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Schwarz symmetrization and comparison results for nonlinear elliptic equations and eigenvalue problems

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Abstract We compare the distribution function and the maximum of solutions of nonlinear elliptic equations defined in general domains with solutions of similar problems defined in a ball using Schwarz symmetrization. As an application, we prove the existence and bound of solutions for some nonlinear equation. Moreover, for some nonlinear problems, we show that if the first p-eigenvalue of a domain is big, the supremum of a solution related to this domain is close to zero. For that we obtain L^∞ estimates for solutions of nonlinear and eigenvalue problems in terms of other L^p norms.

Keywords Schwarz symmetrization · Distribution function · Nonlinear elliptic problem · Eigenvalue problem · Optimal estimates · Degenerate elliptic equation

Mathematics Subject Classification (2000) 35J60 · 35J70 · 35J92

1 Introduction

In this work, we study the L^p-norm and the distribution function of solutions to the Dirichlet problem

$$\begin{cases} -\operatorname{div}(a(u, \nabla u)) = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open-bounded set in $\mathbb{R}^n$, $f : \mathbb{R} \to \mathbb{R}$ and $a : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy some suitable conditions. First we assume the following hypotheses:

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(H1) f is a nonnegative locally Lipschitz function;
(H2) f is nondecreasing;
(H3) $a \in C^0(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n) \cap C^1(\mathbb{R} \times (\mathbb{R}^n \backslash \{0\}); \mathbb{R}^n)$ is given by $a(t, z) = e(t, |z|)z$, where $e \in C^1(\mathbb{R} \times (\mathbb{R}\backslash\{0\}))$ is positive on $\mathbb{R} \times \mathbb{R}\backslash\{0\}$, $a(t, 0) = 0$, $a(t, z) \cdot z$ is convex in the variable $z \in \mathbb{R}^n$ and $\partial_s (|a(t, sz)|) > 0$ for $z \neq 0$ and $s > 0$.
(H4) There exist $p \geq q > 1$, $q_0 > 1$, and positive constants C_s, C_* and C^* s.t.

$$C_s|z|^{q_0} \leq \langle a(t, z), z\rangle \quad \text{for} \quad |z| \leq 1,\ t \in \mathbb{R}$$

and

$$C_*|z|^q \leq \langle a(t, z), z\rangle \leq C^*(|z|^p + |t|^p + 1) \quad \text{for} \quad |z| \geq 1,\ t \in \mathbb{R}.$$

Hence, using that $s \mapsto a(t, sz) \cdot sz$ is increasing and positive,

$$C_*(|z|^q - 1) \leq a(t, z) \cdot z \leq C^*(|z|^p + |t|^p + 1) \quad \text{for} \quad z \in \mathbb{R}^n$$

and

$$C_*(\lambda_B\|w\|_q^q - |\Omega|) \leq \int_\Omega a(w, \nabla w) \cdot \nabla w \, dx \leq C^*(\|\nabla w\|_p^p + \|w\|_p^p + |\Omega|), \quad (1.1)$$

for $w \in W_0^{1,p}(\Omega)$, where λ_B is the first eigenvalue of $-\Delta_q$ in a ball B, which has the same measure as Ω.
(H5) There exist $\beta \geq 0$ and $\alpha < C_*\lambda_B$ such that

$$0 < f(t) \leq \alpha t^{q-1} + \beta \quad \text{for} \quad t > 0.$$

(H6) $|a(t_2, z) - a(t_1, z)| \leq \omega(|t_2 - t_1|)(1 + |z|^{p-1})$ for $t_1, t_2 \in \mathbb{R}$ and $z \in \mathbb{R}^n$, where ω is some nondecreasing modulus of continuity.

At first, our main concern is to compare the maximum and the distribution function of a solution associated to Ω with one associated to B. We can obtain even a priori estimates of solutions for some problems with nonlinear lower order terms and prove the existence of solution. Later on we see also some applications for these estimates, including L^∞ estimates for some eigenvalue and nonlinear problems. So we show that if a domain is "far away" from the ball (i.e., its first p-eigenvalue is big), then the maximum of a solution is small. Indeed, the supremum of a solution is bounded by some negative power of the first p-eigenvalue. This kind of question seems to be new, and the works in the literature normally are focused in comparing solutions with a radial one, disregarding better estimates when the domain is not close to a ball.

More precisely, let B be the open ball in $\mathbb{R}^n$, centered at the origin, such that $|B| = |\Omega|$, where $|C|$ denotes the Lebesgue's measure in $\mathbb{R}^n$ of a measurable set C, and consider the function U_B given by

$$U_B(x) = \sup\{U(x) \mid U \in W_0^{1,p}(B) \text{ is a radial solution of } (\tilde{\mathrm{P}}_B)\}, \quad (1.2)$$

where $(\tilde{\mathrm{P}}_B)$ is the Dirichlet problem

$$\begin{cases} -\mathrm{div}(\tilde{a}(U, \nabla U)) = f(U) & \text{in } B \\ U = 0 & \text{on } \partial B. \end{cases} \quad (\tilde{\mathrm{P}}_B)$$

Let u be a weak solution of

$$\begin{cases} -\mathrm{div}(a(v, \nabla v)) = f(v) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (\mathrm{P}_\Omega)$$

in $W_0^{1,p}(\Omega)$. Observe that u and U_B are nonnegative. Define the distribution function of u by

$$\mu_u(t) = |\{x \in \Omega : u(x) > t\}|.$$

For a, $\tilde{a}$, and f satisfying hypotheses (H1)–(H5) (the constants and powers related to a and $\tilde{a}$ can be different), a or $\tilde{a}$ satisfying (H6), and $\tilde{a}(t, \mathrm{z}) \cdot \mathrm{z} \le a(t, \mathrm{z}) \cdot \mathrm{z}$, we prove that U_B is a solution of ($\tilde{\mathrm{P}}_B$) and, in Theorem 5.6,

$$\mu_u(t) \le \mu_B(t), \quad \forall t \in [0, \max U_B], \tag{1.3}$$

where μ_B is the distribution function of U_B. If Ω is not a ball, $a = a(\mathrm{z})$ and $(a(\mathrm{z}) \cdot \mathrm{z})^{1/r}$ is convex for some $r > 1$, then this inequality is strict.

We also prove some sort of maximum principle with respect to the solutions in the ball in the following sense: if u and U are solutions of (P_Ω) and ($\tilde{\mathrm{P}}_B$), respectively, $u^\sharp \le U$ (not necessarily maximal solution) and $u^\sharp \ne U$, then $u^\sharp < U$ provided f and a satisfy suitable conditions.

These estimates can be applied, for example, to the following problems:

(1) $u \in W_0^{1,p}(\Omega)$ is a weak solution of $-c_1\Delta_p v - c_2\Delta_q v = f(v)$ and $U_B \in W_0^{1,p}(B)$ is the radially symmetric solution of $-d_1\Delta_p V - d_2\Delta_q V = f(V)$, as defined in (1.2), where $c_1 \ge d_1 > 0$, $c_2 \ge d_2 > 0$, and $p \ge q > 1$. The operator $-c_1\Delta_p - c_2\Delta_q$ appears in some general reaction diffusion equations, with applications in physics, biophysics, and chemistry.

(2) $u \in W_0^{1,p}(\Omega)$ and $U_B \in W_0^{1,p}(B)$, $p \ge 18/17$, are solutions of

$$-\mathrm{div}\left(\frac{\nabla v}{(1 + |\nabla v|^2)^{\frac{2-p}{2}}}\right) = f(v).$$

Such restriction on p is due to the convexity requirement on $a(\mathrm{z}) \cdot \mathrm{z} = \mathrm{z}^2(1 + |\mathrm{z}|^2)^{\frac{p-2}{2}}$. The operator on the left-hand side arises in the cracking of plates and the modeling of blast furnaces (see [15,30]).

These comparisons results can be extended to the problem with lower order terms

$$\begin{cases} -\mathrm{div}(a(\nabla u)) - \dfrac{h'(u)}{h(u)} \nabla u \cdot a(\nabla u) = g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $h \in C^1$ is bigger than some positive constant, $f = gh$ and $a_1(t, \mathrm{z}) = h(t)a(\mathrm{z})$ satisfy (H1)–(H5). This holds even if h has a bad growth and a_1 does not satisfy the upper inequality of (1.1). For the special case

$$-\Delta_p - \frac{h'(u)}{h(u)} |\nabla u|^p = g(u)$$

this priori estimate can be used to prove existence of solution.

We get also some result for (P_Ω) even when f is not nondecreasing. Indeed, if f is positive, $f(t)/t^{p-1}$ is decreasing and $a(t, \mathrm{z}) = \tilde{a}(t, \mathrm{z}) = \mathrm{z}|\mathrm{z}|^{p-1}$, we show that

$$\max U_B \ge \max u.$$

This L^∞ estimate can be easily extended to the problem

$$\begin{cases} -\Delta_p v + k(v) = f(v) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where k is positive and nondecreasing, f is positive, and $f(t)/t^{p-1}$ is decreasing. Then we apply these results to prove that if $w \in W_0^{1,p}(\Omega)$ is a solution of $\operatorname{div}(a(x, \nabla w)) = f(w)$ in Ω, where a satisfies some conditions and $f \in C^1(\mathbb{R})$ is bounded by $c|t|^{q-1} + d$, with $1 < q \le p$ and $c, d \ge 0$, then

$$\|w\|_\infty \le C_1 \|w\|_r^{\frac{rp}{n(p-q)+rp}} + C_2 \|w\|_r^{\frac{rp}{n(p-1)+rp}},$$

where $C_1 = C_1(n, p, q, r, \rho, c)$ and $C_2 = C_2(n, p, r, \rho, d)$ are positive constants. In the special case $|\Delta_p w| \le |\lambda||w|^{q-1}$, where $\lambda \in \mathbb{R}$, we have

$$\|w\|_s \le \left[\frac{2}{(\omega_n)^{1/r}} \left(\frac{2(p-1)}{p} \right)^{\frac{n(p-1)}{rp}} \left(\frac{|\lambda|}{n} \right)^{n/rp} \right]^{\frac{s-r}{\kappa s}} \|w\|_r^{\frac{s-r}{\kappa s} + \frac{r}{s}}, \tag{1.6}$$

where $0 < r < s$ and $\kappa = 1 + \frac{n(p-q)}{rp}$. These inequalities imply, according to Corollary 7.4, in a L^∞-norm decay of the solutions of some sublinear equations, when the domain becomes "far away" from a ball with the same volume. Since the ball is the domain of a given measure that maximizes the L^p norms in several problems, it would be interesting to obtain better estimates for solutions that are not defined in a ball. Hence, we need to measure in some way the difference between its domain and the corresponding ball. The first eigenvalue is a possible form of distinction between these sets, which we use to establish some upper bound. Finally, as an application, we prove that $u^\sharp < U$, where u is a solution of (P_Ω) and U a solution of $(\tilde{\mathrm{P}}_B)$, even when f is not monotone, provided the first eigenvalue associated to Ω, $\lambda_p(\Omega)$, is big enough and some conditions on a and f are satisfied.

We point out that we are not interested in establishing existence of solutions for (P_Ω). Our main concern is just to compare these solutions, and we obtain existence results only for the radial case.

Results of this type have been obtained by several authors. In [44], Talenti proved that if u is the weak solution of the Dirichlet problem

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x)u = f(x) \ \text{ in } \ \Omega \quad \text{and} \quad u = 0 \ \text{ on } \ \partial\Omega,$$

where $c(x) \ge 0$, $\sum_{ij} a_{ij}(x)\xi_i\xi_j \ge \xi_1^2 + \cdots + \xi_n^2$ and v is the weak solution of

$$-\Delta v = f^\sharp \ \text{ in } \ B \quad \text{and} \quad v = 0 \ \text{ on } \ \partial B,$$

where B is the ball centered at 0 such that $|B| = |\Omega|$ and $f^\sharp$ is the decreasing spherical rearrangement of f, then $\operatorname{ess\,sup} u \le \operatorname{ess\,sup} v$ and $\mu_u \le \mu_v$. As a consequence, $\|v\|_{L^p}/\|f^\sharp\|_{L^q} \ge \|u\|_{L^p}/\|f\|_{L^q}$. This estimate is an extension of the one previously obtained by Weinberger [49] for the ratio $\|u\|_{L^\infty}/\|f\|_{L^q}$. Further results have been proved for a larger class of linear equations that either satisfy weaker ellipticity conditions (see [7,8]) or contain lower order terms (see [3,5,6,9,19,28,47,48]). Similar problems were studied in [36–38].

As in the linear case, estimates have been obtained for solutions $u \in W_0^{1,p}(\Omega)$ to the nonlinear problem

$$-\sum_{i=1}^{n}(a_i(x,u,\nabla u))_{x_i} - \sum_{i=1}^{n}(b_i(x)|u|^{p-2}u)_{x_i} + h(x,u) = f(x,u) \text{ in}\Omega,$$

comparing the decreasing spherical rearrangement of u with the solution of some nonlinear "symmetrized" problem. For instance, the case $b_i = h = 0$ and $\sum a_i(x,u,\xi)\xi_i \geq A(|\xi|)$, where A is convex and $\lim_{r\to 0} A(r)/r = 0$, is considered in [45]. The problem in a general form is studied in [14], assuming that the coefficients are in suitable spaces and $\sum a_i(x,u,\xi)\xi_i \geq |\xi|^p$. Under similar hypotheses, the case $b_i = 0$ is considered in [26], and different comparison results are obtained. In [1], estimates are proved when the coefficients satisfy $b_i = h = 0, a_i = a_i(Du)$ and $\sum a_i(\xi)\xi_i \geq (H(\xi))^2$, where H is a nonnegative convex function, positively homogeneous of degree 1. Other related results were established in [23,27,41]. Some results also extend to parabolic equations (see e.g., [1,5,10,12]), and some isoperimetric estimates are obtained for the Monge-Ampère equation (see [16,46]).

Usually comparison results are obtained considering a "symmetrized equation" that is different from the original one. In this work, we can keep the original equation and symmetrize only the domain, obtaining sharper estimates. Results similar to ours are established in [11,39] for the laplacian operator, where the authors apply the method of subsolution and supersolution to prove that, for a given symmetric solution U in the ball, there exists some solution in Ω for which the symmetrization is less than U. Indeed, applying the iteration procedure used in those works and the main result of [45], the estimate (1.3) can be obtained in the particular case $-\text{div}(a(\nabla u)) = f(u)$, provided we have some a priori estimate in the L_q norm for subsolutions and the existence of the maximal radial solution U_B. Using different techniques, we prove in Sect. 5 that the symmetrization of any solution of (P_Ω) is bounded by U_B, even in the case $a = a(t,z)$ and $\tilde{a} = \tilde{a}(t,z)$, as long as these functions satisfy (H1)–(H5) and one of them satisfies (H6). In Sect. 2, we review some important concepts and results. Some estimates in this section are interesting by itself. In Sect. 3, we get estimates assuming that $a(\text{z}) = \tilde{a}(\text{z}) = |\text{z}|^{p-2}\text{z}$ and $f(t)/t^{p-1}$ is decreasing. Indeed, we prove that $\max U_B \geq \max u$ even when f is not nondecreasing. Observe that the uniqueness of solution to the problems (P_Ω) and ($\tilde{P}_B$) is proved in [17] for the Laplacian operator when $f(t)/t$ is decreasing. An extension of this is proved to the p-Laplacian in [13]. Hence, some results in this section can be obtained directly from the existence of a solution associated to B that is greater than some solution associated to Ω. In Sect. 4, we study the behavior of solutions in the radial case. In Sect. 6, we obtain a bound to solutions of (1.4), and in some special case, we use this comparison to show the existence of solution. In Sect. 7, we get some inequalities between the L^p norms of solutions of some "eigenvalue problems" and some lower bound for the distribution function of these solutions. For eigenvalue problems, the L^p estimates are established in [2,20], and [21], where the authors obtain sharper estimates, since the constants are optimal. We are not concerned with the best constant but only with the relations between the L^p norms and the real parameter λ. We get an explicit relation for a larger class of equations, and, for the typical eigenvalue problem, the estimate hold not only for the first eigenvalue of the operator but also for the others. Other authors make some similar estimates on manifolds (see e.g., [31,33]) for the classical eigenvalue problem, but the constant depends on the manifold and the boundary. It is also established some L^p estimates for a class of Dirichlet problems and a relation between the norms and the first eigenvalue of the domain.

2 Preliminary results

In this section, we recall some important definitions and useful results. First, if Ω is an open-bounded set in $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ is a measurable function, the distribution function of u is given by

$$\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}| \quad \text{for } t \geq 0.$$

The function μ_u is nonincreasing and right continuous. The decreasing rearrangement of u, also called the *generalized inverse* of μ_u, is defined by

$$u^*(s) = \sup\{t \geq 0 : \mu_u(t) \geq s\}.$$

If $\Omega^\sharp$ is the open ball in $\mathbb{R}^n$, centered at 0, with the same measure as Ω and ω_n is the measure of the unit ball in $\mathbb{R}^n$, the function

$$u^\sharp(x) = u^*(\omega_n |x|^n) \quad \text{for} \quad x \in \Omega^\sharp$$

is the spherically symmetric decreasing rearrangement of u. It is also called the Schwarz symmetrization of u. For an exhaustive treatment of rearrangements, we refer to [4,11,22,32,35, 42]. The next remark reviews important properties of rearrangements and will be necessary through this work.

Remark 2.1 Let v, w be integrable functions in Ω and let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing nonnegative function. Then

$$\int_{\Omega} g(|v(x)|)\,\mathrm{d}x = \int_0^{|\Omega|} g(v^*(s))\,\mathrm{d}s = \int_{\Omega^\sharp} g(v^\sharp(x))\,\mathrm{d}x.$$

Hence, if $\mu_v(t) \geq \mu_w(t)$ for all $t > t_1 > 0$, it follows that

$$\int_{t_1 < v} g(v(x))\,\mathrm{d}x = \int_0^{\mu_u(t_1)} g(v^*(s))\,\mathrm{d}s \geq \int_0^{\mu_w(t_1)} g(w^*(s))\,\mathrm{d}s = \int_{t_1 < w} g(w(x))\,\mathrm{d}x,$$

since $v^*(s) \geq w^*(s)$ for $s \leq \mu_w(t_1)$. Moreover, if $|\{v > t_2\}| \leq |\{w > t_2\}| < \infty$, $|\{v > t_1\}| = |\{w > t_1\}| < \infty$ and $|\{v > t\}| \geq |\{w > t\}|$ for all $t_1 < t < t_2$, then

$$\int_{t_1 < v \leq t_2} g(v(x))\,\mathrm{d}x \;\geq\; \int_{t_1 < w \leq t_2} g(w(x))\,\mathrm{d}x.$$

Finally, an extension of the Pólya-Szegö inequality (see [11,18,35,43]) states that, if $B : [0, \infty) \to [0, \infty)$ is increasing and convex, then

$$\int_{\Omega} B(|\nabla v(x)|)\,\mathrm{d}x \;\geq\; \int_{\Omega^\sharp} B(|\nabla v^\sharp(x)|)\,\mathrm{d}x \quad \text{for} \quad v \geq 0 \;\text{ in } W_0^{1,p}(\Omega).$$

This inequality also holds if we consider $\{t_1 < v < t_2\}$ and $\{t_1 < v^\sharp < t_2\}$ instead of Ω and $\Omega^\sharp$. Indeed, from the coarea formula,

$$\int_{\{v=t\}} \frac{B(|\nabla v(x)|)}{|\nabla v(x)|}\,\mathrm{d}H^{n-1} \geq \int_{\{v^\sharp=t\}} \frac{B(|\nabla v^\sharp(x)|)}{|\nabla v^\sharp(x)|}\,\mathrm{d}H^{n-1}$$

for almost every t, where H^{n-1} is the $(n-1)$-dimensional Hausdorff measure.

Remark 2.2 For any bounded open set Ω' satisfying $|\Omega'| \le |\Omega|$, there exists a constant $C = C(n, q, \alpha, \beta, C_*, |\Omega|, |\Omega'|)$ such that $\sup u \le C$ for any weak solution $u \in W_0^{1,p}(\Omega')$ of $(\mathrm{P}_{\Omega'})$. Moreover, $C = O(|\Omega'|^{\rho})$ as $|\Omega'| \to 0$, where $\rho > 0$ depends only on n and q. This result is a consequence of the following two lemmas.

Lemma 2.3 *Let Ω' be a bounded open set s.t. $|\Omega'| \le |\Omega|$. If $u \in W_0^{1,p}(\Omega')$ is a nonnegative subsolution of $(\mathrm{P}_{\Omega'})$ and conditions (H1),(H5), $C_*(|\mathrm{z}|^q - 1) \le \langle a(t, \mathrm{z}), \mathrm{z}\rangle$ for $\mathrm{z} \in \mathbb{R}^n$, $t \in \mathbb{R}$ are satisfied, then*

$$\|u\|_{L^q} \le M(\Omega') := \left(\frac{2C_*|\Omega'|}{C_*\lambda_{B'} - \alpha}\right)^{1/q} + \left(\frac{2\beta|\Omega'|^{1/q'}}{C_*\lambda_{B'} - \alpha}\right)^{1/(q-1)},$$

where $1/q' + 1/q = 1$, B' is a ball that satisfies $|B'| = |\Omega'|$ and $\lambda_{B'}$ is the first eigenvalue of $-\Delta_q$ in B'.

Proof Multiplying the equation by u and integrating, we get

$$\int_{\Omega'} \nabla u \cdot a(u, \nabla u)\, \mathrm{d}x \le \int_{\Omega'} u f(u)\, \mathrm{d}x \le \alpha\|u\|_q^q + \beta\|u\|_q|\Omega'|^{1/q'}.$$

Since $C_*(|\mathrm{z}|^q - 1) \le \langle a(t, \mathrm{z}), \mathrm{z}\rangle$, the first inequality of (1.1) holds. Hence

$$\|u\|_q\left[(C_*\lambda_{B'} - \alpha)\|u\|_q^{q-1} - \beta|\Omega'|^{1/q'}\right] \le C_*|\Omega'|.$$

Studying the cases $(C_*\lambda_{B'} - \alpha)\|u\|_q^{q-1} - \beta|\Omega'|^{1/q'} \le (C_*\lambda_{B'} - \alpha)\|u\|_q^{q-1}/2$ and $> (C_*\lambda_{B'} - \alpha)\|u\|_q^{q-1}/2$ individually, we get the result. □

Next lemma is a particular result of Theorem 3.11 of [40] in the case $n \ge q$. For $n < q$, the estimate can be obtained following the computations of that theorem and Morrey's inequality. A sketch of the proof is done in the "Appendix".

Lemma 2.4 *Suppose that u satisfies the hypotheses of the preceding lemma. If $n < q$, then*

$$\sup_{\Omega'} u \le C\|u\|_q + D|\Omega'|^{1/q},$$

where $C = C(n, q, \alpha, \beta, C_*)$ and $D = D(n, q, \alpha, \beta, C_*)$.
If $n \ge q$, then

$$\sup_{\Omega'} u \le C(|\Omega'|^{1/n} + 1)^{\rho}\left(\frac{\|u\|_q}{|\Omega'|^{1/q}} + |\Omega'|^{1/n}\right),$$

where $\rho = n/q$ and $C = C(n, q, \alpha, \beta, C_*)$ if $n > q$, and $\rho = \frac{\tilde{q}}{2\tilde{q}-n}$, $\tilde{q} \in (n/2, n)$, and $C = C(n, \alpha, \beta, C_*, \tilde{q})$ if $n = q$.

From these two lemmas we get, for $n < q$, that

$$\sup_{\Omega'} u \le CM(\Omega') + D|\Omega'|^{1/q}, \tag{2.1}$$

where $C = C(n, q, \alpha, \beta, C_*)$ and $D = D(n, q, \alpha, \beta, C_*)$. For $n \ge q$, it follows that

$$\sup_{\Omega'} u \le C(|\Omega'|^{1/n} + 1)^{\rho}\left(\frac{M(\Omega')}{|\Omega'|^{1/q}} + |\Omega'|^{1/n}\right), \tag{2.2}$$

where $C = C(n, q, \alpha, \beta, C_*)$ if $n > q$ and $C = C(n, \alpha, \beta, C_*, \tilde{q})$ if $n = q$. Since $\lambda_{B'} = \lambda_{B_1}/|B'|^{q/n}$, where B_1 is the unit ball, we have

$$M(\Omega') \leq E|\Omega'|^{\frac{1}{q}+\frac{1}{n}} \quad \text{if } |\Omega'| \leq |\Omega|,$$

where E is a constant that depends only on n, q, α, β, C_* and $|\Omega|$. Using this and inequalities (2.1) and (2.2), we obtain

$$\sup u \leq C|\Omega'|^{1/q} \quad \text{for } n < q \quad \text{and} \quad \sup u \leq C|\Omega'|^{1/n} \quad \text{for } n \geq q, \tag{2.3}$$

where C depends only on n, q, α, β, C_*, and $|\Omega|$. Hence, if (Ω_n) is a sequence of domains such that $|\Omega_n| \to 0$ and (u_n) a sequence of solutions of (P_{Ω_n}), then $\sup |u_n| \leq C|\Omega_n|^{\sigma} \to 0$, where $\sigma = 1/q$ or $\sigma = 1/n$.
Now we recall some well-known results that appear in many forms.

Lemma 2.5 *Let u be a weak solution of* (P_Ω) *in $W_0^{1,p}$. Then*

$$\int_{\Omega_t} -uf(u) + \nabla u \cdot a(u, \nabla u)\, \mathrm{d}x = -t \int_{\Omega_t} f(u)\, \mathrm{d}x \quad \forall\, t \geq 0$$

where $\Omega_t = \{x \in \Omega : u(x) > t\}$.

Proof Let $\psi : \mathbb{R} \to \mathbb{R}$ be the function defined by $\psi(s) = (s - t)\chi_{\{s>t\}}(s)$. Consider $\varphi : \Omega \to \mathbb{R}$ given by $\varphi(x) = \psi(u(x))$. Since ψ is a Lipschitz function and $t > 0$, $\varphi \in W_0^{1,p}$. Furthermore,

$$\varphi = (u - t)\chi_{\{u>t\}} \quad \text{and} \quad \nabla\varphi = \chi_{\{u>t\}}\nabla u.$$

Then, since u is a weak solution of (P_Ω),

$$\int_{\Omega} \chi_{\{u>t\}}\nabla u \cdot a(u, \nabla u)\, \mathrm{d}x = \int_{\Omega} f(u)(u - t)\chi_{\{u>t\}}\, \mathrm{d}x$$

proving the lemma. □

Lemma 2.6 *Assuming the same hypotheses as in the last lemma,*

$$\int_{\{u=t\}} \frac{\nabla u \cdot a(u, \nabla u)}{|\nabla u|}\, \mathrm{d}H^{n-1} = \int_{\Omega_t} f(u)\, \mathrm{d}x$$

for almost every $t \geq 0$. If u satisfies $u = c$ on $\partial\Omega$, $c \in \mathbb{R}$, then this identity holds for almost every $t \geq c$.

Proof For $t_1 < t_2$, from Lemma 2.5, we get

$$\int_{A_{t_1 t_2}} -uf(u) + \nabla u \cdot a(u, \nabla u)\, \mathrm{d}x = t_2 \int_{\Omega_{t_2}} f(u)\, \mathrm{d}x - t_1 \int_{\Omega_{t_1}} f(u)\, \mathrm{d}x$$
$$= (t_2 - t_1)\int_{\Omega_{t_2}} f(u)\, \mathrm{d}x - t_1 \int_{A_{t_1 t_2}} f(u)\, \mathrm{d}x,$$

where $A_{t_1 t_2} = \{t_1 < u \leq t_2\}$. Then,

$$\int_{A_{t_1 t_2}} \nabla u \cdot a \, \mathrm{d}x = (t_2 - t_1) \int_{\Omega_{t_2}} f(u) \, \mathrm{d}x + \int_{A_{t_1 t_2}} (u - t_1) f(u) \, \mathrm{d}x. \tag{2.4}$$

Hence, using the coarea formula, we obtain

$$\int_{t_1}^{t_2} \int_{\{u=t\}} \frac{(\nabla u \cdot a)|\nabla u|^{-1}}{t_2 - t_1} \, \mathrm{d}H^{n-1} \mathrm{d}t = \int_{\Omega_{t_2}} f(u) \, \mathrm{d}x + \frac{\int_{A_{t_1 t_2}} (u - t_1) f(u) \, \mathrm{d}x}{t_2 - t_1}.$$

Making $t_2 \to t_1$, the integral in the left-hand side converges to the integrand for almost every t_1 and the integral over Ω_{t_2} converges to a integral over Ω_{t_1}. The last integral goes to zero, since

$$\left| \int_{A_{t_1 t_2}} \frac{(u - t_1)}{t_2 - t_1} f(u) \, \mathrm{d}x \right| < \left| \int_{A_{t_1 t_2}} f(u) \, \mathrm{d}x \right| \leq f(t_2) |A_{t_1 t_2}| \to 0,$$

completing the proof. For the case $u = c$ on $\partial \Omega$, note that $u - c \in W_0^{1,p}(\Omega)$ is a weak solution of $-\mathrm{div}\, \bar{a}(v, \nabla v) = \tilde{f}(v)$, where $\bar{a}(t, \mathrm{z}) = a(t + c, \mathrm{z})$ and $\tilde{f}(t) = f(t + c)$. Then, from the previous case, we get result. □

The following statement is a direct consequence of Brothers and Ziemer's result (see Lemma 2.3 and Remark 4.5 of [18]).

Proposition 2.7 *Let $u \in W_0^{1,p}(\Omega)$ be a nonnegative function and suppose that $a = a(\mathrm{z})$, a satisfy $(H3)$, $a(\mathrm{z}) \cdot \mathrm{z} \in C^2(\mathbb{R}^n \backslash \{0\})$, $(a(\mathrm{z}) \cdot \mathrm{z})^{1/r}$ is convex for some $r > 1$. If the symmetrization $u^\sharp$ is equal to some radial solution of (P_B) on $\Omega^\sharp_{t_1 t_2} = \{x \in \Omega^\sharp \ : \ t_1 < u^\sharp(x) < t_2\}$ and*

$$\int_{t_1 < u < t_2} \nabla u \cdot a(\nabla u) \, \mathrm{d}x = \int_{t_1 < u^\sharp < t_2} \nabla u^\sharp \cdot a(\nabla u^\sharp) \, \mathrm{d}x,$$

for some $0 \leq t_1 < t_2 \leq \max u < +\infty$, then there is a translate of $u^\sharp$ which is almost everywhere equal to u in $\{t_1 < u < t_2\}$. ((P_B) is the problem $(\tilde{\mathrm{P}}_B)$ with $\tilde{a}$ replaced by a).

Proof Let U_1 be the radial solution of (P_B) such that $u^\sharp = U_1$ on $\Omega^\sharp_{t_1 t_2}$. From Lemma 2.6,

$$\int_{\partial B_t} a(\nabla U_1) \cdot n \, \mathrm{d}S = \int_{B_t} f(U_1) \, \mathrm{d}x > 0 \quad \text{for any } t \in [0, \max U_1),$$

where $B_t = \{x : U_1(x) > t\}$. Hence, $a(\nabla U_1) \neq 0$ and, therefore, $\nabla U_1(x) \neq 0$ for any $x \neq 0$. Then $\nabla u^\sharp(x) \neq 0$ on the closure of $\Omega^\sharp_{t_1 t_2}$. Since $|\{\nabla U_1 = 0\}| = 0$, according to a result of Brothers and Ziemer (see Lemma 2.3 and Remark 4.5 of [18]), the equality between the Dirichlet integrals holds only if u is equal to some translate of $u^\sharp$ almost everywhere on $\{t_1 < u < t_2\}$. □

Next we present some comparison results about solutions.

Lemma 2.8 *Consider the radial functions* $u_1(x) = w_1(|x|) \in C^1(B_{R_1})$ *and* $u_2(x) = w_2(|x|) \in H^1(B_{R_2})$, *where* B_{R_i} *is the ball centered at* 0 *with radius* R_i, $w_1 : [0, R_1] \to \mathbb{R}$ *is decreasing,* $w_1'(r) < 0$ *for* $r > 0$, $w_2 : [0, R_2] \to \mathbb{R}$ *is nonincreasing, and* $R_1 > R_2$. *Suppose that* $m = w_1(R_1) = w_2(R_2)$ *and*

$$\int_{\{u_1=t\}} \frac{a(u_1, \nabla u_1) \cdot \nabla u_1}{|\nabla u_1|} \, dH^{n-1} \geq \int_{\{u_2=t\}} \frac{a(u_2, \nabla u_2) \cdot \nabla u_2}{|\nabla u_2|} \, dH^{n-1} \tag{2.5}$$

for almost all $t \in [m, +\infty)$, *where* $a = a(t, z)$ *is a function that satisfies (H3). Then* $u_1 > u_2$ *in* $B_{R_2} \backslash \{0\}$.

Proof We prove by contradiction. So there exists some $r_0 \in (0, R_2)$ such that $w_1(r_0) \leq w_2(r_0)$. The hypotheses imply that

$$w_1(R_2) > w_1(R_1) = w_2(R_2).$$

Hence, from the continuity of w_1 and w_2, we can assume that

$$w_1(r_0) = w_2(r_0) \quad \text{and} \quad w_1(r) > w_2(r) \quad \text{for } r \in (r_0, R_2]. \tag{2.6}$$

Now defining $b(t, |z|) = |a(t, z)|$, we have

$$b(u_i, |\nabla u_i|) = |a(u_i, \nabla u_i)| = \frac{a(u_i, \nabla u_i) \cdot \nabla u_i}{|\nabla u_i|} \quad \text{for } i = 1, 2.$$

Observe that $b = b(t, s) \in C^0(\mathbb{R} \times [0, +\infty)) \cap C^1(\mathbb{R} \times (0, +\infty))$ is positive for $s \neq 0$ and increasing in s. Hence, using (2.5) and $w_i'(|x|) = -|\nabla u_i(x)|$, we get

$$b(t, -w_1'(r_1(t)))\, r_1^{n-1}(t) \geq b(t, -w_2'(r_2(t)))\, r_2^{n-1}(t)$$

a.e. on $I = [m, t_0]$, where $t_0 = w_1(r_0) = w_2(r_0)$ and r_i is some kind of inverse of w_i given by $r_i(t) = \inf\{r \mid w_i(r) \leq t\} = (\mu_{u_i}(t)/\omega_n)^{1/n}$. Notice that r_1 is decreasing and r_2 is nonincreasing and, therefore, they are differentiable a.e. on I with $r_i'(t) = (w_i'(r_i(t))^{-1}$. Then

$$b\left(t, -\frac{1}{r_1'(t)}\right) r_1^{n-1}(t) \geq b\left(t, -\frac{1}{r_2'(t)}\right) r_2^{n-1}(t) \quad \text{a.e. on } I.$$

Defining $d : \mathbb{R} \times (-\infty, 0) \to \mathbb{R}$ by $d(t, y) = [b(t, -1/y)]^{1/(n-1)}$, we obtain

$$d(t, r_1'(t))\, r_1(t) \geq d(t, r_2'(t))\, r_2(t) \quad \text{a.e. on } I$$

and, therefore,

$$d(t, r_1')\, (r_1 - r_2) \geq (d(t, r_2') - d(t, r_1'))\, r_2 \quad \text{a.e. on } I.$$

Since $r_2 \geq r_0 > 0$, $d(t, r_1'(t))$ is continuous and positive in I, and $r_1 - r_2 \geq 0$, there exist $c_1 > 0$ such that

$$c_1\, (r_1 - r_2) \geq (d(t, r_2') - d(t, r_1'))\, r_0 \quad \text{a.e. on } I. \tag{2.7}$$

We prove now that, for some suitable constant $C > 0$,

$$C(r_1 - r_2) \geq r_2' - r_1' \quad \text{a.e. on } I. \tag{2.8}$$

For that note first that if $t \in I$ satisfies $r_2'(t) \leq r_1'(t)$, the inequality is trivial for any $C > 0$ since $r_1 \geq r_2$ on I. In the case $r_2'(t) > r_1'(t)$,

$$d(t, r_2') - d(t, r_1') = \int_{r_1'}^{r_2'} \frac{\partial d}{\partial y}(t, y)\, \mathrm{d}y \geq \int_{r_1'}^{\frac{r_2'+r_1'}{2}} \frac{\left[b\left(t, -\frac{1}{y}\right)\right]^{\frac{2-n}{n-1}}}{n-1} \cdot \frac{b_s\left(t, -\frac{1}{y}\right)}{y^2}\, \mathrm{d}y$$

since the integrand is positive and $(r_2' + r_1')/2 \leq r_2'$. From the C^1 regularity of w_1 and $w_1' < 0$, it follows that the interval $[r_1'(t), r_1'(t)/2]$ is contained in some interval $[y_1, y_2]$, where $y_2 < 0$, for any $t \in I$. Then

$$[r_1', (r_1' + r_2')/2] \subset [r_1', r_1'/2] \subset [y_1, y_2] \subset (-\infty, 0) \quad \text{for any } t \in I,$$

and, using that $|a|$ and $\partial_s |a(t, s\mathrm{z})|$ are positive and continuous for $s, \mathrm{z} \neq 0$, we get

$$b\left(t, -\frac{1}{y}\right) \geq \min_{[y_1, y_2]} b\left(t, -\frac{1}{y}\right) \geq E_1 := \min_{\frac{1}{|y_1|} \leq |\mathrm{z}| \leq \frac{1}{|y_2|}} |a(t, \mathrm{z})| > 0$$

and

$$b_s\left(t, -\frac{1}{y}\right) \geq \min_{[y_1, y_2]} b_s\left(t, -\frac{1}{y}\right) \geq E_2 := \min_{\frac{1}{|y_1|} \leq |\mathrm{z}| \leq \frac{1}{|y_2|}} \partial_s |a(t, s\mathrm{z})|\Big|_{s=1} > 0$$

for $y \in [r_1', (r_1' + r_2')/2]$. Hence,

$$d(t, r_2') - d(t, r_1') \geq \int_{r_1'}^{(r_1'+r_2')/2} \frac{E_1^{\frac{2-n}{n-1}}}{n-1} \cdot \frac{E_2}{y^2}\, \mathrm{d}y \geq \frac{E_1^{\frac{2-n}{n-1}} E_2}{(n-1)\, y_1^2} \cdot \frac{(r_2' - r_1')}{2}.$$

From this and (2.7), we get (2.8) with $C = 2c_1(n-1)y_1^2/(r_0 E_1^{\frac{2-n}{n-1}} E_2)$. Multiplying (2.8) by e^{Ct}, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(r_1 e^{Ct}) \geq \frac{\mathrm{d}}{\mathrm{d}t}(r_2 e^{Ct}) \quad \text{a.e. on } I.$$

Observe that $\int_m^{t_0} (r_2 e^{Ct})' \mathrm{d}t \geq r_2 e^{Ct}\big|_m^{t_0}$, since r_2 is decreasing and e^{Ct} is a C^1 function. To prove that we can split $r_2 e^{Ct}$ into a singular function and an absolutely continuous function, apply the Fundamental Theorem of Calculus, obtaining an identity for the second part and using a sequence of increasing C^1 functions that converges uniformly to r_2, an inequality for the first part.

Therefore,

$$r_1 e^{Ct}\big|_m^{t_0} = \int_m^{t_0} \frac{\mathrm{d}}{\mathrm{d}t}(r_1 e^{Ct})\, \mathrm{d}t \geq \int_m^{t_0} \frac{\mathrm{d}}{\mathrm{d}t}(r_2 e^{Ct})\, \mathrm{d}t \geq r_2 e^{Ct}\big|_m^{t_0}.$$

Hence, using $r_1(t_0) = r_2(t_0) = r_0$, we get $r_1(m) \leq r_2(m)$. But this contradicts $r_1(m) = R_1 > R_2 = r_2(m)$. ◻

3 Comparison results to the p-laplacian

We treat in this section the special case where the differential part of (P_Ω) and ($\tilde{P}_B$) is the p-laplacian operator, and, in addition to the hypotheses (H1) and (H5), we suppose that $f(t)/t^{p-1}$ is decreasing. Then, we can obtain a solution to the problem ($\tilde{P}_B$) minimizing the functional

$$J_B(v) = \int_B \frac{1}{p}|\nabla v|^p - F(v)\,\mathrm{d}x, \tag{3.1}$$

where $F(t) = \int_0^t f(s)\,\mathrm{d}s$. Let $\tilde{U}_B$ be a minimum of J_B. Since $f(t)/t^{p-1}$ is decreasing, $\tilde{U}_B$ is zero or is the unique nontrivial solution to ($\tilde{P}_B$) (see [17] and [13]). Then $\tilde{U}_B = U_B$, where U_B is defined in (1.2). This uniqueness result is applied only in Theorem 3.7.

Definition 3.1 Given a measurable set $E \subset B$, define $J_E : W^{1,p}(B) \to \mathbb{R}$ by

$$J_E(v) = \int_E \frac{1}{p}|\nabla v|^p - F(v)\,\mathrm{d}x.$$

Lemma 3.2 *Let f be a function, possibly non-monotone, that satisfies (H1) and (H5). For any ball $B_R(0) \subset B$ and $h \in \mathbb{R}$, there is a radial minimizer V of the functional J_{B_R} over the space $\mathcal{A}_h = \{w \in W^{1,p}(B_R) : w$ is radial and $w = h$ on $\partial B_R\}$. Moreover, if U and V minimizes J_{B_R} over $\mathcal{A}_{h_1}$ and $\mathcal{A}_{h_2}$, respectively, with $h_1 > h_2 \geq 0$, then $U > V$ in B_R and $J_{B_R}(U) < J_{B_R}(V)$.*

Proof The existence of minimizer in $\mathcal{A}_h$ can be obtained taking a minimizing sequence, observing that it has a weakly convergent subsequence, and using that J_{B_R} is weakly lower semicontinuous since (H5) holds.

To prove that $U > V$, suppose first that the set $A = \{x \in B_R : U(x) < V(x)\}$ is nonempty. Since U and V are radial functions in $W^{1,p}(B_R)$, they are continuous and A is an open set. Note that $w_1 := \max\{U, V\} \in \mathcal{A}_{h_1}$ and $w_2 := \min\{U, V\} \in \mathcal{A}_{h_2}$. Hence, $J_{B_R}(U) \leq J_{B_R}(w_1)$ and, therefore, $J_A(U) + J_{B_R\setminus A}(U) \leq J_A(w_1) + J_{B_R\setminus A}(w_1)$. Using that $U = w_1$ and $\nabla U = \nabla w_1$ a.e. on $B_R\setminus A$, it follows that $J_A(U) \leq J_A(w_1)$. Moreover, using that $w_1 = V$ in A, we get

$$J_A(U) \leq J_A(w_1) = J_A(V).$$

We have also that $J_{B_R}(V) \leq J_{B_R}(w_2)$. Then, using the same argument as before,

$$J_A(V) \leq J_A(U).$$

Hence $J_A(U) = J_A(V) = J_A(w_1)$ and then $U = w_1$ in $B_R\setminus A$ implies that $J_{B_R}(U) = J_{B_R}(w_1)$. Therefore, w_1 is also a minimizer of J_{B_R} and, hence a weak solution of $-\Delta_p v = f(v)$ in B_R.

Notice that for any ring $\mathcal{R} = \{x : r_1 < |x| < R\}$, $r_1 > 0$, taking the radial test function $\varphi_{r,h}(|x|) = \chi_{[0,r-h]}(|x|) + \left(\frac{r+h}{2h} - \frac{|x|}{2h}\right)\chi_{(r-h,r+h]}(|x|)$ with $h > 0$ and $r \in (r_1, R)$, and using that U is a weak solution, we get

$$n\omega_n \int_{r-h}^{r+h} \frac{|\nabla U|^{p-1}}{2h} s^{n-1}\,\mathrm{d}s = \int_{B_r} \nabla\varphi_{r,h} \cdot \nabla U|\nabla U|^{p-2}\,\mathrm{d}x = \int_{B_r} f(U)\varphi_{r,h}\,\mathrm{d}x,$$

where ω_n is the volume of the unit ball. Taking the limit as $h \to 0$, the Lebesgue Differentiation Theorem implies that

$$n\omega_n |\nabla U(r)|^{n-1} r^{n-1} = \int_{B_r} f(U)\,\mathrm{d}x \geq \omega_n R^n \min_{t \geq h_1} f(t) > 0 \tag{3.2}$$

for almost every $r \in (r_1, R)$. The last inequality is a consequence of $U \geq h_1$, which is a result of maximum principle and $-\Delta_p U = f(U) \geq 0$. Therefore, $|\nabla U| \geq c$ a.e. in $\mathcal{R}$, where c is some positive constant that depends on $\mathcal{R}$ and $\min_{t \geq h_1} f(t)$. Thus, U is a solution of a uniformly elliptic equation in this ring and, therefore, a $C^{2,\alpha}$ function in $\mathcal{R}$ for any $\alpha \in (0, 1)$. Since $r_1 < R$ is arbitrary, U is a classical solution of $-\Delta_p U = f(U)$ in $B_R(0)\backslash\{0\}$, with $\nabla U(x) \neq 0$ for $x \neq 0$. By the same argument, w_1 is a classical solution of the same problem. Therefore, U and w_1 are classical solutions of the ordinary differential equation $v'' + (n-1)v'/r = -f(v)$ for $0 < r < R$. Observe now that $\bar{A} \subset B_R$ since $U > V$ on ∂B_R and, then, there exists $x_0 \in \partial A \cap B_R\backslash\{0\}$. Hence $U(x_0) = w_1(x_0)$ and, using that $U \leq w_1$ in B_R and $x_0 \in B_R$, we have $\partial_r U(|x_0|) = \partial_r w_1(|x_0|)$. Then, from the uniqueness result for ODE, $U = w_1$ in some neighborhood of x_0, contradicting that $x_0 \in \partial A$. Therefore, A is empty and $U \geq V$.

If $U(x_0) = V(x_0)$ at some $x_0 \in B_R\backslash\{0\}$, using the same argument as before and $U \geq V$, U and V are classical solutions of some ODE and $\partial_r U(|x_0|) = \partial_r w_1(|x_0|)$. Then $U = V$ in B_R, which contradicts $U > V$ on ∂B_R. Eventually, $U(0) = V(0)$ since we cannot apply the uniqueness result at $r = 0$ for $v'' + (n-1)v'/r = -f(v)$. However, 0 is the maximum point of U and V, and these functions are differentiable at the origin since the first equality of (3.2) implies that $\nabla U(x)$ and $\nabla V(x)$ converge to zero as $x \to 0$. Hence, $U(0) = V(0)$, $\partial_r U(0) = \partial_r V(0) = 0$ and, from the uniqueness result of Proposition A4 of [29], $U = V$ in B_R. Thus, $U > V$ in B_R.

To prove that $J_{B_R}(U) < J_{B_R}(V)$, note that for some positive constant c, the translation $V + c \in \mathcal{A}_{h_1}$. Since U is a minimizer of J_{B_R} over $\mathcal{A}_{h_1}$, $J_{B_R}(U) \leq J_{B_R}(V + c)$. Moreover, as F is strictly increasing on $(0, +\infty)$ and $V + c > V$, $J_{B_R}(V + c) < J_{B_R}(V)$. Hence we conclude the result. □

Theorem 3.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, B be a ball such that $|B| = |\Omega|$, and u be a weak solution of* (P_Ω)*, where $div(a(\nabla u)) = \Delta_p u$ and f is a function that satisfies* (*H1*) *and* (*H5*)*, possibly non-monotone, such that $f(t)/t^{p-1}$ is decreasing on* $(0, +\infty)$*. Then,*

$$\max u \;\leq\; \max U_B,$$

where U_B is the minimizer of the functional given by (3.1)*.*

Proof Let $u^\sharp$ be the Schwarz symmetrization of u. Defining $\Omega_t^\sharp = \{u^\sharp > t\}$, we have that $|\Omega_t^\sharp| = |\Omega_t|$. Therefore, Remark 2.1 implies that

$$\int_{\Omega_t} F(u)\,\mathrm{d}x \;=\; \int_{\Omega_t^\sharp} F(u^\sharp)\,\mathrm{d}x \quad \text{for } t \geq 0.$$

We also know that

$$\int_{\Omega_t} |\nabla u|^p \;\,\mathrm{d}x \;\geq\; \int_{\Omega_t^\sharp} |\nabla u^\sharp|^p\,\mathrm{d}x. \tag{3.3}$$

Then,

$$\int_{\Omega_t} \frac{|\nabla u|^p}{p} - F(u)\,\mathrm{d}x \;\geq\; \int_{\Omega_t^\sharp} \frac{|\nabla u^\sharp|^p}{p} - F(u^\sharp)\,\mathrm{d}x. \tag{3.4}$$

Now suppose that for some $t \geq 0$, we have $|\Omega_t| = |B_t|$, where $B_t = \{U_B > t\}$. In this case, $B_t = \Omega_t^\sharp$ and

$$\int_{\Omega_t^\sharp} \frac{|\nabla u^\sharp|^p}{p} - F(u^\sharp)\,\mathrm{d}x \;\geq\; \int_{B_t} \frac{|\nabla U_B|^p}{p} - F(U_B)\,\mathrm{d}x, \tag{3.5}$$

otherwise the function $\tilde{u} : B \to \mathbb{R}$ given by $\tilde{u} = u^\sharp \chi_{B_t} + U_B \chi_{B_t^c}$ satisfies $J_B(\tilde{u}) < J_B(U_B)$, contradicting that U_B is a minimizer. Then, from (3.4) and (3.5), it follows that

$$\int_{\Omega_t} \frac{|\nabla u|^p}{p} - F(u)\,\mathrm{d}x \;\geq\; \int_{B_t} \frac{|\nabla U_B|^p}{p} - F(U_B)\,\mathrm{d}x.$$

Hence, using Lemma 2.5 and the fact that u and U_B are solutions, we get

$$\int_{\Omega_t} \frac{uf(u) - tf(u)}{p} - F(u)\,\mathrm{d}x \geq \int_{B_t} \frac{U_B f(U_B) - tf(U_B)}{p} - F(U_B)\,\mathrm{d}x. \tag{3.6}$$

Define $h_t : [t, +\infty) \to \mathbb{R}$ by

$$h_t(s) \;=\; \frac{(s-t)f(s)}{p} - F(s). \tag{3.7}$$

Note that $h_t(s)$ is decreasing for $s \geq t$, since

$$h_t'(s) \;=\; \frac{(s-t)f'(s)}{p} - \frac{(p-1)f(s)}{p} \;=\; \frac{(s-t)^p}{p}\left(\frac{f(s)}{(s-t)^{p-1}}\right)' \;<\; 0.$$

Furthermore, as $h_t(t) \leq 0$, $h_t(s) < 0$ for $s > t$. Therefore, from (3.6), we have

$$\int_{\Omega_t} h_t(u)\,\mathrm{d}x \;\geq\; \int_{B_t} h_t(U_B)\,\mathrm{d}x, \tag{3.8}$$

where h_t is decreasing and negative. Suppose that $\max u > \max U_B$. Since $|\Omega| = |B|$, the function $\mu_B(t) = |\{U_B > t\}|$ is continuous and $\mu_u(t)$ is right continuous, there is $t_0 \geq 0$ such that $\mu_u(t_0) = \mu_B(t_0)$ and $\mu_u(t) > \mu_B(t)$ for $t > t_0$. Then,

$$|\{-h_{t_0} \circ u > s\}| \;>\; |\{-h_{t_0} \circ U_B > s\}| \quad \text{for} \quad s > -h_{t_0}(t_0),$$

since $-h_{t_0}$ is a increasing function. Thus, by Fubini's Theorem,

$$-\int_{\Omega_{t_0}} h_{t_0}(u)\,\mathrm{d}x \;>\; -\int_{B_{t_0}} h_{t_0}(U_B)\;\,\mathrm{d}x,$$

contradicting (3.8). □

Remark 3.4 This result can be extended to the problem (1.5) observing first that $q(t) := (f(t) - k(t))/t$ is decreasing. If $q(t) > 0$ for any $t > 0$, it is immediate from the theorem that $\max u \leq \max U$, where u solves (1.5) and $U \in W_0^{1,p}(B)$ is the solution of the symmetrized problem $-\Delta_p V + k(V) = f(V)$ in B. If $q(t_0) = 0$ for some $t_0 \geq 0$, the maximum principle implies that $u, U \leq t_0$. Hence taking u_m and U_m, the sequence of solutions of $-\Delta_p v = \max\{f(v) - k(v), 0\} + 1/m$ in Ω and B, respectively, we have $u_m \leq U_m$, $u_m \to u$ and $U_m \to U$ monotonically, proving the inequality. A related result with this one is stated in [25]. For instance, if f is a positive constant, Theorem 2 of that work give more relations between u and U.

Corollary 3.5 *Assuming the same hypotheses as in Theorem 3.3, if Ω is not a ball and U_B is positive, then*

$$\max u \ < \ \max U_B.$$

Proof If Ω is not a ball, Proposition 2.7 implies that inequalities (3.3) is strict for $t = 0$. Following the same computation as in Theorem 3.3, we have also a strict inequality in (3.8) for $t = 0$, that is

$$\int_\Omega -h_0(u)\,\mathrm{d}x \ < \ \int_B -h_0(U_B)\,\mathrm{d}x. \tag{3.9}$$

As a consequence, we can prove that there is $t > 0$ such that

$$|\Omega_t| \ < \ |B_t|.$$

Indeed, if $\mu_u(t) = |\Omega_t| \ \geq \ |B_t| = \mu_{U_B}(t)$ for any $t > 0$, then, using Remark 2.1 and that $-h_0$ is increasing with $-h(0) = 0$, we obtain the reverse inequality of (3.9), which it is a contradiction.

Now note that the function $v \ = \ u - t$ satisfies

$$\begin{cases} -\Delta_p v = \tilde{f}(v) & \text{in } \Omega_t \\ v = 0 & \text{on } \partial\Omega_t, \end{cases} \tag{3.10}$$

where $\tilde{f}$ is given by $\tilde{f}(s) = f(s + t)$. If B' and B are concentric balls and $|B'| = |\Omega_t|$, then $|B'| < |B_t|$ and $B' \subset B_t$. Since $U_B = t$ on ∂B_t, we get from the maximum principle that $U_B > t$ on $\partial B'$. Hence, using Lemma 3.2, there is a function $w : B' \to \mathbb{R}$ that minimizes $J_{B'}$ under the condition $w \equiv t$ on $\partial B'$, and, therefore, the function $V_{B'} = w - t$ is the solution of (3.10) with Ω_t replaced by B'. Furthermore, $w < U_B$. Since $\tilde{f}$ satisfies all hypotheses required in Theorem 3.3,

$$\max v \ \leq \ \max V_{B'}.$$

Hence,

$$\max_\Omega u \ = \ \max_{\Omega_t}(v + t) \ \leq \ \max_{B'}(V_{B'} + t) \ = \ \max_{B'} w < \max_\Omega U_B$$

proving the result. □

Remark 3.6 Suppose that $u \in W_0^{1,p}(\Omega)$ is a solution of

$$-\operatorname{div}(MDu|Du|^{p-2}) = f(u), \tag{3.11}$$

where $M(x) \ = \ (a_{ij}(x))$ is a matrix with measurable bounded entries such that, $\sum_{ij} a_{ij}(x)\xi_i\xi_j \geq |\xi|^2$. Observing that

$$\tilde{J}(v) := \int_{\Omega} \langle MDv, Dv \rangle \frac{1}{p} |\nabla v|^{p-2} - F(v) \ \mathrm{d}x \geq \int_{\Omega} \frac{1}{p} |\nabla v|^{p} - F(v) \, \mathrm{d}x,$$

and repeating the arguments of Theorem 3.3, we get $\max u \leq \max U_B$. Notice that M can be nonsymmetric.

Next theorem, in the case $p = 2$ and $f(0) > 0$, is a consequence of a result, which establishes that the symmetrization of the minimal solution associated to Ω is smaller or equal than the one associated to the corresponding ball (see [11,39]), and the uniqueness of solution when $f(t)/t$ is decreasing (see [17]). For general p, we can apply a similar argument to compare the minimal solutions (see [34]) and the uniqueness result obtained for the case that $f(t)/t^{p-1}$ is decreasing (see [13]).

Also it can be proved in a independent way using the main result of Sect. 5 and the uniqueness of solution to this problem.

Theorem 3.7 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, B be a ball such that $|B| = |\Omega|$, and u be a weak solution of* (P_Ω)*, where $div(a(\nabla u)) = \Delta_p u$ and f is a nonnegative increasing locally Lipschitz function, such that $f(t)/t^{p-1}$ is decreasing on* $(0, +\infty)$*. Then,*

$$|\{u > t\}| \ < \ |\{U_B > t\}| \quad \forall t \in (0, \max U_B],$$

unless Ω is a ball.

4 Study of the radial solutions

We study now a Dirichlet problem, where the domain is a ball, and we need some additional hypothesis:

(H7) there is some $\mu \in [0, 2)$ such that $\frac{d}{\mathrm{d}s}|a(t, s\mathrm{w})| \geq |a(t, s\mathrm{w})|^{\mu}$ for $s > 0$ small and w unit vector of $\mathbb{R}^n$.

The following theorem is the main result of this section.

Theorem 4.1 *Let $B' = B_{R_0}$ be a open ball in $\mathbb{R}^n$ satisfying $|B'| \leq |\Omega|$ and suppose that $\tilde{a}$ and f satisfy conditions (H1)–(H5) and (H7). If $f(0) > 0$ and $m \geq 0$, then there exists a solution $U_{B'}$ to the problem* ($\tilde{\mathrm{P}}_{B'}$) *with $U_{B'} = m$ on $\partial B'$ such that, for any radial solution U of* ($\tilde{\mathrm{P}}_{B''}$) *with $0 \leq U \leq m$ on $\partial B''$,*

$$U_{B'} > U \quad in \quad B'',$$

where $B'' \subsetneq B'$ are concentric open balls. The same holds in the case $B'' = B'$ if U and $U_{B'}$ are different.

First we have to observe some basic properties of weak solutions and obtain some existence result.

Lemma 4.2 *Assuming the same hypotheses as in the main theorem, if U is a radial weak solution of* ($\tilde{\mathrm{P}}_{B''}$)*, then $U \in C^{2,\alpha}(B''\backslash\{0\}) \cap C^1(B'')$ for any $\alpha < 1$ and U is a classical solution in $B''\backslash\{0\}$.*

Proof First using the ACL characterization of Sobolev functions (see e.g., [50]) and a local diffeomorphism between the Cartesian and the polar system of coordinates, it follows that U is absolutely continuous on closed radial segments that does not contain the origin. Hence, the set $\{U < t\}$ is open in B' for any $t \in \mathbb{R}$. Indeed, these sets are rings of the form $\{x \in B' : r_t < |x| < R_0\}$, otherwise there is a ring $\mathcal{R} = \{r_1 < |x| < r_2\}$ contained in $\{U < t\}$, such that $U = t$ on $\partial\mathcal{R}$, for which the test function $\varphi(x) = (t - U(x))\chi_{\mathcal{R}}(x) \in W_0^{1,p}$ satisfies

$$0 \geq -\int_{\mathcal{R}} \nabla U \cdot \tilde{a}(U, \nabla U)\, \mathrm{d}x = \int_{\mathcal{R}} \nabla\varphi \cdot \tilde{a}(U, \nabla U)\, \mathrm{d}x = \int_{\mathcal{R}} f(U)\varphi\, \mathrm{d}x > 0,$$

that is a contradiction. Hence, U is a nonincreasing radially symmetric function. Observe also that if U is constant in some ring, then taking a nonnegative function with a compact support in this ring, we get a contradiction as before. Then U is strictly decreasing in the radial direction. This conclusion can be obtained more easily for operators where the maximum principle holds.

Notice now that for a given ring $\mathcal{R} = \{r_1 < |x| < r_2\}$, taking the radial test function $\varphi_{R,h}(|x|) = \chi_{[0,R-h]}(|x|) + \left(\frac{R+h}{2h} - \frac{|x|}{2h}\right)\chi_{(R-h,R+h]}(|x|)$, for $h > 0$ and $R \in (r_1, r_2)$, we get

$$n\omega_n \int_{R-h}^{R+h} \frac{b(U, -\partial_r U)}{2h} r^{n-1} dr = \int_{B'} \nabla\varphi_{R,h} \cdot \tilde{a}(U, \nabla U)\, \mathrm{d}x = \int_{B'} f(U)\varphi_{R,h}\, \mathrm{d}x,$$

where $b(t, |z|) = |\tilde{a}(t, z)|$ and ω_n is the volume of the unit ball. Making $h \to 0$, from the Lebesgue Differentiation Theorem, it follows that

$$n\omega_n b(U(R), -\partial_r U(R))R^{n-1} = \int_{B_R} f(U)\, \mathrm{d}x \geq \int_{B_{r_1}} f(0)\, \mathrm{d}x > 0 \tag{4.1}$$

for almost every $R \in (r_1, r_2)$ and then, using $(H3)$, we get that $|\nabla U| \geq c$ a.e. in $\mathcal{R}$, where c is some positive constant that depends on $\mathcal{R}$. Thus, U is a solution of a uniformly elliptic equation in this ring and, therefore, a $C^{2,\alpha}$ function in $\mathcal{R}$ for any $\alpha \in (0, 1)$. Moreover, from (2.3), U is bounded and, from its monotonicity in the radial direction, it can be defined continuously on 0. In fact, using (4.1), we can prove that U is differentiable at the origin and its derivative is zero. □

Lemma 4.3 *Under the same hypotheses as in the main theorem, for any $h > 0$, there exist $R_h > 0$ and a function $U_h \in C^{2,\alpha}(B_{R_h}(0)\backslash\{0\}) \cap C^1(B_{R_h}))$, which is a radial weak solution of $(\tilde{\mathrm{P}}_{B_{R_h}})$ and a classical solution in $B_{R_h}(0)\backslash\{0\}$, such that $U_h(0) = h$. Moreover, such function is unique.*

Proof If such U_h exists, then due to its regularity and $(H3)$, we have $\tilde{a}(U_h, \nabla U_h) = \tilde{e}(U_h, |\nabla U_h|)\nabla U_h$ for some function $\tilde{e} : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ and $U_h(x) = w(|x|)$ for some function $w : [0, R_h] \to \mathbb{R}$ that satisfies, in the classical sense,

$$\begin{cases} (\tilde{e}\, w')' + \dfrac{n-1}{r}\, \tilde{e}\, w' = -f(w) \quad \text{for} \quad r \in [0, R_h] \\ w'(0) = 0 \\ w(R_h) = 0, \end{cases} \tag{4.2}$$

where $'$ denotes $\mathrm{d}/\mathrm{d}r$. To prove the existence of solution to this problem, we consider the following one:

$$\begin{cases} (\tilde{e}(w(r), |w'(r)|)\, w'(r))' + \dfrac{n-1}{r}\tilde{e}\, w'(r) = -f(w(r)) \\ w'(0) = 0 \\ w(0) = h. \end{cases} \tag{4.3}$$

If $\tilde{e}$ depends only on z, according to Proposition A1 of [29], there exists $\delta > 0$ and a positive local solution $w_h : [0, \delta) \to \mathbb{R}$ to (4.3). In the general case, consider first the problem (4.3) with $\tilde{e}$ replaced by $e_0(|z|) = \tilde{e}(h, |z|)$, which has a local solution w_0 defined on $[0, \delta_0)$ as in the previous case. Then, for $k \in \mathbb{N}$, take $\delta_k \le \delta_0$ such that $w_0(r) \ge h - h/k$ on $[0, \delta_k]$ and define e_k such that $\tilde{a}_k(t, z) := e_k(t, |z|)z$ satisfies (H3),(H4),(H7) and

$$e_k(t, |z|) = \begin{cases} e_0(|z|) = \tilde{e}(h, |z|) & \text{for } t \in [h - h/k, +\infty) \\ \tilde{e}(t, |z|) & \text{for } t \in (-\infty, h - 2h/k]. \end{cases}$$

Hence, w_0 is a solution to (4.3) on $[0, \delta_k]$ with $\tilde{e}$ replaced by e_k and, from (4.1), w_0 is decreasing and $\frac{dw_0}{dr}(\delta_k) \ne 0$. Since (H3) implies that $s \to |\tilde{a}_k(t, s\mathrm{w})|$ is increasing for any w, the classical ODE theory implies that we can extend w_0 for a larger interval. Indeed, while some extension is positive, it can be continued to a bigger interval. Since $f(0) > 0$ and $\tilde{a}_k$ satisfies (H4), integrating $(r^{n-1}e_k(w(r), |w'(r)|)w'(r))' = -r^{n-1}f(w(r))$, we conclude that for any positive continuation $\bar{w}_k : [0, \bar{\delta}) \to \mathbb{R}$ of w_0, the right end point satisfies

$$\bar{\delta} \le C := \frac{nC^*}{f(0)}\left[1 + \left(\frac{p}{p-1} \cdot \frac{hf(0)}{nC^*}\right)^{\frac{p-1}{p}}\right]. \tag{4.4}$$

Hence, there exists a continuation $w_k : [0, R_k] \to \mathbb{R}$ such that $w_k(R_k) = 0$ and is positive on $[0, R_k)$. Observe now that, using the same idea as in the estimate (4.1), we get that $|w_k'|$ is uniformly bounded by above. Hence, some subsequence converge uniformly for some non-decreasing function $w_h : [0, R_h] \to \mathbb{R}$ that is positive in $[0, R_h)$ and vanishes at R_h. Indeed, applying again a similar computation as in (4.1) and using the positivity of $|\partial_s \tilde{a}_k(t, sz)|$ for $s, t \ne 0$ from $(H3)$, it follows that w_k' are equicontinuous in compacts sets of $[0, R_h)$ for k large. (More precisely, the Lipschitz norm of w_k' are uniformly bounded in compacts sets of $(0, R_h)$ and $w_k'(r)$ are uniformly close to 0 for r small.) Hence, some subsequence converge uniformly for w_h in the C^1 norm for compact sets of $[0, R_h)$. Hence, due to the regularity of $\tilde{a}$ and the definition of $\tilde{a}_k$, $U_h(x) := w_h(|x|)$ is the weak solution of $-\text{div}\, \tilde{a}(v, \nabla v) = f(v)$ in B_{R_h}. Then, as we observed previously, U_h is a classical solution and satisfies $U_h(0) = h$ since $w_k(0) = h$. Moreover, following the same argument of Proposition A4 of [29] for $\tilde{a}$ that depends also on t, for each $h > 0$, such solution U_h and radius R_h are unique. At this point, we have to use (H7). □

Let us represent the correspondence of this lemma by $\Psi = (\Psi_1, \Psi_2)$, where $\Psi_1(h) = R_h$ and $\Psi_2(h) = U_h$. Observe that $R_h \le C$, where C is given by (4.4). Using this, the equicontinuity of the first derivative of solutions, Arzelà-Ascoli Theorem, and uniqueness for (4.3), we get the following result.

Lemma 4.4 *The function Ψ_1 is continuous on $(0, +\infty)$. Furthermore, for any $h_0 > 0$, $\varepsilon > 0$ and K compact subset of $B_{R_{h_0}}$, there exists $\delta > 0$ such that $\|\Psi_2(h) - \Psi_2(h_0)\|_{C^1(K)} \le \varepsilon$ if $|h - h_0| < \delta$.*

We can also improve estimate (4.4) in the following sense.

Lemma 4.5 *Given $M > 0$, there exists some continuous increasing function $\Theta_M : [0, M] \to \mathbb{R}$ s.t. $\Theta_M(0) = 0$ and $R_h \le \Theta_M(h)$ for $h \le M$, where $R_h = \Psi_1(h)$, i.e., R_h is the point s.t. the nonnegative solution w of (4.3) vanishes.*

Proof Integrating $(r^{n-1}e(w(r), |w'(r)|)w'(r))' = -r^{n-1}f(w(r))$ from 0 to $R \leq R_h$, we get

$$\left|\tilde{a}\left(w(R), |w'(R)|\mathrm{z}\right)\right| = -e(w(R), |w'(R)|)w'(R) \geq \frac{f(0)R}{n}$$

for any $|\mathrm{z}| = 1$. Since $s \mapsto |\tilde{a}(t, s\mathrm{z})|$ is continuous, strictly increasing in $[0, +\infty)$ and vanishes at $s = 0$, where $t \in [0, M]$, the function $\rho(s) := \sup_{t\in[0,M]} |\tilde{a}(t, s\mathrm{z})|$ also satisfies these hypotheses. Using that $w(R) \leq h \leq M$,

$$\rho(-w'(R)) \geq \frac{f(0)R}{n}.$$

Taking the inverse of ρ and integrating from 0 to R_h,

$$h = w(0) - w(R_h) = \int_0^{R_h} -w'(R)\,\mathrm{d}R \geq \int_0^{R_h} \rho^{-1}\left(\frac{f(0)R}{n}\right)\mathrm{d}R.$$

Observe that

$$R_h \mapsto \int_0^{R_h} \rho^{-1}\left(\frac{f(0)R}{n}\right)\,\mathrm{d}R$$

is invertible, since is increasing and continuous. It is also positive and vanishes at 0. Hence, we get the result defining Θ_M as the inverse of this application. □

Lemma 4.6 *Assuming the same hypotheses as in Theorem 4.1, there exists a solution $U_{B'}$ to the problem* ($\mathrm{P}_{B'}$) *with $U_{B'} = 0$ on $\partial B'$, such that*

$$\max U_{B'} \geq \max U,$$

for any radial solution U of ($\mathrm{P}_{B''}$) *satisfying $U = 0$ on $\partial B''$, where $B'' \subset B' = B_{R_0}$ are concentric balls. As a matter of fact, $U_{B'} = \Psi_2(h_0)$, where $h_0 = \max\{h \mid \Psi_1(h) = R_0\}$. Furthermore, the inequality is strict if $U \neq U_{B'}$.*

Proof First we note that Lemma 4.5 implies that

$$\Psi_1(h_1) = R_{h_1} \leq \Theta_1(h_1) < R_0 \quad \text{for small } h_1,$$

since $\Theta_1(h) \to 0$ as $h \to 0$. We can also prove that $\Psi_1(h_2) > R_0$ for a large h_2. Indeed, from (2.3), any solution of ($\mathrm{P}_{B''}$) is bounded by $C|B'|^{1/q}$ if $n < q$ or by $C|B'|^{1/n}$ if $n \geq q$. Hence,

$$\Psi_1(h) > R_0 \quad \text{for} \quad h > M = \max\{C|B'|^{1/q}, C|B'|^{1/n}\}, \tag{4.5}$$

otherwise a ball of radius $\Psi_1(h) \leq R_0$ posses a solution of height $h > M$ contradicting (2.3).

Thus, from the continuity of Ψ_1, the set $A = \{h \mid \Psi_1(h) = R_0\}$ is not empty and is bounded by M. Then, we can define $h_0 = \max A$ and $U_{B'} = \Psi_2(h_0)$. Let U be a radial solution of ($\mathrm{P}_{B''}$) satisfying $U = 0$ on $\partial B''$, where $B'' = B_{\tilde{R}}$ with $\tilde{R} \leq R_0$. Note that $\tilde{R} = \Psi_1(U(0))$ and, thus, inequality (4.5) implies that $U(0) \leq M$. To prove the lemma, we have to show that $U(0) \leq h_0$. Suppose that $U(0) > h_0$. For $h = M+1$, we have $\Psi_1(h) > R_0$ from (4.5). Summarizing,

$$\Psi_1(U(0)) = \tilde{R} \leq R_0 < \Psi_1(h) \quad \text{and} \quad U(0) < h.$$

Therefore, from the continuity of Ψ_1, there exists $h_1 \in [U(0), h)$ such that $\Psi_1(h_1) = R_0$. But this contradicts $h_1 \geq U(0) > h_0$ and the definition of h_0. Hence, $U(0) \leq h_0$. Furthermore, the equality happens only if $U = U_{B'}$, since the solution of (4.3) is unique. □

Proof of Theorem 4.1 Possibility 1: $m = 0$
Let $U_{B'}$ be the function defined in the previous lemma and U a solution of $(P_{B''})$ with $U = 0$ on $\partial B''$, where $B'' \subset B'$ are concentric balls. The set

$$C = \{h > 0 \mid w_h := \Psi_2(h) \geq U_{B'} \text{ in } B' \text{ and } w_h \geq U \text{ in } B''\}$$

is not empty. To prove that, let $h > \max U_{B'}$ such that $h \notin C$. For instance, suppose that w_h does not satisfy $w_h \geq U_{B'}$ in B'. Using that w_h and $U_{B'}$ are continuous radial functions and $w_h(0) = h > U_{B'}(0)$, we conclude that there exists $B'' \subset B'$ such that $w_h > U_{B'}$ in B'' and $w_h = U_{B'}$ in $\partial B''$. Denoting $t_0 = U_{B'}(\partial B'')$, we have $t_0 \leq \max U_{B'} \leq M$, where M is given by (4.5). Hence, the function $\tilde{f}(t) = f(t + t_0)$ satisfies

$$\tilde{f}(t) \leq f(t + M) \leq \alpha(t + M)^{q-1} + \beta \leq \alpha' t^{q-1} + \beta',$$

where α' is any real in $(\alpha, C_* \lambda_B)$ and β' is a constant that depends on α', β, and M. Note that $v = w_h - t_0$ satisfies

$$-\text{div}(\bar{a}(v, \nabla v)) = \tilde{f}(v),$$

where $\bar{a}(t, z) = \tilde{a}(t + t_0, z)$, with the boundary data $v = 0$ on B''. Since $\bar{a}$ and $\tilde{f}$ satisfy (H1)–(H5), it follows from (2.3) that $\sup v \leq \tilde{M}$, where $\tilde{M}$ is a constant that depends on $n, q, \alpha', \beta', C_*$, and $|\Omega|$. Thus $w_h \leq \tilde{M} + M$. This inequality also holds, by the same argument, when condition $w_h \geq U$ in B' is not satisfied. Therefore, $h \in C$ for $h > \tilde{M} + M$, proving that C is not empty.

Let $\alpha_1 = \inf C$. From the continuity of Ψ_1 and the C^1 estimate of Lemma 4.4, $R_1 = \Psi_1(\alpha_1) \geq R_0$, $w_{\alpha_1} = \Psi_2(\alpha_1) \geq U_{B'}$ in B', and $w_{\alpha_1} \geq U$ in B''. Hence, $\alpha_1 = w_{\alpha_1}(0) \geq U_{B'}(0)$. If $\alpha_0 := U_{B'}(0) = \alpha_1$, then $w_{\alpha_1} = U_{B'}$, and, therefore, $U_{B'} \geq U$ proving the theorem. Suppose that $\alpha_1 > \alpha_0$. Then, $R_1 > R_0$, otherwise $R_0 = R_1 = \Psi_1(\alpha_1)$ contradicting $\alpha_1 > \alpha_0 = \max\{\alpha \mid \Psi_1(\alpha) = R_0\}$. Let

$$d_1 = \inf_{x \in B'} (w_{\alpha_1}(x) - U_{B'}(x)) \geq 0 \quad \text{and} \quad d_2 = \inf_{x \in B''} (w_{\alpha_1}(x) - U(x)) \geq 0.$$

If $d_1 = 0$, consider $x_1 \in \bar{B}' \backslash \{0\}$ such that $w_{\alpha_1}(x_1) = U_{B'}(x_1)$. Since $R_1 > R_0$, we have $w_{\alpha_1} > 0$ in $\partial B'$ and, from $U_{B'} = 0$ in $\partial B'$, it follows that $x_1 \in B' \backslash \{0\}$. Observe also that $\nabla w_{\alpha_1}(x_1) = \nabla U_{B'}(x_1)$, since $w_{\alpha_1} \geq U_{B'}$. Then, using that w_{α_1} and $U_{B'}$ are radial, we infer from the uniqueness of solution for ODE that $w_{\alpha_1} = U_{B'}$, contradicting $w_{\alpha_1}(0) = \alpha_1 > \alpha_0 = U_{B'}(0)$. Hence $d_1 > 0$ and, by the same argument, $d_2 > 0$. These contradict Lemma 4.4 and the definition of α_1, proving that $U_{B'} \geq U$.

Possibility 2: $m > 0$
Consider the equation

$$-\text{div}\, \bar{a}(V, \nabla V) = \tilde{f}(V),$$

where $\bar{a}(t, z) = \tilde{a}(t + m, z)$ and $\tilde{f}(t) = f(t + m)$. Notice that $\bar{a}$ and $\tilde{f}$ satisfy (H1)–(H5) and (H7) with the constants $n, p, q, q_0, \alpha', \beta', C_*, C^*, C_s$ and $|\Omega|$, where α' and β' can be chosen, as in Possibility 1, s.t. $\alpha' \in (\alpha, C_* \lambda_B)$ and $\beta' = \beta'(\alpha', \beta, m)$. Then, from Possibility 1, let $\tilde{U} \in W_0^{1,p}(B')$ be the maximal solution associated to this equation. If U is a solution of $(\tilde{P}_{B''})$ with $U \leq m$ on $\partial B''$, then $U - m \leq 0$ or $U - m$ is also a solution of this equation in some ball contained in B''. In both situations, since $\tilde{U}$ is maximal, $\tilde{U} \geq U - m$. So we conclude Possibility 2, taking $U_{B'} = \tilde{U} + m$.

To prove the strict inequality in case $U \not\equiv U_{B'}$, we must observe that if $U(x_0) = U_{B'}(x_0)$ at some $x_0 \in B''$, then $\nabla U(x_0) = \nabla U_{B'}(x_0)$ since $U \leq U_{B'}$. This contradicts the classical results of uniqueness of solution for ODE if $x_0 \neq 0$ and the uniqueness established by Proposition A4 of [29] if $x_0 = 0$. Observe that we need (H7) to apply Proposition A4. □

Theorem 4.7 *Let $B' = B_{R_0}$ be a open ball in $\mathbb{R}^n$ satisfying $|B'| \leq |\Omega|$ and suppose that $\tilde{a}$ and f satisfy conditions (H1)–(H5) and (H7). If $f(0) = 0$ and $m \geq 0$, then there exists a nonnegative solution $U_{B'}$ of $(\tilde{\mathrm{P}}_{B'})$ with $U_{B'} = m$ on $\partial B''$, possibly null, s.t. for any radial solution U of $(\tilde{\mathrm{P}}_{B''})$ with $U \leq m$ on $\partial B''$,*

$$U_{B'} \geq U \quad in \quad B'',$$

where $B'' \subset B'$ are concentric open balls. If $U_{B'}$ is not trivial, then $U_{B'}$ is positive and the inequality is strict unless U and $U_{B'}$ are equal.

Proof Let (t_k) be a sequence of positive reals s.t. $t_k \downarrow 0$, $f_k(t) := f(t + t_k + m)$ and $a_k(t, z) := \tilde{a}(t + t_k + m, z)$. Since a_k and f_k satisfy (H1)–(H7) and $f_k(0) = f(t_k + m) > 0$, we can apply Theorem 4.1 to obtain the maximal solution $U_k \in W_0^{1,p}(B')$ of

$$-\operatorname{div} a_k(v, \nabla v) = f_k(v) \tag{4.6}$$

in B'. Observe that if U is a radial solution of $(\tilde{\mathrm{P}}_{B''})$ satisfying $0 \leq U \leq m$, then $U - t_k - m \leq 0$ or $U - t_k - m$ is also a solution of (4.6) in a ball contained in B'' vanishing on the boundary of this ball. Then, $U_k > U - t_k - m$. Furthermore, since the important constants $(n, q, \alpha', \beta', C_*, |\Omega|)$ associated a_k and f_k can be chosen not depending k, U_k is bounded in the L^∞ norm by the same argument as in Theorem 4.1. Therefore, using (4.1), we get that ∇U_k is a family of equicontinuous functions. Hence, for some subsequence that we denote by U_k, it follows that U_k converges to some function U_0 in the C^1 norm. Therefore, $U_{B'} := U_0 + m$ is a solution of $(\tilde{\mathrm{P}}_{B'})$, with $U_{B'} = m$ on $\partial B'$, and $U_{B'} \geq U$, proving the first part.

Suppose now that $U_{B'}$ is not trivial. According to the proof of Lemma 4.3, $U_{B'} = w_0(|x|)$ for some nonnegative nonincreasing function $w_0 : [0, R_0] \to \mathbb{R}$. If $w_0(r^*) = 0$ for some $r^* \in [0, R_0)$, then $w'(r^*) = 0$ since w is differentiable. But, this contradicts Lemma 2.6 and the fact that $f(U_{B'})$ is positive in some nontrivial set. Then $U_{B'}$ is positive in B'. If U is a radial solution in B'' different from $U_{B'}$, then these functions are different at any point, otherwise $U(\bar{x}) = U_{B'}(\bar{x})$ and $\nabla U(\bar{x}) = \nabla U_{B'}(\bar{x})$ for some $\bar{x} \in B''$ (since $U \leq U_{B'}$) contradicting the uniqueness of solution for ODE. In the case $\bar{x} = 0$, the uniqueness is a consequence of Proposition A4 of [29] that requires (H7). □

Remark 4.8 If (H7) is not satisfied in Theorem 4.1 or 4.7, we still have the existence of U_B such that $U_B \geq U$, as we will see in the next section as a particular case of the main theorem. However, we cannot guarantee the strict inequality. Maybe it is possible that $U_B(0) = U(0)$ and $U_B \not\equiv U$, since (H7) is important for uniqueness of solution for (4.3).

5 Estimates for sublinear equations

The main result in this section is Theorem 5.6. One of the difficulties in proving it is that the operator $w \mapsto -\operatorname{div}(a(w, \nabla w))$ is non-homogeneous in general. For instance, the homogeneity of $-\Delta_p w$ is essential in the proof of (3.6) in Theorem 3.3. So we first present a result, where $|\tilde{a}(t, z)|$ is homogeneous for small z. The idea in its proof is to show that for any solution u and t close to the maximum of u, there exists some radially symmetric solution U_t that is above $u^\sharp$ in $\{u^\sharp \geq t\}$ and $U_t = u^\sharp$ on $\partial\{u^\sharp \geq t\}$. Then, using the results of Sect. 4, we prove that minimum of the set of $t's$, for which such U_t exists, is zero.

Proposition 5.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, B be the ball centered at the origin with $|B| = |\Omega|$, and suppose that a and f satisfy hypotheses (H1)–(H5) and $\tilde{a}$ satisfies (H3)–(H4), possibly with different constants $(\tilde{C}_s, \tilde{C}_*, \tilde{C}^*)$ and different powers $(\tilde{p}, \tilde{q}, \tilde{q}_0)$. Assume also that a or $\tilde{a}$ satisfies (H6), $\tilde{a}(t, z) \cdot z \leq a(t, z) \cdot z$ for any $z \in \mathbb{R}^n$ and $\tilde{a}(t, z) \cdot z = \tilde{C}_s |z|^{\tilde{q}_0}$ for $|z| < \delta$, where $\delta \in (0, 1)$. Then, there exists a radial solution $U_B \in W_0^{1,p}(B)$ of $(\tilde{P}_B)$ s.t. for any solution u of (P_Ω),*

$$U_B \geq u^\sharp \quad \text{in } \Omega^\sharp.$$

If a and $\tilde{a}$ do not satisfy $(H6)$, we can also guarantee the existence of such U_B for the case $\Omega = B$ in the set of radially symmetric solutions.

Remark 5.2 To prove this proposition, we need some existence result to build the solutions U_t that we mentioned before. For that we define in the next lemma a function a^* that depends only on the variable z and which is related to $\tilde{a}$. Hence, we can apply classical techniques to minimize some functional associated to a^*, showing the existence of such U_t.

Lemma 5.3 *There exists a function $a^*(z) \in C^0(\mathbb{R}^n; \mathbb{R}^n) \cap C^1(\mathbb{R}^n \backslash \{0\}; \mathbb{R}^n)$, of the form $a^*(z) = b^*(|z|)z/|z|$, where $b^* \in C^1(\mathbb{R}\backslash\{0\})$ is positive on $\mathbb{R}\backslash\{0\}$, $a^*(0) = 0$, $a^*(z) \cdot z$ is convex, that satisfies*

- $|a^*| \leq |\tilde{a}|$,
- $a^*(z) \cdot z = \tilde{C}_s |z|^{\tilde{q}_0} = \tilde{a}(t, z) \cdot z$ *for* $|z| < \delta$,
- $a^*(z) \cdot z \geq \eta\, \tilde{C}_* |z|^{\tilde{q}}$ *for* $|z| \geq 1$, *where* $\eta \in (0, 1)$,
- $a^*(z) \cdot z = \eta\, \tilde{C}_* |z|^{\tilde{q}}$ *for z large.*

Proof For that, define b^* in $[0, \delta]$ by $b^*(s) = \tilde{C}_s s^{\tilde{q}_0 - 1}$. Then, extend $s\, b^*(s)$ linearly to $[\delta, 1]$ in such a way that it is C^1 in $[0, 1]$. Defining $a^*(z) = b^*(|z|)z/|z|$, we have that $|a^*| \leq |\tilde{a}|$ in $B_1(0)$ from the convexity of $\tilde{a}(t, z) \cdot z$. Let $h = b^*(1)$ and $\eta' < \min\{1, h/\tilde{C}_*\}$. Hence, $s\, b^*(s)|_{s=1} > \eta'\, \tilde{C}_* s^{\tilde{q}}|_{s=1}$ and we can extend $s\, b^*(s)$ linearly until the graph $(s, s\, b^*(s))$ reaches $(s, \eta' \tilde{C}_* s^{\tilde{q}})$ at some point s_0. So define $b^*(s)$ that satisfies $s\, b^*(s) < \eta' \tilde{C}_* s^{\tilde{q}}$ for $s > s_0$, $s\, b^*(s)$ is convex and $s\, b^*(s) = \eta' \tilde{C}_* s^{\tilde{q}}/2$ for s large. Taking $\eta = \eta'/2$, the function $a^*(z)$ defined from b^* as before, fulfills the requirements. □

Lemma 5.4 *Assume the same hypotheses as in the previous proposition, except (H6), and that u is a solution of (P_Ω). Then there exists $t_0 \leq \sup u$, an open ball B^* centered at 0 with the same measure as $\{u \geq t_0\}$, and a radial solution U_{t_0} for*

$$\begin{cases} -\text{div}\, \tilde{a}(V, \nabla V) = f(V) & \text{in } B^* \\ V = t_0 & \text{on } \partial B^* \end{cases} \tag{5.1}$$

such that $U_{t_0} \geq u^\sharp$ in B^*.

Proof Let $M = \text{ess}\sup u > 0$, that is finite by Lemma 2.4.

Possibility 1: $|\{u = M\}| > 0$

Let r_0 be such that the ball $B^* = B_{r_0}(0)$ has the same measure as $\{u = M\}$. Applying Theorem 4.1 or Theorem 4.7 for $B' = B_{r_0}$ and $m = M$, there exists some maximal solution $U_{B'}$ for (5.1) with $t_0 = M$. Then, the result follows taking $t_0 = M$ and $U_{t_0}(x) = U_{B'}$.

Possibility 2: $|\{u = M\}| = 0$

Since f is locally Lipschitz and positive in some neighborhood of M, there exists some $\varepsilon_0 > 0$ such that, for any $\varepsilon \leq \varepsilon_0$, the function

$$G_\varepsilon(t) := \frac{f(t)}{(t-(M-\varepsilon))^{\tilde{q}_0-1}}$$

is decreasing on $(M-\varepsilon, M+\varepsilon_0)$.

Part 1: For $\varepsilon' \leq \varepsilon_0$ small and $t_1 \in (M-\varepsilon', M)$, there is a solution U_{t_1} to the problem (5.1) with t_0 replaced by t_1 such that $|\{U_{t_1} > t_1\}| = \mu_u(t_1)$, $\sup U_{t_1} < M + \varepsilon_0$ and $|\nabla U_{t_1}| \leq \delta$, where δ is given in Proposition 5.1.

To prove this, observe that the definition of M implies that $\mu_u(t) > 0$ for $t \in (M-\varepsilon_0, M)$. For $t_1 \in (M-\varepsilon_0, M)$, let r_1 be such that the ball $B_{r_1}(0)$ satisfies $|B_{r_1}(0)| = \mu_u(t_1)$. Using the same argument as in the Possibility 1, there exists a radial solution U_{t_1} for (5.1) with t_0 and B_{r_0} replaced by t_1 and B_{r_1}. We have that $U_{t_1} - t_1$ is a solution of

$$-\mathrm{div}\, \bar{a}(U, \nabla U) = \tilde{f}(U),$$

where $\bar{a}(t, z) = \tilde{a}(t + t_1, z)$ and $\tilde{f}(t) = f(t + t_1)$, that vanishes on $\partial B_{r_1}(0)$. Note that $\bar{a}$ and $\tilde{f}$ satisfy (H1)–(H5) (the constants associated to $\tilde{f}$ are $\alpha' \in (\alpha, \tilde{C}_*\lambda_B)$ and β' as in the proof of Theorem 4.1). Hence, (2.3) implies that $\sup U_{t_1} - t_1 \leq C|B_{r_1}(0)|^\sigma$, where $C = C(n, \tilde{q}, \alpha', \beta', \frac{\eta\tilde{C}_*}{\tilde{q}_0}, |\Omega|) > 0$, η is associated to a^* from Lemma 5.3, and $\sigma = 1/q$ if $q > n$ or $\sigma = 1/n$ if $q \leq n$. (Since $\eta \in (0, 1)$ and $\tilde{q}_0 > 1$, any operator $\bar{a}$ satisfying $\bar{a}(t, z) \cdot z \geq \tilde{C}_*|z|^{\tilde{q}}$ also satisfies $\bar{a}(t, z) \cdot z \geq \frac{\eta\tilde{C}_*}{\tilde{q}_0}|z|^{\tilde{q}}$. Thus, we can consider $C = C(n, \tilde{q}, \alpha', \beta', \frac{\eta\tilde{C}_*}{\tilde{q}_0}, |\Omega|) \geq C_1 := C_1(n, \tilde{q}, \alpha', \beta', \tilde{C}_*, |\Omega|)$ and we can take C instead C_1). Therefore,

$$\sup U_{t_1} \leq C(\mu_u(t_1))^\sigma + t_1 \leq C(\mu_u(t_1))^\sigma + M.$$

For $\varepsilon_1 \leq \varepsilon_0$ that will be defined later, since

$$\lim_{t\to M^-} \mu_u(t) = |\{u = M\}| = 0,$$

we get $(\mu_u(t))^\sigma < \varepsilon_1/C$ for $t \in (M-\varepsilon', M)$, where $\varepsilon' \leq \varepsilon_0$ is small enough. Thus, $\sup U_{t_1} < M + \varepsilon_0$. For $t \geq t_1$, define $r(t)$ such that $\partial B_{r(t)}(0) = \{U_{t_1} = t\}$. Then, in the case $|\nabla U_{t_1}(x)| \leq 1$, (H4) and Lemma 2.6 imply that

$$\begin{aligned} n\omega_n r(t)^{n-1}\tilde{C}_s|\nabla U_{t_1}(x)|^{\tilde{q}_0} &\leq \int_{\partial B_{r(t)}} |\tilde{a}(U_{t_1}, \nabla U_{t_1})|\, \mathrm{d}H^{n-1} \\ &= \int_{B_{r(t)}(0)} f(U_{t_1})\, \mathrm{d}x \leq \omega_n r(t)^n f(M+\varepsilon_0), \end{aligned}$$

for $x \in \{U_{t_1} = t\}$. From this estimate and $|B_{r(t)}| \leq |B_{r_1}| = \mu_u(t_1) < (\varepsilon_1/C)^{\frac{1}{\sigma}}$,

$$|\nabla U_{t_1}(x)| \leq \left(\frac{\varepsilon_1}{C\omega_n^\sigma}\right)^{\frac{1}{\sigma n\tilde{q}_0}} \left(\frac{f(M+\varepsilon_0)}{n\tilde{C}_s}\right)^{\frac{1}{\tilde{q}_0}} \quad \text{for } x \in B_{r_1}(0).$$

In the case $|\nabla U_{t_1}(x)| > 1$, a similar estimate holds replacing $\tilde{C}_s$ by $\tilde{C}_*$ and $\tilde{q}_0$ by $\tilde{q}$. Any way, taking ε_1 small, $|\nabla U_{t_1}(x)| \leq \delta$, where δ is given in hypothesis of Proposition 5.1. Therefore, U_{t_1} satisfies the $\tilde{q}_0$-Laplacian equation

$$-\tilde{C}_s \Delta_{\tilde{q}_0} U_{t_1} = f(U_{t_1}) \quad \text{in } B_{r_1}. \tag{5.2}$$

Part 2: U_{t_1} is the minimizer of the functional

$$I_{t_1}(V) := \int_{B_{r_1}} \frac{\nabla V \cdot \tilde{a}(V, \nabla V)}{\tilde{q}_0} - \bar{F}(V) \, \mathrm{d}x$$

in the space $E = \{V \in W^{1,\tilde{q}}(B_{r_1}) \mid V = t_1 \text{ on } \partial B_{r_1}\}$, where $\bar{F}(t) = \int_0^t \bar{f}(s)\, \mathrm{d}s$,

$$\bar{f}(s) = \begin{cases} f(s) & \text{if } s \le M + \varepsilon_0 \\ f(M+\varepsilon_0) & \text{if } s > M + \varepsilon_0. \end{cases}$$

For that, consider a^* with the properties stated in the Lemma 5.3. Therefore,

$$I^*_{t_1}(V) \le I_{t_1}(V) \quad \text{for} \quad V \in E,$$

where $I^*_{t_1}$ is defined replacing $\tilde{a}$ by a^* in the definition of I_{t_1}. From the growth conditions on a^* and $\bar{f}$, we can use classical techniques to prove that $I^*_{t_1}$ has a global minimum $U^* \in E$. Moreover, this minimum is a solution of

$$-\mathrm{div}\, \hat{a}(\nabla V) = \bar{f}(V) \quad \text{in} \quad B_{r_1},$$

where

$$\hat{a}(\mathrm{z}) := \frac{a^*(\mathrm{z}) + \mathrm{z} \cdot Da^*(\mathrm{z})}{\tilde{q}_0}.$$

Observe that $\hat{a}(\mathrm{z}) \cdot \mathrm{z} \ge a^*(\mathrm{z}) \cdot \mathrm{z}/\tilde{q}_0$ since $s \mapsto |a^*(s\mathrm{z})|$ is increasing from (H3). Hence $\hat{a}$ and $\bar{f}$ satisfy (H1), (H5), $\eta\tilde{C}_*/\tilde{q}_0(|\mathrm{z}|^q - 1) \le \hat{a}(\mathrm{z}) \cdot \mathrm{z}$ for $\mathrm{z} \in \mathbb{R}^n, t \in \mathbb{R}$ where the important constants in order to apply (2.3) are $n, \tilde{q}, \alpha, \beta, \eta\tilde{C}_*/\tilde{q}_0$ and $|\Omega|$. Then, as in Part 1, $\sup U^* - t_1 < C|B_{r_1}(0)|^\sigma$, where $C = C(n, \tilde{q}, \alpha', \beta', \frac{\eta\tilde{C}_*}{\tilde{q}_0}, |\Omega|)$ is the same constant as before. (Now it is clear why we chose a constant C depending on $\eta\tilde{C}_*/\tilde{q}_0$ instead of $\tilde{C}_*$ at that moment.) Thus, $\sup U^* < M + \varepsilon_0$ and, following the same computations as before, $|\nabla U^*| < \delta$. Then, from $a^*(t, \mathrm{z}) = \tilde{a}(t, \mathrm{z})$ for $|\mathrm{z}| < \delta$, it follows that

$$I^*_{t_1}(U^*) = I_{t_1}(U^*) \tag{5.3}$$

and, therefore, U^* is also a global minimizer of I_{t_1}. From $a^*(t, \mathrm{z}) = \tilde{C}_s|\mathrm{z}|^{\tilde{q}_0-2}\mathrm{z}$ for $|\mathrm{z}| \le \delta$, we have that U^* is also a solution of (5.2). Hence $U_{t_1} - t_1$ and $U^* - t_1$ are solutions of $-\tilde{C}_s\Delta_{\tilde{q}_0} U = \tilde{f}(U)$. Taking $\varepsilon = M - t_1$, we have that $\tilde{f}(t)/t^{\tilde{q}_0-1} = G_\varepsilon(t + t_1)$ that is decreasing on $(0, \varepsilon + \varepsilon_0)$, which contains the range of $U_{t_1} - t_1$ and $U^* - t_1$. From the uniqueness result of [13], $U_{t_1} = U^*$.

Part 3: For $t_1 \in (M - \varepsilon', M)$, there exists $t_0 \ge t_1$ and a solution U of (5.1) s.t. $U \ge u^\sharp$ in $B_{r(t_0)} := \{u^\sharp > t_0\}$, $U = u^\sharp$ on $\partial B_{r(t_0)}$ and $|\{U > t_0\}| = |\{u^\sharp > t_0\}|$.

Using $a^*(\mathrm{z}) \cdot \mathrm{z} \le \tilde{a}(t, \mathrm{z}) \cdot \mathrm{z} \le a(t, \mathrm{z}) \cdot \mathrm{z}$ and $\bar{F}(u^\sharp) = F(u^\sharp)$, the Pólya-Szegö principle for $a^*(\mathrm{z}) \cdot \mathrm{z}$, that $U_{t_1} = U^*$ minimizes I_{t_1}, $I^*_{t_1}$, and (5.3), we get

$$\int_{\Omega_{t_1}} \frac{\nabla u \cdot a(u, \nabla u)}{\tilde{q}_0} - F(u)\,\mathrm{d}x \geq \int_{\Omega_{t_1}} \frac{\nabla u \cdot a^*(\nabla u)}{\tilde{q}_0} - F(u)\,\mathrm{d}x$$
$$\geq \int_{B_{r_1}} \frac{\nabla u^\sharp \cdot a^*(\nabla u^\sharp)}{\tilde{q}_0} - \bar{F}(u^\sharp)\,\mathrm{d}x$$
$$\geq \int_{B_{r_1}} \frac{\nabla U_{t_1} \cdot \tilde{a}(U_{t_1}, \nabla U_{t_1})}{\tilde{q}_0} - \bar{F}(U_{t_1})\,\mathrm{d}x.$$

Hence, from Lemma 2.5 and $\bar{F}(U_{t_1}) = F(U_{t_1})$, we have

$$\int_{\Omega_{t_1}} \frac{(u - t_1) f(u)}{\tilde{q}_0} - F(u)\,\mathrm{d}x \geq \int_{B_{r_1}} \frac{(U_{t_1} - t_1) f(U_{t_1})}{\tilde{q}_0} - F(U_{t_1})\,\mathrm{d}x,$$

that is equal to estimate (3.6). Note also that

$$h_{t_1}(s) = \frac{(s - t_1) f(t)}{\tilde{q}_0} - F(s)$$

is decreasing in $(t_1, M+\varepsilon_0)$ since $G_\varepsilon(s)$ is decreasing in this interval, where $\varepsilon = M - t_1 < \varepsilon_0$. Therefore, using that $U_{t_1}(B_{r_1}), u(B_{r_1}) \subset [t_1, M + \varepsilon_0)$ and an argument similar to the one that come after (3.6), we have

$$\max u \leq \max U_{t_1}.$$

If $u^\sharp \leq U_{t_1}$ in B_{r_1}, Part 3 is proved taking $t_0 = t_1$. Otherwise, there exist $t_2 \in (t_1, M)$ such that $\mu_u(t_2) > \mu_{U_{t_1}}(t_2)$. Therefore, $B' = \{u^\sharp > t_2\}$ and $B'' = \{U_{t_1} > t_2\}$ are concentric balls satisfying $|B'| > |B''|$. Hence, from Theorem 4.1 or 4.7, there exists some solution U_{t_2} of (5.1) with t_0 replaced by t_2, such that $\{U_{t_2} > t_2\} = B'$ and $U_{t_2} > U_{t_1}$ in B''. Since

$$\max u^\sharp \leq \max U_{t_1} < \max U_{t_2},$$

it follows from the right continuity of μ_u and the continuity of $\mu_{U_{t_2}}$ that there exists $t_0 \geq t_2$, such that $|\{U_{t_2} > t_0\}| = |\{u^\sharp > t_0\}|$ and $U_{t_2} \geq u^\sharp$ in $\{u^\sharp > t_0\}$, proving this part.
Part 4: There exists a solution U_{t_0} of (5.1) s.t. $U_{t_0} \geq u^\sharp$ in $B^* := \{u^\sharp \geq t_0\}$, $U = u^\sharp$ on ∂B^* and $|\{U_{t_0} \geq t_0\}| = |\{u^\sharp \geq t_0\}|$.
Let t_0 and U as in Part 3. If $|\{u^\sharp \geq t_0\}| = \mu_u(t_0)$, then the theorem is proved with $B^* = \{u^\sharp > t_0\}$. Otherwise, applying Theorem 4.1 or 4.7 for $B' = \{u^\sharp \geq t_0\}$ and $B'' = \{u^\sharp > t_0\}$, there exists a solution U_{t_0} of (5.1) s.t. $U_{t_0} > U$ in B'', proving the result with $B^* = B'$. □

Now we present a result that resembles a maximum principle for distribution function in the following sense: if the distribution μ_u of a solution satisfies $\mu_u \leq \mu_U$, where μ_U is the distribution of a radial solution, and $\mu_u(t_0) = \mu_U(t_0)$ for some $t_0 \leq \max U$, then $\mu_u(t) = \mu_U(t)$ for any $t \leq t_0$.

Proposition 5.5 *Suppose that* a, $\tilde{a}$, *and* f *satisfy (H2)–(H4), where the constants and powers presented in* $(H4)$ *associated to* $\tilde{a}$ *are given by* $(\tilde{C}_s, \tilde{C}_*, \tilde{C}^*)$ *and* $(\tilde{p}, \tilde{q}, \tilde{q}_0)$, a *or* $\tilde{a}$ *satisfies (H6), and that* $\tilde{a}(t, z) \cdot z \leq a(t, z) \cdot z$ *for any* $z \in \mathbb{R}^n$. *Assume also that* $u \in W_0^{1,p}(\Omega)$ *is a solution of* (P_Ω) *and* $U \in W^{1,p}(B) \cap C^1(B)$ *is a radial positive solution of* $(\tilde{\mathrm{P}}_B)$ *that not necessarily vanishes on* ∂B. *If* $u^\sharp \leq U$ *and* $u^\sharp \not\equiv U$, *then there exists* $t_1 \geq 0$ *such that* $u^\sharp < U$ *in* $\{U > t_1\}$ *and* $u^\sharp = U$ *in* $\{U \leq t_1\}$ *(that can be empty if* $U > 0$ *on* ∂B*).*

This conclusion also holds if a *and* $\tilde{a}$ *does not satisfy* $(H6)$, *but* $\Omega = B$ *and* $u = u^\sharp$.

Moreover, assuming that $u^\sharp \le U$, if f is strictly increasing and $u^\sharp \not\equiv U$, or Ω is not a ball and $a = a(\mathrm{z})$(or $\tilde{a} = \tilde{a}(\mathrm{z})$) satisfies hypotheses of Proposition 2.7, then $u^\sharp < U$ in B.

Proof Since $U \ge u^\sharp$ and f is nondecreasing, we have

$$\int_{\{U>t\}} f(U)\ \mathrm{d}x \ge \int_{\{u^\sharp>t\}} f(u^\sharp)\,\mathrm{d}x = \int_{\{u>t\}} f(u)\,\mathrm{d}x, \tag{5.4}$$

for any $t \ge 0$. Hence, applying Lemma 2.6 for u and U, we get

$$\int_{\{U=t\}} \frac{\tilde{a}(U,\nabla U)\cdot\nabla U}{|\nabla U|}\ \mathrm{d}H^{n-1} \ge \int_{\{u=t\}} \frac{a(u,\nabla u)\cdot\nabla u}{|\nabla u|}\ \mathrm{d}H^{n-1} \tag{5.5}$$

for almost every $t \ge \inf U$. Now suppose that a satisfies (H6). Observe that from the coarea formula,

$$\int_{\{u>t\}} a(u,\nabla u)\cdot\nabla u\ \mathrm{d}x = \int_0^t h_1(s)\ \mathrm{d}s \quad \text{and} \quad \int_{\{u>t\}} |\nabla u| + |\nabla u|^p\ \mathrm{d}x = \int_0^t h_2(s)\ \mathrm{d}s,$$

where

$$h_1(t) = \int_{\{u=t\}} \frac{a(u,\nabla u)\cdot\nabla u}{|\nabla u|}\,\mathrm{d}H^{n-1} \quad \text{and} \quad h_2(t) = \int_{\{u=t\}} 1 + |\nabla u|^{p-1}\,\mathrm{d}H^{n-1}.$$

The same identity holds if we replace u by $u^\sharp$. For this one, denote $h_1^\sharp, h_2^\sharp$ instead of h_1 and h_2. From the Lebesgue Differentiation Theorem, almost every $t \in [0, \sup u]$ is a Lebesgue point of $h_1, h_2, h_1^\sharp$, and $h_2^\sharp$. For such t, define $B(\mathrm{z}) = a(t,\mathrm{z})\cdot\mathrm{z}$. Applying Pólya-Szegö inequality for B, we have

$$\int_{\{t<u\le s\}} a(t,\nabla u)\cdot\nabla u\ \mathrm{d}x \ge \int_{\{t<u^\sharp\le s\}} a(t,\nabla u^\sharp)\cdot\nabla u^\sharp\ \mathrm{d}x \tag{5.6}$$

Moreover, from (H6), we have for $s > t$,

$$\int_{\{t<u\le s\}} \frac{|a(u,\nabla u)\cdot\nabla u - a(t,\nabla u)\cdot\nabla u|}{s-t}\ \mathrm{d}x \le \omega(|s-t|) \int_{\{t<u\le s\}} \frac{|\nabla u| + |\nabla u|^p}{s-t}\ \mathrm{d}x.$$

The integral in the right-hand side converges, since t is a Lebesgue point of h_2. Hence, the right-hand side goes to zero as $s \to t$ and, therefore,

$$\lim_{s\to t} \int_{\{t<u\le s\}} \frac{a(t,\nabla u)\cdot\nabla u}{s-t}\ \mathrm{d}x = \lim_{s\to t} \int_{\{t<u\le s\}} \frac{a(u,\nabla u)\cdot\nabla u}{s-t}\ \mathrm{d}x = h_1(t).$$

In the same way,

$$\lim_{s\to t} \int_{\{t<u^\sharp\le s\}} \frac{a(t,\nabla u^\sharp)\cdot\nabla u^\sharp}{s-t}\ \mathrm{d}x = h_1^\sharp(t).$$

Using these two relations, (5.6) and (5.5), it follows that

$$\int_{\{U=t\}} \frac{\tilde{a}(U,\nabla U)\cdot\nabla U}{|\nabla U|}\,\mathrm{d}H^{n-1} \ge \int_{\{u^\sharp=t\}} \frac{a(u^\sharp,\nabla u^\sharp)\cdot\nabla u^\sharp}{|\nabla u^\sharp|}\,\mathrm{d}H^{n-1} \tag{5.7}$$

for almost every $t \geq \inf U$. Since $\tilde{a}(t, z) \cdot z \leq a(t, z) \cdot z$, we have the same inequality with a or $\tilde{a}$ appearing in both sides. If $\tilde{a}$ satisfies (H6) instead of a, from $\tilde{a}(t, z) \cdot z \leq a(t, z) \cdot z$ we can replace a by $\tilde{a}$ in (5.5). Hence, repeating the argument for $\tilde{a}$, we obtain (5.7) with $\tilde{a}$ in both sides. If a and $\tilde{a}$ does not satisfy ($H6$), but $\Omega = B$ and $u = u^\sharp$, then (5.7) is an immediate consequence of (5.5). In any situation (5.7) holds. Letting $r_1 = (\mu_u(t)/\omega_n)^{1/n}$ and $r_2 = (\mu_U(t)/\omega_n)^{1/n}$, we have some t_0 such that $r_1(t_0) < r_2(t_0)$ since $u^\sharp \not\equiv U$. Hence, Lemma 2.8 implies that $u^\sharp < U$ on $\{U > t_0\}$. Indeed, we can infer that the set of $t's$, for which $r_1(t) = r_2(t)$, is an interval that contains 0. Denoting the supremum of this set by t_1, we have the first part of the result.

Now consider the case f is strictly increasing and $t_1 > 0$. Then we have a strict inequality in (5.4) and, therefore, in (5.7) for any $t \in [0, t_1]$, that contradicts $u^\sharp = U$ in $\{0 \leq U < t_1\}$.

If a is as stated in Proposition 2.7, it follows from $\tilde{a}(z) \cdot z \leq a(z) \cdot z$, (5.4), Lemma 2.6, and Pólya-Szegö principle that

$$\int_{U<t} \nabla U \cdot a(\nabla U)\, dx \geq \int_{u<t} \nabla u \cdot a(\nabla u)\, dx \geq \int_{u^\sharp<t} \nabla u^\sharp \cdot a(\nabla u^\sharp)\, dx,$$

for $t < t_1$. Since $u^\sharp = U$ in $\{U < t_1\}$, the three integrals are equal for $t < t_1$, and therefore, Proposition 2.7 implies that $u^\sharp$ is a translation of u in $\{u < t_1\}$ and Ω is a ball, which is an absurd. Replacing a by $\tilde{a}$, we see that the same conclusion holds if $\tilde{a}$ satisfies the hypotheses of that proposition. □

Proof of Proposition 5.1 Observe that $\tilde{a}$ and f satisfy (H1)–(H5). Furthermore, $\tilde{a}$ also satisfy ($H7$), since $|\tilde{a}(t, z)| = \tilde{C}_s |z|^{\tilde{q}_0 - 1}$ for z small. Then let U_B be the solution stated in Theorem 4.1 or in Theorem 4.7 for $m = 0$. Consider the set

$$A = \{t_0 : \exists \text{ a radial sol. } U_{t_0} \text{ of (5.1) s.t. } U_{t_0} \geq u^\sharp \text{ in } B^* \text{ and } |B^*| = |\{u^\sharp \geq t_0\}|\}.$$

According to the previous lemma, this set is not empty. To prove the theorem, it suffices to show that $0 \in A$. For that we prove the following assertions.

Assertion 1: For any positive $t_1 \in A$, there exists $t' \in A$ such that $t' < t_1$.

From the definition of A, there exists a radial solution U_{t_1} of (5.1) greater than or equal to $u^\sharp$ in $\{u^\sharp \geq t_1\}$. Since U_{t_1} is radial, it can be extended as a positive radial solution of $-\text{div}(\tilde{a}(V, \nabla V)) = f(V)$ in some ball that contains $\{u^\sharp \geq t_1\}$ or in $\mathbb{R}^n$. The maximal extension will be denoted by U_{t_1}. Consider

$$D = \{t \geq 0 : \ |\{U_{t_1} > t\}| = |\{u \geq t\}| \text{ and } |\{U_{t_1} > s\}| \geq |\Omega_s| \text{ for } s > t\},$$

and let $t_2 = \inf D$. Observe that $t_1 \in D$ and so $t_2 \leq t_1$. If $t_2 < t_1$, then there exists $t_3 \in [t_2, t_1) \cap D$. Hence, in this case, our assertion is proved taking $t' = t_3$. Consider now the case $t_2 = t_1$. Thus $0 \notin D$, since $0 < t_1 = t_2$. Therefore, there are two possibilities:

(1) $|\{U_{t_1} > 0\}| > |\Omega|$ and $|\{U_{t_1} > s\}| \geq |\Omega_s|$ for $s > 0$;
(2) $|\{U_{t_1} > s_0\}| < |\Omega_{s_0}|$ for some $s_0 \geq 0$.

Case 1): since $|\{U_{t_1} > s\}| \geq \mu_u(s)$ for $s > 0$, $U_{t_1} \geq u^\sharp$. Then, from the first part of Proposition 5.5, $U_{t_1} = u^\sharp$ in $\{U_{t_1} < t_2\}$, since $U_{t_1} = u^\sharp$ in $\{U_{t_1} = t_2\}$. (This is the only time in this proof we use that either a and $\tilde{a}$ satisfies (H6) or $\Omega = B$ and u is radially symmetric.) However, this contradicts $|\{U_{t_1} > 0\}| > |\Omega|$, and so this case is not possible.

Case 2): from the definition of t_1, it follows that $s_0 < t_1$. Let $B'_{s_0}(0)$ be a ball such that $|B'_{s_0}| = |\{u \geq s_0\}|$. Hence $B' = B'_{s_0}(0)$ and $B'' = \{U_{t_1} > s_0\}$ satisfy $|B'| > |B''|$, and from Theorem 4.1 or 4.7, there exists a solution U_{s_0} of ($\tilde{\mathrm{P}}_{B'}$) with $U_{s_0} = s_0$ on $\partial B'$, such that $U_{s_0} > U_{t_1}$ in B''. Then $U_{s_0} > U_{t_1} \geq u^\sharp$ in $\{U_{t_1} > t_1\}$, and therefore,

$$\mu_{U_{s_0}}(t_1) = |\{U_{s_0} > t_1\}| > |\{U_{t_1} > t_1\}| = |\{u \geq t_1\}| = \mu_u(t_1^-).$$

Since $\mu_{U_{s_0}}$ is continuous and $\mu_u(t_1^-) = \lim_{t\to t_1^-} \mu_u(t)$, we have $\mu_{U_{s_0}}(t) > \mu_u(t)$ for $s_0 < t < t_1$, sufficiently close to t_1. Defining

$$t' = \inf\{t \geq s_0 \ : \ \mu_{U_{s_0}}(s) > \mu_u(s), \ \text{for } t_1 \geq s > t\},$$

it follows that $s_0 \leq t' < t_1$ and $\mu_u(t') \leq \mu_{U_{s_0}}(t') \leq \mu_u(t'^-)$. Observe also that $U_{s_0} > u^\sharp$ in $\{u^\sharp > t'\}$. Hence, this assertion is proved if $\mu_{U_{s_0}}(t') = \mu_u(t'^-)$. If $\mu_{U_{s_0}}(t') < \mu_u(t'^-)$, applying Theorem 4.1 or Theorem 4.7 for the balls $\{U_{s_0} > t'\} \subsetneq \{u^\sharp \geq t'\}$, we get a solution $U_{t'}$ s.t. $U_{t'} > U_{s_0}$ in $\{U_{s_0} > t'\}$ and $|\{U_{t'} > t'\}| = |\{u^\sharp \geq t'\}|$. Then $U_{t'} > u^\sharp$ in $\{u^\sharp \geq t'\}$ and $U_{t'} = u^\sharp$ on $\partial\{u^\sharp \geq t'\}$, completing Assertion 1.

Assertion 2: If $t_1 = \inf A$, then $t_1 \in A$.

We can prove this using the same limit argument as in Lemma 4.4.

These assertions imply that $\inf A = 0$. Then there is a solution U_0 of ($\tilde{\mathrm{P}}_B$) such that $U_0 \geq u^\sharp$. Since U_B is maximal, it follows that $U_0 \leq U_B$, proving the result. □

Theorem 5.6 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, B be a ball centered at the origin with $|B| = |\Omega|$, and suppose that $a, \tilde{a}$, and f satisfy the hypotheses (H1)–(H5), where the constants and powers associated to a and $\tilde{a}$ may be different, and a or $\tilde{a}$ satisfies (H6). If $\tilde{a}(t, \mathrm{z}) \cdot \mathrm{z} \leq a(t, \mathrm{z}) \cdot \mathrm{z}$ for any $\mathrm{z} \in \mathbb{R}^n$, then there exists a radial solution $U_B \in W_0^{1,p}(B)$ of ($\tilde{\mathrm{P}}_B$) such that*

$$U_B \geq u^\sharp \quad in \quad B,$$

where $u^\sharp$ is the symmetrization of any solution u of (P_Ω). U_B does not depend on a.

Furthermore, if Ω is not a ball, one of these solutions is positive, and $a = a(\mathrm{z})$ (or $\tilde{a} = \tilde{a}(\mathrm{z})$) is as stated in Proposition 2.7, then $U_B > u^\sharp$.

Proof For $k \in \mathbb{N}$, let $a_k(t, \mathrm{z}) = b_k(t, |\mathrm{z}|)\mathrm{z}/|\mathrm{z}|$ be a function satisfying (H3) s.t.

- $|a_k| \leq |\tilde{a}|$,
- $a_k(t, \mathrm{z}) \cdot \mathrm{z} = C|\mathrm{z}|^{\tilde{q}_0}$ for some $C > 0$ and $|\mathrm{z}| \leq 1/k$,
- $a_k(t, \mathrm{z}) \cdot \mathrm{z} = \tilde{a}(t, \mathrm{z}) \cdot \mathrm{z}$ for $|\mathrm{z}| \geq 2/k$.

To obtain such a_k, first observe that the convexity of $\tilde{a}(t, \mathrm{z}) \cdot \mathrm{z}$ in z and the relation $\tilde{a}(t, \mathrm{z}) \cdot \mathrm{z} \geq \tilde{C}_s|\mathrm{z}|^{\tilde{q}_0}$ imply that the derivative of $s \mapsto \tilde{a}(t, s\mathrm{w}) \cdot s\mathrm{w}$ is uniformly bounded from below by some $D_k > 0$ for $t \in \mathbb{R}$, $|\mathrm{w}| = 1$ and $s = 1/k$. From $\tilde{a}(t, \mathrm{z}) = \tilde{b}(t, |\mathrm{z}|)\mathrm{z}/|\mathrm{z}|$, we get $\partial_s[\tilde{b}(t, s)\, s] \geq D_k$ for $s = 1/k$ and $t \in \mathbb{R}$. Since $\tilde{a}(t, \mathrm{z}) \cdot \mathrm{z}$ in z is convex, $\partial_s[\tilde{b}(t, s)\, s]$ is increasing in s and then $\partial_s\tilde{b}(t, s)\, s \geq D_k$ for $s = 2/k$. Now define $b_k(t, s)$ in $\mathbb{R} \times [0, 1/k]$ by $b_k(t, s) = C_k|s|^{\tilde{q}_0-1}$, where C_k is such that $\partial_s[b_k(t, s)\, s] = D_k/2$ for $s = 1/k$. (Indeed we can chose $D_k = \tilde{C}_s(1/k)^{\tilde{q}_0-1}$ and $C = C_k = \tilde{C}_s/(2\tilde{q}_0)$.) Hence, it is possible to extend b_k to $\mathbb{R} \times [0, +\infty)$ in such a way that $\partial_s[b_k(t, s)\, s]$ is strictly increasing in s, continuous and $b_k(t, s) = \tilde{b}(t, s)$ for $s \geq 2/k$. The function a_k defined from b_k satisfies the required properties.

Since a, a_k, and f satisfy the hypotheses of Proposition 5.1, there exists some radial solution $U_k \in W_0^{1,p}(B)$ of $-\mathrm{div}\, a_k(V, \nabla V) = f(V)$ in B that satisfies $U_k \geq u^\sharp$, for any solution

u of (P_Ω). Using (2.3), it follows that the sequence (U_k) is bounded in the L^∞ norm and, following the same argument as in Part 1 of Lemma 5.4, the derivative of U_k is also uniformly bounded and equicontinuous. Hence, some subsequence converges to some function U_B that is a weak solution of ($\tilde{\mathrm{P}}_B$), by usual arguments. Moreover, $U_k \geq u^\sharp$ implies that $U_B \geq u^\sharp$, for any solution u of (P_Ω), completing the first part of the theorem.

To see that U_B does not depend on a, it is enough to show that there exists U_B that is the maximal solution of ($\tilde{\mathrm{P}}_B$) according to the definition (1.2). This is a consequence of Theorem 4.1 or Theorem 4.7 if we assume also $(H7)$. In the case $(H7)$ is not satisfied, we can apply Proposition 5.1 for $\tilde{a}$, a_k, f, and $\Omega = B$, just as we did in the previous paragraph. Then, there is a solution U_B of ($\tilde{\mathrm{P}}_B$) that satisfies $U_B \geq U$ for any radial solution U of ($\tilde{\mathrm{P}}_B$). Observe that we can apply Proposition 5.1 even if $\tilde{a}$ and a_k does not obey (H6), since $\Omega = B$ and U is radial.

Suppose now that Ω is not a ball, $a = a(\mathrm{z})$, and u is a solution of (P_Ω). From the first part, $U_B \geq u^\sharp$ and, therefore, applying Proposition 5.5, $U_B > u^\sharp$. □

6 Existence and bound result

First we apply the results of the previous section to prove that the symmetrization of solutions of (1.4) are bounded by a radial solution. Notice that if h is also bounded from above, the result follows immediately from Theorem 5.6 applied to the equation $-\mathrm{div}(h(v)a(\nabla v)) = f(v)$. For h just bounded from below by some positive constant, the proof is given in the next proposition.

Proposition 6.1 *Suppose that $a_1(t, \mathrm{z}) = h(t)a(\mathrm{z})$ and $f(t) = g(t)h(t)$ satisfy (H1)–(H5), where h is a C^1 function bounded from below by some positive constant. Then there exists a radial function U_B, solution of* (1.4) *when the domain is B, such that $U_B \geq u^\sharp$, where $u^\sharp$ is the symmetrization of any solution of* (1.4)*. This is also true even if $a_1(t, \mathrm{z})$ does not satisfy the right inequality of* (1.1)*, provided $a(\mathrm{z})$ obeys the upper bounds of* $(H4)$.

Proof Let $m = \inf h$ and $a_0(t, \mathrm{z}) = m\, a(\mathrm{z})$. Due to the assumption on a, a_0 satisfies (H3), (H4), and (H6). Moreover, the constant C_* of $(H4)$ can be taken as the same for a_0 and a. Theorem 5.6 implies that there exists a maximal solution U_0 for

$$-m\,\mathrm{div}(a(\nabla V)) = f(V) \quad \text{in} \quad B.$$

Let $M_1 = \max U_0$ and observe that, from (2.3), there is a positive constant M_2 that depends only on n, q, α, β, C_*, and $|\Omega|$ such that $\sup u \leq M_2$, where $u \in W_0^{1,p}(\Omega)$ is any solution of $-\mathrm{div}(a_1(v, \nabla v)) = f(v)$ in Ω. Let $M = \max\{M_1, M_2\}$, h_1 be a C^1 function such that $h_1(t) = h(t)$ for $t \leq M$ and $h_1(t) = h(M+1)$ for $t \geq M+1$, and $a_2(t, \mathrm{z}) = h_1(t)a(\mathrm{z})$. Observe that u is solution of $-\mathrm{div}(a_2(v, \nabla v)) = f(v)$ and a_2 satisfies (H1)–(H6). Hence, from Theorem 5.6, there exists a maximal solution U_B of $-\mathrm{div}(a_2(V, \nabla V)) = f(V)$ in B and $U_B \geq u^\sharp$. Moreover $U_B \leq U_0 \leq M$, since $a_2(\mathrm{z}) \cdot \mathrm{z} \geq a_0(\mathrm{z}) \cdot \mathrm{z}$. Therefore, U_B is also a solution of$-\mathrm{div}(h(V)a(\nabla V)) = f(V)$ completing the proof. □

This result gives a priori estimate of a solution u, but does not prove its existence, except for the ball where we obtain the function U_B. We show now an existence result for a particular case, using this estimate.

Theorem 6.2 *Let $a(\mathrm{z}) = \mathrm{z}|\mathrm{z}|^{p-2}$ and suppose that $a_1 = ha$ and $f = gh$ satisfy (H1)–(H5), with the possibility of not fulfillment of the right inequality of* (1.1)*. Then there exists a solution u to the problem* (1.4).

Proof Let M, h_1, and U_B be as defined before. Define the functional

$$J(v) = \int_\Omega (h_1(v))^{\frac{p}{p-1}} \frac{|\nabla v|^p}{p} - \int_0^v f(s)(h_1(s))^{\frac{1}{p-1}}\, \mathrm{d}s\, \mathrm{d}x.$$

Since h_1 is bounded from above and from bellow by some positive constants, conditions (H4) and (H5) hold with $q = q_0 = p$. Then we can minimize J in $W_0^{1,p}(\Omega)$ and obtain a solution u to $-\mathrm{div}(h_1(v)\nabla v|\nabla v|^{p-2}) = f(v)$. From the previous result, we have that u is bounded by U_B and, therefore, is a solution that we are looking for. □

7 Estimates for eigenfunctions

In the next result, the estimate (7.2) and (1.6) were established in [20] and [21] for $p = q = 2$, with the best constant, and extended in [2] for $p = q > 1$, when λ is the first eigenvalue. We use different techniques that can be applied for more general situations.

Theorem 7.1 *Let $\Omega \subset \mathbb{R}^n$ be an open-bounded set and w be a solution of*

$$\begin{cases} -\Delta_p v = \lambda v|v|^{q-2} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{7.1}$$

where $1 < q \le p$ and λ is either a real number if $q < p$ or any eigenvalue of $-\Delta_p$ with trivial boundary data if $q = p$. Then

$$(\max|w|)^{1+\frac{n(p-q)}{rp}} \le \frac{2}{(\omega_n)^{1/r}} \left(\frac{2(p-1)}{p}\right)^{\frac{n(p-1)}{rp}} \left(\frac{\lambda}{n}\right)^{n/rp} \|w\|_r, \tag{7.2}$$

for any $r > 0$. Furthermore,

$$|\tilde{\Omega}_t| \ge \omega_n(\|w\|_\infty - t)^{\frac{n(p-1)}{p}} \left(\frac{p}{p-1}\right)^{\frac{n(p-1)}{p}} \left(\frac{n}{\lambda}\right)^{n/p} \|w\|_\infty^{\frac{n(1-q)}{p}}, \tag{7.3}$$

where $\tilde{\Omega}_t = \{|w| > t\}$, $t \in [0, \max|w|]$.

Proof Let $M = \|w\|_\infty$, $\rho \ge 1$, and $\Omega_2 = \{x : |w(x)| > M/\rho\}$. Then

$$\|w\|_r^r = \int_\Omega |w|^r\, \mathrm{d}x \ge \int_{\Omega_2} |w|^r \mathrm{d}x \ge \left(\frac{M}{\rho}\right)^r |\Omega_2| \tag{7.4}$$

On the other hand,

$$-\Delta_p w = \lambda w|w|^{q-2} \le \lambda M^{q-1}$$

Hence, by the comparison principle of [24], $|w| \le u$ in Ω_2, where u is solution of

$$\begin{cases} -\Delta_p v = \lambda M^{q-1} & \text{in } \Omega_2 \\ v = \frac{M}{\rho} & \text{on } \partial\Omega_2 \end{cases}$$

Let U be the solution of

$$\begin{cases} -\Delta_p V = \lambda M^{q-1} & \text{in } \ B \\ V = \frac{M}{\rho} & \text{on } \ \partial B \end{cases}$$

where B is a ball such that $|B| = |\Omega_2|$. From Theorem 1 of [45] or Theorem 5.6 (or Theorem 3.7), $u^\sharp \le U$. Then

$$M = \max|w| \le \max u = \max u^\sharp \le \max U$$

We can compute U explicitly:

$$U(x) = \frac{p-1}{p}\left(\frac{\lambda}{n}\right)^{\frac{1}{p-1}} M^{\frac{q-1}{p-1}}\left(R^{\frac{p}{p-1}} - |x|^{\frac{p}{p-1}}\right) + \frac{M}{\rho},$$

where $\omega_n R^n = |\Omega_2| = |B|$. Since $M \le \max U = U(0)$,

$$M \le \frac{p-1}{p}\left(\frac{\lambda}{n}\right)^{\frac{1}{p-1}} M^{\frac{q-1}{p-1}} R^{\frac{p}{p-1}} + \frac{M}{\rho}.$$

Hence,

$$R \ge \left[\frac{(\rho-1)p}{\rho(p-1)}\right]^{\frac{p-1}{p}} \left(\frac{n}{\lambda}\right)^{1/p} M^{\frac{p-q}{p}}.$$

Using this and $R = \left(\frac{|\Omega_2|}{\omega_n}\right)^{1/n}$, we get

$$|\Omega_2| \ge \omega_n \left[\frac{(\rho-1)p}{\rho(p-1)}\right]^{\frac{n(p-1)}{p}} \left(\frac{n}{\lambda}\right)^{n/p} M^{\frac{n(p-q)}{p}}.$$

From this, we get the estimate for $|\Omega_t|$ taking $t = M/\rho$. Moreover, applying this inequality with $\rho = 2$ and using (7.4), it follows that

$$\|w\|_r^r \ge \frac{1}{2^r}\omega_n \left(\frac{p}{2(p-1)}\right)^{\frac{n(p-1)}{p}} \left(\frac{n}{\lambda}\right)^{n/p} M^{\frac{n(p-q)}{p}+r}.$$

□

Remark 7.2 The estimates of this theorem still holds if $|\Delta_p w| \le |\lambda w|w|^{q-2}|$ or, equivalently, $-\Delta_p w = \lambda g(w)$, where $|g(w)| \le |w|^{q-1}$. Hence, using the interpolation inequality,

$$\|w\|_s \le \|w\|_\infty^{1-r/s} \|w\|_r^{r/s}, \qquad \text{for } 0 < r < s \le \infty$$

we get (1.6) for solutions of $-\Delta_p w = \lambda g(w)$, where $|g(w)| \le |w|^{q-1}$, with the boundary condition $w = 0$ on $\partial\Omega$. Inequality (7.2) is also true for solutions of $\operatorname{div}(a(x, Dw)) \le |\lambda g(w)|$, provided $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is such that some comparison principle holds. For instance, consider the following hypotheses on a given by [24]:

$$a \in C(\bar{\Omega} \times \mathbb{R}^n; \mathbb{R}^n) \cap C^1(\bar{\Omega} \times (\mathbb{R}^n \setminus \{0\}); \mathbb{R}^n),$$

$$\begin{aligned} a(x, 0) &= 0 \quad \text{for } x \in \Omega, \\ \langle D_z a(x, z)\xi, \xi\rangle &\ge (p-1)|z|^{p-2}|\xi|^2 \quad \text{for } (x, z) \in \Omega \times \mathbb{R}^n\setminus\{0\}, \\ |D_z a(x, z)| &\le C|z|^{p-2} \quad \text{for } (x, z) \in \Omega \times \mathbb{R}^n\setminus\{0\}\ ,\ C > 0. \end{aligned} \tag{7.5}$$

Theorem 7.3 *Let w be a bounded solution of*

$$\begin{cases} -\operatorname{div}(a(x, \nabla v)) = f(v) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.6}$$

where a satisfies (7.5) *and* $f \in C^1(\mathbb{R})$ *satisfies* $|f(t)| \le c|t|^{q-1} + d$*, with* $0 < q \le p$ *and* $c, d \ge 0$*. Then*

$$\|w\|_\infty \le \max\left\{C_1\|w\|_r^{\frac{rp}{n(p-q)+rp}}, C_2\|w\|_r^{\frac{rp}{n(p-1)+rp}}\right\}$$

where $C_1 = C_1(n, p, q, r, \rho, c)$ *and* $C_2 = C_2(n, p, r, \rho, d)$ *are positive constants.*

Proof We use the same ideas of the last theorem. By the comparison principle of [24], $|w| \le u$, where u solves $-\mathrm{div}(a(x, \nabla v)) = cM^{q-1} + d$ in Ω_2 and $u = M/\rho$ on $\partial\Omega_2$. Since the hypotheses on a imply that $\langle a(x, z), z\rangle \ge |z|^p$, using the same argument as in Remark 3.6, we have that $\max u \le \max U$, where U is the solution of $-\Delta_p v = cM^{q-1} + d$ on B and $v = M/\rho$ on ∂B. Notice that U is given by

$$U(x) = \frac{p-1}{p}\left(\frac{1}{n}\right)^{\frac{1}{p-1}}(cM^{q-1} + d)^{\frac{1}{p-1}}\left(R^{\frac{p}{p-1}} - |x|^{\frac{p}{p-1}}\right) + \frac{M}{\rho}.$$

Following the same computations as before, we conclude the proof where the constants are given by

$$C_1 = (2c)^{\frac{n}{n(p-q)+rp}} K^{\frac{p}{n(p-q)+rp}} \quad , \quad C_2 = (2d)^{\frac{n}{n(p-1)+rp}} K^{\frac{p}{n(p-1)+rp}},$$

and

$$K = \frac{1}{\omega_n}\left(1 - \frac{1}{\rho}\right)^{-\frac{n(p-1)}{p}} \rho^r \left(\frac{p-1}{p}\right)^{\frac{n(p-1)}{p}} \left(\frac{1}{n}\right)^{\frac{n}{p}}.$$

□

Using the interpolation inequality observed in Remark 7.2, we can obtain estimates for $\|w\|_s$, where $s \in (r, \infty]$.

Now, we use this theorem to show that the L^p norms of a solution go to zero when its domain becomes "far away" from a ball with the same measure. More precisely, when the first eigenvalue of a domain of a given measure is large, then the L^p norms of solutions in this domain are small.

Corollary 7.4 *Assuming the same hypotheses as in the previous theorem, if* $p = q$ *and* $c < \lambda_p(\Omega)$*, the first eigenvalue of* $-\Delta_p$*, then*

$$\|w\|_\infty \le \max\left\{C_1\left(\frac{d}{\lambda_p(\Omega) - c}\right)^{\frac{1}{p-1}}|\Omega|^{\frac{1}{p}}, C_2\left(\frac{d}{\lambda_p(\Omega) - c}\right)^{\frac{\kappa_1}{p-1}}|\Omega|^{\frac{\kappa_1}{p}}\right\},$$

where $\kappa_1 = p^2/[n(p-1) + p^2]$*. If* $p > q$*, then*

$$\|w\|_\infty \le \max\left\{C_1\tau^{\frac{rp}{n(p-q)+rp}}, C_2\tau^{\frac{rp}{n(p-1)+rp}}\right\},$$

where $\tau = |\Omega|^{1/p}\max\{(2c/\lambda_p(\Omega))^{1/(p-q)}, (2d/\lambda_p(\Omega))^{1/(p-1)}\}$*.*

Proof First note that the growth condition on f and Hölder inequality imply

$$\lambda_p(\Omega)\|w\|_p^p \le \|\nabla w\|_p^p \le \int_\Omega \nabla w \cdot a(\nabla w, x)\,\mathrm{d}x \le c\|w\|_p^q|\Omega|^{\frac{p-q}{p}} + d\|w\|_p|\Omega|^{\frac{p-1}{p}}.$$

The proof for the case $p = q$ follows directly from this and Theorem 7.3. In the case $p > q$, we get from this inequality that $\|w\|_p \le \tau$. Hence, we complete the proof applying Theorem 7.3. □

Corollary 7.5 *Assume the same hypotheses about* a *and* f *as in the previous theorem. Suppose also that* $a = a(z)$, $f(t) > 0$ *for* $t > 0$, $f(t) = 0$ *for* $t \leq 0$ *and there exists some positive solution* U *of* (7.6) *in the ball* B *with the same measure as* Ω. *If* $\lambda_p(\Omega)$ *is sufficiently large, then any solution* u *of* (7.6) *in* Ω *satisfies* $u^\sharp < U$, *where* $u^\sharp$ *is the symmetrization of* u.

The novelty in this corollary is that f does not need to be monotone.

Proof From Hopf lemma, $\partial_n U = c < 0$ on ∂B, and, therefore, there exists some "paraboloid"

$$P(x) = \frac{p-1}{p}\left(\frac{1}{n}\right)^{\frac{1}{p-1}} C^{\frac{1}{p-1}}\left(r^{\frac{p}{p-1}} - |x - x_0|^{\frac{p}{p-1}}\right),$$

where x_0 is the center of B and r is its radius, such that $0 < P < U$ in B. Observe that $-\Delta_p P = C$. Since f is continuous and $f(0) = 0$, let $M > 0$ be such that $f(t) < C$ for $t < M$. Corollary 7.4 implies that $\|u\|_\infty < M$, where u is any solution of (7.6), if $\lambda_p(\Omega)$ is large enough. Then

$$-\mathrm{div}\, a(\nabla u) = f(u) \leq C = -\Delta_p P,$$

and, from Theorem 5.6, $u^\sharp \leq P < U$ proving the result. □

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8 Appendix

We show now Lemma 2.4 with the same arguments as in Theorem 3.11 of [40].

Proof of Lemma 2.4 Let $K > 0$, $\ell > K$, $r \geq 1$, $\gamma = qr - q + 1$,

$$v = P(u) = \min\{(u+K)^r, \ell^{r-1}(u+K)\}$$

and

$$\varphi = G(u) = \min\{(u+K)^\gamma, \ell^{\gamma-1}(u+K)\} - K^\gamma \in W_0^{1,q}(\Omega').$$

Then, using that $a(t, z)\cdot z \geq C_*(|z|^q - 1)$ for all $z \in \mathbb{R}^n$ and $t \in \mathbb{R}$, we get

$$\begin{aligned}\int_{\Omega'} |\nabla v|^q \, dx &\leq \int_{\Omega'} |P'(u)|^q \left(\frac{\nabla u \cdot a(u, \nabla u)}{C_*} + 1\right) dx \\ &\leq \int_{\Omega'} \frac{|P'(u)|^q}{G'(u)} \cdot \frac{\nabla\varphi \cdot a(u, \nabla u)}{C_*}\, dx + \int_{\Omega'} |P'(u)|^q \, dx.\end{aligned}$$

Notice that $|P'(u)|^q / G'(u) = E$, where $E = 1$ if $u + K > \ell$ and $E = r^q/\gamma$ if $u + K < \ell$. Then, $E \leq r^q$ and, using $\nabla\varphi \cdot a(u, \nabla u) \geq 0$,

$$\begin{aligned}\int_{\Omega'} |\nabla v|^q \, dx &\leq \frac{r^q}{C_*}\int_{\Omega'} \nabla\varphi \cdot a(u, \nabla u)\, dx + \int_{\Omega'} |P'(u)|^q \, dx \\ &= \frac{r^q}{C_*}\int_{\Omega'} f(u)G(u)\, dx + \int_{\Omega'} |P'(u)|^q \, dx. \qquad (8.1)\end{aligned}$$

Observe now that for $u + K < \ell$,

$$f(u)G(u) \le (\alpha u^{q-1} + \beta) \cdot (u+K)^\gamma \le \alpha (u+K)^{q-1+\gamma} + \beta \frac{(u+K)^{q-1+\gamma}}{K^{q-1}} \le v^q \left(\alpha + \frac{\beta}{K^{q-1}}\right).$$

In a similar way, we can prove this inequality also for the case for $u + K \ge \ell$. Furthermore, for $u + K \le \ell$,

$$|P'(u)|^q = |r(u+K)^{r-1}|^q = r^q \frac{(u+K)^{rq}}{(u+K)^q} \le r^q \frac{v^q}{K^q},$$

that is also true for $u + K > \ell$. From these two inequalities and (8.1), we get

$$\int_{\Omega'} |\nabla v|^q \, \mathrm{d}x \le \left[\frac{r^q}{C_*} \cdot \left(\alpha + \frac{\beta}{K^{q-1}}\right) + \frac{r^q}{K^q}\right] \int_{\Omega'} v^q \, \mathrm{d}x. \tag{8.2}$$

Now we study the cases $q > n$, $q < n$ and $q = n$ separately.

Case 1 $q > n$.
Observe that for $r = 1$, we get $v = u + K$. Using the Morrey's inequality for $v - K \in W_0^{1,q}$,

$$\|v - K\|_{C^{0,1-n/q}} \le \tilde{C_0} \|v - K\|_{W^{1,q}} \le C_0 (\|v\|_q + K|\Omega'|^{1/q} + \|Dv\|_q),$$

where $C_0 = C_0(n, q)$. From this one and (8.2), we get

$$\sup u = \sup v - K \le \left[C_0 + \left[\frac{1}{C_*} \cdot \left(\alpha + \frac{\beta}{K^{q-1}}\right) + \frac{1}{K^q}\right]^{1/q}\right] \|v\|_q + C_0 K |\Omega'|^{1/q}.$$

Since $\|v\|_q = \|u + K\|_q \le \|u\|_q + K|\Omega'|^{1/q}$, we get $\sup u \le D_1 \|u\|_q + D_2 K |\Omega'|^{1/q}$, where

$$D_1 = C_0 + \left[\frac{1}{C_*} \cdot \left(\alpha + \frac{\beta}{K^{q-1}}\right) + \frac{1}{K^q}\right]^{1/q} \quad \text{and} \quad D_2 = D_1 + C_0.$$

Case 2 $q < n$.
Since $v - K^r \in W_0^{1,q}(\Omega')$, the Sobolev inequality implies

$$\|v - K^r\|_{q*} \le C_0 \|\nabla v\|_q, \tag{8.3}$$

where $q^* = nq/(n-q)$ and $C_0 = \frac{q(n-1)}{n-q}$. Using this and (8.2), we get

$$\|v - K^r\|_{q*} \le C_0 \left[\frac{r^q}{C_*} \cdot \left(\alpha + \frac{\beta}{K^{q-1}}\right) + \frac{r^q}{K^q}\right]^{1/q} \|v\|_q. \tag{8.4}$$

Hence, naming $\chi = n/(n-q)$, it follows that

$$\|v\|_{\chi q} \le C_0 \left[\frac{r^q}{C_*} \cdot \left(\alpha + \frac{\beta}{K^{q-1}}\right) + \frac{r^q}{K^q}\right]^{1/q} \|v\|_q + K^r |\Omega'|^{1/\chi q},$$

that is, $\|v\|_{\chi q} \le D_1 \|v\|_q + D_2$, where

$$D_1 = C_0 \left[\frac{r^q}{C_*} \cdot \left(\alpha + \frac{\beta}{K^{q-1}}\right) + \frac{r^q}{K^q}\right]^{1/q} \quad \text{and} \quad D_2 = K^r |\Omega'|^{1/\chi q}.$$

Since v depends on r and ℓ, we name it by $v_{r,\ell}$. In the same way, $D_1 = D_1(r)$ and $D_2 = D_2(r)$. Hence, the last inequality can be rewritten as

$$\|v_{r,l}\|_{\chi q} \leq D_1(r)\,\|v_{r,l}\|_q + D_2(r). \tag{8.5}$$

Taking $r = 1$, we have $v = u + K$ and, then

$$\|u + K\|_{\chi q} \leq D_1(1)\,\|u + K\|_q + D_2(1).$$

Hence $u+K \in L^{\chi q}$ and, therefore, $(u+K)^\chi \in L^q$. Taking $r = \chi$, we have $|v_{\chi,\ell}| \leq (u+K)^\chi$ for any ℓ. Thus $\|v_{\chi,\ell}\|_q \leq \|(u + k)^\chi\|_q$ and, from (8.5),

$$\|v_{\chi,\ell}\|_{\chi q} \leq D_1(\chi)\|(u + K)^\chi\|_q + D_2(\chi).$$

Using that $v_{\chi,\ell} \uparrow (u + K)^\chi$ as $\ell \to \infty$, we get

$$\|(u + K)^\chi\|_{\chi q} \leq D_1(\chi)\|(u + K)^\chi\|_q + D_2(\chi).$$

Therefore, $u + K \in L^{\chi^2 q}$. More generally, if we take $r = \chi^n$, it follows in a similar way that

$$\|(u + K)^{\chi^n}\|_{\chi q} \leq D_1(\chi^n)\|(u + K)^{\chi^n}\|_q + D_2(\chi^n)$$

and $u + K \in L^{\chi^{n+1} q}$. Thus, $u + K$ is an L^r function for any $r \geq 1$. Hence, making $\ell \to \infty$ in (8.5), we get

$$\|u + K\|^r_{r\chi q} \leq D_1(r)\,\|u + K\|^r_{rq} + D_2(r).$$

Observe now that $D_1(r) = rH$, where

$$H = C_0\left[\frac{1}{C_*}\cdot\left(\alpha + \frac{\beta}{K^{q-1}}\right) + \frac{1}{K^q}\right]^{1/q}.$$

Furthermore,

$$D_2(r) = K^r|\Omega'|^{1/\chi q} \leq \frac{\|u + K\|^r_{rq}}{|\Omega'|^{1/q}}|\Omega'|^{1/\chi q} \leq r\|u + K\|^r_{rq}|\Omega'|^{(1/\chi-1)1/q}.$$

Therefore, the last three relations imply

$$\|u + K\|_{r\chi q} \leq r^{1/r} H_0^{1/r}\|u + K\|_{rq}, \tag{8.6}$$

for $r \geq 1$ and $\chi = n/(n-q)$, where $H_0 = H + |\Omega'|^{(1/\chi-1)1/q}$. Taking $r = \chi^m$ in (8.6), we have

$$\|u + K\|_{\chi^{m+1} q} \leq \chi^{m/\chi^m} H_0^{1/\chi^m}\|u + K\|_{\chi^m q} \quad \text{for } m \in \mathbb{N} \cup \{0\}.$$

Hence, defining $A_m = \sum_{j=0}^m j/\chi^j$ and $B_m = \sum_{j=0}^m 1/\chi^j$, it follows that

$$\|u + K\|_{\chi^{m+1} q} \leq \chi^{A_m} H_0^{B_m}\|u + K\|_q \quad \text{for } m \in \mathbb{N} \cup \{0\}.$$

Since A_m and B_m are convergent series,

$$\sup(u + K) \leq \chi^A H_0^B\|u + K\|_q,$$

where $A = \lim_{m\to\infty} A_m$ and $B = \lim_{m\to\infty} B_m = \frac{\chi}{\chi-1}$. Then

$$\sup u \leq D(H^B + |\Omega'|^{B(1/\chi q-1/q)})(\|u\|_q + \|K\|_q),$$

for $D = \chi^A 2^B$. Observe that $B(\frac{1}{\chi q} - \frac{1}{q}) = -\frac{1}{q}$. Therefore

$$\sup u \le D(H^B + |\Omega'|^{-1/q})(\|u\|_q + K|\Omega'|^{1/q}).$$

Notice that

$$H \le C_0 2^{2/q}\left(\frac{\alpha}{C_*} + \frac{\beta}{C_*} + 1\right)\left(1 + \frac{1}{K}\right).$$

Then, taking $K = |\Omega'|^{1/n}$, it follows that

$$H^B \le C_1(|\Omega'|^{1/n} + 1)^B |\Omega'|^{-1/q},$$

where $C_1 = [C_0 2^{2/q}\,(\alpha/C_* + \beta/C_* + 1)]^B$. Hence

$$\begin{aligned}\sup u &\le 2DC_1\left(|\Omega'|^{1/n} + 1\right)^B |\Omega'|^{-1/q}\left(\|u\|_q + K|\Omega'|^{1/q}\right)\\ &\le C(|\Omega'|^{1/n} + 1)^B\left(|\Omega'|^{-1/q}\|u\|_q + |\Omega'|^{1/n}\right),\end{aligned}$$

proving the result.

Case 3 $q = n$: Taking $\tilde q < q = n$, we get the same estimate as in (8.3) with $\tilde q^*$ instead of q^*. Hence

$$\|v - K^r\|_{\tilde q^*} \le C_0\|\nabla v\|_{\tilde q},$$

where $\tilde q^* = n\tilde q/(n - \tilde q)$ and $C_0 = \frac{\tilde q(n-1)}{n-\tilde q}$. Therefore, from Hölder inequality,

$$\|v - K^r\|_{\tilde q^*} \le C_0\|\nabla v\|_q|\Omega'|^{(q-\tilde q)/q\tilde q}.$$

For $\tilde q > n/2$, we get $\tilde q/(n - \tilde q) > 1$ and, then, $\tilde q^* > n = q$. In this case,

$$\|v - K^r\|_{\chi q} \le C_0\|\nabla v\|_q|\Omega'|^{(q-\tilde q)/q\tilde q},$$

where $\chi = \tilde q^*/q > 1$. Using this and (8.2), it follows that

$$\|v - K^r\|_{\chi q} \le C_0\left[\frac{r^q}{C_*}\cdot\left(\alpha + \frac{\beta}{K^{q-1}}\right) + \frac{r^q}{K^q}\right]^{1/q}\|v\|_q|\Omega'|^{(q-\tilde q)/q\tilde q}.$$

This estimate is basically the same as in (8.4). Hence, taking $K = |\Omega|^{1/n}$ and following the same argument as before we get the result. □

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