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A class of elliptic equations in anisotropic spaces

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Abstract The equation $\Delta u + V(x)u + b(x)u|u|^{\rho-1} + h(x) = 0$ in $\mathbb{R}^n$ is studied in anisotropic Lebesgue spaces. We assume $\frac{n-\theta}{n-2} < \rho < \infty$, with $n \geq 3$ and $0 \leq \theta < 2$, which covers the supercritical range. Our approach relies on estimates of the Riesz potential and allows us to consider a wide class of potentials V, including anisotropic ones. The symmetry and antisymmetry of the solutions are also addressed.

Keywords Elliptic equation · Supercritical range · Anisotropy · Antisymmetry

Mathematics Subject Classification (2000) 35J91 · 35Q40 · 35Q92 · 35QXX

1 Introduction and main results

We are concerned with the semilinear elliptic problem

$$\Delta u + V(x)u + b(x)u|u|^{\rho-1} + h(x) = 0 \quad \text{in} \quad \mathbb{R}^n \tag{1.1}$$

$$u \to 0 \quad \text{as} \quad |x| \to \infty, \tag{1.2}$$

where $n \geq 3$ and $V(x)$, $b(x)$, $h(x)$ are given functions.

Equation (1.1) appears naturally in the study of traveling waves for the Schrödinger equation, standing-wave solutions of the Klein–Gordon equation and quantum mechanics. These equations have been used to describe many physical phenomena, which in general present an

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anisotropic feature, due to the non-homogeneity of the media as well as the complexity of the energy potentials involved. For instance, crystalline matter with presence of multiple dipoles, vibrational spectra of single-crystal, the dynamics of Bose–Einstein condensates under anisotropic potential [5,10,14,17]. In the biological branch, the diffusive logistic equation with harvesting,

$$\frac{\partial u}{\partial t} = \Delta u + V(x)u + b(x)u|u|^{\rho-1} + h(x), \tag{1.3}$$

models fishing or hunting managements where V, b represent competition rates in the environment and h is interpreted as the harvesting rate. We refer to [12] for further historical background and bibliography.

Many authors have studied Eq. (1.1) mainly in Sobolev spaces where the potential V and the range of ρ plays a crucial role. For instance, if the potential V is coercive or has some symmetry properties, several results based on variational methods, such as existence of solutions, are well known (see Strauss [16], Berestycki-Lions [3], Rabinowitz [13] for some recent developments). In [4], the authors consider (1.1), i.e., steady solutions for (1.3), with $V > 0$ and $V \in L^{\frac{n}{2}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

In this paper, we work in anisotropic Lebesgue spaces, and by means of a contraction argument, we find a solution for (1.1)–(1.2) (see Sect. 3). Recall that u belongs to the anisotropic Lebesgue space $L^{\overrightarrow{p}}$ (see [1,2]) with $\overrightarrow{p} = (p_1, p_2, \ldots, p_n)$ and $1 \le p_i \le \infty$, if and only if the norm

$$\|u\|_{\overrightarrow{p}} = \left\| \ldots \left\| \|u\|_{L^{p_1}(dx_1)} \right\|_{L^{p_2}(dx_2)} \cdots \right\|_{L^{p_n}(dx_n)} < \infty. \tag{1.4}$$

The pair $(L^{\overrightarrow{p}}, \|\cdot\|_{\overrightarrow{p}})$ is a Banach space and $(L^{\overrightarrow{p}}, \|\cdot\|_{\overrightarrow{p}}) \equiv (L^p, \|\cdot\|_p)$ when $\overrightarrow{p} = (p, p, \ldots, p)$. These spaces enable us to consider different symmetry properties and decaying behavior depending on axial directions for the weights V, b, and h. Examples of them are

$$V(x) = V_1(x_1)V_2(x_2)\ldots V_n(x_n) \in L^{\overrightarrow{s}}, \tag{1.5}$$

where $x = (x_1, x_2, \ldots, x_n)$, $\overrightarrow{s} = (s_1, s_2, \ldots, s_n)$, and $V_i \in L^{s_i}(\mathbb{R})$ with $s_i \neq s_j$ if $i \neq j$. Indeed, if $\|V\|_{\overrightarrow{s}} = \Pi_{i=1}^n \|V_i\|_{L^{s_i}(\mathbb{R})}$ is small enough, then the potential (1.5) satisfies the hypotheses of Theorem 1.1 below. We are also able to treat V, b, h with changing sign and not belonging to $L^\infty(\mathbb{R}^n)$. The proof of our results is based on careful estimates for the integral operators below (1.10)–(1.12) in anisotropic Lebesgue spaces (see Section 2). As observed in [5], a rich literature deals with Schrödinger equations and operators with isotropic potentials but, in contrast, only a few papers deal with anisotropic ones. In this case, we point out that one cannot perform reduction to spherically symmetric function space which restores the compactness.

In Sect. 4, we show how to extend the results to include negative potentials $V = -\widetilde{V}$ without any smallness assumption on $\widetilde{V}$. For that matter, we consider $\widetilde{V}$ belonging to the reverse Hölder class $\mathcal{H}_m$, with $m \ge n/2$, which contains other types of potentials including $\widetilde{V}$ coercive and $\widetilde{V} = \zeta(\frac{x}{|x|})|x|^{-\alpha}$ with $\alpha < 2$, $\zeta \in L^\infty(\mathbb{S}^{n-1})$ and $\zeta(x) \ge \zeta_0 > 0$. The case $\alpha = 2$ was treated in [5] and [6].

Denoting the area of the unit sphere by ω_n, problem (1.1) can be converted into the following integral equation

$$u(x) = \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} (Vu + bu|u|^{\rho-1} + h)(y)\mathrm{d}y. \tag{1.6}$$

The next step is to find the right spaces to tackle the integral operators in (1.6). The norm $\|\cdot\|_{\overrightarrow{p}}$ presents the dilation property

$$\|u(\sigma x)\|_{\overrightarrow{p}} = \sigma^{-\sum_{i=1}^{n}\frac{1}{p_i}} \|u(x)\|_{\overrightarrow{p}}, \ \ \sigma > 0. \tag{1.7}$$

Let $0 \leq \theta < 2$, $b_\sigma(x) = \sigma^\theta b(\sigma x)$, $V_\sigma(x) = \sigma^2 V(\sigma x)$ and $h_\sigma(x) = \sigma^{\alpha_0+2}h(\sigma x)$ with $\alpha_0 = (2-\theta)/(\rho-1)$. If $u(x)$ is a solution to (1.1), then $u_\sigma(x) = \sigma^{\alpha_0}u(\sigma x)$ solves the rescaling equation

$$\Delta u_\sigma + V_\sigma(x)u_\sigma + b_\sigma(x)u_\sigma|u_\sigma|^{\rho-1} + h_\sigma(x) = 0 \ \ \text{in } \mathbb{R}^n, \tag{1.8}$$

and thus, we consider the following scaling for (1.1):

$$u(x) \to u_\sigma(x) = \sigma^{\alpha_0}u(\sigma x), \quad \sigma > 0. \tag{1.9}$$

In order to prove that the integral operator in (1.6) is well defined, we write it in three parts, namely

$$T_V(u)(x) = \frac{1}{(n-2)\omega_n}\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}}(Vu)(y)\mathrm{d}y, \tag{1.10}$$

$$B_b(u)(x) = \frac{1}{(n-2)\omega_n}\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}}(bu|u|^{\rho-1})(y)\mathrm{d}y, \tag{1.11}$$

$$H(h)(x) = \frac{1}{(n-2)\omega_n}\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}}(h)(y)\mathrm{d}y. \tag{1.12}$$

With this notation, the Eq. (1.6) can be written as

$$u = T_V(u) + B_b(u) + H(h). \tag{1.13}$$

Denote $\overrightarrow{a} \leq \overrightarrow{b}$ when $a_i \leq b_i$ for all $i = 1, \ldots, n$, $\overrightarrow{1} = (1, \ldots, 1)$, $\overrightarrow{\infty} = (\infty, \ldots, \infty)$ and

$$\frac{1}{\overrightarrow{p}} = \left(\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n}\right) \quad \text{for } \overrightarrow{p} = (p_1, \ldots, p_n). \tag{1.14}$$

We assume that the functions V and b satisfy

$$V \in L^{\overrightarrow{s}}(\mathbb{R}^n) \text{ and } b \in L^{\overrightarrow{q}}(\mathbb{R}^n), \tag{1.15}$$

with $\overrightarrow{s} = (s_1, \ldots, s_n)$ and $\overrightarrow{q} = (q_1, \ldots, q_n)$ such that $\Sigma_{i=1}^n \frac{1}{s_i} = 2$ and $\Sigma_{i=1}^n \frac{1}{q_i} < 2$.

In light of dilatation property (1.7), we choose $\theta = \Sigma_{i=1}^n \frac{1}{q_i}$ and consider the following indexes obtained by looking for anisotropic Lebesgue spaces whose norm is invariant by the scaling (1.9):

$$\alpha_0 = \frac{2-\theta}{\rho-1}, \qquad \alpha_h = \alpha_0 + 2 = \frac{2\rho-\theta}{\rho-1},$$

$\overrightarrow{r}_0 = (r_{0,1}, \ldots, r_{0,n})$, $\overrightarrow{r}_1 = (r_{1,1}, \ldots, r_{1,n})$ and $\overrightarrow{d} = (d_1, \ldots, d_n)$ such that

$$\overrightarrow{1} < \overrightarrow{d} < \overrightarrow{r}_0, \quad \overrightarrow{r}_1 < \overrightarrow{\infty}, \quad \Sigma_{i=1}^n \frac{1}{r_{0,i}} = \alpha_0 = \frac{2-\theta}{\rho-1}, \tag{1.16}$$

$$\Sigma_{i=1}^n \frac{1}{d_i} = \alpha_h = \frac{2\rho-\theta}{\rho-1} \quad \text{and} \quad \Sigma_{i=1}^n \frac{1}{r_{1,i}} = \alpha_0 + 1 = \frac{\rho+1-\theta}{\rho-1}. \tag{1.17}$$

We also assume that $\overrightarrow{1} < \overrightarrow{s} < \overrightarrow{\infty}$, $\overrightarrow{1} \le \overrightarrow{q} \le \overrightarrow{\infty}$,

$$\frac{1}{\overrightarrow{s}} = \frac{\rho - 1}{\overrightarrow{r}_0} + \frac{1}{\overrightarrow{q}} \quad \text{and} \quad \frac{1}{\overrightarrow{d}} = \frac{\rho}{\overrightarrow{r}_0} + \frac{1}{\overrightarrow{q}} \le \overrightarrow{1}. \tag{1.18}$$

In what follows, we state the existence of solution for the equation (1.13).

Theorem 1.1 *Let $0 \le \theta < 2$, $\frac{n-\theta}{n-2} < \rho < \infty$, and let $\overrightarrow{r}_0, \overrightarrow{r}_1, \overrightarrow{s}, \overrightarrow{q}, \overrightarrow{d}$ as in (1.16)–(1.18). Assume that $h \in L^{\overrightarrow{d}}(\mathbb{R}^n)$, $V \in L^{\overrightarrow{s}}(\mathbb{R}^n)$, $b \in L^{\overrightarrow{q}}(\mathbb{R}^n)$ with $\Sigma_{i=1}^n \frac{1}{s_i} = 2$ and $\theta = \Sigma_{i=1}^n \frac{1}{q_i}$.*

- *(A) Let C_1 be as in Lemma 2.3. There exists $\varepsilon > 0$ such that if $\eta = C_1 \|V\|_{\overrightarrow{s}} < 1$ and $\|h\|_{\overrightarrow{d}} \le \frac{\varepsilon}{C_1}$, then the integral equation (1.13) has a unique solution $u \in L^{\overrightarrow{r}_0}(\mathbb{R}^n)$ satisfying $\|u\|_{\overrightarrow{r}_0} \le \frac{2\varepsilon}{1-\eta}$. Moreover, $\nabla u \in L^{\overrightarrow{r}_1}(\mathbb{R}^n)$.*
- *(B) Let $\overrightarrow{1} < \overrightarrow{l} < \overrightarrow{r}_2 < \overrightarrow{\infty}$ satisfy $\frac{1}{\overrightarrow{r}_2} = \frac{1}{\overrightarrow{l}} - \frac{1}{\overrightarrow{s}}$. Assume that $h \in L^{\overrightarrow{d}}(\mathbb{R}^n) \cap L^{\overrightarrow{l}}(\mathbb{R}^n)$ and $\overline{\eta} = C_3 \|V\|_{\overrightarrow{s}} < 1$, where C_3 is as in Lemma 2.3. There exists $0 < \overline{\varepsilon} \le \varepsilon$ such that if $\|h\|_{\overrightarrow{d}} \le \frac{\overline{\varepsilon}}{C_1}$ then $u \in L^{\overrightarrow{r}_0}(\mathbb{R}^n) \cap L^{\overrightarrow{r}_2}(\mathbb{R}^n)$.*

Remark 1.1 (Isotropic case) In Theorem 1.1, assume in particular that $q_i = q$, $s_i = s$, $d_i = d$, and $r_{0,i} = r_0$, for all $i = 1, 2, \ldots n$. This corresponds to the isotropic case in which we obtain a solution

$$u \in L^{r_0}(\mathbb{R}^n) \quad \text{with} \quad r_0 = \frac{n(\rho - 1)}{2 - \frac{n}{q}},$$

for $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$, $b \in L^q(\mathbb{R}^n)$ with $\frac{n}{2} < q \le \infty$ and $h \in L^d(\mathbb{R}^n)$ with $d = \frac{n(\rho-1)}{2\rho - \frac{n}{q}}$.

Remark 1.2 (i) The solution u obtained in Theorem 1.1 (A) is a solution in the sense of distributions for (1.1). Moreover, assuming in addition that $V, b, h \in C^{0,\gamma}(\mathbb{R}^n)$ for $0 < \gamma < 1$ (Hölder continuous functions), one can prove that the solution u belongs to $C^2(\mathbb{R}^n)$ and is a classical solution for (1.1) (see [7, Lemma 4.2]).

(ii) (Continuous dependence) Let u_1 and u_2 be two solutions as in Theorem 1.1 (A) corresponding to (h_1, V_1, b) and (h_2, V_2, b), respectively. Let (ε_1, η_1) and (ε_2, η_2) be their respective parameters. Denote $\eta = \max\{\eta_1, \eta_2\}$ and take $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$ sufficiently small so that $\eta + \frac{2^\rho K_1}{(1-\eta)^{\rho-1}} \varepsilon^{\rho-1} \|b\|_{\overrightarrow{q}} < 1$. Then,

$$\|u_1 - u_2\|_{\overrightarrow{r}_0} \le \frac{C_1}{1 - \eta - \frac{2^\rho K_1}{(1-\eta)^{\rho-1}} \varepsilon^{\rho-1} \|b\|_{\overrightarrow{q}}} \|h_1 - h_2\|_{\overrightarrow{d}} + \frac{2\varepsilon C_1}{1-\eta} \|V_1 - V_2\|_{\overrightarrow{s}}, \tag{1.19}$$

where C_1 and K_1 are given in (2.3) and (2.12), respectively.

In order to address symmetry results for Eq. (1.1), we denote by $O(n)$ the orthogonal matrix group in $\mathbb{R}^n$. Let $\mathcal{G}$ be a subset of $O(n)$. We recall that a function u is symmetric under the action of $\mathcal{G}$ when $u(x) = u(T(x))$ for any $T \in \mathcal{G}$. If $u(x) = -u(T^{-1}(x))$ for any $T \in \mathcal{G}$, then u is said to be antisymmetric under $\mathcal{G}$.

Theorem 1.2 *Under hypotheses of Theorem 1.1. Let $\Omega \subset \mathbb{R}^n$ be an arbitrary positive measure set and $\mathcal{G}$ a subset of $O(n)$.*

(A) The solution u is positive (resp. negative) if $V(x), b(x), h(x) \geq 0$ (resp. ≤ 0) a.e. in $\mathbb{R}^n$ and $h(x) > 0$ (resp. < 0) in Ω.
(B) Let $V(x)$ and $b(x)$ be symmetric under the action of $\mathcal{G}$. The solution u is antisymmetric (resp. symmetric) when $h(x)$ is antisymmetric (resp. symmetric) under $\mathcal{G}$.

Remark 1.3 (Special types of symmetry and antisymmetry)

(i) *Let $\mathcal{G} = O(n)$.* If *$V(x), b(x), h(x)$are radially symmetric then uis radially symmetric.*
(ii) *Let $V(x), b(x)$be even functions. The solution uis odd (resp. even) when $h(x)$is odd (resp. even).*

The plan of this paper is as follows. In the next section, we prove estimates in anisotropic Lebesgue spaces for the operators (1.10)–(1.12). Theorems 1.1 and 1.2 are proved in Sect. 3. The results concerning potentials in the reverse Hölder class $\mathcal{H}_m$ are stated and proved in Sect. 4.

2 Estimates in anisotropic spaces

The aim of this section is to obtain estimates for the operators H, T_V, and B_b in anisotropic Lebesgue spaces. We start by recalling the Hölder type inequality in those spaces (see [1]).

Lemma 2.1 *Let $\overrightarrow{1} \leq \overrightarrow{p}, \overrightarrow{p}_j \leq \overrightarrow{\infty}$ for all $j = 1, \ldots, m$. If*

$$\frac{1}{\overrightarrow{p}} = \sum_{j=1}^{m} \frac{1}{\overrightarrow{p}_j}$$

then

$$\left\| \prod_{j=1}^{m} u_j \right\|_{\overrightarrow{p}} \leq \prod_{j=1}^{m} \|u_j\|_{\overrightarrow{p}_j}. \tag{2.1}$$

Below we state the version of the Hardy–Littlewood–Sobolev inequality in anisotropic $L^{\overrightarrow{p}}$ spaces. This estimate was already obtained by [9] in a more general situation for weighted spaces and asymmetric kernels. For completeness, here we present a simpler proof, which is adequate for our purposes.

Lemma 2.2 *Let $\overrightarrow{r} = (r_1, \ldots, r_n)$ and $\overrightarrow{p} = (p_1, \ldots, p_n)$ be such that $\overrightarrow{1} < \overrightarrow{r} < \overrightarrow{p} < \overrightarrow{\infty}$ and $\Sigma_{i=1}^n \frac{1}{p_i} = \Sigma_{i=1}^n \frac{1}{r_i} - \beta$, where $0 < \beta < n$. Then there exists $C = C(\overrightarrow{r}, n, \beta)$ such that*

$$\left\| |x|^{-(n-\beta)} * f \right\|_{\overrightarrow{p}} \leq C \|f\|_{\overrightarrow{r}}, \tag{2.2}$$

for all $f \in L^{\overrightarrow{r}}$.

Proof Let us choose $\overrightarrow{z} = (z_1, z_2, \ldots, z_n)$ with $z_i \geq 0$, $\Sigma_{i=1}^n z_i = 1$, and $z_i(n-\beta) < 1$ for every $i = 1, \ldots, n$, in such a way that

$$\frac{1}{p_i} = \frac{1}{r_i} - (1 - z_i(n-\beta)).$$

For instance,

$$z_i = \frac{1}{n-\beta}\left[1-\left(\frac{1}{r_i}-\frac{1}{p_i}\right)\right].$$

Since $n\,|x| \geq (|x_1| + \ldots + |x_n|)$, we obtain from Young inequality that

$$\frac{1}{|x|^{n-\beta}} \leq \frac{C}{(|x_1| + |x_2| + \ldots + |x_n|)^{n-\beta}} \leq C\Pi_{i=1}^{n} |x_i|^{-z_i(n-\beta)}.$$

Therefore,

$$\left\|\frac{1}{|x|^{n-\beta}} * f\right\|_{\overrightarrow{p}} = \left\|\left\|\ldots\left\|\left\|\frac{1}{|x|^{n-\beta}} * f\right\|_{L^{p_1}(dx_1)}\right\|_{L^{p_2}(dx_2)}\ldots\right\|_{L^{p_{n-1}}(dx_{n-1})}\right\|_{L^{p_n}(dx_n)}$$

$$\leq C\left\||x_n|^{-z_n(n-\beta)} * \left\|\ldots\left\||x_2|^{-z_2(n-\beta)} * f_1(x_2,\ldots,x_n)\right\|_{L^{p_2}(dx_2)}\ldots\right\|_{L^{p_{n-1}}(dx_{n-1})}\right\|_{L^{p_n}(dx_n)}$$

where we define

$$f_1(x_2,\ldots,x_n) = \left\||x_1|^{-z_1(n-\beta)} * f(x_1,x_2,..,x_n)\right\|_{L^{p_1}(dx_1)}.$$

Using L^p-estimates for the Riesz potential $(-\Delta)^{-\frac{\gamma}{2}}$ when $n=1$ (see [8, Theorem 4.5.3 p. 117] and [11]), it follows that

$$\left\||x_1|^{-z_1(n-\beta)} * f(x_1,x_2,\ldots,x_n)\right\|_{L^{p_1}(dx_1)} \leq C\,\|f(x_1,x_2,\ldots,x_n)\|_{L^{r_1}(dx_1)}.$$

Inductively, we obtain

$$\begin{aligned}
&\left\||x|^{-(n-\beta)} * f\right\|_{\overrightarrow{p}}\\
&\leq C\left\||x_n|^{-z_n(n-\beta)} * \left\|\ldots\left\||x_2|^{-z_2(n-\beta)} * f_1(x_2,\ldots,x_n)\right\|_{L^{p_2}(dx_2)}\ldots\right\|_{L^{p_{n-1}}(dx_{n-1})}\right\|_{L^{p_n}(dx_n)}\\
&\leq C\left\||x_n|^{-z_n(n-\beta)} * \left\|\ldots\left\||x_2|^{-z_2(n-\beta)} * \|f(x_1,x_2,\ldots,x_n)\|_{L^{r_1}(dx_1)}\right\|_{L^{p_2}(dx_2)}\ldots\right\|_{L^{p_{n-1}}(dx_{n-1})}\right\|_{L^{p_n}(dx_n)}\\
&\leq C\left\||x_n|^{-z_n(n-\beta)} * \left\|\ldots|x_3|^{-z_3(n-\beta)} * \left[\left\|\|f(x_1,x_2,\ldots,x_n)\|_{L^{r_1}(dx_1)}\right\|_{L^{r_2}(dx_2)}\right]\ldots\right\|_{L^{p_{n-1}}(dx_{n-1})}\right\|_{L^{p_n}(dx_n)}\\
&\ldots\\
&\leq C\left\|\left\|\ldots\left\|\|f(x_1,x_2,\ldots,x_n)\|_{L^{r_1}(dx_1)}\right\|_{L^{r_2}(dx_2)}\ldots\right\|_{L^{r_{n-1}}(dx_{n-1})}\right\|_{L^{r_n}(dx_n)}\\
&= C\,\|f\|_{\overrightarrow{r}},
\end{aligned}$$

which is the desired estimate. □

We now prove a sequence of three lemmas that provide estimates for the operators H, T_V, B_b and their derivatives.

Lemma 2.3 *Let $\overrightarrow{r}_0, \overrightarrow{r}_1, \overrightarrow{r}_2, \overrightarrow{d}$ and $\overrightarrow{l}$ be as in Theorem 1.1. There exist $C_1, C_2, C_3 > 0$ such that*

$$\|H(h)\|_{\vec{r}_0} \le C_1 \|h\|_{\vec{d}}, \; \textit{for all } h \in L^{\vec{d}}, \tag{2.3}$$

$$\|\nabla H(h)\|_{\vec{r}_1} \le C_2 \|h\|_{\vec{d}}, \; \textit{for all } h \in L^{\vec{d}}, \tag{2.4}$$

$$\|H(h)\|_{\vec{r}_2} \le C_3 \|h\|_{\vec{l}}, \; \textit{for all } h \in L^{\vec{l}}. \tag{2.5}$$

Proof It follows from hypotheses that

$$\Sigma_{i=1}^n \frac{1}{d_i} = \frac{2\rho - \Sigma_{i=1}^n \frac{1}{q_i}}{\rho - 1} = \frac{2 - \Sigma_{i=1}^n \frac{1}{q_i}}{\rho - 1} + 2 = \Sigma_{i=1}^n \frac{1}{r_{0,i}} + 2.$$

Applying Lemma 2.2 with $\vec{p} = \vec{r}_0$, $\vec{r} = \vec{d}$ and $\beta = 2$, we obtain

$$\|H(h)\|_{\vec{r}_0} = \frac{1}{(n-2)\omega_n} \left\| \frac{1}{|x|^{n-2}} * h \right\|_{\vec{r}_0} \le C \, \|h\|_{\vec{d}},$$

which proves estimate (2.3). Similarly, since the condition $\frac{1}{\vec{r}_2} = \frac{1}{\vec{l}} - \frac{1}{\vec{s}}$ implies $\Sigma_{i=1}^n \frac{1}{r_{2,i}} = \Sigma_{i=1}^n \frac{1}{l_i} - 2$, the estimate (2.5) follows by using Lemma 2.2 with $(\vec{p}, \vec{r}, \beta) = (\vec{r}_2, \vec{l}, 2)$, namely

$$\|H(h)\|_{\vec{r}_2} = C \left\| \frac{1}{|x|^{n-2}} * h \right\|_{\vec{r}_2} \le C \, \|h\|_{\vec{l}}.$$

In order to prove (2.4), we observe first that

$$\nabla H(h)(x) = \int_{\mathbb{R}^n} \nabla_x \left(\frac{1}{|x-y|^{n-2}} \right) h(y) \mathrm{d}y \tag{2.6}$$

and

$$\left| \nabla_x \left(\frac{1}{|x-y|^{n-2}} \right) \right| \le \frac{C}{|x-y|^{n-1}}.$$

In view of

$$\Sigma_{i=1}^n \frac{1}{r_{1,i}} = \frac{\rho + 1 - \Sigma_{i=1}^n \frac{1}{q_i}}{\rho - 1} = \frac{2\rho - \Sigma_{i=1}^n \frac{1}{q_i}}{\rho - 1} - 1 = \Sigma_{i=1}^n \frac{1}{d_i} - 1,$$

Lemma 2.2 with $\vec{p} = \vec{r}_1$, $\vec{r} = \vec{d}$, $\beta = 1$ yields

$$\begin{aligned} \|\nabla H(h)\|_{\vec{r}_1} &\le C \left\| \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-1}} \, |h(y)| \, \mathrm{d}y \right\|_{\vec{r}_1} \\ &\le C \|h\|_{\vec{d}}, \end{aligned} \tag{2.7}$$

which is the inequality (2.4). □

The next lemma deals with the linear operator T_V.

Lemma 2.4 *Under the hypotheses of Theorem 1.1 and let C_1, C_2 and C_3 be as in Lemma 2.3. We have*

$$\|T_V(u)\|_{\vec{r}_0} \le C_1 \|V\|_{\vec{s}} \, \|u\|_{\vec{r}_0}, \quad \textit{for all } V \in L^{\vec{s}} \quad \textit{and} \quad u \in L^{\vec{r_0}}, \tag{2.8}$$

$$\|\nabla T_V(u)\|_{\vec{r}_1} \le C_2 \|V\|_{\vec{s}} \, \|u\|_{\vec{r}_0}, \quad \textit{for all } V \in L^{\vec{s}} \quad \textit{and} \quad u \in L^{\vec{r_0}}, \tag{2.9}$$

$$\|T_V(u)\|_{\vec{r}_2} \le C_3 \|V\|_{\vec{s}} \, \|u\|_{\vec{r}_2}, \quad \textit{for all } V \in L^{\vec{s}} \quad \textit{and} \quad u \in L^{\vec{r_2}}. \tag{2.10}$$

Proof In view of the relation [see (1.18)]

$$\frac{1}{\overrightarrow{d}} = \frac{\rho}{\overrightarrow{r}_0} + \frac{1}{\overrightarrow{q}} = \frac{1}{\overrightarrow{r}_0} + \frac{1}{\overrightarrow{s}},$$

the Hölder inequality (2.1) yields

$$\|Vu\|_{\overrightarrow{d}} \leq \|V\|_{\overrightarrow{s}} \, \|u\|_{\overrightarrow{r}_0}. \tag{2.11}$$

Since $T_V(u) = H(Vu)$, it follows from (2.3) that

$$\begin{aligned} \|T_V(u)\|_{\overrightarrow{r}_0} &= \|H(Vu)\|_{\overrightarrow{r}_0} \\ &\leq C_1\|Vu\|_{\overrightarrow{d}} \\ &\leq C_1\|V\|_{\overrightarrow{s}} \, \|u\|_{\overrightarrow{r}_0}, \end{aligned}$$

which proves (2.8). Due to the condition $\frac{1}{\overrightarrow{r}_2} = \frac{1}{\overrightarrow{l}} - \frac{1}{\overrightarrow{s}}$, (2.10) can be proved similarly by using (2.5) and Hölder inequality (2.1).

Finally, we deal with (2.9). For that, recall

$$\nabla T_V(u)(x) = \nabla H(Vu) = \int_{\mathbb{R}^n} \nabla_x \left(\frac{1}{|x-y|^{n-2}} \right) (Vu)\mathrm{d}y,$$

apply (2.4) with $h = Vu$ and afterward use (2.11) to obtain

$$\|\nabla T_V(u)\|_{\overrightarrow{r}_1} = \|\nabla H(Vu)\|_{\overrightarrow{r}_1} \leq C_2\|Vu\|_{\overrightarrow{d}} \leq C_2\|V\|_{\overrightarrow{s}} \, \|u\|_{\overrightarrow{r}_0},$$

which is the desired estimate. □

In the sequel, we give estimates for the nonlinear term B_b.

Lemma 2.5 *Under hypotheses of Theorem 1.1. There exist* $K_1, K_2, K_3 > 0$ *such that*

$$\|B_b(u) - B_b(v)\|_{\overrightarrow{r}_0} \leq K_1\|b\|_{\overrightarrow{q}}\|u - v\|_{\overrightarrow{r}_0} \left(\|u\|^{\rho-1}_{\overrightarrow{r}_0} + \|v\|^{\rho-1}_{\overrightarrow{r}_0} \right), \tag{2.12}$$

$$\|\nabla \left[B_b(u) - B_b(v)\right]\|_{\overrightarrow{r}_1} \leq K_2\|b\|_{\overrightarrow{q}}\|u - v\|_{\overrightarrow{r}_0} \left(\|u\|^{\rho-1}_{\overrightarrow{r}_0} + \|v\|^{\rho-1}_{\overrightarrow{r}_0} \right), \tag{2.13}$$

$$\|B_b(u) - B_b(v)\|_{\overrightarrow{r}_2} \leq K_3\|b\|_{\overrightarrow{q}}\|u - v\|_{\overrightarrow{r}_2} \left(\|u\|^{\rho-1}_{r_0} + \|v\|^{\rho-1}_{r_0} \right), \tag{2.14}$$

for all u, v.

Proof We will only prove the estimate (2.12) because (2.13) and (2.14) can be obtained through arguments similar to those used to prove (2.12) and (2.9).

First, recall the pointwise estimate

$$|t|t|^{\rho-1} - s|s|^{\rho-1}| \leq \rho|t - s|(|t|^{\rho-1} + |s|^{\rho-1}) \text{ for all } s, t \in \mathbb{R}, \tag{2.15}$$

and note that

$$B_b(u) - B_b(v) = H[b(u\,|u|^{\rho-1} - v\,|v|^{\rho-1})]. \tag{2.16}$$

Moreover, taking $\overrightarrow{a} = (a_1, \ldots, a_n)$ with $\overrightarrow{a} = (\rho-1)^{-1}\overrightarrow{r}_0$, it follows from (1.18) that

$$\frac{1}{\overrightarrow{d}} = \frac{\rho}{\overrightarrow{r}_0} + \frac{1}{\overrightarrow{q}} = \frac{1}{\overrightarrow{r}_0} + \frac{1}{\overrightarrow{a}} + \frac{1}{\overrightarrow{q}} = \frac{1}{\overrightarrow{r}_0} + \frac{1}{\overrightarrow{c}}, \tag{2.17}$$

where $\overrightarrow{c}$ is such that $\frac{1}{\overrightarrow{c}} = \frac{1}{\overrightarrow{a}} + \frac{1}{\overrightarrow{q}}$. In view of (2.17), estimates (2.1) and (2.15) imply

$$\begin{aligned} \|b(u\,|u|^{\rho-1} - v\,|v|^{\rho-1})\|_{\overrightarrow{d}} &\leq \|b\|_{\overrightarrow{q}} \|u\,|u|^{\rho-1} - v\,|v|^{\rho-1}\,\|_{\frac{\overrightarrow{r}_0}{\rho}}. \\ &\leq \rho \|b\|_{\overrightarrow{q}} \|u - v\|_{\overrightarrow{r}_0} \|\,|u|^{\rho-1} + |v|^{\rho-1}\,\|_{\overrightarrow{a}}. \\ &\leq \rho \|b\|_{\overrightarrow{q}} \|u - v\|_{\overrightarrow{r}_0} (\|u\|^{\rho-1}_{\overrightarrow{r}_0} + \|v\|^{\rho-1}_{\overrightarrow{r}_0}). \end{aligned} \tag{2.18}$$

Finally, (2.16), (2.3), and (2.18) yield the required estimate. □

3 Proof of results

The existence of solutions will be proved by using the previous estimates and a contraction argument.

3.1 Proof of Theorem 1.1.

Part (A): Define the map $\Psi : L_{\overrightarrow{r}_0} \to L_{\overrightarrow{r}_0}$ by

$$\Psi(u) := T_V(u) + B_b(u) + H(h)$$

and consider the ball

$$\mathcal{B}_\varepsilon = \left\{ u \in L_{\overrightarrow{r}_0};\, \|u\|_{\overrightarrow{r}_0} \leq \frac{2\varepsilon}{1-\eta} \right\}$$

endowed with the complete metric $\mathcal{W}(u, v) = \|u - v\|_{\overrightarrow{r}_0}$. We are going to show that $\Psi|_{\mathcal{B}_\varepsilon}$ is a contraction for some $\varepsilon > 0$. Recalling $\eta = C_1\,\|V\|_{\overrightarrow{s}} < 1$, estimates (2.3), (2.8) and (2.12) with $v = 0$ yield

$$\begin{aligned} \|\Psi(u)\|_{\overrightarrow{r}_0} &\leq \|H(h)\|_{\overrightarrow{r}_0} + \|T_V(u)\|_{\overrightarrow{r}_0} + \|B_b(u)\|_{\overrightarrow{r}_0} \\ &\leq C_1\,\|h\|_{\overrightarrow{d}} + C_1\,\|V\|_{\overrightarrow{s}}\,\|u\|_{\overrightarrow{r}_0} + K_1\,\|b\|_{\overrightarrow{q}}\,\|u\|^{\rho}_{\overrightarrow{r}_0} \\ &\leq \varepsilon + \eta \frac{2\varepsilon}{1-\eta} + K_1\,\|b\|_{\overrightarrow{q}}\,\frac{2^\rho \varepsilon^\rho}{(1-\eta)^\rho} \\ &= \left(1 + \eta + K_1\,\|b\|_{\overrightarrow{q}}\,\frac{2^\rho \varepsilon^{\rho-1}}{(1-\eta)^{\rho-1}}\right) \frac{\varepsilon}{1-\eta} \leq \frac{2\varepsilon}{1-\eta}, \end{aligned} \tag{3.1}$$

provided that $\|h\|_{\overrightarrow{d}} \leq \frac{\varepsilon}{C_1}$, $K_1\,\|b\|_{\overrightarrow{q}}\,\frac{2^\rho \varepsilon^{\rho-1}}{(1-\eta)^{\rho-1}} + \eta < 1$ and $u \in \mathcal{B}_\varepsilon$. Therefore, $\mathcal{B}_\varepsilon$ is invariant by Ψ. Since T_V is linear, it follows from (2.8) and (2.12) that

$$\begin{aligned} \|\Psi(u) - \Psi(v)\|_{\overrightarrow{r}_0} &= \|T_V(u - v) + B_b(u) - B_b(v)\|_{\overrightarrow{r}_0} \\ &\leq \eta\,\|u - v\|_{\overrightarrow{r}_0} + K_1\,\|b\|_{\overrightarrow{q}}\,\|u - v\|_{\overrightarrow{r}_0} \left(\|u\|^{\rho-1}_{\overrightarrow{r}_0} + \|v\|^{\rho-1}_{\overrightarrow{r}_0}\right) \\ &\leq \left(\eta + K_1\,\|b\|_{\overrightarrow{q}}\,\frac{2^\rho \varepsilon^{\rho-1}}{(1-\eta)^{\rho-1}}\right) \|u - v\|_{\overrightarrow{r}_0}, \end{aligned} \tag{3.2}$$

for all $u, v \in \mathcal{B}_\varepsilon$. The estimates (3.1) and (3.2) together imply that $\Psi|_{\mathcal{B}_\varepsilon}$ is a contraction. Then, there is a unique solution u of (1.13) satisfying $\|u\|_{\overrightarrow{r}_0} \leq \frac{2\varepsilon}{1-\eta}$, which is the fixed point of Ψ in $\mathcal{B}_\varepsilon$. Moreover, because $u \in L^{\overrightarrow{r}_0}$ and satisfies (1.13), it follows at once from (2.4), (2.9), and (2.13) with $v = 0$ that $\nabla u \in L^{\overrightarrow{r}_1}$.

Part (B): Consider the following interactive sequence

$$u_1 = H(h) \quad \text{and} \quad u_{k+1} = T_V(u_k) + B_b(u_k) + H(h)\ ,\ \ k \in \mathbb{N}. \tag{3.3}$$

Due to the contraction argument performed above, the solution u is the limit of (3.3) in $L^{\vec{r}_0}$. From (2.5), (2.10) and (2.14) with $v = 0$, we deduce

$$\|H(h)\|_{\vec{r}_2} \le C_3 \|h\|_{\vec{l}}$$

and

$$\begin{aligned} \|u_{k+1}\|_{\vec{r}_2} &\le C_3 \|h\|_{\vec{l}} + C_3 \|V\|_{\vec{s}} \|u_k\|_{\vec{r_2}} + K_3 \|b\|_{\vec{q}} \|u_k\|_{\vec{r}_2} \|u_k\|_{\vec{r}_0}^{\rho-1} \\ &\le C_3 \|h\|_{\vec{l}} + \left[C_3 \|V\|_{\vec{s}} + K_3 \|b\|_{\vec{q}} \|u_k\|_{\vec{r}_0}^{\rho-1} \right] \|u_k\|_{\vec{r_2}}\ , \end{aligned} \tag{3.4}$$

because $\frac{1}{\vec{l}} = \frac{1}{\vec{r}_2} + \frac{1}{\vec{s}}$ and $\frac{1}{\vec{s}} = \frac{\rho-1}{\vec{r}_0} + \frac{1}{\vec{q}}$. Let $\overline{\eta} = C_3 \|V\|_{\vec{s}} < 1$ and choose $0 < \overline{\varepsilon} \le \varepsilon$ so that

$$\overline{\eta} + K_3 \|b\|_{\vec{q}} \frac{2^{\rho-1}\overline{\varepsilon}^{\rho-1}}{(1-\eta)^{\rho-1}} < 1. \tag{3.5}$$

Taking $\|h\|_{\vec{d}} \le \frac{\overline{\varepsilon}}{C_1}$, the proof of Part (A) shows that the sequence (3.3) satisfies $\|u_k\|_{\vec{r}_0} \le \frac{2\overline{\varepsilon}}{1-\eta}$ for all $k \in \mathbb{N}$. In view of (3.4),

$$\begin{aligned} \|u_{k+1}\|_{\vec{r}_2} &\le C_3 \|h\|_{\vec{l}} + \left(\overline{\eta} + K_3 \|b\|_{\vec{q}} \frac{2^{\rho-1}\overline{\varepsilon}^{\rho-1}}{(1-\eta)^{\rho-1}} \right) \|u_k\|_{\vec{r_2}} \\ &= C_3 \|h\|_{\vec{l}} + \Gamma \|u_k\|_{\vec{r_2}} \end{aligned}$$

Since $\Gamma < 1$, the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $L^{\vec{r}_2}$; more precisely

$$\|u_k\|_{\vec{r_2}} \le \frac{C_3 \|h\|_{\vec{l}}}{1-\Gamma}, \ \text{ for all } k \in \mathbb{N}.$$

Therefore, up to a subsequence, u_k converges weakly in $L^{\vec{r}_2}$ to $\widetilde{u}$, and in particular, it converges in the sense of distributions. Because $u_k \to u$ in $L^{\vec{r}_0}$ by Part (A), we conclude from the uniqueness of limit in the sense of distributions that $u = \widetilde{u} \in L^{\vec{r}_2}$. □

3.2 Proof of Theorem 1.2

Part (A): Let $h, V, b \ge 0$ a.e. in $\mathbb{R}^n$ and let Ω be a positive-measure set. From expression (1.12), $u_1(x) = H(h) \ge 0$ in $\mathbb{R}^n$ if $h \ge 0$ a.e. in $\mathbb{R}^n$; and $H(h) > 0$ in $\mathbb{R}^n$ if $h(x) > 0$ in Ω. We also have that

$$T_V(u) \ge 0 \quad \text{and} \quad B_b(u) \ge 0 \text{ when } u \ge 0 \text{ a.e. in } \mathbb{R}^n, \tag{3.6}$$

because $V, b \ge 0$ a.e. in $\mathbb{R}^n$. In view of (3.3), an induction procedure shows that, for all $k \in \mathbb{N}$, either $u_k \ge 0$ if $h \ge 0$ a.e. in $\mathbb{R}^n$ or $u_k > 0$ if $h > 0$ in Ω. Since $u_k \to u$ in $L^{\vec{r}_0}$ and the convergence in $L^{\vec{r}_0}$ preserves non-negativity, we get $u \ge 0$ a.e. in $\mathbb{R}^n$. From (1.13) and (3.6),

$$u = T_V(u) + B_b(u) + H(h) \ge H(h) > 0 \text{ a.e. in } \mathbb{R}^n,$$

and we are done. A similar argument works well for the statement concerning negative solutions.

Part (B): We will prove the antisymmetric part of the statement, because the symmetric one is analogous. We claim that if h is antisymmetric under $\mathcal{G}$, then $H(h)$ is also. In fact, given a $T \in \mathcal{G}$, we have

$$\begin{aligned}-H(h)(T^{-1}(x)) &= -\frac{1}{(n-2)\omega_n}\int_{\mathbb{R}^n}\frac{1}{|T^{-1}(x)-y|^{n-2}}h(y)\mathrm{d}y\\ &= -\frac{1}{(n-2)\omega_n}\int_{\mathbb{R}^n}\frac{1}{|T^{-1}(x-T(y))|^{n-2}}h(y)\mathrm{d}y\\ &= -\frac{1}{(n-2)\omega_n}\int_{\mathbb{R}^n}\frac{1}{|x-T(y)|^{n-2}}h(y)\mathrm{d}y.\end{aligned}$$

Performing the change of variables $T(y) = z$ and using that h is antisymmetric, we obtain

$$\begin{aligned}-H(h)(T^{-1}(x)) &= \frac{1}{(n-2)\omega_n}\int_{\mathbb{R}^n}\frac{1}{|x-z|^{n-2}}\left[-h(T^{-1}(z))\right]dz\\ &= \frac{1}{(n-2)\omega_n}\int_{\mathbb{R}^n}\frac{1}{|x-z|^{n-2}}h(z)dz = H(h)(x).\end{aligned}$$

Moreover, $T_V(u) = H(Vu)$ and $B_b(u) = H(bu\,|u|^{\rho-1})$ are antisymmetric whenever u is, because V and b are symmetric. Therefore, through an induction argument, one can prove that each element u_k of the sequence (3.3) is antisymmetric. The convergence $u_k \to u$ in $L^{\vec{r}_0}$ implies (up a subsequence) a.e. pointwise convergence, and so u is also antisymmetric. □

4 Signed potentials

Replacing $V(x) = -\widetilde{V}(x)$, the Eq. (1.1) becomes

$$\Delta u - \widetilde{V}(x)u + b(x)u|u|^{\rho-1} + h(x) = 0 \quad \text{in} \quad \mathbb{R}^n. \tag{4.1}$$

In this section, we restrict our attention to potentials $\widetilde{V}$ in the reverse Hölder class $\mathcal{H}_m$ for $n/2 \leq m < \infty$. We recall that a nonnegative locally L^m integrable function $\widetilde{V}(x)$ in $\mathbb{R}^n$ is said to belong to $\mathcal{H}_m$ if there exists $C = C(m, n, \widetilde{V}) > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}}\widetilde{V}^m\,dx\right)^{1/m} \leq C\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}}\widetilde{V}\,dx\right)$$

holds true for every ball $\mathcal{B}$ in $\mathbb{R}^n$. For example, $\widetilde{V}(x) = |x|^2$ belongs to $\mathcal{H}_m$ for all $m > 1$ and in this case the operator $\mathcal{L}_{\widetilde{V}} = -\Delta + \widetilde{V}$, is the well-known *Hamiltonian harmonic oscillator* or *Hermite operator* that has been widely studied in physics. For instance, in [14], the authors studied the dynamics of Bose–Einstein condensates under the action of potentials with distinct behaviors in the longitudinal axial and transverse radial directions, such as

$$\widetilde{V}(x_1, x_2, x_3) = \lambda\frac{x_1^2}{2} + \frac{x_2^2}{2} + \Theta(x_3), \tag{4.2}$$

where Θ is a bounded function. If $\Theta \in L^{\infty}(\mathbb{R})$ with $\Theta(x) \geq \delta_0 > 0$ then (4.2) belongs to $\mathcal{H}_m$. Another potentials in $\mathcal{H}_m$ are $\widetilde{V} \in L^{\infty}(\mathbb{R}^n)$ with $\widetilde{V}(x) \geq V_0 > 0$ and $\widetilde{V}(x) = \zeta(\frac{x}{|x|})|x|^{-\alpha}$ with $\alpha < \frac{n}{m}$, $\zeta \in L^{\infty}(\mathbb{S}^{n-1})$ and $\zeta(x) \geq \zeta_0 > 0$.

We can convert the problem (4.1) into the integral equation

$$u(x) = \int_{\mathbb{R}^n} G_{\widetilde{V}}(x, y)(bu|u|^{\rho-1} + h)(y)\mathrm{d}y, \tag{4.3}$$

where $G_{\widetilde{V}}(x, y)$ is the Green function in $\mathbb{R}^n$ of the operator $\mathcal{L}_{\widetilde{V}} = -\Delta + \widetilde{V}$. We remark that $G_{\widetilde{V}}$ enjoys some properties, which we will make use of in the sequel. There exists $C_{\widetilde{V}} > 0$ such that

$$0 \leq G_{\widetilde{V}}(x, y) \leq \frac{C_{\widetilde{V}}}{|x-y|^{n-2}}, \tag{4.4}$$

see for example [15, estimate 2.6, p.525] and references therein. This time, the integral equation (1.10) can be written as

$$u = \widetilde{B}_b(u) + \widetilde{H}(h), \tag{4.5}$$

where

$$\widetilde{B}_b(u) = \int_{\mathbb{R}^n} G_{\widetilde{V}}(x, y)(bu|u|^{\rho-1})(y)\mathrm{d}y \quad \text{and} \quad \widetilde{H}(h) = \int_{\mathbb{R}^n} G_{\widetilde{V}}(x, y)h(y)\mathrm{d}y. \tag{4.6}$$

One can adapt the proof of Lemmas 2.3 and 2.5 to obtain the following estimates:

Lemma 4.1 *Let* $0 \leq \theta < 2$, $\frac{n-\theta}{n-2} < \rho < \infty$, *and let* $\overrightarrow{r}_0$, $\overrightarrow{q}$, $\overrightarrow{d}$ *be as in* (1.16)–(1.18) *with* $\theta = \Sigma_{i=1}^{n} \frac{1}{q_i}$. *Suppose that* $\widetilde{V} \geq 0$ *and* $\widetilde{V} \in \mathcal{H}_m$ *with* $m \geq n/2$. *There exist* $\widetilde{C}_1, \widetilde{K}_1 > 0$ *such that*

$$\|\widetilde{H}(h)\|_{\overrightarrow{r}_0} \leq \widetilde{C}_1 \|h\|_{\overrightarrow{d}} \tag{4.7}$$

and

$$\left\|\widetilde{B}_b(u) - \widetilde{B}_b(v)\right\|_{\overrightarrow{r}_0} \leq \widetilde{K}_1 \|b\|_{\overrightarrow{q}} \|u - v\|_{\overrightarrow{r}_0} \left(\|u\|_{\overrightarrow{r}_0}^{\rho-1} + \|v\|_{\overrightarrow{r}_0}^{\rho-1}\right), \tag{4.8}$$

for all h, u, v.

Proof The estimate (4.8) follows from (4.7) and (2.18), because $\widetilde{B}_b(u) = \widetilde{H}(bu|u|^{\rho-1})$ and $\frac{1}{\overrightarrow{d}} = \frac{1}{\overrightarrow{q}} + \frac{\rho}{\overrightarrow{r}_0}$. It remains to prove (4.7). Since $\widetilde{V} \in \mathcal{H}_m$ the estimate (4.4) holds, and then

$$\begin{aligned} \|\widetilde{H}(h)\|_{\overrightarrow{r}_0} &\leq \|\int_{\mathbb{R}^n} G_{\widetilde{V}}(x, y)\, |h(y)|\, dy\|_{\overrightarrow{r}_0} \\ &\leq C_{\widetilde{V}} \|\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}}\, |h(y)|\, dy\|_{\overrightarrow{r}_0} \\ &\leq \widetilde{C}_1 \|h\|_{\overrightarrow{d}}, \end{aligned}$$

where in the last inequality we have used Lemma 2.2. □

We have now the background to undertake potentials in the reverse Hölder class and formulate results for (4.1) in the spirit of Sect. 1.

Theorem 4.2 *Under the hypotheses of Lemma 4.1, let $\Omega \subset \mathbb{R}^n$ be a positive-measure set and assume that $\widetilde{V} \geq 0$ and $\widetilde{V} \in \mathcal{H}_m$ with $m \geq n/2$.*

(A) There exists $\varepsilon > 0$ such that if $\|h\|_{\overrightarrow{d}} \leq \frac{\varepsilon}{\widetilde{C}_1}$, then the integral equation (4.3) has a unique solution $u \in L^{\overrightarrow{r}_0}(\mathbb{R}^n)$ satisfying $\|u\|_{\overrightarrow{r}_0} \leq 2\varepsilon$ where $\widetilde{C}_1$ is as in Lemma 4.1.

(B) Let $b(x), h(x) \geq 0$ (resp. ≤ 0) a.e. in $\mathbb{R}^n$ and $h > 0$ (resp. < 0) in Ω. Then the solution u given in Part (A) is positive. Furthermore, u is radially symmetric when $\widetilde{V}(x), b(x), h(x)$ are radially symmetric.

Remark 4.1 (i) Notice that Theorems 1.1 and 4.2 deal with distinct classes of potentials. Indeed there is no inclusion relation between $\mathcal{H}_m (m \geq n/2)$ and $L^{\overrightarrow{s}}$ with $\Sigma_{i=1}^n \frac{1}{s_i} = 2$. In Theorem 4.2 we deal with sign-defined potentials, but without assuming a smallness condition on $\widetilde{V}$.

(ii) A slight modification in the proof of Theorems 1.1 and 4.2 allows us to consider more general potential such as $V = V_1 - V_2$ where $V_1 \in L^{\overrightarrow{s}}$ is as in Theorem 1.1 and $0 \leq V_2 \in \mathcal{H}_m (m \geq n/2)$. Indeed, in this case we can write (1.1) as

$$-\Delta u + V_2(x)u = V_1(x)u + b(x)u|u|^{\rho-1} + h(x),$$

which can be converted into the integral equation

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^n} G_{V_2}(x, y)(V_1(x)u + bu|u|^{\rho-1} + h)(y)\mathrm{d}y \\ &= \widetilde{\widetilde{T}}_{V_1}(u) + \widetilde{\widetilde{B}}_b(u) + \widetilde{\widetilde{H}}(h), \end{aligned}$$

where G_{V_2} satisfies (4.4).

4.1 Proof of Theorem 4.2

Part (A): The proof also relies upon a fixed point procedure similar to the proof of Theorem 1.1. We define the map $\widetilde{\Psi} : L^{\overrightarrow{r}_0}(\mathbb{R}^n) \to L^{\overrightarrow{r}_0}(\mathbb{R}^n)$ by $\widetilde{\Psi}(u) = \widetilde{B}_b(u) + \widetilde{H}(h)$. Let $\widetilde{K}_1$ be as in (4.8). Fix $0 < \varepsilon < \frac{1}{(2^\rho \widetilde{K}_1 \|b\|_{\overrightarrow{q}})^{1/(\rho-1)}}$ and consider the set

$$\mathcal{B}_\varepsilon = \{u \in L^{\overrightarrow{r}_0}(\mathbb{R}^n) : \|u\|_{\overrightarrow{r}_0} \leq 2\varepsilon\}.$$

From Lemma 4.1, if $u \in \mathcal{B}_\varepsilon$ and $\|h\|_{\overrightarrow{d}} \leq \frac{\varepsilon}{\widetilde{C}_1}$ we obtain

$$\begin{aligned} \left\|\widetilde{\Psi}(u)\right\|_{\overrightarrow{r}_0} &\leq \widetilde{C}_1 \|h\|_{\overrightarrow{d}} + \widetilde{K}_1 \|b\|_{\overrightarrow{q}} \|u\|^\rho_{\overrightarrow{r}_0} \\ &\leq \varepsilon + \widetilde{K}_1 \|b\|_{\overrightarrow{q}} 2^\rho \varepsilon^\rho \\ &\leq \left(1 + \widetilde{K}_1 \|b\|_{\overrightarrow{q}} 2^\rho \varepsilon^{\rho-1}\right) \varepsilon < 2\varepsilon, \end{aligned}$$

which implies that $\widetilde{\Psi}(\mathcal{B}_\varepsilon) \subset \mathcal{B}_\varepsilon$. On the other hand, we have that

$$\begin{aligned} \|\widetilde{\Psi}(u) - \widetilde{\Psi}(v)\|_{\overrightarrow{r}_0} &\leq \|\widetilde{B}_b(u) - \widetilde{B}_b(v)\|_{\overrightarrow{r}_0} \\ &\leq \widetilde{K}_1 \|b\|_{\overrightarrow{q}} \|u - v\|_{\overrightarrow{r}_0} \left(\|u\|^{\rho-1}_{\overrightarrow{r}_0} + \|v\|^{\rho-1}_{\overrightarrow{r}_0}\right) \\ &\leq \widetilde{K}_1 \|b\|_{\overrightarrow{q}} 2^\rho \varepsilon^{\rho-1} \|u - v\|_{\overrightarrow{r}_0}, \end{aligned}$$

for all $u, v \in \mathcal{B}_\varepsilon$, and so it follows that $\widetilde{\Psi}$ is a contraction in $\mathcal{B}_\varepsilon$. Now an application of the Banach fixed point theorem completes the proof.

Part (B): Because the Green function $G_{\widetilde{V}}$ is positive, and radial when $\widetilde{V}$ is radial, the result can be proved by proceeding in parallel to the proof of Theorem 1.2. □

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