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Gevrey global solvability of non-singular real first-order differential operators

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Abstract In this article we deal with Gevrey global solvability of non-singular first-order operators defined on an n-dimensional s-Gevrey manifold, $s > 1$. As done by Duistermaat and Hörmander in the C^∞ framework, we show that Gevrey global solvability is equivalent the existence of a global cross section.

Keywords Global solvability · Gevrey functions · Global transversal · Tubular flow

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1 Introduction

We say that a complex-valued function f is an s-*Gevrey function*, or f is a G^s *function*, on an open subset U of $\mathbb{R}^n$, $s \geq 1$, if f is C^∞ and for every compact subset K of U there exist positive constants C and R such that, for all $\alpha \in \mathbb{Z}^n_+$ and all $x \in K$, one has

$$|\partial^\alpha f(x)| \leq C R^{|\alpha|} \alpha!^s.$$

We denote by $G^s(U)$ the space of all s-Gevrey functions on U.

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It is well known that $G^s(U)$ is a vector space and a ring, with respect to the arithmetic product of functions; moreover, $G^s(U)$ is closed under differentiation and composition (see [7] and [10]).

Let $s \geq 1$ and let M be a Hausdorff topological space, with a countable basis of open sets. A G^s *structure* over M of dimension n is a collection of pairs $\mathcal{A} = \{(\mathcal{U}, x)\}$, each pair is called a *coordinate neighborhood*, where $\emptyset \neq \mathcal{U} \subset M$ is an open set, $x : \mathcal{U} \to \mathbb{R}^n$ is a homeomorphism onto an open subset $x(\mathcal{U})$ of $\mathbb{R}^n$ and the following properties are satisfied:

(i) $\displaystyle\bigcup_{(\mathcal{U},x)\in\mathcal{A}} \mathcal{U} = M$;

(ii) $x(\mathcal{U}\cap\mathcal{W}) \overset{y\circ x^{-1}}{\to} y(\mathcal{U}\cap\mathcal{W})$ is G^s for each pair $(\mathcal{U}, x),\ (\mathcal{W}, y) \in \mathcal{A}$, with $\mathcal{U}\cap\mathcal{W} \neq \emptyset$;

(iii) $\mathcal{A}$ is maximal with respect to (i) and (ii).

Fixed $s \geq 1$, an *n-dimensional* G^s*-manifold* is a Hausdorff topological space M, with a countable basis equipped with a G^s structure of dimension n.

For $s > 1$ the space $G^s(U)$, where U is an open subset of $\mathbb{R}^n$, has the following property: given an open subset V of U and a compact subset $K \subset V$ there exists $f \in G^s(U)$ such that $f \equiv 1$ on K and the support of f is contained in V (see [10]). Hence, we can construct in the standard way partitions of unity on a G^s-manifold M by means of G^s functions with compact support.

Consider $s \geq 1$ and two G^s manifolds M and N. We say that $f : M \to N$ is an *s-Gevrey function*, or a G^s *function*, if for each $p \in M$, there exist coordinate neighborhoods $(\mathcal{U}, x)$ and $(\mathcal{V}, y)$, with $p \in \mathcal{U}$ and $f(p) \in \mathcal{V}$, such that all the components of $y \circ f \circ x^{-1}$ belong to $G^s(x(\mathcal{U}))$. We denote by $G^s(M, N)$ the space of all s-Gevrey functions between M and N; moreover, $G^s(M) := G^s(M, \mathbb{C})$.

Fixed $s \geq 1$, a G^s real vector field on M is a real linear operator

$$L : G^s(M) \to G^s(M)$$

satisfying

$$L(u \cdot v) = Lu \cdot v + u \cdot Lv;$$

locally, $L = \sum_{j=1}^{n} a_j(x)\dfrac{\partial}{\partial x_j}$, where $a_j \in G^s$, $j = 1, \dots, n$.

Let $P : G^s(M) \to G^s(M)$ be a linear first-order operator given by $P = L + a$, where L is a non-singular G^s real vector field defined on M and $a \in G^s(M)$.

We say that P is s*-globally solvable* in M if for each $f \in G^s(M)$ there exists $u \in G^s(M)$ such that $Pu = f$ in M, that is, if $PG^s(M) = G^s(M)$.

Malgrange in [8] and also Duistermaat and Hörmander in [2] characterized the global solvability of P in the C^∞ framework.

In [8], Malgrange showed essentially that $PC^\infty(M) = C^\infty(M)$ is equivalent to the following geometric condition:

(GC) (1) No complete integral curve of L is contained in K.

(2) For every compact subset K of M, there exists a compact subset K' of M, such that every compact interval on an integral curve with end points in K is contained in K'.

In [2], Duistermaat and Hörmander showed that the geometric condition (GC) is equivalent to

($\sharp$) There exists a manifold M_0, an open neighborhood of M_1 of $M_0 \times \{0\}$ in $M_0 \times \mathbb{R}$ which is convex in the $\mathbb{R}$ direction, and a C^∞ diffeomorphism $M \to M_1$ which carries L into the operator $\partial/\partial t$, where points in M_1 are denoted by (x, t).

Now, consider the following example. Let $M = \mathbb{R}^2 \setminus \{(x, 0) \in \mathbb{R}^2;\ x \leq 0\}$ and let $\partial/\partial t$ be a real vector field defined on M. It is easy to see that (GC) is not satisfied. For instance, take $K = ([0, 1]) \times \{1\}) \cup ([0, 1] \times \{-1\})$. Define $f : M \to \mathbb{R}$ by $f(x, t) = \dfrac{1}{x^2 + t^2}$. A simple calculation shows that there is no $u \in C^\infty(M)$ solution of $\partial u/\partial t = f$ in M.

It should be noted that f is real-analytic in M; in particular, $f \in G^s(M)$, for all $s \geq 1$. Of course, there is no $u \in G^s(M)$ solution of $\partial u/\partial t = f$ in M.

Hence, natural questions are as follows: the geometric condition (GC) characterizes the global solvability in a G^s framework? If this is true, can we extend condition ($\sharp$) to G^s framework?

In this article, we will extend the method developed in [2] to study global solvability in G^s classes.

It should be noted that ($\sharp$) is a global version of the tubular flow theorem, and the hard work is the construction of the manifold M_0, which is a global cross section of L. Hence, one of the main tools of this article is the inverse mapping theorem in Gevrey class.

The organization of this paper is as follows. In Sect. 2, we state some useful results about ordinary differential equations in G^s class. In Sect. 3, we deal with the s-global solvability of linear first-order differential equations.

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2 Some useful results in Gevrey classes

In this section, we will give a brief review of basic results of Gevrey spaces and, also, we will present a Gevrey version of tubular flow theorem.

We will start this section proving the following technical result:

Lemma 1 *The Gevrey sequence* $(j!^s)_{j\in\mathbb{N}}$, $s \geq 1$, *satisfies the following property*:

$$k!^{\frac{s-1}{k-1}} \leq \mathrm{e}^{\frac{s-1}{24}}\, j!^{\frac{s-1}{j-1}}, \quad \textit{for all}\ \ 2 \leq k \leq j.$$

Proof For $s = 1$ is trivial. For $s > 1$, the proof follows from the following well-known version of Stirling's Formulae

$$\Gamma(t+1) = t^t \mathrm{e}^{-t}\sqrt{2\pi t}\, \mathrm{e}^{v(t)/(12t)},\ \ t \geq 1,$$

for the Gamma function (or Euler's second integral)

$$\Gamma(t) = \int_0^\infty \mathrm{e}^{-\lambda}\lambda^{t-1}\mathrm{d}\lambda, \quad \text{for } t > 0,$$

where $0 < v(t) < 1$. Since $\Gamma(j+1) = j!, \forall j \in \mathbb{N}$, we have that

$$\frac{k!^{\frac{1}{k-1}}}{j!^{\frac{1}{j-1}}} = \frac{\Gamma(k+1)^{\frac{1}{k-1}}}{\Gamma(j+1)^{\frac{1}{j-1}}} = \frac{\left(k^{k+1/2}\,\mathrm{e}^{-k}\,\sqrt{2\pi}\right)^{\frac{1}{k-1}}\,\mathrm{e}^{v(k)/(12k(k-1))}}{\left(j^{j+1/2}\,\mathrm{e}^{-j}\,\sqrt{2\pi}\right)^{\frac{1}{j-1}}\,\mathrm{e}^{v(j)/(12j(j-1))}}.$$

Note that

$$e^{-v(j)/(12j(j-1))} \geq 1, \quad \text{for all } j \geq 2$$

and

$$e^{v(k)/(12k(k-1))} \leq e^{1/24}, \quad \text{for all } k \geq 2.$$

Hence,

$$\frac{1}{e^{1/24}} \cdot \frac{k!^{\frac{1}{k-1}}}{j!^{\frac{1}{j-1}}} \leq \frac{\left(k^{k+1/2}\, e^{-k}\, \sqrt{2\pi}\right)^{\frac{1}{k-1}}}{\left(j^{j+1/2}\, e^{-j}\, \sqrt{2\pi}\right)^{\frac{1}{j-1}}}, \tag{1}$$

for all $j \in \mathbb{Z}_+$ and for all $k \geq 2$.

Finally, since $x \mapsto \left(x^{x+1/2}\, e^{-x}\, \sqrt{2\pi}\right)^{\frac{1}{x-1}}$ is a monotonically increasing function on $[2, +\infty)$ we have that for $2 \leq k \leq j$ the right hand of (1) is bounded by 1. □

A C^∞ mapping $f : M \to N$ between G^s-manifolds M and N is a G^s *diffeomorphism* when f is a C^∞ diffeomorphism, $f \in G^s(M, N)$ and $f^{-1} \in G^s(N, M)$.

By using Lemma 1, we have from [5] the following s-Gevrey version of inverse mapping theorem:

Theorem 1 (Inverse Mapping Theorem) *Let U be an open subset of $\mathbb{R}^n$, $p \in U$ and $f \in G^s(U, \mathbb{R}^n)$, where $s \geq 1$. If $\mathrm{d}f(p) : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, then there exists a neighborhood V of p and an open subset $W \subset \mathbb{R}^n$ such that $f : V \to W$ is a G^s diffeomorphism.*

By using the theorem above, we have the following version of Theorem 5.5 of [1]:

Theorem 2 *Let $f : N' \to M$ be an imbedding of a G^s-manifold N of dimension n in a G^s manifold M of dimension m. Then $N = f(N')$ is a regular submanifold. As such it is G^s diffeomorphic to N' with respect to the mapping $f : N' \to M$.*

It is easy to see that $p!^2 \leq (p-1)!(p+1)!$. Hence, by using Lemma 1, we have from [6] the following result:

Theorem 3 *Let U be an open subset of $\mathbb{R}^n$ and $T > 0$. If $f : (-T, T) \times U \to \mathbb{R}^n$ is an s-Gevrey function, then for each relatively compact open subset U_1 of U there exists $0 < T_1 \leq T$ such that all the components of the unique solution of*

$$\begin{cases} \varphi'_{x_0}(t) = f(t, x) \\ \varphi_{x_0}(0) = x_0, \end{cases} \tag{2}$$

are also s-Gevrey functions on $(-T_1, T_1) \times U_1$.

Throughout this article φ will denote the flow of L, that is,

$$\begin{aligned} \varphi : D_\varphi \subset \mathbb{R} \times M &\to M \\ (t, x) &\mapsto \varphi(t, x) = \varphi_x(t), \end{aligned}$$

where $\varphi_x(t)$ is the unique solution of (2) with f given by the coefficients of L. It should be noted that since L is a G^s real vector field, Theorem 3 implies that $\varphi \in G^s(D_\varphi)$.

Let M and N be G^s-manifolds and let L and $\tilde{L}$ be G^s real vector fields defined on M and N, respectively; also, denote by φ and $\widetilde{\varphi}$ its respective flows. We say that L and $\widetilde{L}$ are

G^s*-conjugated* if there exists a G^s diffeomorfism $f \in G^s(M, N)$ such that $f(\varphi(t, x)) = \widetilde{\varphi}(t, f(x)), \forall (t, x) \in D_\varphi$.

We now recall the concept of cross section. Let M be an n-dimensional G^s-manifold. Let $p \in M$ and let $U \subset M$ be a neighborhood of p. A *global (local) cross section* of L on M containing p is a codimension one immersed G^s-submanifold Σ of M (of U) such that for all $x \in M$ (for all $x \in U$) there exists a unique $t \in \mathbb{R}$ such that $y = \varphi(t, x) \in \Sigma$ and $T_y(\Sigma) \oplus L(y) = T_y M$ ($T_y(\Sigma) \oplus L(y) = T_y U$) (see [3] and [4]; also, [11]).

It is known that the inverse mapping theorem is the key to prove the tubular flow theorem in the C^k framework. Hence, using Theorems 1 and 3, we can prove the following version:

Theorem 4 (Tubular Flow) *Let L be a G^s real vector field on a G^s-manifold M and let $p \in M$. Assume that L is non-singular at p. Then there exist $\epsilon > 0$ and a neighborhood U of p such that $L_{|U}$ is G^s-conjugated to $\frac{\partial}{\partial x_1}|_{B^n_\epsilon}$, where B^n_ϵ is the ball in the maximum norm $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n;\ |x| < \epsilon\}$. Moreover, if Σ is a cross section of L containing p, we can obtain a G^s-conjugation ψ with the additional property: $\psi(U \cap \Sigma) = \{x \in B^n_\epsilon;\ x = (0, x_2, \ldots, x_n)\}$.*

Corollary 1 *Let L be a G^s real vector field on a G^s-manifold M. Let Σ be a cross section of L. Given $p \in \Sigma$, there are $\epsilon > 0$, a neighborhood V of p and a G^s function $\tau : V \to \mathbb{R}$ such that $\tau(V \cap \Sigma) = 0$ and*

- (i) *for all $q \in V$ the integral curve $\varphi(t, q)$ of $L_{|V}$ is defined and one-to-one on $J_q = (-\epsilon + \tau(q), \epsilon + \tau(q))$;*
- (ii) *$\xi(q) = \varphi(\tau(q), q)$ is the unique point of $\varphi(\cdot, q)_{|J_q}$ in Σ; in particular, $q \in V \cap \Sigma$ if and only if $\tau(q) = 0$;*
- (iii) *$\xi : V \to \Sigma$ is a G^s function and $D\xi(q)$ is surjective for all $q \in V$; moreover, $D\xi(q) \cdot v = 0$ if and only if $v = \alpha L(q)$, for some $\alpha \in \mathbb{R}$.*

As a (non-trivial) consequence of Theorem 4, we have

Theorem 5 (Long Tubular Flow) *Under the hypotheses of Theorem 4, let $\varphi_x : [a, b] \to M$ be a non-closed, compact arc of an integral curve of L. Then there exist a neighborhood U of $\varphi_x([a, b])$, an open interval I containing $[a, b]$ and $\epsilon > 0$ such that $L_{|U}$ is G^s-conjugated to $\frac{\partial}{\partial x_1}|_{I \times B^{n-1}_\epsilon}$. Moreover, the G^s-conjugation ψ has the property: $\psi(\varphi_x([a, b])) = [a, b] \times \{0\} \times \cdots \times \{0\}$.*

Remark 1 Of course, the Corollary 1 can be extended in terms of Theorem 5.

The proof of Theorems 4 and 5 are similar to that of Theorem 1.1 and Proposition 1.1 in [9]—pages 40 and 93—and we will not repeat them here.

3 s-global solvability

Fixed $s > 1$, let M be an n-dimensional G^s-manifold. In this section, we will deal with the s-global solvability of operators of the form $P = L + a$, where L is a G^s real vector field on M and $a \in G^s(M)$.

Our first result (which is a Gevrey version of Theorem 6.4.1 of [2]) is addressed to semiglobal solvability. Denote by $G^s(K)$ the space of all functions of G^s class in some neighborhood of K.

Theorem 6 *Let L be a G^s real vector field on M, and let K be a compact subset of M. Then the following conditions are equivalent*:

a) $LG^s(K) = G^s(K)$.
b) $(L + a)G^s(K) = G^s(K)$, *for every* $a \in G^s(K)$.
c) *There exists* $\psi \in G^s(K)$ *such that* $L^2\psi > 0$ *on* K.
d) *No complete integral curve of* L *is contained in* K.

Proof The arguments to prove that b) $\Leftrightarrow$ a) $\Rightarrow$ c) $\Rightarrow$ d) are analogous to those used by Duistermaat and Hörmander in [2]—Theorem 6.4.1. Here, we will prove that d) $\Rightarrow$ a). The argument is similar that in [2], but with some differences.

As in [2] we have that d) implies

d′) no integral curve of L is contained in K for all positive or negative values of the parameter.

Now, we have that d′) implies that every $y \in K$ lies on a compact interval $\varphi_y : [a, b] \to M$ of an integral curve of L with end points in K^c.

Next, we will denote $x \in \mathbb{R}^n$ by $x = (x_1, x')$, where $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{n-1}$. Applying Theorem 5, we may take new coordinates in a neighborhood $\mathcal{U}$ of $\varphi_y([a, b])$ such that

$$\mathcal{U} = \{a - \delta < x_1 < b + \delta,\ |x'| < \epsilon\},$$
$$\varphi_y(t) = (t, 0),\ a \le t \le b,$$

for a suitable choice of $\delta, \epsilon > 0$, and

$$L = \frac{\partial}{\partial x_1} \quad \text{on } \mathcal{U}.$$

Hence,

$$u(x_1, x') = \int_0^{x_1} f(t, x')\mathrm{d}t$$

is a G^s solution for $Lu = f$ in $\mathcal{U}$; consequently, if f has support sufficiently close to $\varphi_y([a, b])$ then $Lu = f$ can be solved in a neighborhood of K. Finally, using partition of unity (as mentioned in introduction), we can find a desired solution. □

Next, we will present our main result, which is a Gevrey version of Theorem 6.4.2 of [2].

Theorem 7 *Let* L *be a* G^s *real vector field on* M. *The following conditions are equivalent:*

a) $LG^s(M) = G^s(M)$.
b) $(L + a)G^s(M) = G^s(M)$, *for every* $a \in G^s(M)$.
c) *There exists* $\psi \in G^s(M)$ *such that* $L^2\psi > 0$ *and* $\{y \in M;\ \psi(y) \le c\}$ *is compact for every* c.
d) (1) *No complete integral curve of* L *is contained in* K.
   (2) *For every compact subset* K *of* M, *there exists a compact subset* K' *of* M *such that every compact interval on an integral curve with end points in* K *is contained in* K'.
e) *There are no periodic integral curves and the relation*

$$R = \{(x, y) \in M \times M;\ x \text{ and } y \text{ are on the same integral curve of } L\}$$

is a closed G^s *submanifold of* $M \times M$.
f) *There exist a manifold* M_0, *an open neighborhood* M_1 *of* $M_0 \times \{0\}$ *in* $M_0 \times \mathbb{R}$ *which is convex in the* $\mathbb{R}$ *direction, and a* G^s *diffeomorphism* $M \to M_1$ *which carries* L *into the operator* $\partial/\partial t$, *where points in* M_1 *are denoted by* (x, t).

Proof The arguments to prove that f) $\Rightarrow$ a) $\Leftrightarrow$ b), f) $\Rightarrow$ c) $\Rightarrow$ d) and a) $\Rightarrow$ d) $\Rightarrow$ e) are analogous to that used by Duistermaat and Hörmander in [2]—Theorem 6.4.2; however, we will give a special attention to d) $\Rightarrow$ e). As in [2], we can show that the map $(x,t) \mapsto (\varphi(x,t), x)$ is proper; hence, by applying Theorem 2 we have the result.

The proof that e) $\Rightarrow$ f) is a non-trivial adaptation of that in [2]. Let us repeat the arguments here.

e) $\Rightarrow$ f)

Let $\sim$ be the following relation in M: two points $x, y \in M$ are said to be equivalent, $x \sim y$, if x and y lie on the same integral curve of L. Let $M_0 = M/\sim$ be the quotient space and denote by $[x]$ the equivalence class for which x is a representant.

Define $\pi : M \to M_0$ by $\pi(x) = [x]$. It is easy to see that π is an open map. It follows from [12] that $\sim$ on M is open if and only if π is an open map; moreover, if $\sim$ is open and M has open countable basis then $M/\sim$ has open countable basis; also, $R = \{(x,y);\ x \sim y\}$ is closed in $M \times M$ if and only if $M/\sim$ is Hausdorff. Hence, M_0 has open countable basis and is a Hausdorff space.

For each $[p] \in M_0$, let $f_p : A_p \to \Sigma_p$ be the G^s function that define Σ_p as a (local) cross section containing p (we can consider that Σ_p contains at most one point of each integral curve of L). Denote $U_p = \pi(\Sigma_p)$, which is an open subset of M_0.

Define $\gamma_p = \pi^{-1}_{|\Sigma_p} : U_p \to M$. As $\pi_{|\Sigma_p}$ is an open bijective function, we have that γ_p and $f_p^{-1} \circ \gamma_p$ are homeomorphism.

Hence, $\{(U_p, f_p^{-1} \circ \gamma_p),\ p \in M\}$ gives a structure of topological manifold to M_0. As a consequence of Theorem 5, we have that $\gamma_p \circ \gamma_q^{-1} \in G^s(U_p \cap U_q)$; hence, $(f_p^{-1} \circ \gamma_p) \circ (f_q^{-1} \circ \gamma_q)^{-1} = f_p^{-1} \circ (\gamma_p \circ \gamma_q^{-1}) \circ f_q \in G^s(f_q^{-1} \circ \gamma_p(U_p \cap U_q))$. Therefore, $\{(U_p, f_p^{-1} \circ \gamma_p),\ p \in M\}$ gives a structure of G^s manifold to M_0.

Finally, we are ready to construct a global cross section $\gamma : M_0 \to M$. Let $\{V_j\}$ be a locally finite countable refinement with compact closure of $\{U_p\}$. For each V_j, choose one among the sets U_p for which $V_j \subset U_p$ and define $\gamma_j : V_j \to \Sigma_j \subset M$ by $\gamma_j = \gamma_{p_{|V_j}}$. Let $\{\chi_j\}$ be the partition of unit by means of G^s functions subordinated to $\{V_j\}$ (as described in introduction). Let $[y] \in M_0$. Since $\{V_j\}$ is locally finite, there exists a neighborhood W of $[y]$ for which $W \cap V_j \neq \emptyset$ at most for a finite number of index j, say, $\{j_1, \ldots, j_N\}$. Let $\delta_1 = \min\{d([y], \partial V_{j_k})$, for an index j_k for which $[y] \notin \partial V_{j_k}\}$; defining $S(\chi_{j_k})$ the support of χ_{j_k}, let $\delta_2 = \min\{d([y], S(\chi_{j_k}))$, for an index j_k for which $[y] \in \partial V_{j_k}\}$. Finally, let $\delta = \min\{\delta_1, \delta_2\}$. Denote $U_{[y]} = B([y], \delta)$. We have that $U_{[y]} \cap V_\lambda \neq \emptyset$ for $\lambda \in \Lambda_{[y]} \subset \{j_1, \ldots, j_N\}$. Define $\lambda_{[y]} = \min\{\lambda \in \Lambda_{[y]};\ [y] \in V_\lambda\}$. Now, define $t^\lambda_{\lambda_{[y]}} : U_{[y]} \cap V_\lambda \to \mathbb{R}$ so that $\varphi(t^\lambda_{\lambda_{[y]}}[x], \gamma_{\lambda_{[y]}}[x]) \in \Sigma_\lambda$. Of course, $U_{[y]} \subset V_{\lambda_{[y]}}$ and $t^\lambda_{\lambda_{[y]}} \chi_\lambda \in G^s(M_0)$.

Define

$$\begin{aligned} \gamma : M_0 &\to M \\ [y] &\mapsto \varphi\left(\Sigma_{\lambda \in \Lambda_{[y]}} t^\lambda_{\lambda_{[y]}}[y]\chi_\lambda[y], \gamma_{\lambda_{[y]}}[y]\right). \end{aligned}$$

We claim that γ belongs to the G^s class in M_0. First, we must show that γ is well defined in M_0. Let $[y_1], [y_2] \in M_0$. For $j = 1, 2$, let $U_j = U_{[y_j]}$ be a neighborhood of $[y_j]$ and let $\Lambda_{[y_j]}$ be the index set of $[y_j]$ given in the definition of the function γ. Suppose that $U_1 \cap U_2 \neq \emptyset$. Without loss of generality, we can assume that $\lambda_{[y_1]} \leq \lambda_{[y_2]}$. It should be noted that for $\lambda \notin \Lambda_{[y_1]} \cap \Lambda_{[y_2]}$ one has $\chi_\lambda[x] = 0$ for all $[x] \in U_1 \cap U_2$. Hence, for all $[x] \in U_1 \cap U_2$, we have that

$$\begin{aligned}
&\varphi\left(\Sigma_{\lambda'\in\Lambda_2} t^{\lambda'}_{\lambda_{[y_2]}}[x]\chi_{\lambda'}[x], \gamma_{\lambda_{[y_2]}}[x]\right)\\
&\quad=\varphi\left(\Sigma_{\lambda'\in\Lambda_{[y_1]}\cap\Lambda_{[y_2]}} t^{\lambda'}_{\lambda_{[y_2]}}[x]\chi_{\lambda'}[x], \varphi(t^{\lambda_{[y_2]}}_{\lambda_{[y_1]}}[x], \gamma_{\lambda_{[y_1]}}[x])\right)\\
&\quad=\varphi\left(\Sigma_{\lambda'\in\Lambda_{[y_1]}\cap\Lambda_{[y_2]}} t^{\lambda'}_{\lambda_{[y_2]}}[x]\chi_{\lambda'}[x] + t^{\lambda_{[y_2]}}_{\lambda_{[y_1]}}[x]\left(\sum_{j=1}^{\infty}\chi_j[x]\right), \gamma_{\lambda_{[y_1]}}[x])\right)\\
&\quad=\varphi\left(\Sigma_{\lambda'\in\Lambda_{[y_1]}\cap\Lambda_{[y_2]}} t^{\lambda'}_{\lambda_{[y_1]}}[x]\chi_{\lambda'}[x], \gamma_{\lambda_{[y_1]}}[x]\right)=\varphi\left(\Sigma_{\lambda\in\Lambda_{[y_1]}} t^{\lambda}_{\lambda_{[y_1]}}[x]\chi_{\lambda}[x], \gamma_{\lambda_{[y_1]}}[x]\right).
\end{aligned}$$

Therefore, γ is well defined. Finally, since $t^{\lambda}_{\lambda_{[y]}}\chi_\lambda, \gamma_{\lambda_{[y]}} \in G^s(U)$ and $\varphi \in G^s(D_\varphi)$ we have that $\gamma \in G^s(U)$; consequently, $\gamma \in G^s(M_0)$.

By construction, for all $x \in M$, there exists a unique $t \in \mathbb{R}$ such that $p = \varphi(t, x) \in \gamma(M_0)$. We claim that $T_p(\gamma(M_0)) \oplus L(p) = T_pM$, for all $p \in \gamma(M_0)$. Indeed, given $p \in \gamma(M_0)$ and $f_p : A_p \to \Sigma_p$, the cross section containing p let $\xi : V \to I \times B^{n-1}_\epsilon(p) \subset I \times A_p$ be the diffeomorphism given by Theorem 4. Hence, we have that $\gamma(M_0) \cap V = \gamma \circ \gamma_p^{-1} \circ f_p(B^{n-1}_\epsilon(p))$ and also that $g : B^{n-1}_\epsilon(p) \to \mathbb{R}^n$ given by $g(x) = \xi \circ \gamma \circ \gamma_p^{-1} \circ f_p(x) = (g_1(x), x)$ belongs to the G^s class (here g_1 is the 1-coordinate function of g). Of course, for all $x \in B^{n-1}_\epsilon(p)$ the rank of g is equal to $n - 1$; also, for all $0 \neq v \in \mathbb{R}^{n-1}$ and for all $k \in \mathbb{R}$ one has that $Dg(x) \cdot v \neq (k, 0, \dots, 0)$, that is, for all $0 \neq v \in \mathbb{R}^{n-1}$ we have that $Dg(x) \cdot v$ and $(1, 0, \dots, 0)$ are linearly independent. Since for each $y \in V$, $D\xi(y)$ is an isomorphism, it follows that

$$D\left(\gamma \circ \gamma_p^{-1} \circ f_p\right)(x) \cdot v = D\xi^{-1}(\gamma \circ \gamma_p^{-1} \circ f_p(x)) \cdot Dg(x) \cdot v,$$

for all $x \in B^{n-1}_\epsilon(p)$; consequently, the rank of $\gamma \circ \gamma_p^{-1} \circ f_p$ is equal to $n - 1$ and, also,

$$D\left(\gamma \circ \gamma_p^{-1} \circ f_p\right)(x) \cdot v \neq k \cdot \frac{\partial\varphi}{\partial t}\left(0, \gamma \circ \gamma_p^{-1} \circ f_p(x)\right) = k \cdot \frac{\partial\varphi}{\partial t}(0, \gamma[x]),$$

for all $0 \neq v \in \mathbb{R}^{n-1}$ and for all $k \in \mathbb{R}$; that is, for all $0 \neq v \in \mathbb{R}^{n-1}$ one has that $D(\gamma \circ \gamma_p^{-1} \circ f_p)(x) \cdot v$ and $\frac{\partial\varphi}{\partial t}(0, \gamma[x])$ are linearly independent. Hence, for all $p \in \gamma(M_0)$, we have that $T_p(\gamma(M_0)) \oplus L(p) = T_p(M)$.

It is an easy consequence of the discussion above that γ is an immersion.

Therefore, $\gamma : M_0 \to M$ is a global cross section of L.

Consider $M_1 = \{([x], t) \in M_0 \times \mathbb{R}; [x] \in M_0 \text{ and } (t, \gamma[x]) \in D_\varphi\}$, which is an open subset of $M_0 \times \mathbb{R}$.

Define

$$\begin{aligned} h : M_1 &\to M\\ ([x], t) &\mapsto \varphi(t, \gamma[x]). \end{aligned}$$

A simple calculation shows that h is a G^s diffeomorphism; moreover,

$$Dh([x], t) \cdot (0, 0, \dots, 1) = D_n h([x], t) = L\varphi(t, \gamma[x]) = L(h([x], t)).$$

Hence, L is the pushforward of $\partial/\partial t$, via function h. Therefore, the proof is completed.

□

Remark 2 It should be noted that if M is a compact manifold, then $LG^s(M) \subsetneq G^s(M)$.

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