$. Let $s = y^{\alpha_i}$. By [11, A; 7.2(c)], $1 = [xy^{\alpha_i}, y^{\alpha_j}] = [x, y^{\alpha_j}]^s [y^{\alpha_i}, y^{\alpha_j}] = [x, y^{\alpha_j}]^s$. Hence $y^{\alpha_j} = 1$, which contradicts our assumption. Therefore if $G_i \neq G_j$ we have either $G_i = N$ or $G_j = N$. Assume that $M \neq 1$. Then at least one of the factors has a nontrivial Hall p'-subgroup. Without loss of generality, we may suppose that $M \cap G_1 \neq 1$. The above argument implies that either $G_i = G_1$ or $G_i = N$ for every $i \neq 1$. Assume that $G_j = G_1$ for some $j \neq 1$. Applying the above argument, $1 \neq y^{\alpha_j}$ centralizes x. This contradiction shows that $G_2 = \ldots = G_k = N$ and then $G = G_1$.

(ii) Suppose that $N \leq G_i$ for $i = 1, 2, \ldots, r$, $r < k$. Then $N \cap G_j = 1$ for all $j > r$. Assume that $G_j \neq 1$, for some $j > r$. Then $G_j = \langle z \rangle$ is a nontrivial p'-subgroup of G. Suppose there exists $i \leq r$ such that $G_i \neq N$. Then $G_i = N \langle y^{\alpha_i} \rangle$ and $y^{\alpha_i} \neq 1$. Consider $H = \langle xy^{\alpha_i} \rangle$. Arguing as in (i), H is a p'-group. Moreover, as G_i and G_j are totally permutable, H permutes with G_j and HG_j is a p'-subgroup of G and hence HG_j is abelian. Therefore $[xy^{\alpha_i}, z] = 1$. It is also clear that $\langle y^{\alpha_i} \rangle$ permutes with $\langle z \rangle$. Therefore, $\langle y^{\alpha_i} \rangle \langle z \rangle$ is an abelian p'-group. Hence $[y^{\alpha_i}, z] = 1$. Therefore, z centralizes x. This contradiction yields $G_i = N$ for all $i = 1, 2, \ldots, r$. Therefore $G_{r+1} G_{r+2} \ldots G_k$ is a complement of N in G.

(iii) If $N \leq G_i$ for $i = 1, 2, \ldots, r$, $k > r \geq 2$, and $G_j = 1$ for all $j > r$, then we are in case (i). Hence there exists $1 \leq s \leq r$ such that $G = G_s \neq N$ and $G_i = N$ for all $s \neq i \leq r$, or $G = G_1 = G_2 = \ldots = G_r = N$.

The remaining possibility is case (iv). □

Definition 1 If $\mathcal{X}$ is a class of groups, a maximal subgroup M of a group G is called $\mathcal{X}$-abnormal in G if $G/Core_G(M)$ does not belong to $\mathcal{X}$. A subgroup S is called sub-$\mathcal{X}$-abnormal in G if either $G = S$ or there exists a chain

$$S = S_0 \leq S_1 \leq \ldots \leq S_n = G$$

with S_i an $\mathcal{X}$-abnormal maximal subgroup of S_{i+1} for all $i = 0, 1, \ldots, n-1$.

Theorem 3 *Let the group $G = G_1 G_2 \ldots G_k$ be the pairwise totally permutable product of the subgroups $G_1, G_2, \ldots, G_k$. If $\mathcal{F}$ is a saturated formation containing the class $\mathcal{U}$ of all supersoluble groups, then every sub-$\mathcal{F}$-abnormal subgroup of G is factorized.*

Proof We proceed by induction on the index $|G : S|$ of S in G. Suppose that S is a maximal $\mathcal{F}$-abnormal subgroup of G. If $Core_G(S) = 1$, then, by Lemma 3, G is either supersoluble or there exists $i \in \{1, 2, \ldots, k\}$ such that $\prod_{i \neq j} G_j = 1$ and $G = G_i$. Since $\mathcal{F}$ contains $\mathcal{U}$ and S is $\mathcal{F}$-abnormal, it follows that G is not supersoluble. Hence $G = G_i$ for some i. It is clear that in this case S is a factorized subgroup of G. Assume that $D = Core_G(S) \neq 1$. As G/D does not belong to $\mathcal{F}$, it cannot be supersoluble. Applying now Lemma 3 to $G/D = (G_1 D/D)(G_2 D/D) \ldots (G_k D/D)$ we obtain that there exists $i \in \{1, 2, \ldots, k\}$ (we can assume without loss of generality that $i = 1$) such that $G_2 G_3 \ldots G_k$ is contained in D. Then $S = G_2 G_3 \ldots G_k (S \cap G_1) = (S \cap G_2)(S \cap G_3) \ldots (S \cap G_k)(S \cap G_1)$ and $G_i \cap \prod_{j \neq i} G_j \leq S$ for all $i = 1, 2, \ldots, k$. Hence S is factorized.

Assume now that S is not a maximal subgroup of G, and let S_1 be an $\mathcal{F}$-abnormal maximal subgroup of G containing S such that S is sub-$\mathcal{F}$-abnormal in S_1. Then S_1 is a factorized subgroup of G and $|S_1 : S| < |G : S|$. The induction hypothesis implies that S is a factorized subgroup of S_1. Then S is factorized in G by Lemma 2(iv). □

Applying [11, IV; 5.11] and [11, V; 3.10], $\mathcal{F}$-projectors and $\mathcal{F}$-normalizers associated with a saturated formation $\mathcal{F}$ are sub-$\mathcal{F}$-abnormal subgroups. Hence we have:

Corollary 3 *Let the group $G = G_1G_2 \dots G_k$ be a pairwise totally permutable product of the subgroups $G_1, G_2, \dots, G_k$. If $\mathcal{F}$ is a saturated formation containing the class $\mathcal{U}$ of all supersoluble groups, then the $\mathcal{F}$-projectors and $\mathcal{F}$-normalizers of G are factorized subgroups of G.*

For the saturated formation $\mathcal{N}$ of nilpotent groups, we have a much weaker statement.

Theorem 4 *Let the group $G = G_1G_2 \dots G_k$ be the pairwise totally permutable product of the subgroups $G_1, G_2, \dots, G_k$. Then there is a prefactorized Carter subgroup of G.*

Proof Assume the result is not true and let G be a counterexample with $| G | + | G_1 | + | G_2 | + \cdots + | G_k |$ minimal. If G is a nilpotent group, then G is its own Carter subgroup and the conclusion follows. Assume G is not nilpotent, and let C denote a Carter subgroup of G. Then, there exists a non-normal maximal subgroup M of G with $C \leq M$. Denote by $K = Core_G(M)$. If $K = 1$, then G is a primitive group. By Lemma 3, there exists $i \in \{1, 2, \dots, k\}$ such that $\prod_{j \neq i} G_j = 1$ and $G = G_i$ or G is supersoluble. In the first case, it is clear that C is prefactorized. If G is supersoluble, then $G = NM$ with $N = C_G(N) = G^{\mathcal{N}}$ the unique minimal normal subgroup of G, $| N |= p$, p a prime, and M an abelian complement of N with trivial core. By [11, IV; 5.18], the Carter subgroups of G are the complements of N in G. Applying Lemma 4, C is prefactorized. This contradiction yields $K \neq 1$. Then $G/K = (N/K)(M/K)$ is a primitive group, N/K is the unique minimal normal subgroup of G/K, $C_{G/K}(N/K) = N/K$ and M/K is a maximal subgroup of G/K with trivial core. Also, G/K is the pairwise totally permutable product of the subgroups $G_1K/K, G_2K/K, \dots, G_kK/K$. Applying Lemma 3, G/K is either supersoluble or there exists $i \in \{1, 2, \dots, k\}$ (we can assume without loss of generality that $i = 1$), such that $G_2G_3 \dots G_k \leq K \leq M$. Assume that G/K is not supersoluble. Then $M = G_2G_3 \dots G_k(M \cap G_1)$ is a pairwise totally permutable product. By the choice of G, M has a prefactorized Carter subgroup. But C is a Carter subgroup of M. Therefore there exists $m \in M$ such that C^m is prefactorized in M. By Lemma 2(iv), C^m is prefactorized in G. Consequently we may assume that G/K is supersoluble. In particular, $| N/K |= p$ and M/K is abelian of exponent dividing $p - 1$. □

On the other hand, applying [5, Lemma 1], $W = (K \cap G_1)(K \cap G_2) \dots (K \cap G_k)$ is a normal subgroup of G. If $W \neq 1$, then the choice of G implies that G/W has a prefactorized Carter subgroup, C^gW/W, for some $g \in G$. Since W is prefactorized, it follows that C^gW is prefactorized in G by Lemma 2(ii). Now C^gW is a proper subgroup of G because it is contained in M^g. Clearly C^g is a Carter subgroup of C^gW. By the minimal choice of G, there exists $t \in C^gW$ such that C^{gt} is prefactorized in C^gW. Lemma 2(iv) implies that C^{gt} is prefactorized in G. Consequently we may assume that $W = 1$, that is, $K \cap G_i = 1$ for all $i = 1, 2, \dots, k$. It now follows that G_i is supersoluble so that G is also supersoluble by Theorem 3 of [4].

It is clear that p is the largest prime dividing the order of G/K. Assume p is not the largest prime dividing the order of G and let $Q \neq 1$ denote a Sylow q-subgroup of G with q the largest prime divisor of the order of G, $q \neq p$. Then Q is a normal subgroup of G and $Q \leq K$. Moreover, by Proposition 1(i), Q is prefactorized in G. Arguing as above we obtain

$Q = 1$. This contradicts the choice of Q. Thus we may suppose that p is the largest prime dividing the order of G and G has a normal Sylow p-subgroup, P say and $PK/K = N/K$.

Now we apply Lemma 4 to G/K and analyze all the cases appearing there. Assume we are in the hypotheses of Lemma 4(ii). Then $G_1K/K = G_2K/K = \ldots = G_rK/K = N/K$ for some $r < k$ and $(G_{r+1}G_{r+2}\ldots G_k)K/K$ is a complement of N/K in G/K. Since $K \cap G_i = 1$ for all $i = 1, 2, \ldots, k$, we have $\mid G_i \mid= p$ for $i = 1, 2, \ldots, r$ and G_j is an abelian p'-group for $j > r$. Further $P = G_1G_2\ldots G_r$ is the normal Sylow p-subgroup of G and $G_{r+1}G_{r+2}\ldots G_k$ is a Hall p'-subgroup of G. In particular, P is a prefactorized subgroup of G. The above arguments imply that $G = C^gP = CP$. Hence the nilpotent residual $G^{\mathcal{N}}$ of G is contained in P.

On the other hand, $P = G_1G_2\ldots G_r$ with $\mid G_i \mid= p$ for all $i = 1, 2, \ldots, r$. Assume that $G_i = G_j$ for $i \neq j \in \{1, 2, \ldots, r\}$. Then G can be regarded as a pairwise totally permutable product of less than k factors. The choice of G implies that there is a conjugate of C which is prefactorized with respect to this new factorization. This clearly implies that this conjugate is actually prefactorized with respect to $G = G_1G_2\ldots G_k$. Hence we may assume that $G_i \neq G_j$ for $i \neq j \in \{1, 2, \ldots, r\}$. This clearly implies that $[G_i, G_j] = 1$ for all $i \neq j$, $i, j \in \{1, 2, \ldots, r\}$, that is, P is elementary abelian. Next we prove that $P = G_1 \times G_2 \times G_3 \times \ldots \times G_r$. Assume that $G_1 \cap (G_2G_3\ldots G_r) \neq 1$. Then G_1 is contained in $G_2G_3\ldots G_r$ and then $G = G_2G_3\ldots G_k$. By the choice of G, there exists a Carter subgroup T of G which is prefactorized with respect to this new factorization. This clearly implies that T is prefactorized in $G = G_1G_2\ldots G_k$. Hence we have that $G_1 \cap (G_2G_3\ldots G_r) = 1$. Repeating the argument with the other factors, we have that P is the direct product of $G_1, G_2, \ldots, G_r$. In particular, $G^{\mathcal{N}}$ is abelian and so it is complemented in G by C (see [11, IV; 5.18]). If $P = G^{\mathcal{N}}$, then C would be a Hall p'-subgroup of G and, by Proposition 1(ii), C would have a prefactorized conjugate, contrary to hypothesis. Hence $G^{\mathcal{N}}$ is a proper subgroup of P. By Theorem 2, $G^{\mathcal{N}} = (G^{\mathcal{N}} \cap G_1)(G^{\mathcal{N}} \cap G_2)\ldots(G^{\mathcal{N}} \cap G_r)$. Since all the factors G_i have order p, it follows that $G^{\mathcal{N}}$ is the product of all factors G_j which are contained in $G^{\mathcal{N}}$. Without loss of generality, we may assume that $G^{\mathcal{N}} = G_1G_2\ldots G_h$ for some $h < r$. Let $T = (G_{h+1}G_{h+2}\ldots G_r)(G_{r+1}G_{r+2}\ldots G_k)$. Thus $G = G^{\mathcal{N}}T$ and $G^{\mathcal{N}} \cap T = 1$. Furthermore T is a Carter subgroup of G which is clearly prefactorized, against supposition.

Now we assume G/K satisfies Lemma 4(i). If $G_1K/K = G_2K/K = \ldots G_kK/K = N/K$, then $G/K = N/K$ is abelian. Hence $M/K = 1$, that is, $M = K$, contrary to assumption. Therefore there exists $i \in \{1, 2, \ldots, k\}$, we suppose $i = 1$, such that $G/K = G_1K/K$ and $G_iK/K = N/K$ for all $i \neq 1$. Since $K \cap G_i = 1$ for all i, we have $\mid G_i \mid= p$ for all $i \neq 1$, $G = G_1K$ and $M = K(M \cap G_1)$. Thus $M \cap G_1$ is a p'-group. Moreover $N = G_2G_3\ldots G_k(N \cap G_1)$. Now $(N \cap G_1)K/K$ is a subgroup of N/K which is of order p. Hence either $(N \cap G_1)K/K = 1$ or $N = (N \cap G_1)K$. If $(N \cap G_1)K/K = 1$, then $N \cap G_1 \leq K \cap G_1 = 1$. Therefore $N = G_2G_3\ldots G_k$ and $G = G_1N$. Furthermore G/N is abelian. Thus G_1 is also abelian. But then G/K is abelian and $M = K$, contrary to supposition. Consequently $N = (N \cap G_1)K$ and $|N \cap G_1| = p$. In particular, N and K are both p-groups. Now $G = NM = NK(M \cap G_1) = N(M \cap G_1)$. Thus N is the Sylow p-subgroup of G and $M \cap G_1$ is a Hall p'-subgroup of G. Also $N \cap G_1$ is a Sylow p-subgroup of G_1, and $G_1 = (N \cap G_1)(M \cap G_1)$. Arguing as in case (ii), we obtain that N is elementary abelian. We prove that $N = (N \cap G_1) \times G_2 \times G_3 \times \ldots \times G_k$. Suppose that $G_2 \leq (N \cap G_1)G_3\ldots G_k$, then $G = G_1G_3\ldots G_k$ and, by the choice of G, there exists a Carter subgroup prefactorized with respect to $G = G_1G_3\ldots G_k$. Now an standard argument shows that this Carter subgroup is factorized with respect to $G = G_1G_2\ldots G_k$. This is a contradiction. If $N \cap G_1 \leq G_2G_3\ldots G_k$, then $G = (M \cap G_1)G_2G_3\ldots G_k$. By the choice of

G, there exists a Carter subgroup T of G such that $T = (T \cap M \cap G_1)(T \cap G_2) \dots (T \cap G_k) \leq (T \cap G_1)(T \cap G_2) \dots (T \cap G_k) \leq T$, and T is prefactorized, against supposition. Note that this argument could be used with every factor of the decomposition of N. Consequently $N = (N \cap G_1) \times G_2 \times G_3 \times \dots \times G_k$. In particular, $G^{\mathcal{N}} \leq N$ is abelian and so it is complemented in G by C (see [11, IV; 5.18]). If $N = G^{\mathcal{N}}$, then C would be a Hall p'-subgroup of G and, by Proposition 1(ii), C would have a prefactorized conjugate, contrary to hypothesis. Hence $G^{\mathcal{N}}$ is a proper subgroup of N. By Theorem 2, $G^{\mathcal{N}} = (G^{\mathcal{N}} \cap G_1)(G^{\mathcal{N}} \cap G_2) \dots (G^{\mathcal{N}} \cap G_k) = (G^{\mathcal{N}} \cap N \cap G_1)(G^{\mathcal{N}} \cap G_2) \dots (G^{\mathcal{N}} \cap G_k)$. Assume first that $G^{\mathcal{N}} \cap G_1 = 1$, $G_i \leq G^{\mathcal{N}}$ for $i = 1, 2, \dots, h$ and $G_j \cap G^{\mathcal{N}} = 1$ for $i = h+1, \dots, k$. Then $G = G^{\mathcal{N}}(G_{h+1} \dots G_k)(N \cap G_1)(M \cap G_1) = (G_2 \dots G_h)(G_{h+1} \dots G_k)(N \cap G_1)(M \cap G_1)$. Now let $T = (G_{h+1} \dots G_k)(N \cap G_1)(M \cap G_1)$. Arguing as in case (ii), $G^{\mathcal{N}} \cap T = 1$ (recall that N is a direct product of the subgroups $(N \cap G_1), G_2, \dots, G_k$). Therefore, T is a Carter subgroup of G and by construction it is prefactorized. On the other hand, if $N \cap G_1 \leq G^{\mathcal{N}}$, then a similar argument yields $G^{\mathcal{N}} = (N \cap G_1)G_2 \dots G_h$ and $G = G^{\mathcal{N}}T$ where $T = G_{h+1} \dots G_k(M \cap G_1)$ is a Carter subgroup of G and it is clearly prefactorized in G. We reach a contradiction in both cases.

Assume we are in case (iii) of Lemma 4. Note that if $G_jK/K = 1$ for $j \geq r+1$, then as $G_j \cap K = 1$, we obtain $G_j = 1$ for all $j \geq r+1$. Therefore we will be in case (i).

Finally, if we are in the hypotheses of (iv) in Lemma 4, we have $G/K = G_iK/K$ for some i and $G_jK/K = 1$ for all $j \neq i$. As before, this means that $G_j = 1$ for all $j \neq i$. Therefore $G = G_i$ and a Carter subgroup of G is clearly prefactorized, the final contradiction.

Theorem 5 *Let the group $G = G_1G_2 \dots G_k$ be the pairwise totally permutable product of the subgroups $G_1, G_2, \dots, G_k$. Then G has a prefactorized system normalizer.*

Proof Denote by r the nilpotent length of the soluble group G. Then $G \in \mathcal{N}^r$, the class of soluble groups with nilpotent length at most r. If $r = 2$, then by [11, V; 4.2], the system normalizers of G coincide with the Carter subgroups of G. Therefore, the conclusion follows by Theorem 4. Thus, we may assume that $r > 2$. Set $G = H_0$ and denote by H_i an $\mathcal{N}^{r-1-i}\mathcal{N}$-projector of H_{i-1} for $i = 1, 2, \dots, r-1$. By [11, V; 4.3(b)], H_{r-1} is a system normalizer of G. We have the chain:

$$1 = H_r \leq H_{r-1} \leq \dots \leq H_1 \leq H_0 = G$$

Since the class $\mathcal{U}$ of all supersoluble groups is contained in $\mathcal{N}^2 \subseteq \mathcal{N}^r$, we can apply Corollary 3 and Lemma 2(iv) to conclude that H_{r-2} is factorized in G. Now H_{r-1} is a Carter subgroup of H_{r-2}. Therefore, by Theorem 4, there exists $h \in H_{r-2}$ such that H_{r-1}^h is prefactorized in H_{r-2}. By Lemma 2(iv), H_{r-1}^h is prefactorized in G. The proof of the theorem is now complete. □

The following example shows that a Carter subgroup of a group need not be factorized.

Example 5 Consider the group

$$G = \langle a, b, c \mid a^3 = b^7 = c^7 = [b, c] = 1,\ a^2ba = b^2,\ a^2ca = c^2 \rangle$$

G is the totally permutable product of the subgroups $A = G$ and $B = \langle b \rangle$. Moreover, the Carter subgroups of G (which coincide with the system normalizers of G) are the conjugates of $\langle a \rangle$, which are not factorized in G.

In the final part of the section, we study $\mathcal{F}$-projectors and $\mathcal{F}$-normalizers in mutually permutable products of two factors and obtain that they are not in general factorized although they are always prefactorized subgroups of the group.

Lemma 5 [7, Lemmas 1 and 2] *Assume G is the product of two mutually permutable subgroups A and B. Then:*

(i) If N is a minimal normal subgroup of G, then $\{N \cap A, N \cap B\} \subseteq \{N, 1\}$.
(ii) If $N \leq A$ is a minimal normal subgroup of G and $N \cap B = 1$, then either $N \leq C_G(A)$ or $N \leq C_G(B)$; if furthermore N is noncyclic, then $N \leq C_G(B)$.
(iii) If N is a minimal normal subgroup of G and $N \cap A = N \cap B = 1$, then $\mid N \mid= p$, where p is a prime, and either $N \leq C_G(A)$ or $N \leq C_G(B)$.

Theorem 6 *Let the group $G = AB$ be the mutually permutable product of the subgroups A and B. If $\mathcal{F}$ is a saturated formation containing the class $\mathcal{U}$ of all supersoluble groups, then every sub-$\mathcal{F}$-abnormal subgroup of G is prefactorized.*

Proof Assume the result is false and let G be a counterexample of minimal order. Then there exists a sub-$\mathcal{F}$-abnormal subgroup S of G which is not prefactorized. Let us take S of maximal order. If S is not a maximal subgroup of G, then there exists M an $\mathcal{F}$-abnormal maximal subgroup of G with $S \leq M$ and S is sub-$\mathcal{F}$-abnormal in M. By the choice of S, $M = (M \cap A)(M \cap B)$. Now the minimality of G implies S is prefactorized in M and therefore in G by Lemma 2(iv). Consequently, we may suppose that S is an $\mathcal{F}$-abnormal maximal subgroup of G. □

Then G satisfies the following properties:

(i) $Core_G(A \cap B) = 1$.
Assume there exists a minimal normal subgroup N of G such that $N \leq A \cap B$. Suppose that $N \leq S$. Now S/N is an $\mathcal{F}$-abnormal subgroup of G/N. By induction, S/N is prefactorized in G/N. But N is prefactorized in G. By Lemma 2(ii), S is prefactorized, contrary to supposition. Therefore $G = SN$, $S \cap N = 1$, $A = N(S \cap A)$ and $B = N(S \cap B)$. Consequently $G = NS = N(S \cap A)(S \cap B)$. Therefore $S = (S \cap A)(S \cap B)$, which contradicts the choice of G.

(ii) Every minimal normal subgroup of G is contained in S.
Assume there exists a minimal normal subgroup M of G which is not contained in S. Then $G = SM$.

(a) Suppose that $M \cap A = M \cap B = 1$. Then by Lemma 5(iii) $|M| = p$, p a prime number. Therefore $G/C_G(M)$ is an $\mathcal{F}$-group and $G^{\mathcal{F}} \leq C_G(M)$. Applying [11, IV; 1.17(b)], $S^{\mathcal{F}} \leq G^{\mathcal{F}}$ and so $S^{\mathcal{F}}$ is normal in G. If $S^{\mathcal{F}} = 1$, then $S \in \mathcal{F}$. In this case, $G = SM$ would be the totally permutable product of two $\mathcal{F}$-subgroups. Therefore, by [3, Lemma 4], $G \in \mathcal{F}$, contrary to the fact that S is $\mathcal{F}$-abnormal in G. Hence $S^{\mathcal{F}} \neq 1$. On the other hand, $G/S^{\mathcal{F}}$ is the totally permutable product of $S/S^{\mathcal{F}}$ and $MS^{\mathcal{F}}/S^{\mathcal{F}}$ and both are $\mathcal{F}$-groups. A new application of [3, Lemma 4] yields $G/S^{\mathcal{F}}$ is an $\mathcal{F}$-group and so $G^{\mathcal{F}} = S^{\mathcal{F}}$. The minimality of G implies that $S/S^{\mathcal{F}}$ is prefactorized in $G/S^{\mathcal{F}}$ and, by Theorem 2, $S^{\mathcal{F}}$ is prefactorized in G. By Lemma 2(ii), S is prefactorized in G, against supposition.

(b) Suppose that $M = (M \cap A)(M \cap B)$. We know by (i) that M is not contained in $A \cap B$. Assume that $M \leq A$ and $M \cap B = 1$. Now $A = M(S \cap A)$. We claim that $S \cap A \neq 1$. Suppose, arguing by contradiction, that $S \cap A = 1$. Then $A = M$, $G = AB = MB = MS$ and $A \cap B = M \cap B = 1$. Let X be a subgroup of A of prime order. Then $X = XB \cap M$ is normalized by B. Consequently, X is normal in G and $X = M$, that is, $|M| = p$, where p is a prime. Now consider the quotient group $G/Core_G(S) = (MCore_G(S)/Core_G(S))(S/Core_G(S))$. It is a primitive group, and $MCore_G(S)/Core_G(S)$ has order p. Thus $G/Core_G(S)$ is supersoluble, contradicting the fact that S is $\mathcal{F}$-abnormal in G. Hence $S \cap A \neq 1$. On the other hand, applying Lemma 5(ii), either M is cyclic of prime order or $M \leq C_G(B)$. We prove that $S \cap A$ is a maximal subgroup of A. If M has prime order, it is clear that $S \cap A$ is a maximal subgroup of A. Assume that $M \leq C_G(B)$. Then, M is a minimal normal subgroup of A, and $S \cap A$ is a proper subgroup of A because M is not contained in S. Thus, $S \cap A$ is a maximal subgroup of A. Suppose that $A \cap B$ is not contained in $S \cap A$. Then $A = (S \cap A)(A \cap B)$, since $A \cap B$ is a permutable subgroup of A. Thus $G = (S \cap A)(A \cap B)B = (S \cap A)B$ and $S = (S \cap A)(S \cap B)$, contrary to the choice of S. Consequently we may assume that $A \cap B$ is contained in S. On the other hand, by [9, 3.5], $A \cap B$ is a subnormal subgroup of G. Therefore M normalizes $A \cap B$ by [11, A; 14.3]. Then $T = (A \cap B)^G = (A \cap B)^S \leq S$. If $A \cap B = 1$, then the product of A and B is totally permutable and S is factorized by Theorem 3. This contradiction yields $A \cap B \neq 1$. Since $A \cap B$ is contained in $(T \cap A)(T \cap B)$, it follows that $(T \cap A)(T \cap B)$ is a nontrivial factorized normal subgroup of G contained in S. By induction, $S/(T \cap A)(T \cap B)$ is prefactorized in $G/(T \cap A)(T \cap B)$. Applying Lemma 2(ii), S is prefactorized. This contradiction proves (ii).

Let N be a minimal normal subgroup of G. Then $N \leq S$. If $N = (N \cap A)(N \cap B)$, the minimal choice of G implies that S/N is prefactorized in G/N. By lemma 2(ii), the same is true for S. This contradiction yields $N \cap A = N \cap B = 1$ for all minimal normal subgroups of G. This is to say that $Core_G(A)Core_G(B) = 1$, which contradicts [8, Theorem 1].

Our last example shows that there exist mutually permutable products in which the $\mathcal{U}$-projectors ($\mathcal{U}$-normalizers) are not factorized.

Example 6 Consider $G \simeq \Sigma_4 = [C_2 \times C_2][C_3]C_2$ the symmetric group of degree 4. Then $G = AB$ is the mutually permutable product of A and B where A denotes the alternating group of degree 4 and $B \simeq [C_2 \times C_2]C_2$. Now consider $H \simeq [C_3]C_2$ a subgroup of G isomorphic to the symmetric group of degree 3. Then it is an $\mathcal{U}$-projector of G (it is also an $\mathcal{U}$-normalizer of G) which is prefactorized in G but not factorized.

We bring the paper to a close with the following.

Open questions

(a) Can Theorems 4 and 5 be extended to pairwise mutually permutable products?
(b) Is Theorem 6 true for mutually permutable products with more than two factors?

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References

1. Amberg, B., Franciosi, S., de Giovanni, F.: Products of Groups. Clarendon Press, Oxford (1992)
2. Ballester-Bolinches, A., Pedraza-Aguilera, M.C., Pérez-Ramos, M.D.: Totally and Mutually Permutable Products of Finite Groups, Groups St. Andrews 1997 in Bath I. London Math. Soc. Lecture Note Ser. **260**, 65–68. Cambridge University Press, Cambridge (1999)
3. Ballester-Bolinches, A., Pedraza-Aguilera, M.C., Pérez-Ramos, M.D.: On finite products of totally permutable groups. Bull. Aust. Math. Soc. **53**, 441–445 (1996)

4. Ballester-Bolinches, A., Pedraza-Aguilera, M.C., Pérez-Ramos, M.D.: Finite groups which are products of pairwise totally permutable subgroups. Proc. Edinb. Math. Soc. **41**, 567–572 (1998)
5. Ballester-Bolinches, A., Beidleman, J.C., Heineken, H., Pedraza-Aguilera, M.C.: On pairwise mutually permutable products. Forum Math. **21**, 1081–1090 (2009)
6. Ballester-Bolinches, A., Beidleman, J.C., Heineken, H., Pedraza-Aguilera, M.C.: Local classes and pairwise mutually permutable products of finite groups. Documenta Math. **15**, 255–265 (2010)
7. Beidleman, J.C., Heineken, H.: Mutually permutable subgroups and group classes. Arch. Math. **85**, 18–30 (2005)
8. Beidleman, J.C., Heineken, H.: Group classes and mutually permutable products. J. Algebra **297**, 409–416 (2006)
9. Carocca, A.: p-supersolvability of factorized groups. Hokkaido Math. J. **21**, 395–403 (1992)
10. Carocca, A., Maier, R.: Theorems of Kegel-Wielandt Type, Groups St. Andrews 1997 in Bath I. London Math. Soc. Lecture Note Ser. **260**, 195–201. Cambridge University Press, Cambridge, (1999)
11. Doerk, K., Hawkes, T.: Finite Soluble Groups. Walter De Gruyter, Berlin (1992)
12. Maier, R., Schmid, P.: The embedding of quasinormal subgroups in finite groups. Math. Z. **131**, 269–272 (1973)