

On the curve diffusion flow of closed plane curves

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Abstract In this paper, we consider the steepest descent H^{-1} -gradient flow of the length functional for immersed plane curves, known as the curve diffusion flow. It is known that under this flow there exist both initially immersed curves that develop at least one singularity in finite time and initially embedded curves that self-intersect in finite time. We prove that under the flow closed curves with initial data close to a round circle in the sense of normalised L^2 oscillation of curvature exist for all time and converge exponentially fast to a round circle. This implies that for a sufficiently large ‘waiting time’, the evolving curves are strictly convex. We provide an optimal estimate for this waiting time, which gives a quantified feeling for the magnitude to which the maximum principle fails. We are also able to control the maximum of the multiplicity of the curve along the evolution. A corollary of this estimate is that initially embedded curves satisfying the hypotheses of the global existence theorem remain embedded. Finally, as an application we obtain a rigidity statement for closed planar curves with winding number one.

Keywords Geometric evolution equations · Higher order quasilinear partial differential equations · Differential geometry of plane curves · Curve diffusion flow · Surface diffusion flow

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1 Introduction

Suppose $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}^2$ is an immersed closed plane curve of period P and consider the energy

$$L(\gamma) = \int_0^P |\gamma_u| \, du,$$

where $\gamma_u = \partial_u \gamma$. We wish to deform γ towards a minimiser of L , and for this purpose we shall consider the steepest descent gradient flow of L in H^{-1} . There are some advantages in choosing H^{-1} instead of L^2 . One is that for any initial curve the signed area is constant under the flow, which implies that if the signed area of the initial curve is nonzero, then the flow is never asymptotic to a lower dimensional subset of \mathbb{R}^2 .

The Euler-Lagrange operator of L in H^{-1} is

$$\text{grad}_{H^{-1}} L(\gamma) = k_{ss},$$

where $k = \langle \gamma_{ss}, \nu \rangle$ is the curvature of γ , ν a unit normal vector field on γ , and s denotes arc-length. The curve diffusion flow is the one-parameter family of immersed curves $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ with normal velocity equal to $-\text{grad}_{H^{-1}}(L(\gamma))$, that is

$$\partial_t^\perp \gamma = -k_{ss}. \tag{CD}$$

The curve diffusion flow is a degenerate system of quasilinear fourth-order parabolic partial differential equations, and as such it is not expected that a maximum or comparison principle holds. Indeed, Giga and Ito [25] provided the first example of a simple, closed, strictly embedded planar curve that develops a self-intersection in finite time under the flow. They also gave [26] the first example of a simple, closed, strictly convex planar curve that becomes non-convex in finite time. Furthermore, Elliot and Maier-Paape showed [15] that the curve diffusion flow may drive an initial graph to become non-graphical in finite time. It was eventually shown by Blatt [8] that non-preservation of convexity and non-preservation of embeddedness are basic properties of a large class of general higher order hypersurface flows.

It is also known (see Polden [36] for the first example and Escher and Ito [16] for many others) that the curve diffusion flow can from smooth immersed initial data develop finite time curvature singularities. In contrast, our goal in this paper is to demonstrate a new class of initial data (generalising [14, Theorem 6.1]), which gives rise to an immortal solution converging exponentially fast to a simple round circle.

The curve diffusion flow has been considered for some time in the literature. The first point to note is that for regular enough initial data $\gamma_0 : \mathbb{S}^1 \rightarrow \mathbb{R}^2$, there is a maximal $T \in (0, \infty]$ and corresponding solution $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$, which satisfies (CD). Local existence, although technical and sometimes tricky, is by now standard—in this paper we state a version (Theorem 2.1), which is a combination of Elliot and Garcke [14] and Dziuk et al. [11, Theorem 3.1], although similar results appeared earlier, see [6, 9, 14, 39] for example. It is also quite standard regardless: as mentioned, the evolution Eq. (CD) is a degenerate fourth-order quasilinear parabolic system, and local existence can be obtained, for example, through the method of semigroups (Angenent [4], Amann [1–3], Escher et al. [17, 19], and Lunardi [35] are good references), the Nash-Moser inverse function theorem (see Hamilton [27, 28], and Gage and Hamilton [22]) or through more classical methods such as can be found in Polden [36] and Huisken and Polden [29] (see also Sharples [37] and the books [12, 13, 21]). The local existence theorem we use requires that the curvature of γ lies in L^2 . One should note

that there are local existence results which do not require any control of curvature, instead requiring Lipschitz with small Lipschitz constant or slightly more regularity than C^1 for the initial data, see Koch and Lamm [30], Escher and Mucha [18], and Asai [5] for example.

The analysis we present here is direct and geometric in nature, and should be compared with [7, 10, 11, 28, 31–33, 36, 40]. It rests on the observation that the normalised oscillation of curvature

$$K_{osc}(\gamma(\cdot, t)) = L(\gamma(\cdot, t)) \int_{\gamma} (k - \bar{k})^2 ds,$$

where \bar{k} denotes the average of the curvature, is in many respects a natural ‘energy’ for the flow. The only stationary solutions of (CD) are lines and multiply covered circles, for which $K_{osc} = 0$. Further, for arbitrary smooth initial data $\int_0^t K_{osc} d\tau \leq L^4(\gamma(\cdot, 0))/16\pi^2$ (see Lemma 3.2), that is, $K_{osc} \in L^1([0, T])$.

We prove that if K_{osc} is initially small and the isoperimetric ratio $I = L^2/4\pi A$ is initially close to one, then they remain so. This is enough to begin a ‘bootstrapping’ style procedure, in which we use interpolation inequalities as in [11] to obtain uniform bounds for all higher derivatives of curvature. These observations and some extra arguments give the global existence result of this paper.

Theorem 1.1 *Suppose $\gamma_0 : \mathbb{S}^1 \rightarrow \mathbb{R}^2$ is a regular smooth immersed closed curve with $A(\gamma_0) > 0$ and*

$$\int_{\gamma_0} k ds = 2\pi. \tag{1}$$

There exists a constant $K^ > 0$ such that if*

$$K_{osc}(\gamma_0) < K^*, \quad \text{and} \quad I(\gamma_0) < \exp\left(\frac{K^*}{8\pi^2}\right), \tag{2}$$

then the curve diffusion flow $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ with γ_0 as initial data exists for all time and converges exponentially fast to a round circle with radius $\sqrt{\frac{A(0)}{\pi}}$.

Remark 1.2 One advantage of our direct method is that we are able to easily find an allowable choice for the constant K^* above; in particular, one may select

$$K^* = \frac{2\pi + 12\pi^2 - 4\pi\sqrt{3}\pi\sqrt{1+3\pi}}{3} \simeq \frac{1}{18}.$$

Remark 1.3 So long as $A(\gamma_0) \neq 0$, one may always guarantee $A(\gamma_0) > 0$ by reversing the orientation of ν , since (CD) is invariant under change of orientation.

Remark 1.4 As can be seen from the proof of Proposition 3.7, the smallness condition (2) could be weakened to

$$K_{osc}(\gamma_0) + 8\pi^2 \log \sqrt{I(\gamma_0)} \leq 2K^* - \delta,$$

for any $\delta > 0$. We do not expect this to be optimal, however. At this time, it is not known if there exists any smooth plane curve satisfying (1) which gives rise to a curve diffusion flow with finite maximal existence time. Without at least one such singular example, it is difficult to even conjecture on what an optimal form of (2) may be.

It is clear that Theorem 1.1 implies $k(\cdot, t) \rightarrow \sqrt{\frac{\pi}{A(0)}}$, and so after a fixed time translation, we have

$$k(\cdot, t) \geq \sqrt{c} > 0$$

for any $c \in (0, \frac{\pi}{A(0)})$ (cf. [31, Lemma 5.5] and [41] for the Willmore flow and surface diffusion flow of surfaces, respectively). In other words, after some finite time, the curvature becomes positive and remains so. This can be thought of as ‘eventual positivity’ and is reminiscent of the situation considered in [20,23,24]. There, using very different techniques, eventual local positivity and other related qualitative properties are observed for biharmonic parabolic equations under certain conditions. To further quantify the size of the ‘waiting time’, we present the following.

Proposition 1.5 *Suppose $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ solves (CD) and satisfies the assumptions of Theorem 1.1. Then,*

$$\mathcal{L}\{t \in [0, \infty) : k(\cdot, t) \not\geq 0\} \leq \left(\frac{L(\gamma_0)}{2\pi}\right)^4 - \left(\frac{A(\gamma_0)}{\pi}\right)^2.$$

In the above, $k(\cdot, t) \not\geq 0$ means that there exists a p such that $k(p, t) \leq 0$. This estimate is optimal in the sense that the right-hand side is zero for a simple circle.

It is not clear at all from Theorem 1.1 if initially embedded curves remain so, nor even if we can control the maximum of the multiplicity (the number of times the curve intersects itself in one point) of the evolving curve. We do have good control of the oscillation of curvature, however, and in the spirit of [34, Theorem 6] (see also the monotonicity formula in [38] and appendix of [33]), present the following theorem to address this issue.

Theorem 1.6 *Suppose $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}^2$ is a smooth immersed curve with winding number ω and let m denote the maximum number of times γ intersects itself in any one point; that is*

$$\int_{\gamma} k \, ds = 2\omega\pi \quad \text{and} \quad m(\gamma) = \sup_{x \in \mathbb{R}^2} |\gamma^{-1}(x)|.$$

Then,

$$K_{osc}(\gamma) \geq 16m^2 - 4\omega^2\pi^2.$$

When combined with Proposition 3.7, we obtain the following.

Corollary 1.7 *Any curve diffusion flow $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ with initial data $\gamma_0 : \mathbb{S}^1 \rightarrow \mathbb{R}^2$ satisfying the assumptions of Theorem 1.1 with*

$$K^* < 64 - 4\pi^2 \simeq 24.5$$

remains embedded for all time.

Note that in particular the allowable choice for K^* given in Remark 1.2 is smaller than $64 - 4\pi^2$.

Theorem 1.1 gives a one-parameter family of smooth diffeomorphisms connecting the initial data γ_0 with a round circle. This implies the following rigidity result.

Corollary 1.8 *Let $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}^2$ be a regular closed immersed curve satisfying the assumptions of Theorem 1.1. Then, γ is diffeomorphic to a round circle.*

This paper is organised as follows. In Sect. 2, we fix our notation, state the local existence theorem, and prove some elementary Sobolev-Poincaré-Wirtinger inequalities. Section 3 contains estimates for the curvature in L^2 and the isoperimetric ratio under various assumptions, which forms the bulk of the work involved in proving Theorem 1.1. The theorem itself and Proposition 1.5 are proved in Sect. 4. We finish the paper by proving Theorem 1.6 and Corollary 1.7 in Sect. 5.

2 Preliminaries

Suppose $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ is a regular smooth immersed plane curve. We say that γ is periodic with period P if there exists a vector $V \in \mathbb{R}^2$ and a positive P such that for all $m \in \mathbb{N}$

$$\gamma(u + P) = \gamma(u) + V, \quad \text{and} \quad \partial_u^m \gamma(u + P) = \partial_u^m \gamma(u).$$

If $V = 0$, then γ is closed. In this case, γ is an immersed circle, $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}^2$. The length of γ is

$$L(\gamma) = \int_0^P |\gamma_u| \, du,$$

and the signed enclosed area is

$$A(\gamma) = -\frac{1}{2} \int_0^P \langle \gamma, \nu \rangle |\gamma_u| \, du, \tag{3}$$

where ν is a unit normal vector field on γ . Throughout the paper, we keep γ parametrised by arc-length s , where $ds = |\gamma_u| \, du$. Integrals over γ are to be interpreted as integrals over the interval of periodicity.

Consider the one-parameter family of immersed curves $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ with normal velocity equal to $-\text{grad}_{H^{-1}}(L(\gamma))$, that is

$$\partial_t^\perp \gamma = -k_{ss}. \tag{CD}$$

The following theorem is standard. The uniqueness below is understood modulo the natural group of invariances enjoyed by (CD): rotations, translations, changes of orientation, and so on, as is customary for geometric flows.

Theorem 2.1 (Local existence) *Suppose $\gamma_0 : \mathbb{R} \rightarrow \mathbb{R}^2$ is a periodic regular curve parametrised by arc-length and of class $C^1 \cap W^{2,2}$ with $\|k\|_2 < \infty$. Then, there exists a $T \in (0, \infty]$ and a unique one-parameter family of immersions $\gamma : \mathbb{R} \times [0, T) \rightarrow \mathbb{R}^2$ parametrised by arc-length such that*

- (i) $\gamma(0, \cdot) = \gamma_0$;
- (ii) $\partial_t^\perp \gamma = -k_{ss}$;
- (iii) $\gamma(\cdot, t)$ is of class C^∞ and periodic of period $L(\gamma(\cdot, t))$ for every $t \in (0, T)$;
- (iv) T is maximal.

Theorem 2.1 justifies the use of smooth calculations in the derivation of our estimates. When we use the expression “ $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ solves (CD)”, we are invoking Theorem

2.1 in the special case where the initial data is assumed to be closed, but not necessarily embedded.

We will need the following elementary Sobolev-Poincaré-Wirtinger inequalities.

Lemma 2.2 *Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is absolutely continuous and periodic with period P . Then if $\int_0^P f \, dx = 0$, we have*

$$\int_0^P f^2 \, dx \leq \frac{P^2}{4\pi^2} \int_0^P |f_x|^2 \, dx,$$

with equality if and only if $f(x) = a \sin(2x\pi/P + b)$.

Proof Expand f as a Fourier series and then use Parseval’s identity. □

Corollary 2.3 *Under the assumptions of Lemma 2.2,*

$$\|f\|_\infty^2 \leq \frac{P}{2\pi} \|f_x\|_2^2.$$

Proof As f has zero average, there exist p_1, p_2 such that $f(p_1) = f(p_2) = 0$ and $0 \leq p_1 < p_2 < P$. Thus, since f is absolutely continuous and periodic,

$$f^2(x) = \int_{p_1}^x f(u) f_x(u) \, du - \int_x^{p_2} f(u) f_x(u) \, du.$$

Therefore,

$$f^2(x) \leq \int_{p_1}^{p_2} |f(u) f_x(u)| \, du \leq \int_0^P |f(u) f_x(u)| \, du.$$

Now Hölder’s inequality and Lemma 2.2 above implies

$$\|f\|_\infty^2 \leq \|f\|_2 \|f_x\|_2 \leq \frac{P}{2\pi} \|f_x\|_2^2,$$

as required. □

As most of our analysis is based on integral estimates, it is efficient to first compute the derivative of an integral along the flow in general.

Lemma 2.4 *Suppose $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ solves (CD), and $f : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}$ is a periodic function with the same period as γ . Then*

$$\frac{d}{dt} \int_\gamma f \, ds = \int_\gamma f_t + f k k_{ss} - f_s (\partial_t^\top \gamma) \, ds.$$

Proof First note that $\tau = \gamma_u / |\gamma_u| = \gamma_s$ is a unit tangent vector field along γ . We compute

$$\begin{aligned} \frac{\partial}{\partial t} |\gamma_u|^2 &= 2 \langle \gamma_{ut}, \gamma_u \rangle = 2 \left\langle \partial_u ((\partial_t^\perp \gamma)v) + (\partial_t^\top \gamma)\tau, |\gamma_u| \tau \right\rangle \\ &= 2\partial_t^\perp \gamma \langle \partial_u v, |\gamma_u| \tau \rangle + 2|\gamma_u| \partial_u \partial_t^\top \gamma \\ &= -2\partial_t^\perp \gamma |\gamma_u| \langle k|\gamma_u| \tau, \tau \rangle + 2|\gamma_u| \partial_u \partial_t^\top \gamma \\ &= -2k\partial_t^\perp \gamma |\gamma_u|^2 + 2(\partial_s \partial_t^\top \gamma) |\gamma_u|^2. \end{aligned}$$

Therefore,

$$\frac{\partial}{\partial t} ds = k k_{ss} ds + (\partial_s \partial_t^\top \gamma) ds.$$

Using this, we differentiate the integral to find

$$\begin{aligned} \frac{d}{dt} \int_\gamma f ds &= \frac{d}{dt} \int_0^P f |\gamma_u| du \\ &= \int_0^P f_t ds + \int_0^P f (k k_{ss} + \partial_s \partial_t^\top \gamma) ds + f(P, t) |\gamma_u(P)| P' \\ &= \int_\gamma f_t + f k k_{ss} - f_s (\partial_t^\top \gamma) ds. \end{aligned}$$

We obtained the last equality using integration by parts and the periodicity of γ with the identity

$$\partial_t^\top \gamma(u, t) - \partial_t^\top \gamma(u + P(t), t) = |\gamma_u(u, t)| P'(t),$$

which in turn follows from the definition of γ . (Note in particular that the tangential velocity $\partial_t^\top \gamma$ is not periodic.) □

3 Curvature estimates in L^2 and the isoperimetric ratio

The evolution Eq. (CD) is particularly natural as solutions decrease in length while keeping enclosed area fixed. This is only necessarily true for curves immersed in \mathbb{R}^2 , and in fact this is the chief reason why we consider plane curves as opposed to curves in \mathbb{R}^n or immersed in a manifold.

Lemma 3.1 *Suppose $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ solves (CD). Then,*

$$\frac{d}{dt} L = - \int_\gamma k^2 ds, \quad \text{and} \quad \frac{d}{dt} A = 0.$$

In particular, the isoperimetric ratio decreases in absolute value with velocity

$$\frac{d}{dt} I = - \frac{2I}{L} \int_\gamma k^2 ds.$$

Proof Lemma 2.4 with $f \equiv 1$ gives

$$\frac{d}{dt} L = \int_\gamma k k_{ss} ds = - \int_\gamma k_s^2 ds,$$

where we used integration by parts and the periodicity of the curve. For the area, we first note that

$$\tau_s = k \nu, \quad \nu_s = -k \tau, \quad \nu_t = k_{sss} \tau - (\partial_t^\top \gamma) k \tau. \tag{4}$$

The first two relations are immediate from differentiating $\langle \tau, \nu \rangle = 0$ and using the definition of the curvature. For the third, we first compute the commutator of the arc-length and time derivatives:

$$\begin{aligned} \partial_{ts} &= \partial_t(|\gamma_u|^{-1}\partial_u) = |\gamma_u|^{-1}\partial_{tu} - |\gamma_u|^{-2}(\partial_t|\gamma_u|)\partial_u \\ &= \partial_{st} + k(\partial_t^\perp\gamma)\partial_s - (\partial_s\partial_t^\top\gamma)\partial_s \\ &= \partial_{st} - kk_{ss}\partial_s - (\partial_s\partial_t^\top\gamma)\partial_s. \end{aligned} \tag{5}$$

Using this and the first two equalities in (4), we compute the evolution of the unit tangent vector field τ .

$$\begin{aligned} \partial_t\tau &= \partial_{ts}\gamma \\ &= \partial_s((\partial_t^\perp\gamma)\nu + (\partial_t^\top\gamma)\tau) - kk_{ss}\gamma_s - (\partial_s\partial_t^\top\gamma)\gamma_s \\ &= -k_{sss}\nu - k_{ss}\nu_s + (\partial_t^\top\gamma)\gamma_{ss} - kk_{ss}\gamma_s \\ &= -k_{sss}\nu + (\partial_t^\top\gamma)\gamma_{ss}. \end{aligned} \tag{6}$$

Noting that $|\nu|^2 = 1$ implies ν_t has no normal component, we obtain the final equality in (4) by differentiating $\langle \nu, \tau \rangle$

$$\begin{aligned} \langle \partial_t\nu, \tau \rangle &= -\langle \nu, \partial_t\tau \rangle = -\langle \nu, -k_{sss}\nu + (\partial_t^\top\gamma)\gamma_{ss} \rangle \\ &= k_{sss} - k(\partial_t^\top\gamma). \end{aligned}$$

Returning to the area functional, we can now directly evaluate the derivative.

$$\begin{aligned} \frac{d}{dt}A &= -\frac{1}{2}\frac{d}{dt}\int_\gamma \langle \gamma, \nu \rangle ds \\ &= -\frac{1}{2}\int_\gamma -k_{ss} + \langle \gamma, \nu_t \rangle + \langle \gamma, \nu \rangle kk_{ss} - (\partial_t^\top\gamma)\langle \gamma, \nu_s \rangle ds \\ &= -\frac{1}{2}\int_\gamma \langle \gamma, k_{sss}\tau - k(\partial_t^\top\gamma)\tau \rangle + \langle \gamma, \nu \rangle kk_{ss} - (\partial_t^\top\gamma)\langle \gamma, -k\tau \rangle ds \\ &= -\frac{1}{2}\int_\gamma -k_{ss} - \langle \gamma, \tau_s \rangle k_{ss} + \langle \gamma, \nu \rangle kk_{ss} ds = 0, \end{aligned}$$

where we used Lemma 2.4 with $f = \langle \gamma, \nu \rangle$ in the second line and integration by parts, the periodicity of γ and the formula (4) throughout. □

We now turn our attention to the scale-invariant quantity

$$K_{osc} = L \int_\gamma (k - \bar{k})^2 ds,$$

where

$$\bar{k} = \frac{1}{L} \int_\gamma k ds.$$

Note that we have (and will continue to) suppressed the dependence of K_{osc} and L on $\gamma(\cdot, t)$. When we must indicate the dependence of K_{osc} and L on $\gamma(\cdot, t)$, we shall use the notation $K_{osc}(t) = K_{osc}(\gamma(\cdot, t))$ and $L(t) = L(\gamma(\cdot, t))$.

A fundamental observation is that Lemmas 3.1 and 2.2 together imply $K_{osc} \in L^1([0, T])$.

Lemma 3.2 *Suppose $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ solves (CD). Then,*

$$\|K_{osc}\|_1 < L^4(0)/16\pi^2.$$

Proof Applying Lemma 2.2 with $f = k - \bar{k}$ and recalling Lemma 3.1, we have

$$K_{osc} \leq \frac{L^3}{4\pi^2} \|k_s\|_2^2 = -\frac{1}{16\pi^2} \frac{d}{dt} L^4,$$

so

$$\int_0^t K_{osc} d\tau \leq \frac{L^4(0)}{16\pi^2}.$$

□

The above lemma holds regardless of initial data and appears to indicate that the quantity K_{osc} is a natural ‘energy’ for the flow.

Remark 3.3 A similar argument as above also shows that $\|k_s\|_2^2 \in L^1([0, T))$ with the estimate $\|\|k_s\|_2^2\|_1 \leq L(0)$. Although we will not need this fact, it does suggest that $\|k_s\|_2^2$ is another well-behaved quantity under the flow.

There exists an $\omega \in \mathbb{R}$ satisfying

$$\int_{\gamma} k ds \Big|_{t=0} = 2\omega \pi. \tag{7}$$

In the case where the solution is a family of closed curves, ω is the winding number of $\gamma(\cdot, 0)$. Since the solution is a one-parameter family of smooth diffeomorphisms and the winding number is a topological invariant, the winding number of the curves $\gamma(\cdot, t)$ remains constant. This can also be directly proven as in the lemma below.

Lemma 3.4 *Suppose $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ solves (CD) and*

$$\int_{\gamma} k ds \Big|_{t=0} = 2\omega \pi.$$

Then,

$$\int_{\gamma} k \, ds = 2\omega \pi .$$

In particular, the average curvature increases in absolute value with velocity

$$\frac{d}{dt} \bar{k} = \frac{2\omega \pi}{L^2} \|k_s\|_2^2.$$

Proof Differentiating $|v|^2 = 1$ and $|\tau|^2 = 1$ gives that $\langle v_t, \tau_s \rangle = 0$ and $\langle v, \tau_{ss} \rangle = 0$. Using this and (4), (6), we compute the evolution of the curvature as

$$\begin{aligned} \frac{\partial}{\partial t} k &= \partial_t \langle v, \gamma_{ss} \rangle = \langle \partial_t v, \gamma_{ss} \rangle + \langle v, \partial_t \gamma_{ss} \rangle \\ &= \langle v, \partial_t \tau_s \rangle \\ &= \left\langle v, \partial_s \tau_t - k k_{ss} \tau_s - (\partial_s \partial_t^\top \gamma) \tau_s \right\rangle \\ &= -k^2 k_{ss} - k (\partial_s \partial_t^\top \gamma) + \langle v, \partial_s \tau_t \rangle \\ &= -k^2 k_{ss} - \langle v, \partial_s (k_{sss} v) \rangle + (\partial_t^\top \gamma) \langle v, \gamma_{sss} \rangle \\ &= -k_{sss} - k^2 k_{ss} + (\partial_t^\top \gamma) k_s. \end{aligned} \tag{8}$$

Therefore, applying Lemma 2.4 with $f = k$, we have

$$\frac{d}{dt} \int_{\gamma} k \, ds = - \int_{\gamma} k_{sss} + k^2 k_{ss} - k^2 k_{ss} + (\partial_t^\top \gamma) k_s - (\partial_t^\top \gamma) k_s \, ds = 0,$$

using integration by parts and the periodicity of γ . This completes the proof. □

We now compute the evolution of K_{osc} .

Lemma 3.5 *Suppose $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ solves (CD). Then,*

$$\begin{aligned} \frac{d}{dt} K_{osc} + K_{osc} \frac{\|k_s\|_2^2}{L} + 2L \|k_{ss}\|_2^2 \\ = 3L \int_{\gamma} (k - \bar{k})^2 k_s^2 \, ds + 6\bar{k}L \int_{\gamma} (k - \bar{k}) k_s^2 \, ds + 2\bar{k}^2 L \|k_s\|_2^2. \end{aligned}$$

Proof This is a direct computation.

$$\begin{aligned} \frac{d}{dt} K_{osc} &= - \int_{\gamma} k_s^2 \, ds \int_{\gamma} (k - \bar{k})^2 \, ds + 2L \int_{\gamma} (k - \bar{k}) (-k_{sss} - k^2 k_{ss} + (\partial_t^\top \gamma) k_s) \, ds \\ &\quad + L \int_{\gamma} k k_{ss} (k - \bar{k})^2 - 2k_s (k - \bar{k}) (\partial_t^\top \gamma) \, ds \\ &= -K_{osc} \frac{\|k_s\|_2^2}{L} - 2L \|k_{ss}\|_2^2 + 2L \int_{\gamma} k^2 k_s^2 \, ds + 4L \int_{\gamma} k (k - \bar{k}) k_s^2 \, ds \\ &\quad - L \int_{\gamma} (k - \bar{k})^2 k_s^2 \, ds - 2L \int_{\gamma} k (k - \bar{k}) k_s^2 \, ds \end{aligned}$$

$$\begin{aligned}
 &= -K_{osc} \frac{\|k_s\|_2^2}{L} - 2L \|k_{ss}\|_2^2 + 4L \int_{\gamma} k^2 k_s^2 ds - 2\bar{k}L \int_{\gamma} k k_s^2 ds \\
 &\quad - L \int_{\gamma} (k - \bar{k})^2 k_s^2 ds.
 \end{aligned}$$

Rearranging, we have

$$\begin{aligned}
 &\frac{d}{dt} K_{osc} + K_{osc} \frac{\|k_s\|_2^2}{L} + 2L \|k_{ss}\|_2^2 \\
 &= 4L \int_{\gamma} k^2 k_s^2 ds - 2\bar{k}L \int_{\gamma} (k - \bar{k}) k_s^2 ds - 2\bar{k}^2 L \int_{\gamma} k_s^2 ds - L \int_{\gamma} (k - \bar{k})^2 k_s^2 ds \\
 &= 3L \int_{\gamma} (k - \bar{k})^2 k_s^2 ds + 6\bar{k}L \int_{\gamma} (k - \bar{k}) k_s^2 ds + 2\bar{k}^2 L \int_{\gamma} k_s^2 ds.
 \end{aligned}$$

This proves the lemma. □

Although K_{osc} is a priori controlled in L^1 , we need much finer control on K_{osc} before we can assert control on other curvature quantities and deduce global existence. (Indeed, global existence is not true in the class of solutions given by Theorem 2.1.)

While K_{osc} is small, we do have the desired control. The following proposition gives us a pointwise estimate.

Proposition 3.6 *Suppose $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ solves (CD). If there exists a T^* such that for $t \in [0, T^*)$, we have*

$$K_{osc}(t) \leq \frac{4\pi + 24\pi^2\omega^2 - 8\pi\sqrt{3\pi}\sqrt{\omega^2 + 3\pi\omega^4}}{3} = 2K^*,$$

then during this time the estimate

$$K_{osc} + 8\omega^2\pi^2 \log L + \int_0^t K_{osc} \frac{\|k_s\|_2^2}{L} d\tau \leq K_{osc}(0) + 8\omega^2\pi^2 \log L(0)$$

holds.

Proof Lemma 3.4, Corollary 2.3 and Hölder’s inequality implies

$$3L \int_{\gamma} (k - \bar{k})^2 k_s^2 ds \leq \frac{3L}{2\pi} K_{osc} \|k_{ss}\|_2^2,$$

and

$$6\bar{k}L \int_{\gamma} (k - \bar{k}) k_s^2 ds \leq 6\omega L \sqrt{K_{osc}} \|k_{ss}\|_2^2.$$

Thus, by Lemma 3.5

$$\frac{d}{dt} K_{osc} + K_{osc} \frac{\|k_s\|_2^2}{L} + \left(2 - K_{osc} \frac{3}{2\pi} - 6\sqrt{K_{osc}}\omega\right) L \|k_{ss}\|_2^2 \leq 2\bar{k}^2 L \|k_s\|_2^2.$$

Now from Lemmas 3.1 and 3.4, we know

$$2\bar{k}^2 L \|k_s\|_2^2 = \frac{8\omega^2 \pi^2}{L} \|k_s\|_2^2 = -8\omega^2 \pi^2 \frac{d}{dt} \log L.$$

Using the smallness of K_{osc} and integrating finish the proof.

It is clear that even with K_{osc} initially small, the estimate given by Proposition 3.6 is useless if we cannot also exert good control on the ratio $L(0)/L(t)$. In particular, we require that it remains only slightly larger than one. This is easily achieved in the case we are interested in, that of closed curves with $\omega = 1$, by an application of the isoperimetric inequality and Lemma 3.1. Observe

$$\frac{L(0)}{L(t)} \leq \frac{L(0)}{\sqrt{4\pi A(t)}} = \frac{L(0)}{\sqrt{4\pi A(0)}} = \sqrt{I(0)},$$

and this can be made arbitrarily close to one.

Proposition 3.7 *Suppose $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ solves (CD) and satisfies (1). Then,*

$$K_{osc}(0) < K^*, \quad \text{and} \quad I(0) < \exp\left(\frac{K^*}{8\pi^2}\right),$$

implies

$$K_{osc} \leq 2K^*$$

for all $t \in [0, T)$.

Proof From Theorem 2.1 and the smallness assumption, there exists a maximal $T^* > 0$ such that $K_{osc}(t) \leq 2K^*$ for $t \in [0, T^*)$. Suppose $T^* < T$. Applying Proposition 3.6, Lemma 3.1, the isoperimetric inequality and the smallness assumption, we have

$$K_{osc} \leq K_{osc}(0) + 8\pi^2 \log \sqrt{I} < \frac{3K^*}{2},$$

for all $t \in [0, T^*)$. Taking $t \rightarrow \delta$, we arrive at a contradiction, and so $\delta = T$. This finishes the proof. □

4 Global existence

We shall first prove Theorem 1.1. There are two parts to this theorem: long time existence ($T = \infty$) and convergence ($\gamma(\mathbb{S}^1)$ approaches a round circle exponentially fast). Given Proposition 3.7 and the blowup criterion from [11], it is rather straightforward to conclude the first part of Theorem 1.1.

Corollary 4.1 *Suppose $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ solves (CD) and satisfies the assumptions of Theorem 1.1. Then, $T = \infty$.*

Proof Suppose $T < \infty$. Then by [11, Theorem 3.1], $\|k\|_2^2 \rightarrow \infty$ as $t \rightarrow T$. However,

$$K_{osc} = L \int_{\gamma} (k - \bar{k})^2 ds = L \|k\|_2^2 - L^2 \bar{k}^2 = L \|k\|_2^2 - 4\pi^2,$$

and since $2\sqrt{\pi A(\gamma_0)} \leq L \leq L(\gamma_0)$, this means $K_{osc} \rightarrow \infty$ as $t \rightarrow T$. This is in direct contradiction with Proposition 3.7. □

It remains to classify the limit. First observe that if we can show $K_{osc} \rightarrow 0$, then we will obtain the desired convergence result (in a weaker topology), as curves with constant curvature in the plane are either circles or straight lines. Lemma 3.2 and Corollary 4.1 imply $K_{osc} \in L^1([0, \infty))$, and since $K_{osc} \geq 0$, to conclude this a sufficient condition is that K'_{osc} is uniformly bounded.

Proposition 4.2 *Suppose $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ solves (CD) and satisfies the assumptions of Theorem 1.1. Then there exists a constant $c_1 \in [1, \infty)$ depending only on γ_0 such that*

$$\|k_s\|_2^2 \leq c_1.$$

Proof Let us assume there exist $\delta_0 \geq 0, \delta_0 < \delta_1$ such that $\|k_s\|_2^2 > 1$ for $t \in [\delta_0, \delta_1]$. (Note that $\delta_0 = 0$ is allowed.) Outside of such intervals, we may take $c_1 = 1$. We apply Lemma 2.4 to k_s^2 to obtain

$$\frac{d}{dt} \int_{\gamma} k_s^2 ds = \int_{\gamma} (k_s^2)_t + k k_{ss} k_s^2 - (k_s^2)_s (\partial_t^\top \gamma) ds.$$

Now the interchange formula (5) and the evolution of the curvature (8) imply

$$\begin{aligned} k_{st} &= k_{ts} - k k_{ss} k_s - (\partial_t^\top \gamma)_s k_s \\ &= (-k_{ssss} - k^2 k_{ss} + (\partial_t^\top \gamma) k_s)_s - k k_{ss} k_s - (\partial_t^\top \gamma)_s k_s \\ &= -k_{sssss} - 2k k_s k_{ss} - k^2 k_{sss} + (\partial_t^\top \gamma)_s k_s + (\partial_t^\top \gamma) k_{ss} - k k_{ss} k_s - (\partial_t^\top \gamma)_s k_s \\ &= -k_{sssss} - 2k k_s k_{ss} - k^2 k_{sss} + (\partial_t^\top \gamma) k_{ss} - k k_{ss} k_s. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d}{dt} \int_{\gamma} k_s^2 ds &= \int_{\gamma} (k_s^2)_t + k k_{ss} k_s^2 ds \\ &= -2 \int_{\gamma} k_s k_{sssss} + 2k k_s^2 k_{ss} + k^2 k_s k_{sss} ds - \int_{\gamma} k k_s^2 k_{ss} ds \\ &= -2 \|k_{sss}\|_2^2 + \frac{5}{3} \int_{\gamma} k_s^4 ds - 2 \int_{\gamma} k^2 k_s k_{sss} ds \\ &= -2 \|k_{sss}\|_2^2 + \frac{5}{3} \int_{\gamma} k_s^4 ds + 2 \int_{\gamma} k^2 k_{ss}^2 ds + 4 \int_{\gamma} k k_s^2 k_{ss} ds. \end{aligned} \tag{9}$$

Since

$$\int_{\gamma} k_s^4 ds = -3 \int_{\gamma} k k_s^2 k_{ss} ds \leq \frac{1}{2} \int_{\gamma} k_s^4 ds + \frac{9}{2} \int_{\gamma} k^2 k_{ss}^2 ds,$$

we have

$$\frac{5}{3} \int_{\gamma} k_s^4 ds + 4 \int_{\gamma} k k_s^2 k_{ss} ds \leq 27 \int_{\gamma} k^2 k_{ss}^2 ds.$$

Combining this with (9) gives

$$\frac{d}{dt} \int_{\gamma} k_s^2 ds \leq -2 \|k_{sss}\|_2^2 + 27 \int_{\gamma} k^2 k_{ss}^2 ds.$$

Rewriting the second term, integrating by parts, and using Lemma 2.2, we have

$$\begin{aligned}
 \int_{\gamma} k^2 k_{ss}^2 ds &= \int_{\gamma} (k - \bar{k})^2 k_{ss}^2 ds + 2\bar{k} \int_{\gamma} k k_{ss} ds - \bar{k}^2 \int_{\gamma} k_{ss}^2 ds \\
 &\leq 2 \int_{\gamma} (k - \bar{k})^2 k_{ss}^2 ds + 2\bar{k}^2 \int_{\gamma} k_{ss}^2 ds \\
 &\leq \frac{K_{osc}}{\pi} \|k_{sss}\|_2^2 + 2\bar{k}^2 \|k_s\|_2 \|k_{sss}\|_2 \\
 &\leq \frac{K_{osc}}{\pi} \|k_{sss}\|_2^2 + \frac{1}{54} \|k_{sss}\|_2^2 + 54\bar{k}^4 \|k_s\|_2^2 \\
 &\leq \frac{K_{osc}}{\pi} \|k_{sss}\|_2^2 + \frac{1}{54} \|k_{sss}\|_2^2 + \frac{27\bar{k}^4 \sqrt{LK_{osc}}}{\pi} \|k_{sss}\|_2 \\
 &\leq \|k_{sss}\|_2^2 \left(\frac{1}{27} + \frac{K_{osc}}{\pi} \right) + \frac{27^3 \bar{k}^8 LK_{osc}}{2\pi^2}.
 \end{aligned}$$

Noting that $K_{osc} < \frac{\pi}{27}$ by Proposition 3.7 (in fact $K_{osc} \lesssim 0.106$), we find

$$\frac{d}{dt} \int_{\gamma} k_s^2 ds + \|k_{sss}\|_2^2 \leq \frac{27^3 \bar{k}^8 LK_{osc}}{2\pi^2}.$$

Observe that Lemma 3.2, the isoperimetric inequality, and $\|k_s\|_2^2 \geq 1$ imply

$$\int_{\delta_0}^t \frac{27^3 \bar{k}^8 LK_{osc}}{2\pi^2 \|k_s\|_2^2} \leq \int_{\delta_0}^t \frac{27^3 \bar{k}^8 LK_{osc}}{2\pi^2} \leq c \|K_{osc}\|_1 \leq c,$$

where $c = c(\gamma_0)$ is a universal constant. Lemmas 2.2, 3.1 and Proposition 3.7 combined with the above and integrating now gives

$$\int_{\gamma} k_s^2 ds \leq c e^{-\frac{16\pi^4}{L^4(0)}(t-\delta_0)},$$

where $t \in [\delta_0, \delta_1]$.

As Proposition 4.2 gives us good control over a quantity so long as it is larger than one, we name it a *dissipation* estimate. With this in hand, convergence to a round circle now follows quite easily.

Corollary 4.3 *Suppose $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ solves (CD) and satisfies the assumptions of Theorem 1.1. Then, $\gamma(\mathbb{S}^1)$ approaches a round circle with radius $\sqrt{\frac{A(0)}{\pi}}$.*

Proof Corollary 4.1 implies $T = \infty$, and Lemma 3.2 gives $K_{osc} \in L^1([0, \infty))$, so if K'_{osc} is bounded, we shall be able to directly conclude $K_{osc} \rightarrow 0$ and obtain the desired statement.

Recall Lemma 3.5 and estimate

$$\begin{aligned} \left| \frac{d}{dt} K_{osc} \right| &\leq 3L \int_{\gamma} (k - \bar{k})^2 k_s^2 ds + 6\bar{k}L \int_{\gamma} |k - \bar{k}| k_s^2 ds + 2\bar{k}^2 L \|k_s\|_2^2 \\ &\leq \frac{3L^2(0)}{2\pi} \|k_s\|_2^4 + 12\omega\pi \sqrt{\frac{L(0)}{2\pi}} \|k_s\|_2^3 + 6\omega\pi \sqrt{\omega\pi A(0)} \|k_s\|_2^2 \\ &\leq c(\omega, L(0), A(0))(c_1 + c_1^2), \end{aligned}$$

where we used Proposition 4.2. This finishes the proof.

Combining the convergence result with a short computation allows us to estimate the measure of the set of times during which the curvature is not strictly positive.

Proof (Proof of Proposition 1.5) Rearranging γ in time if necessary, we may assume that

$$\begin{aligned} k(\cdot, t) &\not\geq 0, \quad \text{for all } t \in [0, t_0) \\ k(\cdot, t) &> 0, \quad \text{for all } t \in [t_0, \infty) \end{aligned}$$

where $t_0 > \left(\frac{L(0)}{2\pi}\right)^4 - \left(\frac{A(0)}{\pi}\right)^2$, otherwise we have nothing to prove. However, in this case, we have

$$\frac{d}{dt} L = -\|k_s\|_2^2 \leq -\frac{\pi^2}{L^2} \|k\|_2^2 \leq -\frac{4\pi^4}{L^3}, \quad \text{for } t \in [0, t_0),$$

where we used the fact that γ is closed and that the curvature has a zero. This implies

$$L^4(t) \leq -16t\pi^4 + L^4(0), \quad \text{for } t \in [0, t_0),$$

and thus $L^4(t_0) < 16\pi^2 A^2(0)$. This is in contradiction with the isoperimetric inequality. \square

In the spirit of classical interpolation, we now obtain exponential decay of the L^2 norm of k_{ss} .

Proposition 4.4 *Suppose $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ solves (CD) and satisfies the assumptions of Theorem 1.1. Then, there exists a constant $c_2 \in [1, \infty)$ depending only on γ_0 such that*

$$\|k_{ss}\|_2^2 \leq c_2 e^{-\frac{4\pi^4}{L^4(0)}t}.$$

Proof We compute and estimate

$$\begin{aligned} \frac{d}{dt} \int_{\gamma} k_{ss}^2 ds &= -2 \int_{\gamma} k_{ssss}^2 ds + 3 \int_{\gamma} k^2 k_{sss}^2 ds + 2 \int_{\gamma} k^2 k_{ss} k_{ssss} ds \\ &\leq - \int_{\gamma} k_{ssss}^2 ds + 3 \|k\|_{\infty}^2 \int_{\gamma} k_{sss}^2 ds + \|k\|_{\infty}^4 \int_{\gamma} k_{ss}^2 ds \\ &\leq -\frac{1}{2} \int_{\gamma} k_{ssss}^2 ds + \frac{11}{4} \|k\|_{\infty}^4 \int_{\gamma} k_{ss}^2 ds \\ &\leq -\frac{1}{4} \int_{\gamma} k_{ssss}^2 ds + \frac{121 \|k\|_{\infty}^8}{16L} K_{osc}. \end{aligned}$$

After a fixed time translation, we have $\|k\|_\infty^8 \leq 2\left(\frac{\omega\pi}{A(0)}\right)^4 \leq c$ and so

$$\frac{d}{dt} \int_\gamma k_{ss}^2 ds \leq -\frac{4\pi^4}{L^4(0)} \int_\gamma k_{ss}^2 ds + cK_{osc},$$

where we also used Lemma 2.2. Since $K_{osc} \in L^1([0, \infty))$, we apply Gronwall’s inequality to obtain

$$\int_\gamma k_{ss}^2 ds \leq c_2 e^{-\frac{4\pi^4}{L^4(0)}t},$$

as required. □

We finish by giving uniform estimates for the higher derivatives of curvature. For this, there are two obvious approaches. With the previous estimate in hand, we have in fact shown that $\|k_s\|_\infty \rightarrow 0$ along a subsequence of times $t_j \rightarrow \infty$. This allows us to employ classical methods to obtain the exponential convergence; see [14] for example.

However, the desired decay estimates also follow by combining our work here with the powerful interpolation inequalities in [11]. Although not as transparent and elementary as our analysis above, it is very efficient.

Proposition 4.5 *Suppose $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ solves (CD) and satisfies the assumptions of Theorem 1.1. Then, for each $m \in \mathbb{N}$, there exist constants $c_m, c'_m \in (0, \infty)$ depending only on γ_0 such that*

$$\|\partial_s^m k\|_2^2 \leq c_m e^{-tc'_m}, \quad \text{and} \quad \|\partial_s^m k\|_\infty \leq \sqrt{L(0)c_{m+1}} e^{-\frac{1}{2}c'_m t}.$$

Proof Recall Equation (3.2) in [11]:

$$\frac{d}{dt} \int_\gamma (\partial_s^m k)^2 ds + \int_\gamma (\partial_s^{m+2} k)^2 ds \leq c\|k\|_2^{4m+10}, \tag{10}$$

where c is a constant depending only on m . Noting that Proposition 4.2 and Corollary 4.3 imply $\|k\|_2^2$ is uniformly bounded, and we combine (10) with the simple interpolation inequality

$$\|\partial_s^m k\|_2^2 \leq c (\|\partial_s^{m+2} k\|_2^2 + \|k\|_2^2)$$

to obtain

$$\frac{d}{dt} \int_\gamma (\partial_s^m k)^2 ds + c \int_\gamma (\partial_s^m k)^2 ds \leq c_0,$$

where c and c_0 are absolute constants depending only on m and m, γ_0 , respectively. Therefore,

$$\int_\gamma (\partial_s^m k)^2 ds \leq c_0 e^{-ct},$$

and the L^∞ estimates follow immediately. □

The proof of Theorem 1.1 is now complete.

5 Controlling density with K_{osc}

We first treat a self-intersection as a singularity and ‘pull’ information from it in a manner analogous to the proof of Simon’s monotonicity formula [38]. See also Theorem 6 in [34] and the appendix of [33].

Lemma 5.1 *Suppose $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}^2$ is a smooth immersed curve with $x \in \gamma(\mathbb{S}^1)$. Then,*

$$8 |\gamma^{-1}(x)| = \int_{\gamma} (k^2 - k_0^2) |\gamma| ds$$

where

$$k_0 = 2 \left| \frac{\langle \gamma, \nu \rangle}{|\gamma|^2} + \frac{k}{2} \right|.$$

Proof The right hand side of the equality above is translation invariant, so we may assume without loss of generality that $x = (0, 0)$ is the origin. Let $\varepsilon > 0$ and consider the test function $\eta : \mathbb{S}^1 \rightarrow \mathbb{R}$ defined by

$$\eta(s) = \min \left\{ \frac{1}{\varepsilon}, \frac{1}{|\gamma(s)|} \right\}.$$

Note that

$$\int_{\gamma} \eta_s^2 ds \leq \int_{\gamma} \frac{1}{|\gamma|^4} ds \leq \frac{L}{\varepsilon^4} < \infty,$$

and so

$$\int_{\gamma} \eta_s \langle \gamma, \gamma_s \rangle ds = - \int_{\gamma} \eta \langle \gamma, \gamma_s \rangle_s ds. \tag{11}$$

Computing, we have

$$\int_{\gamma} \eta_s \langle \gamma, \gamma_s \rangle ds = - \int_{\gamma^{-1}(\{|\gamma| \geq \varepsilon\})} \frac{\langle \gamma, \gamma_s \rangle^2}{|\gamma|^3} ds$$

and

$$- \int_{\gamma} \eta \langle \gamma, \gamma_s \rangle ds = - \int_{\gamma^{-1}(\{|\gamma| < \varepsilon\})} \frac{1 + k \langle \gamma, \nu \rangle}{\varepsilon} ds - \int_{\gamma^{-1}(\{|\gamma| \geq \varepsilon\})} \frac{1 + k \langle \gamma, \nu \rangle}{|\gamma|} ds.$$

Inserting these into (11) and simplifying gives

$$\begin{aligned} \frac{L(|\gamma| < \varepsilon)}{\varepsilon} + \frac{1}{\varepsilon} \int_{\gamma^{-1}(\{|\gamma| < \varepsilon\})} k \langle \gamma, \nu \rangle ds &= \int_{\gamma^{-1}(\{|\gamma| \geq \varepsilon\})} \frac{\langle \gamma, \gamma_s \rangle^2}{|\gamma|^3} - \frac{1 + k \langle \gamma, \nu \rangle}{|\gamma|} ds \\ &= \int_{\gamma^{-1}(\{|\gamma| \geq \varepsilon\})} -\frac{\langle \gamma, \nu \rangle^2}{|\gamma|^3} - k \frac{\langle \gamma, \nu \rangle}{|\gamma|} ds \\ &= - \int_{\gamma^{-1}(\{|\gamma| \geq \varepsilon\})} \left(\frac{\langle \gamma, \nu \rangle}{|\gamma|^2} + \frac{k}{2} \right)^2 |\gamma| ds + \frac{1}{4} \int_{\gamma^{-1}(\{|\gamma| \geq \varepsilon\})} k^2 |\gamma| ds. \end{aligned}$$

Since $k \langle \gamma, \nu \rangle = \langle \gamma, \gamma_{ss} \rangle$ we have

$$\frac{L(|\gamma| < \varepsilon)}{\varepsilon} + \frac{1}{\varepsilon} \int_{\gamma^{-1}(|\gamma| < \varepsilon)} k \langle \gamma, \nu \rangle ds = \frac{\langle \gamma, \gamma_s \rangle}{\varepsilon} \Big|_{\partial(|\gamma| < \varepsilon)}.$$

Thus,

$$\frac{4 \langle \gamma, \gamma_s \rangle}{\varepsilon} \Big|_{\partial(|\gamma| < \varepsilon)} = \int_{\gamma^{-1}(|\gamma| \geq \varepsilon)} (k^2 - k_0^2) |\gamma| ds. \tag{12}$$

Noting that γ is C^1 in $(0, 0)$ and taking $\varepsilon \rightarrow 0$ finishes the proof. □

Remark 5.2 The above proof requires only that $\gamma \in W^{2,2}$, as then (using Corollary 2.3 for example) $\gamma \in C^1$, and the L^2 norm of k and k_0 is well-defined.

We shall also need the following well-known inequality for the i -th elementary symmetric functions

$$\pi_i(l_1, \dots, l_n) = \sum_{1 \leq k_1 < \dots < k_i \leq n} l_{k_1} \dots l_{k_n}.$$

Lemma 5.3 (Newton’s inequality) *Let $l \in \mathbb{R}^n$ be a vector of positive real numbers. Then,*

$$\frac{\pi_{i+1}(l)}{\pi_{i+2}(l)} \geq \frac{\pi_i(l)}{\pi_{i+1}(l)} \frac{\binom{n}{i+1}^2}{\binom{n}{i} \binom{n}{i+2}}.$$

Proof (Proof of Theorem 1.6) Since γ is compact, there exists an $x_0 \in \mathbb{R}^2$ such that

$$m(\gamma) = \sup_{x \in \mathbb{R}^2} |\gamma^{-1}(x)| = |\gamma^{-1}(x_0)|.$$

Applying Lemma 5.1 in x_0 , we have

$$8m = 8|\gamma^{-1}(x_0)| = \int_{\gamma} (k^2 - k_0^2) |\gamma| ds \leq \int_{\gamma} k^2 |\gamma| ds. \tag{13}$$

We now decompose γ into m closed arcs $\gamma_1, \dots, \gamma_m$, each smooth outside of the point x_0 . Let l_i denote the length of the arc γ_i . Our goal is to apply (13) to each arc γ_i ; however, these curves are not regular enough at x_0 . To ameliorate this point, consider an associated curve $\tilde{\gamma}_i : [0, l_i] \rightarrow \mathbb{R}^2$, also with length l_i , which is without self-intersections, smooth outside x_0 and satisfies

$$\begin{aligned} \tilde{\gamma}_i(0) &= \gamma_i(0) = x_0 \\ \lim_{\varepsilon \searrow 0} \partial_s \tilde{\gamma}_i(\varepsilon) &= \lim_{\varepsilon \searrow 0} \partial_s \gamma_i(\varepsilon) \\ \int_{\tilde{\gamma}_i} \tilde{k}^2 |\tilde{\gamma}| ds &\leq \int_{\gamma_i} k^2 |\gamma_i| ds. \end{aligned}$$

This is realised for example by reflecting γ_i across the line

$$x_0 + r \operatorname{rot}_{\pi/2} \left(\partial_s \gamma_i(0) + \partial_s \gamma_i(l_i) \right), \quad r \in \mathbb{R}.$$

We now consider the extension $\tilde{\gamma}_i$ of each arc $\gamma_i : [0, l_i) \rightarrow \mathbb{R}^2$ defined by

$$\hat{\gamma}_i(s) = \begin{cases} \gamma_i(s) & \text{for } s \in [0, l_i), \\ \tilde{\gamma}_i(s - l_i) & \text{for } s \in [l_i, 2l_i). \end{cases}$$

Note that $m(\hat{\gamma}_i) = 2$, and $|\hat{\gamma}_i| \leq l_i/4$. Applying (13) to $\hat{\gamma}_i$, we have

$$8 \leq \int_{\hat{\gamma}_i} \hat{k}^2 |\hat{\gamma}_i| ds \leq 2 \int_{\gamma_i} k^2 |\gamma_i| ds \leq \frac{l_i}{2} \int_{\gamma_i} k^2 ds.$$

Therefore,

$$\int_{\gamma} k^2 ds = \sum_{i=1}^m \int_{\gamma_i} k^2 ds \geq 16 \sum_{i=1}^m \frac{1}{l_i} = 16 \frac{\pi_{m-1}(l_1, \dots, l_m)}{\pi_m(l_1, \dots, l_m)}. \tag{14}$$

Iterating Lemma 5.3 ($m - 1$)-times over gives the estimate

$$\frac{\pi_{m-1}(l_1, \dots, l_m)}{\pi_m(l_1, \dots, l_m)} \geq \frac{m^2}{\pi_1(l_1, \dots, l_m)}.$$

Since $\pi_1(l_1, \dots, l_m) = L$, combining this estimate with (14) implies

$$L \int_{\gamma} k^2 ds = K_{osc} - 4 \pi^2 \omega^2 \geq 16m^2 - 4 \pi^2 \omega^2,$$

as required. □

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