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Homogenization with an oscillating drift: from L^2-bounded to unbounded drifts, 2D compactness results, and 3D nonlocal effects

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Abstract This paper extends results obtained by Tartar (1977, 1986) and revisited in Briane and Gérard (Ann Scuola Norm Sup Pisa, to appear), on the homogenization of a Stokes equation perturbed by an oscillating drift. First, a N-dimensional scalar equation, for $N \geq 3$, and a tridimensional Stokes equation are considered in the periodic framework only assuming the L^2-boundedness of the drift and so relaxing the equi-integrability condition of Briane and Gérard (Ann Scuola Norm Sup Pisa, to appear). Then, it is proved that the L^2-boundedness can be removed in dimension two, provided that the divergence of the drift has a sign. On the contrary, nonlocal effects are derived in dimension three with a free divergence drift that is only bounded in L^1.

Keywords Homogenization · Second-order elliptic equations · Stokes equation · Drift

Mathematics Subject Classification (2000) 35B27 · 76M50

1 Introduction

At the end of the seventies, Tartar [15,16] (see also [17,18]) studied the homogenization of the following Stokes equation perturbed by an oscillating drift term (modeling the Coriolis force) in a bounded domain Ω of $\mathbb{R}^3$,

$$\begin{cases} -\Delta u_\varepsilon + \operatorname{curl}(v_\varepsilon) \times u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega \\ \operatorname{div}(u_\varepsilon) = 0 & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where v_ε is a sequence in $L^\infty(\Omega)^3$ and f belongs to $H^{-1}(\Omega)^3$. Assuming that the sequence v_ε converges weakly to some v in $L^3(\Omega)^3$, and applying his method of oscillating test functions (introduced in the Appendix of [14]) with the parameterized functions

$$\begin{cases} -\Delta w_\varepsilon^\lambda + \operatorname{curl}(v_\varepsilon)\times\lambda + \nabla q_\varepsilon^\lambda = f & \text{in } \Omega \\ \operatorname{div}\left(w_\varepsilon^\lambda\right) = 0 & \text{in } \Omega \quad \text{for } \lambda\in\mathbb{R}^3, \\ w_\varepsilon^\lambda = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

he proved that the limit equation of (1.1) is the Brinkman [8] type equation

$$\begin{cases} -\Delta u + \operatorname{curl}(v)\times u + \nabla p + M^* u = f & \text{in } \Omega \\ \operatorname{div}(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where M^* is the positive definite symmetric matrix-valued function defined by the convergences

$$(Dw_\varepsilon^\lambda)^T\, v_\varepsilon \rightharpoonup M^*\lambda \quad \text{weakly in } L^{\frac{3}{2}}(\Omega)^3, \quad \text{for any } \lambda\in\mathbb{R}^3. \tag{1.4}$$

Since the energy density associated with the Stokes equation (1.1) is reduced to $|Du_\varepsilon|^2$, we may introduce the equivalent scalar equation in a bounded open set Ω of $\mathbb{R}^N$, with a drift term and the same density energy $|\nabla u_\varepsilon|^2$, namely

$$\begin{cases} -\Delta u_\varepsilon + b_\varepsilon\cdot\nabla u_\varepsilon + \operatorname{div}(b_\varepsilon\, u_\varepsilon) = f & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where b_ε is a sequence in $L^\infty(\Omega)^N$ and f belongs to $H^{-1}(\Omega)$. Consider the Hodge decomposition of b_ε in $L^2(\Omega)^N$, that is, $b_\varepsilon = \nabla w_\varepsilon + \xi_\varepsilon$ where w_ε belongs to $H_0^1(\Omega)$ and ξ_ε is a divergence free function in $L^2(\Omega)^N$.

Assuming that b_ε is bounded in $L^2(\Omega)^N$, with ∇w_ε equi-integrable in $L^2(\Omega)^N$, and using an alternative method based on a parametrix of the Laplace operator, we obtained in [6] the homogenized equation

$$\begin{cases} -\Delta u + b\cdot\nabla u + \operatorname{div}(b\, u) + \mu_*\, u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $b = \nabla w + \xi$ is the weak limit of b_ε in $L^2(\Omega)^N$ and μ_* is the function given by

$$|\nabla w_\varepsilon - \nabla w|^2 \rightharpoonup \mu_* \quad \text{weakly in } L^1(\Omega). \tag{1.7}$$

This approach was also applied in [6] to the two-dimensional Stokes equation, assuming similarly that the drift v_ε is equi-integrable in $L^2(\Omega)^2$. Moreover, we showed that the function μ_* of (1.7) and the Brinkman matrix M^* of (1.4) are not in general the correct defect measures involving in the extra order term of the homogenized equations when the equi-integrability condition does not hold. Therefore, it seems natural to address the following question:

Do the homogenization results of [6] subsist when the drift is only bounded in $L^2(\Omega)^N$, or is not bounded in $L^2(\Omega)^N$? In the present paper, we provide a rather complete answer in the periodic framework.

First, in Sect. 2, we study the case where the drift is bounded in $L^2(\Omega)^N$, with $N \leq 3$, both for the N-dimensional scalar Eq. (1.5) and for the tridimensional Stokes equation (1.1). The homogenized equations are of the same type as (1.6) and (1.3), but in general with a constant $\mu \neq \mu_*$ (see Theorem 2.1) and a Brinkman matrix $M \neq M^*$ (see Theorem 2.2). The proof is based on oscillating test functions that differ from the functions (1.2) used by Tartar and are similar to the functions introduced by Dal Maso and Garroni [10] for studying the homogenization of second-order elliptic equations without drift term. We also show that the Tartar formula (1.4) for the Brinkman homogenized matrix is actually valid, that is, $M = M^*$, when the drift v_ε is bounded and equi-integrable in $L^{12/5}(\Omega)^3$.

Then, in Sect. 3, restricting ourselves to the scalar Eq. (1.5), we consider the case of drifts that are not necessarily bounded in $L^2(\Omega)^N$. On the one hand, in dimension two assuming that the divergence of the drift b_ε has a sign, with no prescribed bound for b_ε, we prove (see Theorem 3.1) that homogenized equation is still of the type (1.6), including the degenerate equation $u = 0$ associated with some effective drift of infinite norm. So, the nature of the equation with a zero-order term is preserved in the homogenization process. On the contrary, in dimension three, there is no such compactness result since nonlocal effects may appear (see Theorem 4.2) with divergence-free drifts that are only bounded in $L^1(\Omega)^3$. This gap between dimension two and dimension three is reminiscent with the homogenization of purely diffusive equations of the type

$$-\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f \quad \text{in } \Omega, \tag{1.8}$$

where A_ε is an equi-coercive sequence of symmetric matrix-valued functions which are not necessarily equi-bounded from above. Indeed, the family of equations (1.8) is shown to be stable by homogenization in dimension two at least in the periodic case (see, e.g., [3,4]), while dimension three may induce nonlocal effects for suitable sequences A_ε that are only bounded in $L^1(\Omega)^{3\times 3}$ (see, e.g., [1,11,7]). However, in the present case, nonlocal effects are due to a quite different coupling between a second-order equation and a first-order equation induced by the drift term.

Notations

- For $N \geq 2$, I is the unit matrix of $\mathbb{R}^{N\times N}$.
- For any $A, B \in \mathbb{R}^{N\times N}$, A^T is the transposed matrix of A, and $A : B := \operatorname{tr}(A^T B)$.
- The conjugate exponent of $p \geq 1$ is denoted by $p' := \dfrac{p}{p-1}$.
- For $u : \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $Du := \left(\dfrac{\partial u_i}{\partial x_j}\right)_{1\leq i,j\leq N}$.
- For $\Sigma : \mathbb{R}^N \longrightarrow \mathbb{R}^{N\times N}$, $\operatorname{Div}(\Sigma) := \left(\sum_{j=1}^N \dfrac{\partial \Sigma_{ij}}{\partial x_j}\right)_{1\leq i\leq N}$.
- $H^1_\sharp(Y)$, with $Y := \left[-\frac{1}{2}, \frac{1}{2}\right]^N$, denotes the space of the Y-periodic functions on $\mathbb{R}^N$ which belong to $H^1_{\mathrm{loc}}(\mathbb{R}^N)$.

2 Homogenization with a L^2-boundedness drift

2.1 The scalar equation

Along this section, Ω is a bounded open set of $\mathbb{R}^N$, with $N \leq 3$, and Y denotes the unit cube $\left[-\frac{1}{2}, \frac{1}{2}\right]^N$, which is identified to the N-dimensional torus $\mathbb{R}^N/\mathbb{Z}^N$. Consider a Y-periodic vector-valued function B_ε in $L^\infty_\sharp(Y)^N$ satisfying

$$B_\varepsilon \rightharpoonup B \quad \text{weakly in } L^2_\sharp(Y)^N, \quad \text{with} \quad \bar{B} := \int_Y B \, \mathrm{d}y, \tag{2.1}$$

and the oscillating drift b_ε defined by

$$b_\varepsilon(x) := B_\varepsilon\left(\frac{x}{\varepsilon}\right), \quad \text{for } x \in \Omega. \tag{2.2}$$

Also consider the solution W_ε in $H^1_\sharp(Y)/\mathbb{R}$ of the equation

$$\Delta W_\varepsilon = \operatorname{div}(B_\varepsilon) \quad \text{in } \mathbb{R}^N, \tag{2.3}$$

and assume that there exists

$$\mu_* := \lim_{\varepsilon \to 0} \int_Y |\nabla W_\varepsilon|^2 \, \mathrm{d}y. \tag{2.4}$$

Note that $\mu_* < \infty$ since W_ε is bounded in $H^1_\sharp(Y)/\mathbb{R}$.

We have the following homogenization result for the scalar drift problem (1.5):

Theorem 2.1 *There exists a subsequence of ε, still denoted by ε, and a constant $\mu \in [0, \mu_*]$ such that for any $f \in H^{-1}(\Omega)$, the solution u_ε of the problem (1.5) with the drift (2.2) converges weakly in $H^1_0(\Omega)$ to the solution u of the equation*

$$-\Delta u + 2\,\bar{B} \cdot \nabla u + \mu\, u = f \quad \text{in } \Omega. \tag{2.5}$$

Moreover, if the limit B of (2.1) is not divergence free in $\mathbb{R}^N$, then $\mu > 0$.
On the other hand, if the sequence $\nabla W_\varepsilon\left(\frac{x}{\varepsilon}\right)$ is equi-integrable in $L^2_{\mathrm{loc}}(\Omega)^N$, then $\mu = \mu_*$.

Proof First of all, putting u_ε in Eq. (1.5), we obtain the energy equality

$$\int_\Omega |\nabla u_\varepsilon|^2 \, \mathrm{d}x = \langle f, u_\varepsilon \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}, \tag{2.6}$$

which implies that the sequence u_ε is bounded in $H^1_0(\Omega)$ and thus converges, up to a subsequence, to some function u in $H^1_0(\Omega)$.

We will apply the Tartar oscillating test function method to the following functions. By the Lax–Milgram theorem, for any $\delta > 0$, there exists a unique solution $Z_{\delta,\varepsilon}$ in $H^1_\sharp(Y)$ of the equation

$$-\frac{1}{\varepsilon^2}\Delta Z_{\delta,\varepsilon} - \frac{1}{\varepsilon} B_\varepsilon \cdot \nabla Z_{\delta,\varepsilon} - \frac{1}{\varepsilon}\operatorname{div}\left(B_\varepsilon\, Z_{\delta,\varepsilon}\right) + \delta\, Z_{\delta,\varepsilon} = 1 \quad \text{in } \mathbb{R}^N, \tag{2.7}$$

the variational formulation of which is given for any $\varphi \in H^1_\sharp(Y)$, by

$$\frac{1}{\varepsilon^2}\int_Y \nabla Z_{\delta,\varepsilon}\cdot\nabla\varphi\,\mathrm{d}y - \frac{1}{\varepsilon}\int_Y B_\varepsilon\cdot\nabla Z_{\delta,\varepsilon}\,\varphi\,\mathrm{d}y + \frac{1}{\varepsilon}\int_Y B_\varepsilon\cdot\nabla\varphi\, Z_{\delta,\varepsilon}\,\mathrm{d}y + \int_Y \delta\, Z_{\delta,\varepsilon}\,\varphi\,\mathrm{d}y$$
$$= \int_Y \varphi\,\mathrm{d}y. \tag{2.8}$$

Hence, the function defined by

$$z_{\delta,\varepsilon}(x) := Z_{\delta,\varepsilon}\left(\frac{x}{\varepsilon}\right), \quad \text{for } x \in \mathbb{R}^N, \tag{2.9}$$

satisfies the rescaled equation

$$-\ \Delta z_{\delta,\varepsilon} - b_\varepsilon\cdot\nabla z_{\delta,\varepsilon} - \operatorname{div}\big(b_\varepsilon\, z_{\delta,\varepsilon}\big) + \delta\, z_{\delta,\varepsilon} = 1 \quad \text{in } \mathbb{R}^N. \tag{2.10}$$

Putting $Z_{\delta,\varepsilon}$ as test function in (2.8), we have

$$\frac{1}{\varepsilon^2}\int_Y |\nabla Z_{\delta,\varepsilon}|^2\,\mathrm{d}y + \delta\int_Y Z_{\delta,\varepsilon}^2\,\mathrm{d}y = \bar{Z}_{\delta,\varepsilon} := \int_Y Z_{\delta,\varepsilon}\,\mathrm{d}y, \tag{2.11}$$

which implies that $Z_{\delta,\varepsilon}$ converges weakly, up to a subsequence, to a constant $\bar{Z}_\delta$ in $H^1_\sharp(Y)$. Using a diagonal extraction, there exists a subsequence of ε, still denoted by ε, such that the previous convergence holds for any δ in a dense countable set D of $(0,\infty)$.

Note that $\bar{Z}_\delta \geq 0$, since by the maximum principle $Z_{\delta,\varepsilon} \geq 0$ a.e. in Y. Moreover, if $\bar{Z}_\delta = 0$, then by (2.11), the sequence $\frac{1}{\varepsilon}|\nabla Z_{\delta,\varepsilon}| + Z_{\delta,\varepsilon}$ converges strongly to 0 in $L^2(Y)$. But, putting $\varphi = 1$ as test function in (2.8), we get that

$$1 = -\frac{1}{\varepsilon}\int_Y B_\varepsilon\cdot\nabla Z_{\delta,\varepsilon}\,\mathrm{d}y + \delta\,\bar{Z}_{\delta,\varepsilon} = -\frac{1}{\varepsilon}\int_Y \nabla W_\varepsilon\cdot\nabla Z_{\delta,\varepsilon}\,\mathrm{d}y + \delta\,\bar{Z}_{\delta,\varepsilon}, \tag{2.12}$$

which yields to the contradiction $1 = 0$. Therefore, we have $\bar{Z}_\delta > 0$. By estimate (2.11) we also have $\delta\,\bar{Z}_\delta^2 \leq \bar{Z}_\delta$, hence $\bar{Z}_\delta \in \big(0, \frac{1}{\delta}\big]$.

Now, apply the Tartar oscillation test functions. Let $f \in H^{-1}(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$. Putting $\varphi z_{\delta,\varepsilon}$ in (1.5), φu_ε in (2.10), and equating the two formulas, we get that

$$\int_\Omega \nabla u_\varepsilon\cdot\nabla\varphi\, z_{\delta,\varepsilon}\,\mathrm{d}x - \int_\Omega \nabla z_{\delta,\varepsilon}\cdot\nabla\varphi\, u_\varepsilon - 2\int_\Omega b_\varepsilon\cdot\nabla\varphi\, u_\varepsilon\, z_{\delta,\varepsilon}\,\mathrm{d}x - \delta\int_\Omega \varphi\, z_{\delta,\varepsilon}\, u_\varepsilon\,\mathrm{d}x$$
$$= \int_\Omega f\,\varphi\, z_{\delta,\varepsilon}\,\mathrm{d}x - \int_\Omega \varphi\, u_\varepsilon\,\mathrm{d}x. \tag{2.13}$$

Moreover, by (2.1), the sequence b_ε of (2.2) weakly converges to the constant $\bar{B}$ in $L^2(\Omega)^N$, $z_{\delta,\varepsilon}$ weakly converges to the constant $\bar{Z}_\delta > 0$ in $H^1(\Omega)$ for any $\delta \in D$, and up to a subsequence u_ε converges weakly to u in $H^1_0(\Omega)$. Hence, by the Rellich compactness theorem, the sequence $u_\varepsilon\, z_{\delta,\varepsilon}$ converges strongly to $u\,\bar{Z}_\delta$ in $L^2(\Omega)$ ($N \leq 3$). Therefore, passing to the limit in (2.13), it follows that

$$\int_\Omega \nabla u\cdot\nabla\varphi\,\bar{Z}_\delta\,\mathrm{d}x - 2\int_\Omega \bar{B}\cdot\nabla\varphi\, u\,\bar{Z}_\delta\,\mathrm{d}x + \int_\Omega \varphi\,(1-\delta\,\bar{Z}_\delta)\,u\,\mathrm{d}x = \int_\Omega f\,\varphi\,\bar{Z}_\delta\,\mathrm{d}x.$$

Therefore, u is the solution in $H_0^1(\Omega)$ of the equation

$$-\ \Delta u + 2\,\bar{B}\cdot\nabla u + \mu_\delta\, u = f \quad \text{in } \Omega, \quad \text{where} \quad \mu_\delta := \frac{1}{\bar{Z}_\delta} - \delta \geq 0. \tag{2.14}$$

Since (2.14) has a unique solution u for given $f \in H^{-1}(\Omega)$ and $\delta > 0$, u_ε converges weakly to u in $H_0^1(\Omega)$ for the whole sequence ε which ensures the weak convergences of B_ε in $L^2_\sharp(Y)^N$ and of $Z_{\delta,\varepsilon}$ in $H^1_\sharp(Y)$ for any $\delta \in D$. In particular choosing $f = 1$, we deduce from (2.14) that for any $\delta, \delta' \in D$, $\mu_\delta\, u = \mu_{\delta'}\, u$, which implies that $\mu_\delta = \mu_{\delta'}$ since u is clearly not the zero function in Ω. The nonnegative constant $\mu := \mu_\delta$ is thus independent of $\delta \in D$.

On the other hand, applying the Cauchy–Schwarz inequality in (2.12) and using (2.11), we have

$$\begin{aligned} 1 - \delta\,\bar{Z}_{\delta,\varepsilon} &\leq \left(\int_Y |\nabla W_\varepsilon|^2\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon^2}\int_Y |\nabla Z_{\delta,\varepsilon}|^2\,\mathrm{d}y\right)^{\frac{1}{2}} \\ &= \left(\int_Y |\nabla W_\varepsilon|^2\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\bar{Z}_{\delta,\varepsilon} - \delta\int_Y Z^2_{\delta,\varepsilon}\,\mathrm{d}y\right)^{\frac{1}{2}}, \end{aligned}$$

Hence, passing to the limit as $\varepsilon \to 0$ with (2.4) we get that

$$\mu = \frac{1}{\bar{Z}_\delta} - \delta \leq \mu_*^{\frac{1}{2}}\left(\frac{1}{\bar{Z}_\delta} - \delta\right)^{\frac{1}{2}} = \mu_*^{\frac{1}{2}}\,\mu^{\frac{1}{2}},$$

which yields the inequality $\mu \leq \mu_*$.

Now, assume that $\mu = 0$. Then, by (2.11) we have

$$\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon^2}\int_Y |\nabla Z_{\delta,\varepsilon}|^2\,\mathrm{d}y\right) = \bar{Z}_\delta - \delta\,\bar{Z}_\delta^2 = \bar{Z}_\delta^2\,\mu = 0. \tag{2.15}$$

Putting $\varphi \in H^1_\sharp(Y)$ in (2.8) and multiplying by ε the equality, we obtain that

$$\frac{1}{\varepsilon}\int_Y \nabla Z_{\delta,\varepsilon}\cdot\nabla\varphi\,\mathrm{d}y - \int_Y B_\varepsilon\cdot\nabla Z_{\delta,\varepsilon}\,\varphi\,\mathrm{d}y + \int_Y B_\varepsilon\cdot\nabla\varphi\, Z_{\delta,\varepsilon}\,\mathrm{d}y = O(\varepsilon).$$

Therefore, passing to the limit as $\varepsilon \to 0$ in the previous equality together with (2.1) and (2.15) we obtain that

$$\bar{Z}_\delta \int_Y B\cdot\nabla\varphi\,\mathrm{d}y = 0, \quad \forall\,\varphi \in H^1_\sharp(Y),$$

which implies that B is divergence free in $\mathbb{R}^N$, since $\bar{Z}_\delta > 0$. Conversely, if B is not divergence free in $\mathbb{R}^N$, then μ is positive.

Finally, assume that the sequence $\nabla W_\varepsilon\left(\frac{x}{\varepsilon}\right)$ is equi-integrable in $L^2(\Omega)^N$, and choose W_ε such that $\int_Y W_\varepsilon(y)\,\mathrm{d}y = 0$. Then, by the Poincaré–Wirtinger inequality combined with $\mu_* < \infty$, the sequence W_ε is bounded in $H^1_\sharp(Y)$. Now, consider the εY-periodic function defined by $w_\varepsilon^\sharp(x) := \varepsilon\, W_\varepsilon(\frac{x}{\varepsilon})$, and the solution w_ε in $H_0^1(\Omega)$ of the equation $\Delta w_\varepsilon = \operatorname{div}(b_\varepsilon)$

in Ω. By (2.2) and (2.3), the function $w_\varepsilon - w_\varepsilon^\sharp$ is harmonic in Ω, and $w_\varepsilon^\sharp$ converges weakly to 0 in $H^1(\Omega)$, so does w_ε in $H_0^1(\Omega)$. Let $\varphi \in C_c^\infty(\Omega)$, we have

$$\begin{aligned}
&\int_\Omega |\nabla w_\varepsilon - \nabla w_\varepsilon^\sharp|^2 \,\varphi \,\mathrm{d}x \\
&\quad = \int_\Omega \nabla(w_\varepsilon - w_\varepsilon^\sharp)\cdot\nabla\big(\varphi\,(w_\varepsilon - w_\varepsilon^\sharp)\big)\,\mathrm{d}x - \int_\Omega \nabla(w_\varepsilon - w_\varepsilon^\sharp)\cdot\nabla\varphi\,(w_\varepsilon - w_\varepsilon^\sharp)\,\mathrm{d}x \\
&\quad = -\int_\Omega \nabla(w_\varepsilon - w_\varepsilon^\sharp)\cdot\nabla\varphi\,(w_\varepsilon - w_\varepsilon^\sharp)\,\mathrm{d}x \underset{\varepsilon\to 0}{\longrightarrow} 0,
\end{aligned}$$

hence

$$|\nabla w_\varepsilon - \nabla w_\varepsilon^\sharp|^2 \longrightarrow 0 \quad \text{strongly in } L^1_{\mathrm{loc}}(\Omega). \tag{2.16}$$

On the other hand, the equi-integrability and the εY-periodicity of $\nabla w_\varepsilon^\sharp$ combined with (2.4) imply that $|\nabla w_\varepsilon^\sharp|^2$ converges weakly to μ_* in $L^1_{\mathrm{loc}}(\Omega)$. This combined with (2.16) yields that $|\nabla w_\varepsilon|^2$ also converges weakly to μ_* in $L^1_{\mathrm{loc}}(\Omega)$. Therefore, thanks to Theorem 2.4 of [6] the homogenized equation reads as (2.5) with $\mu = \mu_*$. □

2.2 The Stokes equation

In this section, $N = 3$, and Ω is a regular connected bounded open set of $\mathbb{R}^3$. Consider a Y-periodic vector-valued function V_ε in $L^\infty_\sharp(Y)^3$, satisfying

$$V_\varepsilon \rightharpoonup V \quad \text{weakly in } L^2_\sharp(Y)^3, \quad \text{with} \quad \bar{V} := \int_Y V\,\mathrm{d}y, \tag{2.17}$$

such that there exists

$$\mu_* := \lim_{\varepsilon\to 0} \int_Y |V_\varepsilon|^2\,\mathrm{d}y, \tag{2.18}$$

and consider the associated oscillating function v_ε defined by

$$v_\varepsilon(x) := V_\varepsilon\left(\frac{x}{\varepsilon}\right), \quad \text{for } x \in \Omega. \tag{2.19}$$

Let $f \in H^{-1}(\Omega)^3$. Our aim is to study the homogenization of the perturbed Stokes problem

$$\begin{cases}
-\Delta u_\varepsilon + \mathrm{curl}\,(v_\varepsilon)\times u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega \\
\mathrm{div}\,(u_\varepsilon) = 0 & \text{in } \Omega \\
u_\varepsilon = 0 & \text{on } \partial\Omega.
\end{cases} \tag{2.20}$$

Problem (2.20) can be regarded as a drift problem due to the following weak formulation of the zero-order term

$$\mathrm{curl}\,(v_\varepsilon)\times u_\varepsilon = \mathrm{Div}\,(v_\varepsilon\otimes u_\varepsilon) + (Du_\varepsilon)^T\,v_\varepsilon - \nabla\,(v_\varepsilon\cdot u_\varepsilon)\,, \tag{2.21}$$

where the vector-valued function v_ε plays the same role as the drift b_ε in the scalar problem (1.5). Indeed, the term $\mathrm{Div}\,(v_\varepsilon\otimes u_\varepsilon) + (Du_\varepsilon)^T\,v_\varepsilon$ is similar to $\mathrm{div}\,(b_\varepsilon\,u_\varepsilon) + b_\varepsilon\cdot\nabla u_\varepsilon$ in (1.5), while $\nabla\,(v_\varepsilon\cdot u_\varepsilon)$ can be included in the pressure term of (2.20).

Since Ω is a connected regular bounded open set and $v_\varepsilon \in L^\infty(\Omega)^N$ for a fixed ε, we easily deduce from the Lax–Milgram Theorem applied to the Hilbert space of the free divergence functions in $H_0^1(\Omega)^3$ that there exist a unique $u_\varepsilon \in H_0^1(\Omega)^3$ and a unique $p_\varepsilon \in L^2(\Omega)/\mathbb{R}$ solutions of the Stokes problem (1.1).

Tartar [15,16] (see also [18], Chapter 19) derived a Stokes equation with a Brinkman law assuming that v_ε is bounded in $L^3(\Omega)^3$. To this end, he introduced a family of oscillating test functions parameterized by a vector $\lambda \in \mathbb{R}^3$, the equivalent of which in the periodic case is defined by

$$w_\varepsilon^\lambda(x) := W_\varepsilon^\lambda\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad q_\varepsilon^\lambda(x) := Q_\varepsilon^\lambda\left(\frac{x}{\varepsilon}\right), \quad \text{for } x \in \mathbb{R}^3, \tag{2.22}$$

where $W_\varepsilon^\lambda \in H_\sharp^1(Y)^3$ and $Q_\varepsilon^\lambda \in L_\sharp^2(Y)/\mathbb{R}$ are the solutions of the Stokes problem

$$\begin{cases} -\dfrac{1}{\varepsilon}\,\Delta W_\varepsilon^\lambda + \operatorname{curl}(V_\varepsilon)\times\lambda + \nabla Q_\varepsilon^\lambda = 0 & \text{in } \mathbb{R}^3 \\[2ex] \operatorname{div}\left(W_\varepsilon^\lambda\right) = 0 & \text{in } \mathbb{R}^3 \\[2ex] \displaystyle\int_Y W_\varepsilon^\lambda\, dy = 0. \end{cases} \tag{2.23}$$

The functions w_ε^λ and p_ε^λ are solutions of the rescaled problem

$$\begin{cases} -\Delta w_\varepsilon^\lambda + \operatorname{curl}(v_\varepsilon)\times\lambda + \nabla q_\varepsilon^\lambda = 0 & \text{in } \mathbb{R}^3 \\[2ex] \operatorname{div}\left(w_\varepsilon^\lambda\right) = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.24}$$

Tartar obtained a homogenized problem with a Brinkman force of the type $M^* u$, where u is the limit velocity and M^* is the constant matrix in $\mathbb{R}^{3\times 3}$ defined, up to a subsequence, by

$$M^*\lambda := \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\int_Y \left(DW_\varepsilon^\lambda\right)^T V_\varepsilon\, dy\right), \quad \text{for } \lambda \in \mathbb{R}^3. \tag{2.25}$$

Note that the matrix M^* is symmetric and nonnegative, since by putting W_ε^μ in (2.23) and using the weak formulation (2.21), we have

$$M^*\lambda\cdot\mu = \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon^2}\int_Y DW_\varepsilon^\lambda : DW_\varepsilon^\mu\, dy\right), \quad \forall\,\lambda,\mu \in \mathbb{R}^3. \tag{2.26}$$

In the present context, the sequence v_ε is only assumed to be bounded in $L^2(\Omega)^3$. Hence, we propose an approach based on different oscillating test functions that are similar to the functions (2.7) of the scalar case. More precisely, consider for fixed $\delta > 0$ and $\lambda \in \mathbb{R}^3$ the solution $Z_{\delta,\varepsilon}^\lambda$ in $H_\sharp^1(Y)^3$ of the Stokes problem

$$\begin{cases} -\dfrac{1}{\varepsilon^2}\,\Delta Z_{\delta,\varepsilon}^\lambda - \dfrac{1}{\varepsilon}\operatorname{curl}(V_\varepsilon)\times Z_{\delta,\varepsilon}^\lambda + \dfrac{1}{\varepsilon}\,\nabla P_{\delta,\varepsilon}^\lambda + \delta\, Z_{\delta,\varepsilon}^\lambda = \lambda & \text{in } \mathbb{R}^3 \\[2ex] \operatorname{div}\left(Z_{\delta,\varepsilon}^\lambda\right) = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{2.27}$$

the variational formulation of which is given by

$$\frac{1}{\varepsilon^2}\int_Y DZ^\lambda_{\delta,\varepsilon} : D\Phi\,\mathrm{d}y + \frac{1}{\varepsilon}\int_Y (V_\varepsilon \otimes Z^\lambda_{\delta,\varepsilon}) : D\Phi\,\mathrm{d}y - \frac{1}{\varepsilon}\int_Y (DZ^\lambda_{\delta,\varepsilon})^T V_\varepsilon \cdot \Phi\,\mathrm{d}y$$
$$- \frac{1}{\varepsilon}\int_Y P^\lambda_{\delta,\varepsilon}\,\mathrm{div}\,(\Phi)\,\mathrm{d}y + \int_Y (\delta\, Z^\lambda_{\delta,\varepsilon} - \lambda)\cdot\Phi\,\mathrm{d}y = 0, \quad \forall\,\Phi \in H^1_\sharp(Y)^3. \tag{2.28}$$

The functions defined by

$$z^\lambda_{\delta,\varepsilon}(x) := Z^\lambda_{\delta,\varepsilon}\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad p^\lambda_{\delta,\varepsilon}(x) := P^\lambda_{\delta,\varepsilon}\left(\frac{x}{\varepsilon}\right), \quad \text{for } x \in \mathbb{R}^3, \tag{2.29}$$

are solutions of the rescaled Stokes problem

$$\begin{cases} -\,\Delta z^\lambda_{\delta,\varepsilon} - \mathrm{curl}\,(v_\varepsilon) \times z^\lambda_{\delta,\varepsilon} + \nabla p^\lambda_{\delta,\varepsilon} + \delta\, z^\lambda_{\delta,\varepsilon} = \lambda & \text{in } \mathbb{R}^3 \\ \mathrm{div}\left(z^\lambda_{\delta,\varepsilon}\right) = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.30}$$

We have the following result:

Theorem 2.2 *There exist a subsequence of ε, still denoted by ε, and a nonnegative matrix $M \in \mathbb{R}^{3\times 3}$ satisfying the inequality*

$$|M^T\lambda|^2 \le \mu_*\, M^T\lambda\cdot\lambda, \quad \forall\,\lambda \in \mathbb{R}^3, \quad \textit{where } \mu_* \textit{ is the limit } (2.18), \tag{2.31}$$

such that for any $f \in H^{-1}(\Omega)^3$, the solution u_ε of (2.20) *weakly converges in $H^1_0(\Omega)^3$ to the solution u of the Brinkman problem*

$$\begin{cases} -\,\Delta u + \nabla p + Mu = f & \textit{in } \Omega \\ \mathrm{div}\,(u) = 0 & \textit{in } \Omega. \end{cases} \tag{2.32}$$

Moreover, if the limit V of (2.17) *satisfies the condition*

$$\forall\,\mu \in \mathbb{R}^3 \setminus \{0\}, \quad \mathrm{curl}\,\big(\mathrm{curl}\,(V) \times \mu\big) \in \mathcal{D}'(\mathbb{R}^3) \setminus \{0\}, \tag{2.33}$$

then M is positive definite.
On the other hand, if the sequence v_ε of (2.19) *is bounded and equi-integrable in $L^{12/5}(\Omega)^3$, then M agrees with the Tartar matrix M^* of* (2.25).

Remark 2.3 The matrix M of the Brinkman problem is not necessarily symmetric. In [6], we gave an example of a nonsymmetric matrix for a two-dimensional Stokes equation.

Proof The proof is divided into three steps.

First step: Derivation of a priori estimates.
First of all, putting u_ε in Eq. (2.20), we obtain the energy equality

$$\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x = \langle f, u_\varepsilon\rangle_{H^{-1}(\Omega)^3, H^1_0(\Omega)^3}, \tag{2.34}$$

which implies that the sequence u_ε is bounded in $H^1_0(\Omega)^3$. Then, by the embedding of $H^1(\Omega)$ into $L^6(\Omega)$, the sequences $|v_\varepsilon \otimes u_\varepsilon|$ and $v_\varepsilon \cdot u_\varepsilon$ are bounded in $L^{\frac{3}{2}}(\Omega)$, while the sequence $\left|(Du_\varepsilon)^T v_\varepsilon\right|$ is bounded in $L^1(\Omega)$. Moreover, the Eq. (1.1) combined with the weak formulation (2.21) yields

$$\nabla p_\varepsilon = \Delta u_\varepsilon - \mathrm{Div}\,(v_\varepsilon \otimes u_\varepsilon) - (Du_\varepsilon)^T v_\varepsilon + \nabla(v_\varepsilon \cdot u_\varepsilon) + f$$

which is thus bounded in $W^{-1,r'}(\Omega)$ for any $r \in \left(1, \frac{3}{2}\right)$, due to the embedding of $L^1(\Omega)$ into $W^{-1,r'}(\Omega)$. Hence, p_ε is bounded in $L^r(\Omega)/\mathbb{R}$ for any $r \in \left(1, \frac{3}{2}\right)$, by the classical estimate of the pressure (see, e.g., [12]). Therefore, up to extract a subsequence u_ε and p_ε satisfy for any $r \in \left(1, \frac{3}{2}\right)$, the convergences

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } H^1(\Omega)^3 \quad \text{and} \quad p_\varepsilon \rightharpoonup p \quad \text{weakly in } L^r(\Omega)/\mathbb{R}. \tag{2.35}$$

Now, let us derive estimates satisfied by the test function (2.27). Let $\delta > 0$ and $\lambda \in \mathbb{R}^3$. Putting $\Phi = Z^\lambda_{\delta,\varepsilon}$ as test function in (2.28), we have

$$\frac{1}{\varepsilon^2}\int_Y |DZ^\lambda_{\delta,\varepsilon}|^2\,dy + \delta\int_Y |Z^\lambda_{\delta,\varepsilon}|^2\,dy = \lambda\cdot\bar{Z}^\lambda_{\delta,\varepsilon}, \quad \text{where} \quad \bar{Z}^\lambda_{\delta,\varepsilon} := \int_Y Z^\lambda_{\delta,\varepsilon}\,dy. \tag{2.36}$$

By this energy estimate, the linearity of $\lambda \mapsto Z^\lambda_{\delta,\varepsilon}$, and using a diagonal extraction, there exist a subsequence of ε, still denoted by ε, and a constant vector $\bar{Z}_\delta \in \mathbb{R}^3$, such that for a given countable dense set D of $(0,\infty)$, we have

$$Z^\lambda_{\delta,\varepsilon} \rightharpoonup \bar{Z}^\lambda_\delta \quad \text{weakly in } H^1_\sharp(Y)^3, \quad \forall\,\lambda\in\mathbb{R}^3,\ \forall\,\delta\in D. \tag{2.37}$$

Moreover, proceeding as for the pair $(u_\varepsilon, p_\varepsilon)$, by the weak formulation (2.21) and the convergences (2.17), (2.37), the sequence $\mathrm{curl}\,(V_\varepsilon)\times Z^\lambda_{\delta,\varepsilon}$ is bounded in $W^{-1,r'}_\sharp(Y)^3$ for any $r \in \left(1, \frac{3}{2}\right)$, while by (2.36), the sequence $\frac{1}{\varepsilon}\,\Delta Z^\lambda_{\delta,\varepsilon}$ is bounded in $H^{-1}_\sharp(Y)^3$. Hence, the pressure in (2.27) satisfies for any $r \in \left(1, \frac{3}{2}\right)$,

$$P^\lambda_{\delta,\varepsilon} \rightharpoonup P^\lambda_\delta \quad \text{weakly in } L^r_\sharp(Y)/\mathbb{R}, \quad \forall\,\lambda\in\mathbb{R}^3,\ \forall\,\delta\in D. \tag{2.38}$$

Therefore, rescaling the convergence (2.37) combined with the boundedness of $\frac{1}{\varepsilon}\,|DZ^\lambda_{\delta,\varepsilon}|$ in $L^2_\sharp(Y)$ and the convergence (2.38), we get that the εY-periodic sequences $z^\lambda_{\delta,\varepsilon}$, $p^\lambda_{\delta,\varepsilon}$ defined by (2.29) satisfy for any $r \in \left(1, \frac{3}{2}\right)$,

$$z^\lambda_{\delta,\varepsilon} \rightharpoonup \bar{Z}^\lambda_\delta \quad \text{weakly in } H^1(\Omega)^3, \quad p^\lambda_{\delta,\varepsilon} \rightharpoonup 0 \quad \text{weakly in } L^r(\Omega)/\mathbb{R}, \quad \forall\,\lambda\in\mathbb{R}^3,\ \forall\,\delta\in D. \tag{2.39}$$

Let us conclude this first step by some properties satisfied by the limit $\bar{Z}^\lambda_\delta$. Note that by linearity, there exists a matrix $N_\delta \in \mathbb{R}^{3\times3}$ such that $\bar{Z}^\lambda_\delta = N_\delta\lambda$ for any $\lambda\in\mathbb{R}^3$. Passing to the limit $\varepsilon \to 0$ in (2.36), we get the inequality

$$N_\delta\lambda\cdot\lambda \geq \delta\,|Z^\lambda_\delta|^2 = \delta\,|N_\delta\lambda|^2, \quad \forall\,\lambda\in\mathbb{R}^3. \tag{2.40}$$

Moreover, putting $\Phi = \lambda \in \mathbb{R}^3$ as test function in (2.28), we have

$$-\frac{1}{\varepsilon}\int_Y \left(\mathrm{curl}\,(V_\varepsilon)\times Z^\lambda_{\delta,\varepsilon}\right)\cdot\lambda\,dy + \delta\,\bar{Z}^\lambda_{\delta,\varepsilon}\cdot\lambda = -\frac{1}{\varepsilon}\int_Y \left(DZ^\lambda_{\delta,\varepsilon}\right)^T V_\varepsilon\cdot\lambda\,dy + \delta\,\bar{Z}^\lambda_{\delta,\varepsilon}\cdot\lambda = |\lambda|^2. \tag{2.41}$$

If $N_\delta\lambda = 0$, then by (2.36), the sequence $\frac{1}{\varepsilon}\,|DZ^\lambda_{\delta,\varepsilon}|$ converges strongly to 0 in $L^2_\sharp(Y)$. Hence, passing to the limit as $\varepsilon \to 0$ in (2.41), we deduce that $\lambda = 0$. Therefore, the matrix N_δ is invertible and by (2.40) satisfies the inequality (in the sense of the quadratic forms)

$$N_\delta^{-1} \geq \delta\, I. \tag{2.42}$$

On the other hand, assume that there exists $\mu \in \mathbb{R}^3 \setminus \{0\}$ such that $N_\delta^{-1}\mu \cdot \mu = \delta\,|\mu|^2$. Then, the vector $\lambda := N_\delta^{-1}\mu$ satisfies the equality $N^\delta\lambda \cdot \lambda = \delta\,|N_\delta\lambda|^2$. This combined with equality (2.36) thus yields

$$\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon^2}\int_Y |DZ^\lambda_{\delta,\varepsilon}|^2\,\mathrm{d}y\right) = N_\delta\lambda\cdot\lambda - \delta\,|N_\delta\lambda|^2 = 0.$$

Therefore, passing to the limit in the variational formulation (2.28) multiplied by ε, and using convergences (2.37) and (2.38), we obtain that

$$-\int_Y \big(\mathrm{curl}\,(V)\times N_\delta\lambda\big)\cdot\Phi\,\mathrm{d}y = -\int_Y \big(\mathrm{curl}\,(V)\times\mu\big)\cdot\Phi\,\mathrm{d}y = 0, \quad \forall\,\Phi\in H^1_\sharp(Y)^3,\ \mathrm{div}\,(\Phi)=0,$$

which by periodicity implies that $\mathrm{curl}\,\big(\mathrm{curl}\,(V)\times\mu\big) = 0$ in $\mathcal{D}'(\mathbb{R}^3)$. Conversely, if (2.33) holds, then

$$N_\delta^{-1}\mu\cdot\mu > \delta\,|\mu|^2, \quad \forall\,\mu\in\mathbb{R}^3\setminus\{0\},$$

namely $N_\delta^{-1} - \delta\, I$ is positive definite.

Second step: Derivation of the limit problem.

Let us apply the Tartar method with the oscillating test functions $z^\lambda_{\delta,\varepsilon}$. Let $\lambda \in \mathbb{R}^3$ and $\varphi \in C^\infty_c(\Omega)$. Putting $\varphi\, u_\varepsilon$ in (2.30) and $\varphi\, z^\lambda_{\delta,\varepsilon}$ in (2.20), and equating the two formulas, we have

$$\begin{aligned}
&\int_\Omega Du_\varepsilon : (z^\lambda_{\delta,\varepsilon}\otimes\nabla\varphi)\,\mathrm{d}x - \int_\Omega Dz^\lambda_{\delta,\varepsilon} : (u_\varepsilon\otimes\nabla\varphi)\,\mathrm{d}x\\
&\quad - \int_\Omega p_\varepsilon\,\nabla\varphi\cdot z_{\delta,\varepsilon}\,\mathrm{d}x + \int_\Omega p^\lambda_{\delta,\varepsilon}\,\nabla\varphi\cdot u_\varepsilon\,\mathrm{d}x - \delta\int_\Omega \varphi\, z_{\delta,\varepsilon}\cdot u_\varepsilon\,\mathrm{d}x\\
&\quad = \int_\Omega \varphi\, f\cdot z_{\delta,\varepsilon}\,\mathrm{d}x - \int_\Omega \varphi\,\lambda\cdot u_\varepsilon\,\mathrm{d}x.
\end{aligned}$$

Hence, passing to the limit as $\varepsilon \to 0$ in the previous equality owing to convergences (2.35) and (2.39) and to the strong convergences of u_ε, $z^\lambda_{\delta,\varepsilon}$ in $L^s(\Omega)$ for any $s \in [1, 6)$, we obtain that

$$\int_\Omega Du : (N_\delta\lambda\otimes\nabla\varphi)\,\mathrm{d}x - \int_\Omega p\,\nabla\varphi\cdot N_\delta\lambda\,\mathrm{d}x + \int_\Omega \varphi\,(\lambda - \delta\, N_\delta\lambda)\cdot u\,\mathrm{d}x = \int_\Omega \varphi\, f\cdot N_\delta\lambda\,\mathrm{d}x. \tag{2.43}$$

Set $\lambda_i := N_\delta^{-1}\mathrm{e}_i$, for $i = 1, 2, 3$, where $(\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$ is the canonic basis of $\mathbb{R}^3$, and let $\Phi = (\varphi_1, \varphi_2, \varphi_3) \in C^\infty_c(\Omega)^3$. Using $\lambda = \lambda_i$ and the functions $\varphi = \varphi_i$, for $i = 1, 2, 3$, in

(2.43), and adding the three equations, it follows that u satisfies the variational problem

$$\int_\Omega Du : D\Phi \,\mathrm{d}x - \int_\Omega p \,\mathrm{div}\,(\Phi)\,\mathrm{d}x + \int_\Omega (N_\delta^{-1} - \delta\, I)^T u \cdot \Phi \,\mathrm{d}x = \int_\Omega f \cdot \Phi \,\mathrm{d}x. \tag{2.44}$$

On the other hand, proceeding as in the scalar case, it is easy to check that by (2.42) the nonnegative matrix

$$M := (N_\delta^{-1} - \delta\, I)^T \tag{2.45}$$

is independent of δ. Therefore, (2.44) is the variational formulation of the Stokes problem (2.32) with the nonnegative matrix M. Moreover, for the whole sequence ε such that the convergences (2.17), (2.37), and (2.38) hold, the sequences u_ε, p_ε converge according to (2.35), respectively, to the solutions u, p of problem (2.32)

Let us prove the inequality (2.31). Let $\lambda \in \mathbb{R}^3$. Using the Cauchy–Schwarz inequality combined with estimate (2.36) in the equality (2.41), we have

$$\begin{aligned}|\lambda|^2 - \delta\, \bar{Z}^\lambda_{\delta,\varepsilon} \cdot \lambda &= -\frac{1}{\varepsilon} \int_Y (DZ^\lambda_{\delta,\varepsilon})^T V_\varepsilon \cdot \lambda \,\mathrm{d}y \le \left(\int_Y |V_\varepsilon|^2 \,\mathrm{d}y \right)^{\frac12} \left(\frac{1}{\varepsilon^2} \int_Y |DZ^\lambda_{\delta,\varepsilon}|^2 \,\mathrm{d}y \right)^{\frac12} |\lambda| \\ &= \left(\int_Y |V_\varepsilon|^2 \,\mathrm{d}y \right)^{\frac12} \left(\lambda \cdot \bar{Z}^\lambda_{\delta,\varepsilon} - \delta \int_Y |Z^\lambda_{\delta,\varepsilon}|^2 \,\mathrm{d}y \right)^{\frac12} |\lambda|.\end{aligned}$$

Hence, by convergences (2.37) and (2.18), we get that

$$\begin{aligned}(I - \delta\, N_\delta)\lambda \cdot \lambda = |\lambda|^2 - \delta\, N_\delta \lambda \cdot \lambda &\le \mu_*^{\frac12} \left(N_\delta \lambda \cdot \lambda - \delta\, |N_\delta \lambda|^2 \right)^{\frac12} |\lambda| \\ &= \mu_*^{\frac12} \left((I - \delta\, N_\delta)\lambda \cdot N_\delta \lambda \right)^{\frac12} |\lambda|\end{aligned}$$

Set $\mu := N_\delta \lambda$. Therefore, from the previous inequality and the definition (2.45) of M, we deduce that

$$M^T \mu \cdot (M + \delta\, I)^T \mu \le \mu_*^{\frac12} \,(M^T \mu \cdot \mu)^{\frac12} \left| (M + \delta\, I)^T \mu \right|,$$

which implied the desired inequality (2.31) as δ tends to 0.

Third step: Comparison with the Tartar result.

It remains to prove the equality $M = M^*$ assuming that v_ε (2.19) is bounded and equi-integrable in $L^{12/5}(\Omega)^3$. First of all, thanks to the regularity result of the Stokes equation (see, e.g., [12] Theorem 2, p. 67) there exists a subsequence of ε, still denoted by ε, such that the solutions w^μ_ε, q^μ_ε of (2.24) satisfy the convergences

$$w^\mu_\varepsilon \rightharpoonup 0 \quad \text{weakly in } W^{1,12/5}(\Omega)^3, \quad q^\mu_\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^{12/5}(\Omega)/\mathbb{R}, \quad \forall\, \mu \in \mathbb{R}^3. \tag{2.46}$$

Let $\lambda, \mu \in \mathbb{R}^3$ and $\varphi \in C_c^\infty(\Omega)$. Putting $\varphi\, w^\mu_\varepsilon$ in (2.30) and $\varphi\, z^\lambda_{\delta,\varepsilon}$ in (2.24) (with μ), using the weak formulation (2.21), the weak convergences (2.39) and (2.46) of the velocities and the pressures, and equating the two formulas we have

$$\int_Y (v_\varepsilon \otimes z^\lambda_{\delta,\varepsilon}) : Dw^\mu_\varepsilon\,\varphi\,dx - \int_\Omega (Dz^\lambda_{\delta,\varepsilon})^T v_\varepsilon \cdot w^\mu_\varepsilon\,\varphi\,dx + \int_Y (v_\varepsilon \otimes \mu) : Dz^\lambda_{\delta,\varepsilon}\,\varphi\,dx \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{2.47}$$

In the first term of (2.47), v_ε is bounded and equi-integrable in $L^{12/5}(\Omega)^3$, by the Sobolev embedding $z^\lambda_{\delta,\varepsilon}$ is bounded in $L^6(\Omega)^3$ and converges a.e. in Ω to $\bar{Z}^\lambda_\delta$ up to a subsequence, and Dw^μ_ε is bounded in $L^{12/5}(\Omega)^{3\times 3}$, with $\frac{5}{12}+\frac{1}{6}+\frac{5}{12}=1$. Hence, by the Egorov theorem, we have

$$(v_\varepsilon \otimes z^\lambda_{\delta,\varepsilon}) : Dw^\mu_\varepsilon - (v_\varepsilon \otimes \bar{Z}^\lambda_\delta) : Dw^\mu_\varepsilon \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\Omega).$$

This combined with the εY-periodicity of the functions and the definition (2.25) of M^* yields, up to a subsequence,

$$(v_\varepsilon \otimes z^\lambda_{\delta,\varepsilon}) : Dw^\mu_\varepsilon \rightharpoonup \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\int_Y (V_\varepsilon \otimes \bar{Z}^\lambda_\delta) : DW^\mu_\varepsilon\,dy\right) = M^*\mu\cdot\bar{Z}^\lambda_\delta \quad \text{in } \mathcal{D}'(\Omega). \tag{2.48}$$

Similarly for the second term of (2.47), since w^μ_ε is bounded in $L^{12}(\Omega)^3$ by (2.46) combined with the Sobolev embedding, and converges to 0 a.e. in Ω up to a subsequence, we have

$$(Dz^\lambda_{\delta,\varepsilon})^T v_\varepsilon \cdot w^\mu_\varepsilon \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{2.49}$$

For the third term of (2.47), the εY periodicity implies that, up to a subsequence,

$$(v_\varepsilon \otimes \mu) : Dz^\lambda_{\delta,\varepsilon} \rightharpoonup \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\int_Y (V_\varepsilon \otimes \mu) : DZ^\lambda_{\delta,\varepsilon}\,dy\right) \quad \text{in } \mathcal{D}'(\Omega). \tag{2.50}$$

On the other hand, putting $\Phi = \mu$ in (2.28), it follows that

$$-\frac{1}{\varepsilon}\int_Y (V_\varepsilon \otimes \mu) : DZ^\lambda_{\delta,\varepsilon}\,dy = -\frac{1}{\varepsilon}\int_Y (DZ^\lambda_{\delta,\varepsilon})^T V_\varepsilon \cdot \mu\,dy = \lambda\cdot\mu - \delta\,\bar{Z}^\lambda_{\delta,\varepsilon}\cdot\mu.$$

Finally, using the convergence (2.47) combined with (2.48), (2.49), (2.50), and the previous equality, and taking into account (2.25), (2.45) and the symmetry of M^*, we obtain that

$$M^*\mu\cdot\bar{Z}^\lambda_\delta = M^* N_\delta\lambda\cdot\mu = (I-\delta\,N_\delta)\lambda\cdot\mu = M^T N_\delta\lambda\cdot\mu, \quad \forall\,\lambda,\mu\in\mathbb{R}^3,$$

which implies that $M^* = M^T = M$, since N_δ is invertible. Therefore, we derive the same Brinkman matrix that in the Tartar approach. □

3 Homogenization with a large drift in dimension two

Let $Y := \left[-\frac{1}{2},\frac{1}{2}\right]^2$, let B_ε be a Y-periodic vector-valued function in $L^\infty_\sharp(Y)^2$, and let b_ε be the oscillating drift defined by

$$b_\varepsilon(x) := B_\varepsilon\left(\frac{x}{\varepsilon}\right), \quad \text{for } x\in\mathbb{R}^2. \tag{3.1}$$

Consider the solution Z_ε in $H^1_\sharp(Y)$ of the equation

$$-\frac{1}{\varepsilon^2}\,\Delta Z_\varepsilon - \frac{1}{\varepsilon}\,B_\varepsilon\cdot\nabla Z_\varepsilon - \frac{1}{\varepsilon}\,\mathrm{div}\,(B_\varepsilon\,Z_\varepsilon) + Z_\varepsilon = 1 \quad \text{in } \mathbb{R}^2, \tag{3.2}$$

or equivalently in the torus, for any $V \in H^1_\sharp(Y)$,

$$\frac{1}{\varepsilon^2}\int_Y \nabla Z_\varepsilon \cdot \nabla V \,\mathrm{d}y - \frac{1}{\varepsilon}\int_Y B_\varepsilon \cdot \nabla Z_\varepsilon \, V \,\mathrm{d}y + \frac{1}{\varepsilon}\int_Y B_\varepsilon \cdot \nabla V \, Z_\varepsilon \,\mathrm{d}y + \int_Y Z_\varepsilon \, V \,\mathrm{d}y = \int_Y V \,\mathrm{d}y. \tag{3.3}$$

By the maximum principle Z_ε is nonnegative. Moreover, taking $V = Z_\varepsilon$ in (3.3), we get that

$$\frac{1}{\varepsilon^2}\int_Y |\nabla Z_\varepsilon|^2 \,\mathrm{d}y + \int_Y Z_\varepsilon^2 \,\mathrm{d}y = \int_Y Z_\varepsilon \,\mathrm{d}y. \tag{3.4}$$

Hence, up to extract a subsequence, we can assume that

$$Z_\varepsilon \rightharpoonup \bar{Z} \;\text{ weakly in } H^1_\sharp(Y), \quad \text{where} \quad \lim_{\varepsilon\to 0}\left(\int_Y Z_\varepsilon \,\mathrm{d}y\right) = \bar{Z} \in [0,1]. \tag{3.5}$$

Define the sequence ξ_ε in $\mathbb{R}^2$ by

$$\xi_\varepsilon := \int_Y B_\varepsilon \, Z_\varepsilon \,\mathrm{d}y. \tag{3.6}$$

As in the previous section, we will use the oscillating test function defined by $z_\varepsilon(x) := Z_\varepsilon(\frac{x}{\varepsilon})$, for $x \in \mathbb{R}^2$, which is solution of the rescaled equation from (3.3),

$$-\,\Delta z_\varepsilon - b_\varepsilon \cdot \nabla z_\varepsilon - \operatorname{div}(b_\varepsilon \, z_\varepsilon) + z_\varepsilon = 1 \quad \text{in } \mathbb{R}^2. \tag{3.7}$$

Then, we have the following homogenization result for the scalar drift problem (1.5) with no prescribed bound on b_ε, but assuming a sign of its divergence:

Theorem 3.1 *Assume that the drift b_ε satisfies*

$$\operatorname{div}(b_\varepsilon) \geq 0 \;\; \text{in } \mathcal{D}'(\mathbb{R}^2) \;\; \text{or} \;\; -\operatorname{div}(b_\varepsilon) \geq 0 \;\; \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{3.8}$$

Let Ω be a bounded open set of $\mathbb{R}^2$, and let $f \in H^{-1}(\Omega)$. Then, we have the following alternative:

- *If the sequence $|B_\varepsilon Z_\varepsilon|$ is bounded in $L^1(Y)$ and $\bar{Z} > 0$, then, up to a subsequence, ξ_ε converges to some ξ in $\mathbb{R}^2$, and the solution u_ε of the problem (1.5) with the drift (3.1) converges weakly in $H^1_0(\Omega)$ to the solution u of the equation*

$$-\,\Delta u - \frac{2}{\bar{Z}}\,\xi \cdot \nabla u + \left(\frac{1}{\bar{Z}} - 1\right) u = f \quad \text{in } \Omega. \tag{3.9}$$

- *If $\|B_\varepsilon Z_\varepsilon\|_{L^1(Y)^2}$ converges to ∞ with the extra assumption*

$$\liminf_{\varepsilon\to 0} \frac{|\xi_\varepsilon|}{\|B_\varepsilon \, Z_\varepsilon\|_{L^1(Y)^2}} > 0, \tag{3.10}$$

  or $\bar{Z} = 0$, then u_ε converges strongly to 0 in $H^1_0(\Omega)$.

Proof First of all, for a given $f \in H^{-1}(\Omega)$, the solution u_ε of (1.5) satisfies the energy equality

$$\int_\Omega |\nabla u_\varepsilon|^2 \,\mathrm{d}x = \langle f, u_\varepsilon \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}, \tag{3.11}$$

which implies that u_ε is bounded in $H_0^1(\Omega)$ and thus converges weakly, up to a subsequence, to some u in $H_0^1(\Omega)$. Using a density argument in (3.11), we can also assume that the right-hand side f belongs to $L^\infty(\Omega)$.

Let $\varphi \in C_c^\infty(\Omega)$. Putting $\varphi\, z_\varepsilon$ in (1.5), $\varphi\, u_\varepsilon$ in equation of (3.7), and equating the two formulas, we get that

$$\begin{aligned}&\int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi\, z_\varepsilon\, \mathrm{d}x - \int_\Omega \nabla z_\varepsilon \cdot \nabla\varphi\, u_\varepsilon\, \mathrm{d}x - 2\int_\Omega b_\varepsilon \cdot \nabla\varphi\, u_\varepsilon\, z_\varepsilon\, \mathrm{d}x - \int_\Omega \varphi\, z_\varepsilon\, u_\varepsilon\, \mathrm{d}x \\ &\quad = \int_\Omega f\,\varphi\, z_\varepsilon\, \mathrm{d}x - \int_\Omega \varphi\, u_\varepsilon\, \mathrm{d}x.\end{aligned} \tag{3.12}$$

The delicate term is the third one with the product $b_\varepsilon\, z_\varepsilon\, u_\varepsilon$. It is enough to derive its limit locally thanks to a partition of the unity. To this end, consider an open square $Q \Subset \Omega$, and, for ε small enough, the smallest square Q_ε composed of an entire number of cells $\varepsilon\,(k+Y)\,, k \in \mathbb{Z}^2$, such that $Q \subset Q_\varepsilon \Subset \Omega$. Note that $Q_\varepsilon = R_\varepsilon(Q)$, where R_ε is an affine mapping converging uniformly to Identity locally in $\mathbb{R}^2$. Let $w_\varepsilon \in H_0^1(Q_\varepsilon)$ be the solution of the equation

$$\Delta w_\varepsilon = \operatorname{div}(b_\varepsilon\, z_\varepsilon) \quad \text{in } Q_\varepsilon. \tag{3.13}$$

The following result holds:

Lemma 3.2 *Under assumption (3.8), the sequence $\|\nabla w_\varepsilon\|_{L^p(Q_\varepsilon)^2}$ is bounded for any $p \in [1,2)$.*

Let $\tilde{w}_\varepsilon \in H^1(Q_\varepsilon)$ be the stream function with zero Q_ε-average, defined from (3.13) by

$$b_\varepsilon\, z_\varepsilon = \nabla w_\varepsilon + J\nabla\tilde{w}_\varepsilon, \quad \text{where} \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}. \tag{3.14}$$

First, assume that the sequence $|B_\varepsilon\, Z_\varepsilon|$ is bounded in $L^1(Y)$. Hence, by periodicity, the sequence $b_\varepsilon\, z_\varepsilon$ is bounded in $L^1(Q)^2$. Moreover, by Lemma 3.2 and (3.14), the sequence $\tilde{w}_\varepsilon$ is bounded in $W^{1,1}(Q)$. For $\varphi \in C_c^\infty(Q)$, we have by an integration by parts

$$\int_Q b_\varepsilon \cdot \nabla\varphi\, u_\varepsilon\, z_\varepsilon\, \mathrm{d}x = \int_Q \nabla w_\varepsilon \cdot \nabla\varphi\, u_\varepsilon\, \mathrm{d}x - \int_Q J\nabla\tilde{w}_\varepsilon \cdot \nabla u_\varepsilon\, \varphi\, \mathrm{d}x \tag{3.15}$$

On the one hand, thanks to Lemma 3.2 and to the weak convergence of u_ε to u in $H_0^1(\Omega)$, which is compactly embedded in $L^q(\Omega)$ for any $q \in [1,\infty)$, we clearly have, up to a subsequence,

$$\lim_{\varepsilon\to 0}\int_Q \nabla w_\varepsilon \cdot \nabla\varphi\, u_\varepsilon\, \mathrm{d}x = \int_Q \nabla w \cdot \nabla\varphi\, u\, \mathrm{d}x, \tag{3.16}$$

where w is the weak limit of w_ε in $L^p(Q)$, for some $p \in (1,2)$. On the other hand, up to a subsequence $\tilde{w}_\varepsilon$ converges weakly to some $\tilde{w}$ in $BV(Q)$. Due to the periodicity of $b_\varepsilon\, z_\varepsilon$ and Lemma 3.2, the sequence $\nabla\tilde{w}_\varepsilon$ converges weakly-$*$ in $\mathcal{M}(Q)^2$ to $\nabla\tilde{w}$ satisfying

$$\xi = \nabla w + J\nabla\tilde{w}, \tag{3.17}$$

where ξ is the limit of ξ_ε (3.6) in $\mathbb{R}^2$, so that $\nabla\tilde{w} \in L^p(Q)^2$. Moreover, again by periodicity and up to a subsequence we have

$$|b_\varepsilon\, z_\varepsilon| \rightharpoonup \lim_{\varepsilon\to 0}\left(\int_Y |B_\varepsilon\, Z_\varepsilon|\,\mathrm{d}y\right) \quad \text{weakly-}* \text{ in } \mathcal{M}(Q), \tag{3.18}$$

and $|\nabla w_\varepsilon|$ converges weakly in $L^p(Q)$. This combined with $|J\nabla\tilde{w}_\varepsilon| \le |b_\varepsilon\, z_\varepsilon| + |\nabla w_\varepsilon|$ implies that the sequence $|J\nabla\tilde{w}_\varepsilon|$ converges weakly-$*$ in $\mathcal{M}(Q)$ to a function of $L^p(Q)$. Therefore, since ∇u_ε converges weakly to ∇u in $L^2(\Omega)^2$, $J\nabla\tilde{w}_\varepsilon$ is divergence free and converges weakly-$*$ to $J\nabla w$ in $\mathcal{M}(\Omega)^2$, and since there is no concentration effects for the sequence $|J\nabla\tilde{w}_\varepsilon - J\nabla\tilde{w}|$, the div-curl result of [5] (Theorem 3.1 and Remark 3.2) combined with integrations by parts and the fact that $\nabla\tilde{w} \in L^p(Q)^2$ yields

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_Q J\nabla\tilde{w}_\varepsilon\cdot\nabla u_\varepsilon\,\varphi\,\mathrm{d}x &= \left\langle J\nabla\tilde{w}\cdot\nabla u,\varphi\right\rangle_{\mathcal{D}'(Q),C_c^\infty(Q)}\\ &= \left\langle \operatorname{div}\left(u\,J\nabla\tilde{w}\right),\varphi\right\rangle_{\mathcal{D}'(Q),C_c^\infty(Q)} = -\int_Q J\nabla\tilde{w}\cdot\nabla\varphi\,u\,\mathrm{d}x.\end{aligned} \tag{3.19}$$

Hence, passing to the limit in (3.15) together with (3.16), (3.19), and (3.17), it follows that

$$\lim_{\varepsilon\to 0}\int_Q b_\varepsilon\cdot\nabla\varphi\,u_\varepsilon\,z_\varepsilon\,\mathrm{d}x = \int_Q \xi\cdot\nabla\varphi\,u\,\mathrm{d}x. \tag{3.20}$$

Moreover, by estimates (3.5), the sequence z_ε converges weakly to $\bar{Z}$ in $H^1(\Omega)$. Therefore, passing to the limit in (3.12) together with (3.20), we obtain that

$$\int_Q \nabla u\cdot\nabla\varphi\,\bar{Z}\,\mathrm{d}x - 2\int_Q \xi\cdot\nabla\varphi\,u\,\mathrm{d}x - \int_Q \varphi\,\bar{Z}\,u\,\mathrm{d}x = \int_Q f\,\varphi\,\bar{Z}\,\mathrm{d}x - \int_Q \varphi\,u\,\mathrm{d}x, \tag{3.21}$$

which is the variational formulation of Eq. (3.9) if $\bar{Z} > 0$. Using a uniqueness argument, the sequence u_ε converges weakly to the solution u of (3.9) for any sequence ε satisfying the convergence of ξ_ε to ξ and (3.5)

Otherwise, $\bar{Z} = 0$ and equality (3.21) implies that for any $\varphi \in H_0^1(\Omega)$,

$$\int_\Omega \varphi\,u\,\mathrm{d}x - 2\int_\Omega \xi\cdot\nabla\varphi\,u\,\mathrm{d}x = 0.$$

Taking $\varphi = u$ and integrating by parts the second integral, it follows that $u = 0$. This combined with the energy equality (3.11) implies that u_ε converges strongly to 0 in $H_0^1(\Omega)$.

Now, assume that $\|B_\varepsilon Z_\varepsilon\|_{L^1(Y)^2}$ converges to ∞. Dividing the equality (3.12) by $\|B_\varepsilon Z_\varepsilon\|_{L^1(Y)^2}$ and taking into account that the sequences $u_\varepsilon, z_\varepsilon$ are bounded in $H^1(\Omega)$, we get that

$$\lim_{\varepsilon\to 0}\int_\Omega \zeta_\varepsilon\cdot\nabla\varphi\,u_\varepsilon\,\mathrm{d}x = 0, \quad \text{where} \quad \zeta_\varepsilon := \frac{B_\varepsilon\,Z_\varepsilon}{\|B_\varepsilon\,Z_\varepsilon\|_{L^1(Y)^2}}\left(\frac{x}{\varepsilon}\right). \tag{3.22}$$

By virtue of the εY-periodicity, the sequence ζ_ε is bounded in $L^1(\Omega)^2$, and, up to a subsequence, converges weakly-$*$ in $\mathcal{M}(\Omega)$ to some vector $\zeta \in \mathbb{R}^2$, with $\zeta \neq 0$ by (3.10).

Replacing $b_\varepsilon\, z_\varepsilon$ by ζ_ε, and repeating the procedure of the previous case, we obtain similarly to (3.20) that

$$\lim_{\varepsilon\to 0}\int_\Omega \zeta_\varepsilon\cdot\nabla\varphi\, u_\varepsilon\,\mathrm{d}x=\int_\Omega \zeta\cdot\nabla\varphi\, u\,\mathrm{d}x.$$

This combined with equality (3.22) yields that $\zeta\cdot\nabla u=0$ a.e. in Ω. However, since ζ is a nonzero vector of $\mathbb{R}^2$, there exists a constant $C>0$ such that the following Poincaré–Wirtinger inequality holds (it is enough to extend $u\in H_0^1(\Omega)$ by zero in an open square containing $\bar\Omega$, a side of which is parallel to ζ):

$$\int_\Omega u^2\,\mathrm{d}x\le C\int_\Omega |\zeta\cdot\nabla u|^2\,\mathrm{d}x=0.$$

Therefore, $u=0$ and by (3.11) u_ε converges strongly to 0 in $H_0^1(\Omega)$. □

Proof of Lemma 3.2 For any $k>0$, define the truncation $w_\varepsilon^k:=(-k\vee w_\varepsilon)\wedge k$. Putting w_ε^k in equations (3.7), (3.13) and integrating by parts, we have

$$\int_{Q_\varepsilon}\nabla z_\varepsilon\cdot\nabla w_\varepsilon^k\,\mathrm{d}x-\int_{Q_\varepsilon} b_\varepsilon\cdot\nabla z_\varepsilon\, w_\varepsilon^k\,\mathrm{d}x+\int_{Q_\varepsilon}|\nabla w_\varepsilon^k|^2\,\mathrm{d}x=\int_{Q_\varepsilon}(1-z_\varepsilon)\,w_\varepsilon^k\,\mathrm{d}x. \quad (3.23)$$

Again using (3.13), the second integral of (3.23) can be written

$$\begin{aligned}\int_{Q_\varepsilon} b_\varepsilon\cdot\nabla z_\varepsilon\, w_\varepsilon^k\,\mathrm{d}x&=\int_{Q_\varepsilon} b_\varepsilon\cdot\nabla(z_\varepsilon\, w_\varepsilon^k)\,\mathrm{d}x-\int_{Q_\varepsilon} b_\varepsilon\cdot\nabla w_\varepsilon^k\, z_\varepsilon\,\mathrm{d}x\\&=\int_{Q_\varepsilon} b_\varepsilon\cdot\nabla(z_\varepsilon\, w_\varepsilon^k)\,\mathrm{d}x-\int_{Q_\varepsilon}|\nabla w_\varepsilon^k|^2\,\mathrm{d}x.\end{aligned}$$

Substituting this equality in (3.23), it follows that

$$\begin{aligned}2\int_{Q_\varepsilon}|\nabla w_\varepsilon^k|^2\,\mathrm{d}x+\int_{Q_\varepsilon}\nabla z_\varepsilon\cdot\nabla w_\varepsilon^k\,\mathrm{d}x&=\int_{Q_\varepsilon} b_\varepsilon\cdot\nabla(z_\varepsilon\, w_\varepsilon^k)\,\mathrm{d}x+\int_{Q_\varepsilon}(1-z_\varepsilon)\,w_\varepsilon^k\,\mathrm{d}x\\&=\int_{Q_\varepsilon} b_\varepsilon\cdot\nabla\big(z_\varepsilon\,(w_\varepsilon^k\mp k)\big)\,\mathrm{d}x+\int_{Q_\varepsilon} b_\varepsilon\cdot\nabla(\pm k\, z_\varepsilon)\,\mathrm{d}x+\int_{Q_\varepsilon}(1-z_\varepsilon)\,w_\varepsilon^k\,\mathrm{d}x. \quad (3.24)\end{aligned}$$

Then, by the inequality (3.8), that is, $\mp\,\mathrm{div}\,(b_\varepsilon)\ge 0$, combined with the fact that $\mp\, z_\varepsilon\,(w_\varepsilon^k\mp k)$ is Q_ε-periodic and nonnegative, equality (3.24) implies that

$$2\int_{Q_\varepsilon}|\nabla w_\varepsilon^k|^2\,\mathrm{d}x+\int_{Q_\varepsilon}\nabla z_\varepsilon\cdot\nabla w_\varepsilon^k\,\mathrm{d}x\le\pm k\int_{Q_\varepsilon} b_\varepsilon\cdot\nabla z_\varepsilon\,\mathrm{d}x+\int_{Q_\varepsilon}(1-z_\varepsilon)\,w_\varepsilon^k\,\mathrm{d}x. \quad (3.25)$$

However, using the definition of Q_ε combined with the εY-periodicity of $b_\varepsilon\cdot\nabla z_\varepsilon$, and putting $V=1$ in Eq. (3.3), we obtain that

$$\left|\int_{Q_\varepsilon} b_\varepsilon\cdot\nabla z_\varepsilon\,\mathrm{d}x\right|=\frac{|Q_\varepsilon|}{\varepsilon}\left|\int_Y B_\varepsilon\cdot\nabla Z_\varepsilon\,\mathrm{d}y\right|=|Q_\varepsilon|\left|\int_Y (Z_\varepsilon-1)\,\mathrm{d}y\right|\le c.$$

Therefore, since z_ε is bounded in $H^1(\Omega)$ inequality (3.25) and the previous estimate imply that there exist two constants $k_0, C > 0$ such that

$$\sup_{\varepsilon>0} \int_{Q_\varepsilon} |\nabla w_\varepsilon^k|^2 \,\mathrm{d}x \le C\,k, \quad \forall\, k \ge k_0. \tag{3.26}$$

Now, passing from the square Q_ε to the fixed square $Q = R_\varepsilon^{-1}(Q_\varepsilon)$ through the affine mapping R_ε (the gradient of which converges to I_2 in $\mathbb{R}^{2\times 2}$), we can apply Lemma 4.2 of [2] to the sequence $w_\varepsilon \circ R_\varepsilon \in H_0^1(Q)$ satisfying the estimate (3.26) with the fixed square Q. As a consequence, $\nabla(w_\varepsilon \circ R_\varepsilon)$ is bounded in any Marcinkiewicz space of exponent $q \in [1, 2)$ in the bounded open set Q; hence, $\nabla(w_\varepsilon \circ R_\varepsilon)$ is bounded in $L^p(Q)^2$ for any $p \in [1, 2)$. Therefore, for any $p \in [1, 2)$, there exists a constant $C_p > 0$ such that

$$\sup_{\varepsilon>0} \int_{Q_\varepsilon} |\nabla w_\varepsilon|^p \,\mathrm{d}x \le C_p,$$

which concludes the proof. □

4 Nonlocal effects in dimension three

Let $Y := \left[-\frac{1}{2}, \frac{1}{2}\right]^2$, and let $\Omega := \omega \times (0, 1)$ be a vertical cylinder the basis of which ω is a regular bounded open set of $\mathbb{R}^2$. Let $\omega_\varepsilon \subset \omega$ be the εY-periodic lattice composed of the small disks centered at the points εk, $k \in \mathbb{Z}^2$, and of radius $\varepsilon r_\varepsilon$, that is,

$$\omega_\varepsilon = \omega \cap \bigcup_{k\in\mathbb{Z}^2} (\varepsilon\, k + \varepsilon\, Q_\varepsilon)\,, \tag{4.1}$$

where Q_ε is the closed disk centered at the origin and of radius $r_\varepsilon \to 0$. Consider the oscillating drift b_ε defined by

$$b_\varepsilon(x) := \frac{\beta_\varepsilon}{2}\,\frac{1_{\omega_\varepsilon}(x')}{|Q_\varepsilon|}\,\mathrm{e}_3 = \frac{\beta_\varepsilon}{2}\,\frac{1_{\Omega_\varepsilon}(x)}{|Q_\varepsilon|}\,\mathrm{e}_3, \quad \text{for } x = (x', x_3) \in \Omega, \tag{4.2}$$

where β_ε is a positive sequence, $\mathrm{e}_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and $\Omega_\varepsilon := \omega_\varepsilon \times (0, 1)$. Note that b_ε is divergence free in Ω. Assume that

$$\lim_{\varepsilon\to 0} \frac{2\pi}{\varepsilon^2 |\ln r_\varepsilon|} = \gamma \in [0, \infty] \quad \text{and} \quad \lim_{\varepsilon\to 0} \beta_\varepsilon = \beta \in [0, \infty]. \tag{4.3}$$

To state the homogenization result, we need the following preliminary lemma:

Lemma 4.1 *Let z_ε be the solution in $H_0^1(\Omega)$ of the equation*

$$-\,\Delta z_\varepsilon - b_\varepsilon \cdot \nabla z_\varepsilon - \operatorname{div}(b_\varepsilon\, z_\varepsilon) = -\,\Delta z_\varepsilon - \beta_\varepsilon\, \frac{1_{\Omega_\varepsilon}}{|Q_\varepsilon|}\,\frac{\partial z_\varepsilon}{\partial x_3} = 1 \quad \textit{in } \Omega. \tag{4.4}$$

Assume that $\gamma \in (0, \infty)$. Then, there exist a subsequence of ε, still denoted by ε, and two nonnegative functions $z \in H_0^1(\Omega) \cap C^0(\Omega)$ and $\bar{z} \in H^1\big(0, 1; L^2(\omega)\big)$ such that

$$z_\varepsilon \rightharpoonup z \;\; \textit{weakly in } H_0^1(\Omega) \quad \textit{and} \quad \frac{1_{\Omega_\varepsilon}}{|Q_\varepsilon|}\, z_\varepsilon \rightharpoonup \bar{z} \;\; \textit{weakly-}{*} \textit{ in } \mathcal{M}(\bar{\Omega}), \tag{4.5}$$

with

$$\begin{cases} z > 0 & \textit{in } \Omega, \\ \displaystyle\int_0^1 \bar{z}(\cdot, t)\, \mathrm{d}t \leq \int_0^1 z(\cdot, t)\, dt & \textit{a.e. in } \omega, \\ \bar{z}(\cdot, 1) \leq \bar{z}(\cdot, 0) & \textit{a.e. in } \omega. \end{cases} \tag{4.6}$$

Moreover,

– *if $\beta < \infty$, z and $\bar{z}$ are solutions of the coupled system*

$$\begin{cases} -\Delta z + \gamma\,(z - \bar{z}) = 1 & \textit{in } \Omega \\ \beta\, \dfrac{\partial \bar{z}}{\partial x_3} + \gamma\,(z - \bar{z}) = 0 & \textit{in } \Omega, \end{cases} \tag{4.7}$$

– *if $\beta = \infty$, z and $\bar{z}$ are solutions of*

$$\begin{cases} -\Delta z + \gamma\,(z - \bar{z}) = 1 & \textit{in } \Omega \\ \bar{z} - \bar{z}(\cdot, 0) = 0 & \textit{in } \Omega. \end{cases} \tag{4.8}$$

In the sequel, we will use a subsequence ε defined as in Lemma 4.1. Consider for $f \in H^{-1}(\Omega)$ and the vector-valued function b_ε defined by (4.2) the drift problem

$$\begin{cases} -\Delta u_\varepsilon + b_\varepsilon \cdot \nabla u_\varepsilon + \operatorname{div}(b_\varepsilon\, u_\varepsilon) = -\Delta u_\varepsilon + \beta_\varepsilon \dfrac{1_{\Omega_\varepsilon}}{|Q_\varepsilon|} \dfrac{\partial u_\varepsilon}{\partial x_3} = f & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.9}$$

Then, we have the following homogenization result:

Theorem 4.2 *The solution u_ε of (4.9) converges weakly in $H_0^1(\Omega)$ to the solution u of one of the following equations in Ω:*

– *if $\gamma, \beta \in (0, \infty)$,*

$$-\Delta u + \gamma\, u - \frac{\gamma^2}{\beta}\, \mathrm{e}^{-\frac{\gamma}{\beta} x_3} \int_0^{x_3} \mathrm{e}^{\frac{\gamma}{\beta} t}\, u(x', t)\, \mathrm{d}t - \gamma\, \frac{\mathrm{e}^{-\frac{\gamma}{\beta} x_3}\, \bar{z}(x', 1)}{\int_0^1 \mathrm{e}^{\frac{\gamma}{\beta}(1-t)} z(x', t)\, \mathrm{d}t} \int_0^1 \mathrm{e}^{\frac{\gamma}{\beta} t}\, u(x', t)\, \mathrm{d}t = f, \tag{4.10}$$

– *if $\gamma \in (0, \infty)$ and $\beta = \infty$,*

$$-\Delta u + \gamma\, u - \gamma\, \frac{\bar{z}(x')}{\int_0^1 z(x', t)\, \mathrm{d}t} \int_0^1 u(x', t)\, \mathrm{d}t = f, \tag{4.11}$$

– *if $\gamma = \infty$ and $\beta < \infty$,*

$$-\Delta u + \beta\, \frac{\partial u}{\partial x_3} = f, \tag{4.12}$$

– *if* $\gamma = 0$ *or* $\beta = 0$,

$$- \Delta u = f, \tag{4.13}$$

– *if* $\gamma = \beta = \infty$,

$$u = 0. \tag{4.14}$$

Proof We will need the following auxiliary results:

Lemma 4.3 *There exists a constant* $C > 0$ *such that*

$$\forall\, v \in H_0^1(\Omega), \quad \fint_{\Omega_\varepsilon} v^2\,\mathrm{d}x \le C\left(1 + \varepsilon^2 |\ln r_\varepsilon|\right) \int_\Omega |\nabla v|^2\,\mathrm{d}x. \tag{4.15}$$

Lemma 4.4 *Let* $\hat{v}_\varepsilon$ *be the* x_3*-independent and* εY*-periodic function defined by*

$$\hat{v}_\varepsilon(x) := \hat{V}_\varepsilon\left(\frac{x'}{\varepsilon}\right), \quad \textit{for } x = (x', x_3) \in \mathbb{R}^3, \quad \textit{where}$$

$$\hat{V}_\varepsilon(y) := \begin{cases} 0 & \textit{if } r = \sqrt{y_1^2 + y_2^2} \le r_\varepsilon \\[2ex] \dfrac{\ln r_\varepsilon - \ln r}{\ln r_\varepsilon + \ln 2} & \textit{if } r \in (r_\varepsilon, 1/2) \qquad\qquad \textit{for } y \in Y. \\[2ex] 1 & \textit{elsewhere}, \end{cases} \tag{4.16}$$

Then, there exists a constant $C > 0$ *such that for any* $v \in H_0^1(\Omega)$,

$$\left| \int_\Omega \nabla \hat{v}_\varepsilon \cdot \nabla v\,\mathrm{d}x - \frac{2\pi}{\varepsilon^2 |\ln(2r_\varepsilon)|} \left(\int_\Omega v\,\mathrm{d}x - \int_\Omega \frac{1_{\Omega_\varepsilon}}{|Q_\varepsilon|}\, v\,\mathrm{d}x \right) \right| \le \frac{C}{\varepsilon\,|\ln r_\varepsilon|}\, \|\nabla v\|_{L^2(\Omega)}. \tag{4.17}$$

Moreover, if $\gamma < \infty$, *the sequence* $\hat{v}_\varepsilon$ *converges weakly to* 1 *in* $H^1_{\mathrm{loc}}(\mathbb{R}^2)$.

Lemma 4.3 is a straightforward consequence of [13] (Lemma 3), and Lemma 4.4 is an easy adaptation of [9] (Example 2.1) and [7] (Lemma 1).

First of all, as in the previous section, the energy equality (3.11) holds, the sequence u_ε weakly converges, up to a subsequence, to some function u in $H_0^1(\Omega)$, and we can assume that the right-hand side f of (4.9) belongs to $L^\infty(\Omega)$.

Assume that $\gamma > 0$, and set

$$\bar{u}_\varepsilon := \frac{1_{\Omega_\varepsilon}}{|Q_\varepsilon|}\, u_\varepsilon. \tag{4.18}$$

Note that $|\Omega_\varepsilon| \approx |\Omega|\,|Q_\varepsilon|$. Hence, by the Cauchy–Schwarz inequality and the estimate (4.15) with $\gamma > 0$, we have for any $\varphi \in C^0(\bar{\Omega})$,

$$\limsup_{\varepsilon\to 0} \int_\Omega |\varphi\, \bar{u}_\varepsilon|\,\mathrm{d}x \le c\, \limsup_{\varepsilon\to 0} \left[\left(\fint_{\Omega_\varepsilon} u_\varepsilon^2\,\mathrm{d}x \right)^{\frac{1}{2}} \left(\fint_{\Omega_\varepsilon} \varphi^2\,\mathrm{d}x \right)^{\frac{1}{2}} \right] \le c\, \|\varphi\|_{L^2(\Omega)}. \tag{4.19}$$

Taking $\varphi = 1$ in (4.19), we get that the sequence $\bar{u}_\varepsilon$ is bounded in $L^1(\Omega)$, and thus, up to a subsequence, $\bar{u}_\varepsilon$ converges weakly-$*$ to some $\bar{u}$ in $\mathcal{M}(\bar{\Omega})$. Again using (4.19) with an arbitrary $\varphi \in C^0(\bar{\Omega})$, the Riesz representation theorem implies that $\bar{u} \in L^2(\Omega)$.

Now, assume that $\gamma > 0$ and $\beta < \infty$. Passing to the limit in Eq. (4.9), we obtain that

$$-\ \Delta u + \beta\,\frac{\partial \bar{u}}{\partial x_3} = f \quad \text{in } \mathcal{D}'(\Omega). \tag{4.20}$$

Assume in addition that $\gamma < \infty$. Let $\varphi \in C_c^\infty(\Omega)$. Putting $\varphi\,\hat{v}_\varepsilon$ (which vanishes in Ω_ε) in Eq. (4.9) and $\varphi\,u_\varepsilon$ in estimate (4.17) it follows that

$$\begin{aligned}\int_\Omega \nabla u_\varepsilon \cdot \nabla(\varphi\,\hat{v}_\varepsilon)\,\mathrm{d}x - \int_\Omega \nabla \hat{v}_\varepsilon \cdot \nabla(\varphi\,u_\varepsilon)\,\mathrm{d}x &= \int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi\,\hat{v}_\varepsilon\,\mathrm{d}x - \int_\Omega \nabla \hat{v}_\varepsilon \cdot \nabla\varphi\,u_\varepsilon\,\mathrm{d}x\\ = \int_\Omega f\,\varphi\,\hat{v}_\varepsilon\,\mathrm{d}x - \frac{2\pi}{\varepsilon^2|\ln(2r_\varepsilon)|}\int_\Omega \varphi\,(u_\varepsilon - \bar{u}_\varepsilon)\,\mathrm{d}x + o(1).\end{aligned}$$

This combined with the weak convergences of $\hat{v}_\varepsilon$ to 1 in $H^1(\Omega)$, of u_ε to u in $H_0^1(\Omega)$, and of $\hat{u}_\varepsilon$ to $\bar{u}$ in $\mathcal{M}(\Omega)$ implies that

$$\int_\Omega \nabla u \cdot \nabla\varphi\,\mathrm{d}x = \int_\Omega f\,\varphi\,\mathrm{d}x - \int_\Omega \gamma\,(u - \bar{u})\,\varphi\,\mathrm{d}x,$$

which yields the equation

$$-\ \Delta u + \gamma\,(u - \bar{u}) = f \quad \text{in } \mathcal{D}'(\Omega). \tag{4.21}$$

Equating (4.20) and (4.21), we also have

$$\beta\,\frac{\partial \bar{u}}{\partial x_3} + \gamma\,(\bar{u} - u) = 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{4.22}$$

Since $u, \bar{u} \in L^2(\Omega)$, we deduce from (4.22) that $\bar{u} \in H^1\big(0, 1; L^2(\omega)\big)$.

Now, it remains to distinguish the different cases.

Case $\gamma, \beta \in (0, \infty)$:

To derive the limit Eq. (4.10), it is enough to determine the trace $\bar{u}(\cdot, 0)$ by virtue of the first-order Eq. (4.22). To this end, we use the sequence z_ε of Lemma 4.1. Let $\varphi \in C_c^\infty(\omega)$ ($\varphi = \varphi(x')$). Putting $\varphi\,z_\varepsilon$ in Eq. (4.9), $\varphi\,u_\varepsilon$ in Eq. (4.4), equating the two equations, and integrating by parts the term with $\frac{\partial u_\varepsilon}{\partial x_3}$ (using that φ is independent of x_3), we get that

$$\int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi\,z_\varepsilon - \int_\Omega \nabla z_\varepsilon \cdot \nabla\varphi\,u_\varepsilon = \int_\Omega f\,\varphi\,z_\varepsilon\,\mathrm{d}x - \int_\Omega \varphi\,u_\varepsilon\,\mathrm{d}x,$$

which yields at the limit

$$\int_\Omega \nabla u \cdot \nabla\varphi\,z - \int_\Omega \nabla z \cdot \nabla\varphi\,u\,\mathrm{d}x = \int_\Omega f\,\varphi\,z\,\mathrm{d}x - \int_\Omega \varphi\,u\,\mathrm{d}x. \tag{4.23}$$

On the other hand, putting $\varphi\,z$ in Eq. (4.21), $\varphi\,u$ in the first Eq. of (4.7), and equating the two equations, we obtain that

$$\int_\Omega \nabla u \cdot \nabla\varphi\, z + \int_\Omega \gamma\,(u-\bar u)\,\varphi\, z\,\mathrm{d}x - \int_\Omega \nabla z\cdot\nabla\varphi\, u\,\mathrm{d}x - \int_\Omega \gamma\,(z-\bar z)\,\varphi\, u\,\mathrm{d}x$$
$$= \int_\Omega f\,\varphi\, z\,\mathrm{d}x - \int_\Omega \varphi\, u\,\mathrm{d}x,$$

which combined with (4.23) implies that

$$\int_\Omega \gamma\,\bar u\, z\,\varphi\,\mathrm{d}x = \int_\Omega \gamma\,\bar z\, u\,\varphi\,\mathrm{d}x.$$

Therefore, due to $\gamma > 0$ and the arbitrariness of $\varphi = \varphi(x')$, we deduce that

$$\int_0^1 u(\cdot, x_3)\,\bar z(\cdot, x_3)\,\mathrm{d}x_3 = \int_0^1 z(\cdot, x_3)\,\bar u(\cdot, x_3)\,\mathrm{d}x_3 \quad \text{a.e. in } \omega. \tag{4.24}$$

Moreover, using that $\bar u\,\bar z \in W^{1,1}\big(0, 1; L^2(\omega)\big)$, the second Eq. of (4.7), (4.22), and (4.24) gives

$$\bar u(\cdot,1)\,\bar z(\cdot,1) - \bar u(\cdot,0)\,\bar z(\cdot,0) = \int_0^1 \frac{\partial(\bar u\,\bar z)}{\partial x_3}\,\mathrm{d}x_3 = \frac{\gamma}{\beta}\int_0^1 \big[(u-\bar u)\,\bar z - (z-\bar z)\,\bar u\big]\,\mathrm{d}x_3 = 0. \tag{4.25}$$

On the other hand, integrating the first-order Eq. (4.22) with respect to the variable x_3, we have

$$\bar u(\cdot, x_3) = \frac{\gamma}{\beta}\,\mathrm{e}^{-\frac{\gamma}{\beta}x_3}\int_0^{x_3} \mathrm{e}^{\frac{\gamma}{\beta}t}\, u(\cdot, t)\,\mathrm{d}t + \bar u(\cdot, 0)\,\mathrm{e}^{-\frac{\gamma}{\beta}x_3} \quad \text{a.e. in } \omega, \tag{4.26}$$

and similarly with the second Eq. of (4.7)

$$\bar z(\cdot, x_3) = -\frac{\gamma}{\beta}\,\mathrm{e}^{\frac{\gamma}{\beta}x_3}\int_0^{x_3} \mathrm{e}^{-\frac{\gamma}{\beta}t}\, z(\cdot, t)\,\mathrm{d}t + \bar z(\cdot, 0)\,\mathrm{e}^{\frac{\gamma}{\beta}x_3} \quad \text{a.e. in } \omega. \tag{4.27}$$

Combining (4.25) and (4.26), (4.27) for $x_3 = 1$, and taking into account (4.6), we get the formula

$$\bar u(\cdot, 0) = \frac{\bar z(\cdot,1)}{\int_0^1 \mathrm{e}^{\frac{\gamma}{\beta}(1-t)} z(\cdot,t)\,\mathrm{d}t}\int_0^1 \mathrm{e}^{\frac{\gamma}{\beta}t}\, u(\cdot,t)\,\mathrm{d}t \quad \text{a.e. in } \omega. \tag{4.28}$$

Therefore, putting (4.26) and (4.28) in (4.21), we obtain the desired nonlocal Eq. (4.10).

To conclude, we need to prove that Eq. (4.10), or equivalently the system (4.21), (4.22), and (4.28), admits a unique solution u in $H_0^1(\Omega)$. Due to the linearity, it is enough to show that $u = 0$ when the right-hand side $f = 0$ in Ω. In this case, putting u in (4.21), $\bar u$ in (4.22), integrating by parts, and adding the two equations, it follows that

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x + \gamma\int_\Omega (u-\bar u)^2\,\mathrm{d}x + \frac{\beta}{2}\int_\omega \big(\bar u^2(x',1) - \bar u^2(x',0)\big)\,\mathrm{d}x' = 0. \tag{4.29}$$

Moreover, taking into account the equalities (4.26) and (4.27) (as by-products of (4.22) and the second Eq. of (4.7), respectively), it is easy to check that conversely equality (4.28) implies relation (4.25). Now, if $\bar{u}(x',0)=0$, then we clearly have $\bar{u}^2(x',1)-\bar{u}^2(x',0)\geq 0$. Otherwise, by (4.28) we have both $\bar{u}(x',0)\neq 0$ and $\bar{z}(x',1)\neq 0$, hence from (4.25) and the third inequality of (4.6) we deduce that

$$\left|\frac{\bar{u}(x',1)}{\bar{u}(x',0)}\right| = \frac{\bar{z}(x',0)}{\bar{z}(x',1)} \geq 1,$$

which implies that $\bar{u}^2(x',1)-\bar{u}^2(x',0)\geq 0$ a.e. in ω. Therefore, using this inequality in (4.29), we get that $u=0$ a.e. in Ω, which establishes the uniqueness for Eq. (4.10) As a consequence, u_ε converges to the solution u of (4.10) for the whole sequence ε defined in Lemma 4.1.

Case $\gamma\in(0,\infty)$ and $\beta=\infty$:

Since $\gamma\in(0,\infty)$, the limit Eq. (4.21) and the relation (4.24) are still valid. Moreover, dividing equations (4.4) and (4.9) by β_ε and passing to the limit as $\varepsilon\to 0$, it follows that

$$\frac{\partial\bar{z}}{\partial x_3}=\frac{\partial\bar{u}}{\partial x_3}=0 \quad \text{in } \Omega.$$

Hence, the functions $\bar{z}$ and $\bar{u}$ are independent of the coordinate x_3. This combined with (4.24) implies that

$$\bar{u}=\frac{\bar{z}}{\int_0^1 z(\cdot,t)\,\mathrm{d}t}\int_0^1 u(\cdot,t)\,\mathrm{d}t \quad \text{a.e. in } \omega.$$

Putting this relation in (4.21), we thus get the nonlocal Eq. (4.11).

To conclude as in the previous case, it remains to prove the uniqueness in Eq. (4.11). Assume that $f=0$ in Ω. Then, putting u in Eq. (4.11) and integrating by parts, we get that

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x+\gamma\int_\omega\left[\int_0^1 u^2(x',x_3)\,\mathrm{d}x_3-\frac{\bar{z}(x')}{\int_0^1 z(x',t)\,\mathrm{d}t}\left(\int_0^1 u(x',x_3)\,\mathrm{d}x_3\right)^2\right]\mathrm{d}x'=0. \tag{4.30}$$

However, using successively the Cauchy–Schwarz inequality and the second inequality of (4.6), we have

$$\int_0^1 u^2(x',x_3)\,\mathrm{d}x_3\geq\left(\int_0^1 u(x',x_3)\,\mathrm{d}x_3\right)^2\geq\frac{\bar{z}(x')}{\int_0^1 z(x',t)\,\mathrm{d}t}\left(\int_0^1 u(x',x_3)\,\mathrm{d}x_3\right)^2 \quad \text{a.e. } x'\in\omega.$$

Therefore, using this inequality in (4.30), we obtain that $u=0$ a.e. in Ω, which implies the uniqueness property.

Case $\gamma=\infty$ and $\beta<\infty$:

The function $\hat{v}_\varepsilon$ defined by (4.16) satisfies the estimate

$$\|\nabla\hat{v}_\varepsilon\|_{L^2(\Omega)}\leq\frac{c}{\sqrt{\varepsilon^2|\ln r_\varepsilon|}}. \tag{4.31}$$

Then, multiplying inequality (4.17) by $\varepsilon^2|\ln r_\varepsilon|$ (which tends to 0 since $\gamma = \infty$), and passing to the limit with $v = \varphi\, u_\varepsilon$, for $\varphi \in C_c^\infty(\Omega)$, we obtain that

$$\int_\Omega \varphi\,(u - \bar{u})\,\mathrm{d}x = 0,$$

hence $\bar{u} = u$. Therefore, Eq. (4.20) yields the local Eq. (4.12).

Case $\gamma = 0$ or $\beta = 0$:

If $\gamma = 0$, by Lemma 4.4 and estimate (4.31), the sequence $\hat{v}_\varepsilon$ converges strongly to 1 in $H^1(\Omega)$. Let $\varphi \in C_c^\infty(\Omega)$. Then, putting $\varphi\,\hat{v}_\varepsilon$ (which vanishes in Ω_ε) in Eq. (4.9) and passing to the limit, we get that

$$\int_\Omega \nabla u \cdot \nabla\varphi\,\mathrm{d}x = \int_\Omega f\,\varphi\,\mathrm{d}x,$$

which yields the limit Eq. (4.13) whatever the asymptotic behavior of β_ε.

If $\gamma > 0$ and $\beta = 0$, Eq. (4.20) gives the limit Eq. (4.13).

Case $\gamma = \beta = \infty$:

As in the third case, we have $\bar{u} = u$. Moreover, dividing the Eq. (4.9) by β_ε and passing to the limit, it follows that

$$\frac{\partial \bar{u}}{\partial x_3} = \frac{\partial u}{\partial x_3} = 0 \quad \text{in } \Omega.$$

Since $u \in H_0^1(\Omega)$, this implies that $u = 0$ a.e. in Ω. □

Proof Lemma 4.1 Proceeding as in the proof of Theorem 4.2, the convergences (4.5) hold true up to a subsequence, with $\bar{z} \in H^1\big(0, 1; L^2(\omega)\big)$. Moreover, the functions z and $\bar{z}$ are solutions of the coupled system (4.7) (if $\beta < \infty$) or (4.8) (if $\beta = \infty$). The De Giorgi, Stampacchia regularity result for the second-order elliptic equations also implies that $z \in C^0(\Omega)$. On the other hand, putting the negative part z_ε^- of z_ε in Eq. (4.4), we deduce immediately that z_ε, and thus, the limits $z, \bar{z}$ are nonnegative in Ω. Then, since

$$-\,\Delta z + \gamma\, z = 1 + \gamma\,\bar{z} \geq 1 = -\,\Delta\zeta + \gamma\,\zeta \quad \text{in } \Omega, \quad \text{where } \zeta \in H_0^1(\Omega),$$

the strong maximum principle applied to ζ implies that $z \geq \zeta > 0$ in Ω.

It remains to prove the second part of (4.6). To this end, consider for a nonnegative function $\varphi \in C_c^\infty(\omega)$, the solution w_ε^φ of the problem

$$\begin{cases} -\,\Delta w_\varepsilon^\varphi + \beta_\varepsilon\,\dfrac{1_{\Omega_\varepsilon}}{|Q_\varepsilon|}\,\dfrac{\partial w_\varepsilon^\varphi}{\partial x_3} = 0 & \text{in } \Omega \\[2ex] w_\varepsilon^\varphi - (1 - \hat{v}_\varepsilon)\,\varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.32}$$

Putting the function $(w_\varepsilon^\varphi)^- \in H_0^1(\Omega)$ in (4.32), we deduce that $w_\varepsilon^\varphi \geq 0$ a.e. in Ω. Then, putting $w_\varepsilon^\varphi - (1 - \hat{v}_\varepsilon)\,\varphi$ in (4.4), z_ε in (4.32) and noting that $(1 - \hat{v}_\varepsilon)\,\varphi$ is independent of x_3, we have

$$\begin{aligned} -\int_\Omega (1 - \hat{v}_\varepsilon)\,\varphi\,\mathrm{d}x &\leq \int_\Omega \big(w_\varepsilon^\varphi - (1 - \hat{v}_\varepsilon)\,\varphi\big)\,\mathrm{d}x \\ &= \int_\Omega \nabla z_\varepsilon \cdot \nabla\big(w_\varepsilon^\varphi - (1 - \hat{v}_\varepsilon)\,\varphi\big)\,\mathrm{d}x - \int_\Omega \beta_\varepsilon\,\frac{1_{\Omega_\varepsilon}}{|Q_\varepsilon|}\,\frac{\partial z_\varepsilon}{\partial x_3}\,\big(w_\varepsilon^\varphi - (1 - \hat{v}_\varepsilon)\,\varphi\big)\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= - \int_\Omega \nabla z_\varepsilon \cdot \nabla\big((1-\hat{v}_\varepsilon)\,\varphi\big)\,\mathrm{d}x + \int_\Omega \nabla w_\varepsilon^\varphi \cdot \nabla z_\varepsilon\,\mathrm{d}x + \int_\Omega \beta_\varepsilon\,\frac{1_{\Omega_\varepsilon}}{|Q_\varepsilon|}\,\frac{\partial w_\varepsilon^\varphi}{\partial x_3}\,z_\varepsilon\,\mathrm{d}x \\
&= - \int_\Omega \nabla z_\varepsilon \cdot \nabla\big((1-\hat{v}_\varepsilon)\,\varphi\big)\,\mathrm{d}x.
\end{aligned}
$$

Hence, thanks to the weak convergence of $\hat{v}_\varepsilon$ to 1 in $H^1(\Omega)$ and to estimate (4.17), we get the inequality

$$
0 \le \int_\Omega \nabla\hat{v}_\varepsilon \cdot \nabla z_\varepsilon\,\varphi\,\mathrm{d}x + o(1) = \int_\Omega \gamma\,(z-\bar{z})\,\varphi\,\mathrm{d}x + o(1),
$$

which, due to $\gamma > 0$ and the arbitrariness of $\varphi = \varphi(x') \ge 0$, implies that

$$
0 \le \int_0^1 \big(z(\cdot, x_3)\,\mathrm{d}x_3 - \bar{z}(\cdot, x_3)\big)\,\mathrm{d}x_3 \quad \text{a.e. in } \omega, \tag{4.33}
$$

which yields the second inequality of (4.6). Finally, if $\beta \in (0, \infty)$, inequality (4.33) combined with the second Eq. of (4.7) gives

$$
\bar{z}(\cdot, 0) - \bar{z}(\cdot, 1) = -\int_0^1 \frac{\partial\bar{z}}{\partial x_3}\,\mathrm{d}x_3 = \frac{\gamma}{\beta}\int_0^1 \big(z(\cdot, x_3) - \bar{z}(\cdot, x_3)\big)\,\mathrm{d}x_3 \ge 0 \quad \text{a.e. in } \omega,
$$

which establishes the third inequality of (4.6). The case $\beta \in \{0, \infty\}$ is straightforward. □

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References

1. Bellieud, M., Bouchitté, G.: Homogenization of elliptic problems in a fiber reinforced structure, Nonlocal effects. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **26**(4), 407–436 (1998)
2. Bénilan, P., Boccardo, L., Gallouët, T., Gariepy, R., Pierre, M., Vazquez, J.-L.: An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations. Ann. Scuola Norm. Sup. Pisa, Serie IV **22**(2), 241–273 (1995)
3. Briane, M.: Nonlocal effects in two-dimensional conductivity. Arch. Rat. Mech. Anal. **182**(2), 255–267 (2006)
4. Briane, M., Casado-Díaz, J.: Asymptotic behavior of equicoercive diffusion energies in dimension two. Calc. Var. PDE'S **29**(4), 455–479 (2007)
5. Briane, M., Casado-Díaz, J., Murat, F.: The div-curl lemma Òtrente ans après: an extension and an application to the G-convergence of unbounded monotone operators. J. Math. Pures Appl. **91**, 476–494 (2009)
6. Briane, M., Gérard, P.: A drift homogenization problem revisited. Ann. Scuola Norm. Sup. Pisa (to appear)
7. Briane, M., Tchou, N.: Fibered microstructures for some nonlocal Dirichlet forms. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **30**(4), 681–711 (2001)
8. Brinkman, H.C.: A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles. Appl. Sci. Res. A **1**, 27–34 (1947)
9. Cioranescu, D., Murat, F.: Un terme étrange venu d'ailleurs, I & II, Nonlinear partial differential equations and their applications, Collège de France Seminar, **II & III**, ed. by H. Brezis & J.-L. Lions, Research Notes in Math., **60 & 70**, Pitman, London, 1982, pp. 98–138 & pp. 154–178. English translation: A strange term coming from nowhere. in Topics in the mathematical modelling of composite materials, ed. by A. Cherkaev & R.V. Kohn., Progress in nonlinear differential equations and their applications, **31**, Birkhäuser, Boston 1997, pp. 44–93
10. Dal Maso, G., Garroni, A.: New results on the asymptotic behavior of Dirichlet problems in perforated domains. Math. Mod. Meth. Appl. Sci. **3**, 373–407 (1994)

11. Fenchenko, V.N., Khruslov, E.Ya.: Asymptotic of solution of differential equations with strongly oscillating matrix of coefficients which does not satisfy the condition of uniform boundedness. Dokl. AN Ukr. SSR. **4**: (1981)
12. Ladyzhenskaya, O.A.: The mathematical theory of viscous incompressible flow, translated from the Russian, Mathematics and its Applications, **2**, Gordon and Breach, Science Publishers, New York-London-Paris (1969)
13. Pideri, C., Seppecher, P.: A second gradient material resulting from the homogenization of an heterogeneous linear elastic medium. Continuum Mech. Thermodyn. **9**(5), 241–257 (1997)
14. Sanchez-Palencia E. (1980) Non homogeneous materials and vibration theory. Monogr. Phys. **127**. Springer, Berlin
15. Tartar: "Homogénéisation en hydrodynamique", singular perturbation and boundary layer theory. Lecture Notes Math **597**, Springer, Berlin, Heidelberg, 474-481 (1977)
16. Tartar: "Remarks on homogenization", Homogenization and effective moduli of materials and media. IMA Vol. Math. Appl., **1**, Springer, New-York, 228–246 (1986)
17. Tartar: H-measures, a new approach for studying homogenization, oscillations and concentration effects in partial differential equations. Proc. Roy. Soc. Edinburgh **115A**, 193–230 (1990)
18. Tartar: The general theory of homogenization: a personalized introduction. Lecture Notes of the Unione Matematica Italiana, Springer, Berlin, Heidelberg, p. 471 (2009)