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Eigenvalue decay rates for positive integral operators

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Abstract We present decay rates for the eigenvalues of positive integral operators with smooth kernels on special metric spaces endowed with a strictly positive measure. The smoothness is defined by either differentiability conditions or inequalities of Lipschitz type. We use the decay rates to place the operators in some Schatten p-classes.

Keywords Eigenvalue decay rates · Positive definiteness · Mercer's theorem

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1 Introduction

Integral operators acting on L^2 spaces appear quite naturally in many branches of mathematics. If the operator has countably many eigenvalues, the search for decay rates for them, under a variety of assumptions, is a problem that has attracted attention for decades. Some classical references on this topic are König [11], and Pietsch [15].

The present work has its primary motivation in some recent results described in the papers Buescu [1], Buescu and Paixão [2–4] where the case in which the generating kernel of the operator is a smooth and positive definite element of $L^2(I^2)$, I being an interval, was detailed covered. The crucial difference between the results proved in these references and others proved earlier (see Reade [16] for example) is that the compactness of the interval was no longer used at the cost of an additional assumption of the generating kernel.

Before we describe what we intend to do, we would like to mention other relevant earlier contributions. Reference Han [10] treated the very same problem replacing the interval with

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a cube. In the papers [12,13], the interval was replaced with a special compact metric space; the analysis was developed endowing the space with a finite and strictly positive measure. Decay rates in the case when I is replaced with the unit sphere in Euclidean space were obtained in Ferreira et al. [6] using the very same techniques introduced by Chang and Ha [5], coupled with a sharp result on spherical quadrature. In Ferreira and Menegatto [7], we adapted some of the techniques introduced in Buescu [1], Buescu and Paixão [2,3,16] to study decay rates for the eigenvalues of operators generated by kernels defined on metric spaces having a special structure, typical examples being convex open sets, usual spheres, and tori.

Among the many tools used in the analysis of decay rates are extensions and generalizations of the so-called Mercer's theorem. Novitskiĭ's paper [14] is an important reference where a quite general version of Mercer's theorem was proved. For recent applications, see Sun [17] and references quoted there.

In Sect. 2 of the present paper, we will develop a basic Mercer's theory in the case when X is a general metric space endowed with a strictly positive measure and the restriction of the generating kernel of the operator is integrable in the diagonal of $X \times X$. As far as we know, this is a new contribution along the lines of Mercer's theory. In the rest of the paper, we will refine some of the results in Ferreira and Menegatto [7] and generalize others. Precisely, we will describe decay rates for the eigenvalues of the integral operator when the generating kernel is positive definite on a certain metric space and satisfies smoothness conditions of Lipschitz type. In Sect. 3, we define the category of metric spaces we will consider. At the end of the section, we introduce the smoothness conditions we intend to use in the formulation of the main results. Section 4 is entirely composed of preparatory technical results while Sect. 5 contains the description of the decay rates. They are used to identify a sharp real number q so that the operator can be included in the Schatten p-class, $p > q$.

2 Background and Mercer-like results

Throughout the section, X will denote a metric space endowed with a measure ν. A kernel $K : X \times X \to \mathbb{C}$ is $L^2(X, \nu)$-*positive definite* when the associated integral operator

$$\mathcal{K}(f)(x) := \int_X K(x, y) f(y)\, \mathrm{d}\nu(y), \quad f \in L^2(X, \nu),\ x \in X,$$

is positive, that is,

$$\langle \mathcal{K}(f), f\rangle_2 := \int_X \left(\int_X K(x, y) f(y)\, \mathrm{d}\nu(y) \right) \overline{f(x)}\, \mathrm{d}\nu(x) \geq 0, \quad f \in L^2(X, \nu).$$

The set of all such kernels will be written as $L^2PD(X, \nu)$. If ν is *strictly positive*, that is, it is a (complete) Borel measure such that every open nonempty subset of X has positive measure and every $x \in X$ belongs to an open subset of X of finite measure, then a continuous element of $L^2PD(X, \nu)$ is necessarily positive definite in the usual sense:

$$\sum_{i,j=1}^{n} \overline{c_i} c_j K(x_i, x_j) \geq 0,$$

holds for all $n \geq 1$, $\{x_1, x_2, \ldots, x_n\} \subset X$ and any scalars $c_1, c_2, \ldots, c_n$ [7, Theorem 2.3]. For use ahead, we inform the reader that the previous inequality implies that K is hermitian, nonnegative in the diagonal of X and

$$|K(x, y)|^2 \leq K(x, x)K(y, y), \quad x, y \in X.$$

We now introduce the smoothness notion we will use in our results. We will write $\mathcal{A}(X, \nu)$ to denote the subset of $C(X \times X) \cap L^2PD(X, \nu)$ formed by the kernels possessing an *integrable diagonal*, that is, those kernels K for which $x \in X \to K(x, x)$ belongs to $L^1(X, \nu)$. Since the integral operator $\mathcal{K}$ generated by a kernel in $L^2PD(X, \nu) \cap L^2(X \times X, \nu \times \nu)$ is compact and self-adjoint [9, p. 70] so is the integral operator generated by a kernel in $\mathcal{A}(X, \nu)$.

The following lemma settles the remaining doubts.

Lemma 2.1 *Assume ν is strictly positive. If $K \in \mathcal{A}(X, \nu)$ then K has a $L^2(X \times X, \nu \times \nu)$-convergent series representation in the form*

$$K(x, y) = \sum_{n=1}^{\infty} \lambda_n(\mathcal{K})\phi_n(x)\overline{\phi_n(y)}, \quad x, y \in X,$$

in which $\{\phi_n\}$ is $L^2(X, \nu)$-orthonormal and $\{\lambda_n(\mathcal{K})\}$ decreases to 0. The series is uniformly convergent on compact subsets of $X \times X$. If $\lambda_n(\mathcal{K}) > 0$ then ϕ_n is a continuous eigenfunction of $\mathcal{K}$. In particular, $\mathcal{K}$ is compact.

Proof If $K \in \mathcal{A}(X, \nu)$ then the inequality $|K(x, y)|^2 \leq K(x, x)K(y, y), x, y \in X$ implies that K is in $L^2(X \times X, \nu \times \nu)$ so that $\mathcal{K}$ is compact. To complete the proof, it suffices to adapt arguments from the proof of Theorem 2.4 in Ferreira and Menegatto [7]. □

We now move to the main result in this section, first introducing some terminology and notation. If T is a compact operator on a Hilbert space $\mathcal{H}$ then $|T| := (T^*T)^{1/2}$ is compact, positive and self-adjoint. As so, denoting by $\{s_n(T)\}$ the sequence of eigenvalues of $|T|$, each repeated as often as its multiplicity, we say that an operator T is in the *Schatten p-class* $\mathcal{S}_p$ [9, p. 49, 87] when

$$\sum_{n=1}^{\infty} s_n(T)^p < \infty.$$

Elements of $\mathcal{S}_1$ are usually called *trace-class* (or nuclear) operators while the class $\mathcal{S}_2$ is called the *Hilbert–Schmidt* class. If T is trace-class then the sum

$$\sum_{f \in \mathcal{B}} \langle T(f), f\rangle_{\mathcal{H}},$$

is independent of the choice of the orthonormal basis $\mathcal{B}$ of $(\mathcal{H}, \langle\cdot, \cdot\rangle_{\mathcal{H}})$ [9, p. 63]. This sum is called the *trace* of T and denoted here by $\mathrm{tr}(T)$. The trace acts linearly over the vector subspace $\mathcal{S}_1$ of the space of all bounded linear operators on $\mathcal{H}$. If, in addition, T is either self-adjoint or normal then $s_n(T) = |\lambda_n(T)|$, where $\{\lambda_n(T)\}$ is the sequence of eigenvalues of T and

$$\mathrm{tr}(T) = \sum_{n=1}^{\infty} \lambda_n(T).$$

An important family of trace-class operators is that containing all finite rank operators on $\mathcal{H}$. For $p \geq 1$, the class $\mathcal{S}_p$ is a Banach space when one uses the norm given by the formula

$$\|T\|_p := \left(\sum_{n=1}^{\infty} (s_n(T))^p \right)^{1/p}.$$

In particular,

$$\langle S, T \rangle := \mathrm{tr}(ST^*), \quad S, T \in \mathcal{S}_2,$$

defines an inner product and $(\mathcal{S}_2, \langle \cdot, \cdot \rangle)$ is a Hilbert space [9, p. 66].

The last concept we need is that of support of a measure. If ν is also a Borel measure then the support $supp(\nu)$ of ν is the set of all points x of X for which every open neighborhood of x has positive measure. We record that the condition $\nu(X \setminus supp(\nu)) = 0$ mentioned in the next result holds when ν is a Radon measure on X.

In Theorem 2.2 below, we write $Y := supp(\nu)$.

Theorem 2.2 *Assume ν is a Borel measure, the restriction of ν to Y is strictly positive and $\nu(X \setminus Y) = 0$. Let K be a kernel in $C(X \times X) \cap L^2(X \times X, \nu \times \nu)$ possessing an integrable diagonal. Finally, assume $\mathcal{K}$ possesses a $L^2(X, \nu)$-convergent spectral representation in the form*

$$\mathcal{K}(f) = \sum_{n=1}^{\infty} \lambda_n(\mathcal{K}) \langle f, \phi_n \rangle_2 \phi_n, \quad f \in L^2(X, \nu), \tag{2.1}$$

in which $\{\phi_n\}$ is an orthonormal subset of $L^2(X, \nu)$ and the sequence $\{\lambda_n(\mathcal{K})\}$ is a subset of a circle sector from the origin of $\mathbb{C}$ with central angle less than π. Then, the restriction of K to $Y \times Y$ has a $L^2(Y \times Y, \nu \times \nu)$-convergent series representation in the form

$$K(x, y) = \sum_{n=1}^{\infty} \lambda_n(\mathcal{K}) \phi_n(x) \overline{\phi_n(y)}, \quad x, y \in Y, \tag{2.2}$$

with uniform convergence on compacts sets. In particular, $\mathcal{K}$ is compact and $\lambda_n(\mathcal{K})\phi_n \in C(X)$ for all n.

Proof Let $\mathcal{K}_1$ be the integral operator generated by the restriction of K to $Y \times Y$. Since the functions in $L^2(X, \nu)$ differ from those in $L^2(Y, \nu)$ by a set of measure zero, it is quite clear that $\mathcal{K}_1$ has the same series representation $\mathcal{K}$ has. Now, we define an operator T by choosing two numbers $\alpha \in [0, 2\pi]$ and $l > 0$, so that the set $\{\mathrm{e}^{i\alpha}\lambda_n(\mathcal{K}) : n = 1, 2, \ldots\}$ of eigenvalues of the operator $P := \mathrm{e}^{i\alpha}\mathcal{K}$ belongs to the circle sector from the origin with central angle $\pi - 2\arctan l$, bounded by the rays $t = \pm ls$, $s \geq 0$. Next, we write $\alpha_n = \mathrm{e}^{i\alpha}\lambda_n(\mathcal{K})$, $n = 1, 2, \ldots$ and put $T := (P + P^*)/2$. Clearly, T is an integral operator generated by the kernel

$$L(x, y) = \frac{\mathrm{e}^{i\alpha} K(x, y) + \mathrm{e}^{-i\alpha} \overline{K(x, y)}}{2}, \quad x, y \in Y,$$

having a spectral representation in the form

$$T(f) = \sum_{n=1}^{\infty} (Re\, \alpha_n) \langle f, \phi_n \rangle_2 \phi_n, \quad f \in L^2(Y, \nu).$$

As so, L is an element of $\mathcal{A}(Y, \nu)$. Lemma 2.1 is applicable, a consequence being the formula

$$\sum_{n=1}^{\infty}(Re\,\alpha_n)|\phi_n(x)|^2 = L(x, x), \quad x \in Y, \tag{2.3}$$

with uniform convergence on compact subsets of Y. The same lemma implies that each function $\lambda_n(\mathcal{K})\phi_n$ is continuous. To finish the proof, we use the estimate

$$|\lambda_n(\mathcal{K})|^2 = |\alpha_n|^2 \leq (Re\,\alpha_n)^2(1 + l^2), \quad n = 1, 2, \ldots, \tag{2.4}$$

and the Cauchy–Schwarz inequality to deduce that

$$\left|\sum_{n=p}^{q} \lambda_n(\mathcal{K})\phi_n(x)\overline{\phi_n(y)}\right|^2 \leq (1 + l^2) \sum_{n=p}^{q}(Re\,\alpha_n)|\phi_n(x)|^2 \sum_{n=p}^{q}(Re\,\alpha_n)|\phi_n(y)|^2,$$

for all $x, y \in Y$, whenever $q \geq p \geq 1$. The above inequalities guarantee both the uniform convergence of the series (2.2) to a necessarily continuous kernel and the compactness of $\mathcal{K}$. Now, standard analysis arguments plus the use of (2.1) show that the series converges to K. □

Additional information regarding the operator $\mathcal{K}_1$ appearing in the previous proof is the content of our next result.

Theorem 2.3 *Under the conditions in Theorem 2.2, the following statements hold:*

(i) The range of $\mathcal{K}_1$ is a subset of $C(Y) \cap L^2(Y, \nu)$;
(ii) The operator $\mathcal{K}_1$ is normal and compact, and the series (2.1) is uniformly convergent on compact subsets of Y.

Proof The beginning of the proof of Theorem 2.2 implies that $\mathcal{K}_1$ is normal and compact. Next, let Z be a compact subset of Y. Inequality (2.4) shows that

$$\left|\sum_{n=p}^{p+q} \lambda_n(\mathcal{K})\langle f, \phi_n\rangle_2\phi_n(x)\right|^2 \leq \sqrt{1 + l^2}\, \|\mathcal{K}\| \sup_{z\in Z} L(z, z) \sum_{n=p}^{p+q} |\langle f, \phi_n\rangle_2|^2, \quad x \in Z,$$

whenever $f \in L^2(X, \nu)$. So, Bessel's inequality and the usual Cauchy's criterion for convergence imply the uniform convergence of the series in Z. □

Remark 2.4 All three results above hold when X is a first countable topological space. In order to see that it suffices to verify that Theorem 2.4 in Ferreira and Menegatto [7] still holds with this new assumption on X. Indeed, continuity and sequential continuity are equivalent in a first countable topological space.

Remark 2.5 There are other alternative settings for the previous results. Just to give an example, we mention [17] where another version of Lemma 2.1 can be found. If X is a locally compact topological space and ν is as before, a different setting can be obtained by replacing the integrability of K in the diagonal with the following smoothness condition: we write K_x, $x \in X$, to denote the function $K_x : X \to \mathbb{C}$ defined by the formula $K_x(y) := K(x, y)$, $y \in X$ and similarly, we introduce the symbol $K^y := K(x, y)$, $y \in X$. The kernel K is termed *smooth* whenever it is continuous and both functions $x \in X \to K_x \in L^2(X, \nu)$

and $y \in X \to K^y \in L^2(X, \nu)$ belong to $C(X, L^2(X, \nu))$. For a smooth kernel K, the formula

$$\mathcal{K}(\phi)(x) = \int_X K(x, y)\phi(y)\, \mathrm{d}\nu(y)$$
$$= \int_X \phi(y) K_x(y)\, \mathrm{d}\nu(y) = H_\phi\left(\overline{K_x}\right), \quad \phi \in L^2(X, \nu)$$

in which $H_\phi(\psi) = \langle \phi, \psi\rangle_2$, $\psi \in L^2(X, \mu)$ shows that the range of $\mathcal{K}$ is a subset of $C(X)$. In addition, if $\mathcal{K}$ is compact, the proof of Lemma 2.1 can be adapted to hold in this context. Also, $\mathcal{K}$ is trace-class if and only if $x \in X \to K(x, x)$ belongs to $L^1(X, \nu)$.

Remark 2.6 The arguments in the proof of Lemma 2.1 show that the condition $K \in \mathcal{A}(X, \nu)$ implies the smoothness of K, as described above.

The following result, another consequence of Theorem 2.2, gives some information on the decay rates for eigenvalues of the integral operator.

Theorem 2.7 *Under the conditions in Theorem 2.2, the following additional statements hold:*

(i) *There exist* $\alpha \in [0, 2\pi]$ *and* $l > 0$ *such that*

$$\sum_{n=1}^{\infty} |\lambda_n(\mathcal{K})| \le (1 + l^2)^{1/2} \int_X Re\,(\mathrm{e}^{i\alpha} K(x, x))\mathrm{d}\nu(x);$$

(ii) *The operator* $\mathcal{K}$ *is trace-class. In particular,* $\mathcal{K}$ *is an element of* $\cap_{p\ge 1}\mathcal{S}_p$*;*
(iii) *If the eigenvalues of* $\mathcal{K}$ *are arranged so that* $|\lambda_n(\mathcal{K})| \ge |\lambda_{n+1}(\mathcal{K})|$, $n = 1, 2, \ldots$, *then*

$$|\lambda_n(\mathcal{K})| \le \frac{(1 + l^2)^{1/2}}{n} \int_X Re\,(\mathrm{e}^{i\alpha} K(x, x))\mathrm{d}\nu(x), \quad n = 1, 2, \ldots.$$

(iv) *Under the conditions in* (iii), $\lambda_n(\mathcal{K}) = o(n^{-1})$ *as* $n \to \infty$.

Proof Choosing α and l as in the proof of Theorem 2.2 and integrating Formula (2.3), we have

$$\sum_{n=1}^{\infty} Re\,\alpha_n \le \int_Y L(x, x)\mathrm{d}\nu(x) = \int_X L(x, x)\mathrm{d}\nu(x), \quad n = 1, 2, \ldots.$$

Combining this with inequality (2.4), we conclude that

$$|\lambda_1(\mathcal{K})| + \cdots + |\lambda_n(\mathcal{K})| \le (1 + l^2)^{1/2}(Re\,\alpha_1 + \cdots + Re\,\alpha_n) \le (1 + l^2)^{1/2} \int_X L(x, x)\mathrm{d}\nu(x)$$

This implies the formula in (i). Statement (ii) is a consequence of the formula in (i). If an arrangement as described in (iii) holds, then the previous inequality can be refined to

$$n|\lambda_n(\mathcal{K})| \leq (1+l^2)^{1/2} \int_X L(x,x)\mathrm{d}\nu(x),$$

which implies the inequality in (iii). Statement (iv) follows from (i) and Lemma 5.1 ahead. □

Remark 2.8 Under the conditions in the previous theorem, we immediately have that $\mathcal{K}_1$ is trace-class and $\mathrm{tr}(\mathcal{K}_1) = \mathrm{tr}(\mathcal{K})$.

Remark 2.9 The representation (2.1) holds automatically when $\mathrm{e}^{i\alpha}\mathcal{K}$ is positive and compact for some $\alpha \in [0, 2\pi]$. Indeed, since positive operators on complex Hilbert spaces are self-adjoint, the representation follows from the usual spectral theorem.

3 Technical and Lipschitz conditions

This section is essentially preparatory for what comes in the forthcoming sections. Unless stated otherwise, (X, d) will be a metric space and ν a measure on X. The following definition is very close to another introduced in Ferreira and Menegatto [7] and settles the class of metric spaces we will be interested in. The symbol $B[y, r]$ will indicate the closed ball of radius r centered at y.

Let q be a positive integer and t a positive real. We call the triple (X, d, ν) a (q, t)-compact space when there exist $x_0 \in X$ and positive real numbers a, b, c, e, and r_0 fulfilling the following condition: if $N \in \mathbb{Z}_+$ and $r \geq r_0$ then there exists a family $\{C_n^r : n = 1, 2, \ldots, k(N)\}$ of subsets of X such that:

(i) $\nu(C_n^r \cap C_l^r) = 0, n \neq l$;
(ii) $d(x, y) \leq ar^t N^{-t}, x, y \in C_n^r$, and $\nu(C_n^r) \leq er^q N^{-q}, 1 \leq n \leq k(N)$;
(iii) $k(N) \leq bN^q$;
(iv) $B[x_0, r\,c] := \{x \in X : d(x, x_0) \leq r\,c\} = \cup_{n=1}^{k(N)} C_n^r$.

Remark 3.1 A comparison with the setting presented in Ferreira and Menegatto [7] reveals that the actual condition (ii) is more stringent. The reader is advised that this additional requirement will be not needed in all the arguments ahead (see Theorem 4.4 for instance). The alert reader may also observe that the definition can be put into the language of covering numbers of large balls in (X, d). The gain would be the reduction of the number of parameters and the drawback would be the use of a concept not too familiar to many readers. Examples fitting the description covered by the definition can be found in Ferreira and Menegatto [7].

Lemma 3.2 below describes a basic property of the context we are adopting.

Lemma 3.2 *Write X as a finite union of subspaces, say, $X = \cup_{j=1}^m X_j$. Let q be a positive integer and t a positive real number. Then, (X, d, ν) is (q, t)-compact if and only if every (X_j, d, ν) is (q, t)-compact.*

Proof Assume (X, d, ν) is (q, t)-compact and fix $j \in \{1, 2, \ldots, m\}$. Let $x_0 \in X$ and the constants a, b, c, e, and r_0 as described in the previous definition. Set

$$\alpha = \max\{d(x_0, X_j) : j = 1, 2, \ldots, m\}$$

and choose, as we can, $x_0^j \in B[x_0, \alpha + 1] \cap X_j$. Next, define $B_j[x_0^j, r] := B[x_0^j, r] \cap X_j$. If $r \geq \alpha + 1$ and $x \in B_j[x_0^j, r]$ then

$$d(x, x_0) \leq d(x, x_0^j) + d(x_0^j, x_0) \leq r + \alpha \leq 2r,$$

that is, $B_j[x_0^j, r] \subset B[x_0, 2r]$. If $r \geq \max\{r_0, \alpha+1\}$ then we may take the family of sets $\{C_n^{2r} : n = 1, 2, \ldots, k(N)\}$, $k(N) \leq bN^q$, from the (q,t)-compactness applied to (X, d, ν) and define

$$C_n^r(j) := C_n^{2r} \cap B_j[x_0^j, rc], \quad n = 1, 2, \ldots, k(N).$$

Clearly, $\nu(C_n^r(j) \cap C_i^r(j)) = 0$, $n \neq i$. Also

$$d(x, y) \leq a(2r)^t N^{-t} = (a2^t) r^t N^{-t}, \quad x, y \in C_n^r(j),\ n = 1, 2, \ldots, k(N),$$

and

$$\nu(C_n^r(j)) \leq \nu(C_n^{2r}) \leq (e2^q) r^q N^{-q}, \quad n = 1, 2, \ldots, k(N).$$

Finally, the equality

$$\cup_{n=1}^{k(N)} C_n^r(j) = \left(\cup_{n=1}^{k(N)} C_n^{2r}\right) \cap B_j[x_0^j, rc] = B[x_0, 2r\,c] \cap B_j[x_0^j, rc] = B_j[x_0^j, rc]$$

completes the proof of the (q,t)-compactness of (X_j, d, ν).

Conversely, assume each (X_j, d, ν) is (q,t)-compact. Let $x_0^j \in X_j$ and $a_j,\ b_j,\ c_j, e_j$, and r_0^j obtained from the (q,t)-compactness in each case. We will ratify the (q,t)-compactness of (X, d, ν) via the following constants: $a = \max\{a_j : j = 1, 2, \ldots, m\}$, $b = \max\{b_j : j = 1, 2, \ldots, m\}$, $c = \min\{c_j : j = 1, 2, \ldots, m\}$, and $e = \max\{e_j : j = 1, 2, \ldots, m\}$. To do that, first define $\beta = \max\{d(x_0^1, x_0^j) : j = 1, 2, \ldots, m\}$ and $r_0 = \max\{r_0^j : j = 1, 2, \ldots, m\}$. If both r and rc are at least $\beta + r_0$ and $x \in B[x_0^1, rc] \cap X_j$, for some j, then $d(x, x_0^j) \leq d(x, x_0^1) + d(x_0^1, x_0^j) \leq rc + \beta \leq 2rc$, that is, $x \in B_j[x_0^j, 2rc] \subset B_j[x_0^j, 2rc_j]$. Hence,

$$B[x_0^1, rc] \subset \cup_{j=1}^m B_j[x_0^j, 2rc] \subset \cup_{j=1}^m B_j[x_0^j, 2rc_j].$$

Fix a positive integer N and define $x_0 := x_0^1$. Keeping r subject to the choice above, select families $\{C_n^{2r}(j) : n = 1, 2, \ldots, k_j(N)\}$ fulfilling the requirements in the (q,t)-compactness of the $(X_j, d, , \nu)$. Define

$$\hat{C}_n^r(1) := C_n^{2r}(j) \cap B[x_0, rc], \quad n = 1, 2, \ldots, k_1(N),$$

and, inductively,

$$\hat{C}_n^r(j) := \left(C_n^{2r}(j) \cap B[x_0, rc]\right) \setminus \cup_{l=1}^{j-1} \cup_{i=1}^{k_l(N)} \hat{C}_i^r(l),$$

for $j = 2, 3, \ldots, m$ and $n = 1, 2, \ldots k_j(N)$. We now consider the family $\mathcal{F}$ of all the $\hat{C}_n^r(j)$ which are nonempty. Clearly, $\mathcal{F}$ can be indexed by a set of cardinality at most mbN^q. Any two distinct sets from $\mathcal{F}$ are disjoint. If $\hat{C}_n^r(j) \in \mathcal{F}$ then $d(x, y) \leq 2^t ar^t N^{-t}$, $x, y \in \hat{C}_n^r(j)$. Also, $\nu(\hat{C}_n^r(j)) \leq 2^q er^q N^{-q}$. Finally,

$$\begin{aligned}
\cup_{j=1}^m \cup_{n=1}^{k_j(N)} \hat{C}_n^r(j) &= \cup_{j=1}^m \cup_{n=1}^{k_j(N)} \left[(C_n^{2r}(j) \cap B[x_0, rc]) \setminus \cup_{l=1}^{j-1} \cup_{i=1}^{k_l(N)} \hat{C}_i^r(l)\right] \\
&= \cup_{j=1}^m \left[\cup_{n=1}^{k_j(N)} C_n^{2r}(j)\right] \cap B[x_0, rc] \\
&= \left(\cup_{j=1}^m B_j[x_0^j, 2rc_j]\right) \cap B\left[x_0^1, rc\right] = B\left[x_0^1, r\,c\right] = B[x_0, r\,c].
\end{aligned}$$

This shows (X, d, ν) is (q,t)-compact. □

Next, we introduce a Lipschitz condition we will adopt.

Definition 3.3 Let $\alpha > 0$ and $s \geq 0$ be constants. A kernel K belongs to the Lipschitz class $Lip^{\alpha,s}(X, \nu)$ when the following conditions hold:

(i) There exist $\delta > 0$ and a locally integrable function $A : X \to [0, +\infty]$ such that

$$|K(x, x) - K(x, y)| \leq A(x)d(x, y)^{\alpha}, \quad x, y \in X, \ d(x, y) < \delta;$$

(ii) There exists $B \geq 0$ such that

$$\limsup_{r\to\infty} r^{-s} \int\limits_{B[y,r]} A(x)\, \mathrm{d}\nu(x) < B, \quad y \in X.$$

Definitions very close to this one can be found in other sources in the literature (see Buescu and Paixão [2], and Ferreira and Menegatto [7] for example).

Next, we introduce an adapted to our purposes notion of uniform continuity [2]. Let V be a normed linear space. A hermitian kernel $G : X \times X \to V$ is said to be *uniformly continuous in the diagonal* of $X \times X$ if for every real number $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\|G(x, y) - G(x, x)\| < \epsilon$ whenever $x, y \in X$ and $d(x, y) < \delta$. Clearly, an uniformly continuous hermitian function is uniformly continuous in the diagonal. The previous definition implies that

$$\left\| \frac{G(x, x) + G(y, y)}{2} - G(y, x) \right\| < \epsilon,$$

as long as $x, y \in X$ and $d(x, y) < \delta$. With this in mind, we have the following definition.

Definition 3.4 Let $\alpha > 0$ and G a hermitian kernel on $X \times X$. We will say that G has the α-average property whenever we can find a constant $M > 0$ fulfilling the following requirement: for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$\left| \frac{G(x, x) + G(y, y)}{2} - G(y, x) \right| < Md(x, y)^{\alpha}\epsilon,$$

whenever $x, y \in X$ and $d(x, y) < \delta$.

Example 3.5 Let U be an open and connected subset of $\mathbb{R}^m$ and G a real hermitian kernel on $U \times U$ for which $\partial G/\partial x$ is uniformly continuous in the diagonal of $S \times S$, in which S is both a subset of U and a connected C^1 surface in $\mathbb{R}^m$ endowed with its geodesic distance d. If $\epsilon > 0$ has been fixed, there exists a $\delta = \delta(\epsilon) > 0$ such that

$$\left\| \frac{\partial G}{\partial x}(x, y) - \frac{\partial G}{\partial x}(x, x) \right\| < \epsilon, \quad x, y \in S, \ d(x, y) < \delta. \tag{3.1}$$

Since G is real and hermitian,

$$\frac{\partial G}{\partial y}(x, y) = \frac{\partial G}{\partial x}(y, x), \quad x, y \in U. \tag{3.2}$$

Next, we intend to apply the mean value inequality to the kernel $H : U \times U \to \mathbb{C}$ given by

$$H(x, y) = \frac{G(x, x) + G(y, y)}{2} - G(y, x), \quad x, y \in U.$$

Clearly, H is differentiable and $H(x, x) = 0$, $x \in U$. As so, we deduce that

$$|H(x, x) - H(x, y)| = \left| \frac{G(x, x) + G(y, y)}{2} - G(y, x) \right| \leq d(x, y) \sup_{z\in\overline{xy}} \|H'(x, z)\|,$$

for all $x, y \in S$, in which $\overline{xy}$ is a geodesic line connecting x and y and H' is the total derivative of H. Due to (3.2), it can be easily seen that

$$H'(x, y) = \left(\frac{\partial G}{\partial x}(x, x) - \frac{\partial G}{\partial x}(x, y)\right) + \left(\frac{\partial G}{\partial x}(y, y) - \frac{\partial G}{\partial x}(y, x)\right).$$

Recalling (3.1), we can see that $|H(x, x) - H(x, y)| \leq 2d(x, y)\epsilon$, when $x, y \in S$ and $d(x, y) < \delta$. In particular, the restriction of G to $S \times S$ has the 1-average property. The same conclusion can be reached for the restriction of G to a subset of $S \times S$. Also, if we assume that $S \cap U = \cup_{i=1}^{n} S_i$ (disjoint) then the same arguments may be used to conclude that the restriction of G to $S_i \times S_i$ has the 1-average property. These facts motivate the context described in Theorem 4.2 ahead.

A weak version of the previous definition is as follows.

Definition 3.6 Let $\alpha > 0$ and $s \geq 0$ be constants and G a kernel on $X \times X$. We will say that G has the *$Lip^{\alpha,s}(X, \nu)$-average property* when:

(i) There exist $\delta > 0$ and a locally integrable function $A : X \to [0, +\infty]$ such that

$$\left|\frac{G(x, x) + G(y, y)}{2} - G(y, x)\right| \leq A(x)d(x, y)^{\alpha}, \quad x, y \in X, \; d(x, y) < \delta;$$

(ii) There exists $B \geq 0$ such that

$$\limsup_{r \to \infty} r^{-s} \int_{B[y,r]} A(x)\, d\nu(x) < B, \quad y \in X.$$

The following result establishes two connection among the concepts introduced above. The proof is left to the readers.

Theorem 3.7 *Let (X, d, ν) be (q, t)-compact.*

(i) *If a kernel K has the α-average property for some $\alpha > 0$ then it has the $Lip^{\alpha,q}(X, \nu)$-average property.*
(ii) *If K is a hermitian element from $Lip^{\alpha,s}(X, \nu)$ and the associated function A is a constant then K has the $Lip^{\alpha,s}(X, \nu)$-average property.*

4 Estimates on the sums of the eigenvalues

The setting adopted in this section needs to be so that the main results listed in Sect. 2 hold. As so, we will assume (X, d) is a metric space endowed with a strictly positive measure ν and K is an element of $\mathcal{A}(X, \nu)$. However, the reader may decide either to use a different setting or to alter the one just described. The sequence of eigenvalues of $\mathcal{K}$ will be written as $\lambda_1(\mathcal{K}) \geq \lambda_2(\mathcal{K}) \geq \cdots$, taking into account multiplicities.

The following result is the key technical step in the deduction of all the results to come.

Theorem 4.1 *Let H be the kernel defined by the formula*

$$H(x, y) := \frac{K(x, x) + K(y, y)}{2} - K(y, x), \quad x, y \in X.$$

Then

$$\sum_{n=\Gamma+1}^{\infty} \lambda_n(\mathcal{K}) \leq \sum_{n=1}^{\Gamma} \frac{1}{\nu(C_n)} \int_{C_n}\int_{C_n} Re\, H(x,y)\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y) + \int_{X\setminus\cup_{n=1}^{\Gamma} C_n} K(x,x)\,\mathrm{d}\nu(x),$$

whenever $\{C_n : n = 1, 2, \ldots, \Gamma\}$ *is a family of measurable subsets of* X *such that* $0 < \nu(C_n) < \infty$, $n = 1, 2, \ldots, \Gamma$, *and* $\nu(C_n \cap C_l) = 0$, $n \neq l$.

Proof It can be adapted from the proof of Theorem 4.6 in Ferreira and Menegatto [7]. The crucial step in the proof is to observe that

$$\int_{C_n}\int_{C_n} H(y,x)\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y) = \int_{C_n}\int_{C_n} Re\, H(y,x)\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y)$$

under the conditions in the statement of the theorem. □

Theorem 4.2 *Assume* (X, d, ν) *is* (q, t)*-compact. In addition, assume* X *has a decomposition* $X = \cup_{j=1}^{m} X_j$ *so that the restriction of* $Re\, K$ *to every* $X_j \times X_j$ *has the* α*-average property, for some* $\alpha > 0$ *fixed. Further, assume there exist* $\beta > 0$ *and* $C \geq 0$ *such that*

$$\limsup_{r\to\infty} r^{\beta} \int_{X\setminus B[y,r]} K(x,x)\,\mathrm{d}\nu(x) \leq C, \quad y \in X.$$

Define $\gamma := \beta t\alpha(\beta + q + t\alpha)^{-1}$. *If* n *is large enough then there exist* $b > 0$, *a positive integer* $k(n) \in \{0, 1, \ldots, mbn^q\}$ *and a constant* $C_1 > 0$ *such that*

$$\sum_{j=k(n)+1}^{\infty} n^{\gamma} \lambda_j(\mathcal{K}) \leq C_1.$$

Proof Using Definition 3.4 with $\epsilon = 1$, we can select $\delta > 0$ and $M > 0$ so that

$$\left| Re \left[\frac{K(x,x) + K(y,y)}{2} - K(y,x) \right] \right| < M d(x,y)^{\alpha}, \quad x, y \in X_j,$$

for $j = 1, 2, \ldots, m$, as long as $d(x,y) \leq \delta$. Recalling the proof of Lemma 3.2, we can select $x_0^j \in X_j$ and positive real numbers a, b, c, e, and r_0 for which the following statement holds: if $N \in \mathbb{Z}_+$, $r \geq r_0$, and $j \in \{1, 2, \ldots, m\}$, there exists a family $\{C_n^r(j) : n = 1, 2, \ldots, k_j(N)\}$ of subsets of X_j such that

- $\nu(C_n^r(j) \cap C_l^r(j)) = 0$, $n \neq l$;
- $d(x,y) \leq ar^t N^{-t}$, $x, y \in C_n^r(j)$ and $\nu(C_n^r(j)) \leq er^q N^{-q}$, $n = 1, 2, \ldots, k_j(N)$;
- $k_j(N) \leq bN^q$;
- $B_j[x_0^j, r\,c] = \cup_{n=1}^{k_j(N)} C_n^r(j)$.

We intend to apply the preceding argument with a specific choice for r. Precisely, after N has been fixed, we intend to use the conclusion above with $r = r(N) := N^{t\alpha/(\beta+q+t\alpha)}$. This choice has the following features: $r(N) \to \infty$ and $N^{-1}r(N) \to 0$, as $N \to \infty$, while $r(N) > r_0$ and $aN^{-t}r(N)^t < \delta$, when N is large enough. Applying Theorem 4.1, we deduce that

$$\sum_{j=\Gamma+1}^{\infty}\lambda_j(\mathcal{K})=\sum_{j=1}^{m}\sum_{n=1}^{k_j(N)}\frac{1}{\nu(C_n^r(j))}\int_{C_n^r(j)}\int_{C_n^r(j)} Re\,H(x,y)\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y)$$
$$+\int_{X\setminus\cup_{j=1}^m B_j[x_0^j,rc]} K(x,x)\,\mathrm{d}\nu(x),$$

where $\Gamma := \sum_{j=1}^m k_j(N) \le mbN^q$ and

$$H(x,y) := \frac{K(x,x)+K(y,y)}{2} - K(y,x), \quad x, y \in X.$$

Note that we may assume $\nu(C_n^r(j)) > 0$ for all $n = 1, 2, \ldots, k_j(N)$ in the equality above. We bound the double sum, which we call S_1, as follows:

$$\begin{aligned}|S_1| &\le \sum_{j=1}^{m}\sum_{n=1}^{k_j(N)}\frac{1}{\nu(C_n^r(j))}\int_{C_n^r(j)}\int_{C_n^r(j)} Md(x,y)^\alpha\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y)\\ &\le \sum_{j=1}^{m}\sum_{n=1}^{k_j(N)}\frac{1}{\nu(C_n^r(j))}\int_{C_n^r(j)}\int_{C_n^r(j)} Ma^\alpha r^{t\alpha}N^{-t\alpha}\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y)\\ &\le \sum_{j=1}^{m}\sum_{n=1}^{k_j(N)} Ma^\alpha r^{t\alpha}N^{-t\alpha}\nu(C_n^r(j))\\ &\le mbeMa^\alpha r^{q+t\alpha}N^{-t\alpha}\end{aligned}$$

Using the limsup assumption to bound the remaining integral, we deduce that

$$\int_{X\setminus\cup_{j=1}^m B_j[x_0^j,rc]} K(x,x)\,\mathrm{d}\nu(x) \le \int_{X\setminus B[x_0^1,rc/2]} K(x,x)\,\mathrm{d}\nu(x) \le \frac{2^\beta Cc^{-\beta}}{r^\beta}.$$

Introducing our choice for r, we finally obtain

$$\begin{aligned}\sum_{j=k(N)+1}^{\infty}\lambda_j(\mathcal{K}) &\le mbea^\alpha M\frac{(N^{t\alpha/(\beta+q+t\alpha)})^{q+t\alpha}}{N^{t\alpha}} + \frac{2^\beta Cc^{-\beta}}{(N^{t\alpha/(\beta+q+t\alpha)})^\beta}\\ &= \frac{mbea^\alpha M}{N^{t\alpha\beta/(\beta+q+t\alpha)}} + \frac{2^\beta Cc^{-\beta}}{N^{t\alpha\beta/(\beta+q+t\alpha)}},\end{aligned}$$

as long as N is large enough. In particular,

$$\sum_{j=k(N)+1}^{\infty}\lambda_j(\mathcal{K}) \le \frac{C_1}{N^\gamma},$$

in which $C_1 = mbea^\alpha M + 2^\beta Cc^{-\beta}$ and $k(N) \le mbN^q$. □

The previous theorem becomes simpler when stronger conditions on either X or K are assumed. In that case, the limsup assumption is no longer needed.

Theorem 4.3 *Assume (X, d, ν) is (q, t)-compact. Further, assume X has a decomposition $X = \cup_{j=1}^m X_j$ so that the restriction of Re K to every $X_j \times X_j$ has the α-average property, for some $\alpha > 0$. If either X or the support of K is bounded then there exist a constant $b > 0$ and an integer $k(n) \in \{0, 1, \ldots, mbn^q\}$ such that*

$$\lim_{n\to\infty} \sum_{j=k(n)+1}^{\infty} n^{t\alpha} \lambda_j(\mathcal{K}) = 0.$$

Proof If the additional assumption on either X or K holds then there exists $r > 0$ such that

$$\int\limits_{X\setminus B[y,rc]} K(x, x)\, \mathrm{d}\nu(x) = 0.$$

Since K is nonnegative in the diagonal of X, the same equality holds if we increase r. To proceed, we repeat the arguments from the proof of Theorem 4.2, but applying the α-average property with a general ϵ. The estimation for S_1 becomes $|S_1| \le \epsilon mbeMa^\alpha r^{q+t\alpha} N^{-t\alpha}$. Adjusting r so that both the estimation on S_1 and the equality above hold for the same values of r, we can find a number $k(N)$ in $\{0, 1, \ldots, mbN^q\}$ such that

$$\sum_{j=k(N)+1}^{\infty} N^{t\alpha} \lambda_j(\mathcal{K}) \le mbea^\alpha M r^{q+t\alpha} \epsilon,$$

as long as N is large enough. □

The next result is a version of Theorem 4.2, replacing the α-average property assumption with a Lipschitz condition.

Theorem 4.4 *Assume (X, d, ν) is (q, t)-compact. In addition, assume X has a decomposition $X = \cup_{j=1}^m X_j$ so that the restriction of Re K to every $X_j \times X_j$ has the $Lip^{\alpha,s}(X_j, \nu)$-average property, for some $\alpha > 0$ and $s \ge 0$. Further, assume there exist $\beta > 0$ and $C \ge 0$ so that*

$$\limsup_{r\to\infty} r^\beta \int\limits_{X\setminus B[y,r]} K(x, x)\, \mathrm{d}\nu(x) \le C, \quad y \in X.$$

Define $\gamma := \beta t\alpha(\beta + s + t\alpha)^{-1}$. If n is large enough then there exist a constant $b > 0$, a positive integer $k(n) \in \{0, 1, \ldots, mbn^q\}$ and a constant $C_1 > 0$ such that

$$\sum_{j=k(n)+1}^{\infty} n^\gamma \lambda_j(\mathcal{K}) \le C_1.$$

Proof The proof requires an elementary adaptation of the proof of Theorem 4.2. The details will be omitted. □

Remark 4.5 The proof of Theorem 4.4 does not use the condition $\nu(C_n^r(j)) \le er^q N^{-q}$ provided by the definition of (q, t)-compactness. However, the ν-finiteness of such sets cannot be discarded.

Theorem 4.6 *Assume (X, d, ν) is (q, t)-compact. Further, assume X has a decomposition $X = \cup_{j=1}^m X_j$ so that the restriction of Re K to every $X_j \times X_j$ has the $Lip^{\alpha,s}(X_j, \nu)$-average property, for some $\alpha > 0$ and $s \ge 0$. If either X or the support of K is bounded and n is large* *enough then there exist positive constants b and C_1 and an integer $k(n) \in \{0, 1, \ldots, mbn^q\}$ such that*

$$\sum_{j=k(n)+1}^{\infty} n^{t\alpha} \lambda_j(\mathcal{K}) \leq C_1.$$

Proof It suffices to adapt the proof of Theorem 4.3. □

5 The main results

In this section, we use the results proved in Sect. 4 to express decay rates for the eigenvalues of the integral operator $\mathcal{K}$. As so, the basic assumptions on (X, d), ν and K continue here.

The results in this section depend upon the two technical known results listed below, updated versions of Lemma 6.1 in Ferreira et al. [6].

Lemma 5.1 *Let $\{a_n\}$ be a nonincreasing sequence of nonnegative real numbers. Let l, q, and n_0 be nonnegative integers, $\overline{p}$ a positive integer at least 1 and $\gamma \in \mathbb{R}$. Suppose there exists a constant $C > 0$ satisfying the following property: if $n \geq n_0$, there exists $k(n) \leq \overline{p}n^q$ such that*

$$\sum_{j=k(n)+l+1}^{\infty} n^{\gamma} a_j \leq C.$$

Then, the set $\{n^{1+\gamma/q} a_n : n = 1, 2, \ldots\}$ is bounded. In particular, we conclude that $a_n = O(n^{-1-\gamma/q})$ as $n \to \infty$.

Lemma 5.2 *Let $\{a_n\}$ be a nonincreasing sequence of nonnegative real numbers. If there exist positive integers $\overline{p}$, l, and q, and a real number γ such that*

$$\lim_{n\to\infty} \sum_{j=k(n)+l+1}^{\infty} n^{\gamma} a_j = 0,$$

for some $k(n) \in \{0, 1, \ldots, \overline{p}n^q\}$, then $a_n = o(n^{-1-\gamma/q})$ as $n \to \infty$.

As far as we can see, the results we obtain below can be interpreted as generalizations of others found in Buescu and Paixão [2], and Ferreira and Menegatto [7]. The setting in which they are presented has two virtues: it is more general and the assumptions used provide other paths one could follow.

Theorem 5.3 *Assume (X, d, ν) is (q, t)-compact. In addition, assume X has a decomposition $X = \cup_{j=1}^{m} X_j$ so that the restriction of Re K to every $X_j \times X_j$ has the $Lip^{\alpha,s}(X_j, \nu)$-average property, for some α and s. Further, assume there exist $\beta > 0$ and $C \geq 0$ so that*

$$\limsup_{r\to\infty} r^{\beta} \int_{X\setminus B[y,r]} K(x, x)\, \mathrm{d}\nu(x) \leq C, \quad y \in X. \tag{5.1}$$

Define $\gamma := \beta t\alpha(\beta + s + t\alpha)^{-1}$. Then, $\lambda_n(\mathcal{K}) = O(n^{-1-\gamma/q})$ as $n \to \infty$ and $\mathcal{K} \in \mathcal{S}_p$, whenever $p > (1 + \gamma/q)^{-1}$.

Proof The first assertion follows from Theorem 4.4 and Lemma 5.1. Now, if $C > 0$ is a constant such that $\lambda_n(\mathcal{K}) \le Cn^{-1-\gamma/q}$ then

$$\sum_{n=1}^{\infty}(\lambda_n(\mathcal{K}))^p \le C^p \sum_{n=1}^{\infty} n^{-(1+\gamma/q)p} < \infty,$$

as long as $p > (1+\gamma/q)^{-1}$. □

Remark 5.4 If $X = [-1, 1]$ is endowed with the usual Lebesgue measure, it is easy to see that

$$K(x, y) := \sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)} \cos(n\pi x)\cos(n\pi y), \quad x, y \in X,$$

generates an operator $\mathcal{K}$ in $\mathcal{S}_p$, $p > 1$, but not in $\mathcal{S}_1$. Also, $\lambda_n(\mathcal{K}) = o(n^{-1})$ as $n \to \infty$. This implies that both conditions $\lambda_n(\mathcal{K}) = o(n^{-p})$ as $n \to \infty$ and $\lambda_n(\mathcal{K}) = O(n^{-p})$ as $n \to \infty$ are weaker than the condition $\mathcal{K} \in \mathcal{S}_p$. Conditions to place an integral operator $\mathcal{K}$ in $\mathcal{S}_1$ were discussed in Ferreira et al. [8] in the case when X is a subset of the Euclidian space.

Theorem 5.5 *Assume (X, d, ν) is (q, t)-compact. Further, assume X has a decomposition $X = \cup_{j=1}^m X_j$ so that the restriction of Re K to every $X_j \times X_j$ has the $Lip^{\alpha,s}(X_j, \nu)$-average property, for some α and s. If for each $\beta > 0$, there exists $C = C(\beta) \ge 0$ such that*

$$\limsup_{r\to\infty} r^{\beta} \int_{X\setminus B[y,r]} K(x, x)\, d\nu(x) \le C, \quad y \in X, \tag{5.2}$$

then $\lambda_n(\mathcal{K}) = o(n^{-1-\theta/q})$, $\theta \in [0, t\alpha)$, as $n \to \infty$ and $\mathcal{K} \in \mathcal{S}_p$, whenever $p > (1+t\alpha/q)^{-1}$.

Proof The function

$$\psi(\beta) := \frac{t\alpha\beta}{\beta + s + t\alpha}, \quad \beta \in [0, \infty),$$

is continuous with range $[0, t\alpha)$. As so, Theorem 5.3 implies that $\lambda_n(\mathcal{K}) = O(n^{-1-\theta/q})$ as $n \to \infty$, whenever $\theta \in [0, t\alpha)$. If $\lambda_n(\mathcal{K}) \ne o(n^{-1-\gamma_0/q})$ as $n \to \infty$, for some $\gamma_0 \in [0, t\alpha)$, then there would exist $C > 0$ such that $\limsup_{n\to\infty} n^{-1-\gamma_0/q}\lambda_n(\mathcal{K}) \ge C$. But this would lead to a unbounded sequence $\{n^{1+\theta/q}\lambda_n(\mathcal{K})\}$ when $\theta \in (\gamma_0, t\alpha)$, a contradiction. □

Theorem 4.6 and Lemma 5.1 yield the following result.

Theorem 5.6 *Assume (X, d, ν) is (q, t)-compact. Further, assume X has a decomposition $X = \cup_{j=1}^m X_j$ so that the restriction of Re K to every $X_j \times X_j$ has the $Lip^{\alpha,s}(X_j, \nu)$-average property, for some α and s. If either X or the support of K is bounded then $\lambda_n(\mathcal{K}) = O(n^{-1-t\alpha/q})$ as $n \to \infty$ and $\mathcal{K} \in \mathcal{S}_p$, whenever $p > (1+t\alpha/q)^{-1}$.*

Remark 5.7 The previous three theorems still hold when the $Lip^{\alpha,s}(X_j, \nu)$-average property on K is replaced with $Lip^{\alpha,s}(X_j, \nu)$. Details are left to the reader and a help of results described in Ferreira and Menegatto [7] is needed.

Recalling Theorem 3.7, we point that the previous theorems yield estimates on the eigenvalues for integral operators generated by kernels in $\mathcal{A}(X, \nu)$ when they possess the α-average property. The next results provides sharper estimates in this very same setting. As a matter of fact, they were the real motivation for the introduction of the α-average property the way we did.

Theorem 5.8 *Assume (X, d, ν) is (q, t)-compact. Assume X has a decomposition $X = \cup_{j=1}^m X_j$ so that the restriction of Re K to every $X_j \times X_j$ has the α-average property, for some $\alpha > 0$. If*

$$\lim_{r\to\infty} r^\beta \int\limits_{X\setminus B[y,r]} K(x,x)\,\mathrm{d}\nu(x) = 0, \quad \beta > 0, \ y \in X, \tag{5.3}$$

then $\lambda_n(\mathcal{K}) = o(n^{-1-\gamma/q})$ as $n \to \infty$, in which $\gamma := \beta t\alpha(\beta + q + t\alpha)^{-1}$.

Proof The proof of Theorem 4.2 implies that for every $\epsilon > 0$ there exist $N(\epsilon)$ and $k(N) \leq mbN^q$ so that

$$\sum_{j=k(N)+1}^{\infty} N^\gamma \lambda_j(\mathcal{K}) \leq mbea^\alpha M\epsilon + \epsilon, \quad N \geq N(\epsilon), \tag{5.4}$$

where the constants a, b, and e comes from the (q, t)-compactness. Hence, the result follows from Lemma 5.2. □

The use of Theorem 4.3 and Lemma 5.2 produces the following consequence of the previous result.

Theorem 5.9 *Assume (X, d, ν) is (q, t)-compact. Further, assume X has a decomposition $X = \cup_{j=1}^m X_j$ so that the restriction of Re K to every $X_j \times X_j$ has the α-average property, for some $\alpha > 0$. If either X or the support of K is bounded then $\lambda_n(\mathcal{K}) = o(n^{-1-t\alpha/q})$ as $n \to \infty$.*

Depending on the context, the limsup condition (5.1) can be obtained from the existence of a constant $\beta > m$ such that

$$\limsup_{\substack{|x|\to\infty \\ x\in X}} |x|^\beta K(x,x) < \infty.$$

Similarly, if the same limsup is zero then (5.3) holds. Also, if this inequality holds for all $\beta > m$ then (5.2) holds.

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