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Complex vector fields, unique continuation and the maximum modulus principle

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Abstract We prove unique continuation and maximum modulus principle for solutions to systems of differential equations and inequalities, involving complex vector fields, under conditions that generalize some weak-pseudoconcavity assumptions for the tangential Cauchy-Riemann complex.

Keywords Complex vector fields · Maximum modulus principle · Weak unique continuation · Abstract CR manifold

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0 Introduction

The purpose of this paper is to prove results concerning unique continuation and the maximum modulus principle for solutions to systems of differential equations, and differential inequalities, involving complex linear partial differential operators of the first order. These results extend similar results obtained in [3,5,6] for the system of complex vector fields associated with abstract almost CR manifolds, under pseudoconcavity assumptions. Here, we consider more general systems of complex vector fields, and weaken the pseudoconcavity assumptions, in the spirit of [2].

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Throughout the paper, we shall consistently use the following notation:

$$\begin{aligned}
M &= \text{a smooth real manifold of dimension } m,\\
\mathcal{C}^\infty(M) &= \text{smooth complex valued functions on } M,\\
\mathfrak{X}(M) &= \text{smooth real vector fields on } M,\\
\mathfrak{X}^{\mathbb{C}}(M) &= \text{smooth complex vector fields on } M,\\
\mathfrak{Z}(M) &= \text{some left } \mathcal{C}^\infty(M) - \text{submodule of} \mathfrak{X}^{\mathbb{C}}(M),\\
\mathbb{H}_{\mathfrak{Z}}(M) &= \{\operatorname{Re} Z \mid Z \in \mathfrak{Z}(M)\} \subset \mathfrak{X}(M),\\
\mathbf{Z}_p M &= \{Z_p \mid Z \in \mathfrak{Z}(M)\}.
\end{aligned}$$

1 The maximum modulus principle

We denote by $\mathcal{O}_{\mathfrak{Z}}(U)$ the set of weak L^2_{loc}-solutions of

$$Zu = 0 \quad \text{on} \quad U, \quad \forall Z \in \mathfrak{Z}(M). \tag{1.1}$$

This means that $u \in L^2_{loc}(U)$ and

$$\int_U u \cdot Z^* \phi \, d\mu = 0, \quad \forall Z \in \mathfrak{Z}(M), \ \forall \phi \in \mathcal{C}_0^\infty(U).$$

1.1 Local maximum modulus principle

Lemma 1.1 *For any open subset U of M, the space $\mathcal{O}_{\mathfrak{Z}}(U) \cap W^{1,\infty}_{loc}(U)$ of Lipschitz continuous solutions of (1.1) is a $\mathbb{C}$-algebra.*

Moreover, if W is an open set in $\mathbb{C}$, $F \in \mathcal{O}(W)$ a holomorphic function on W, $u \in \mathcal{O}_{\mathfrak{Z}}(U) \cap W^{1,\infty}_{loc}(U)$ and $u(U) \subset W$, then also the composition $F \circ u$ belongs to $\mathcal{O}_{\mathfrak{Z}}(U) \cap W^{1,\infty}_{loc}(U)$.

Proof Indeed, $W^{1,\infty}_{loc}(U)$ is a $\mathbb{C}$-algebra and $Z(uv) = vZu + uZv \in L^\infty_{loc}(U)$ for all complex vector fields $Z \in \mathfrak{X}^{\mathbb{C}}(M)$. If $u, v \in \mathcal{O}_{\mathfrak{Z}}(U) \cap W^{1,\infty}_{loc}(U)$, then $Z(uv) = 0$ a.e. in U, and hence, $uv \in L^2_{loc}(U)$ is a weak solution of (1.1).

The last statement follows from the fact that $\mathcal{O}_{\mathfrak{Z}}(U) \cap W^{1,\infty}_{loc}(U)$ is a closed subspace of $W^{1,\infty}_{loc}(U)$. □

Let $\mathfrak{Y}(M)$ be any set of smooth real vector fields on M and for each $p \in M$ let $\mathbf{Y}_p M = \{Y_p \mid Y \in \mathfrak{Y}(M)\}$. Let U be an open subset of M. A $\mathfrak{Y}(M)$-path in U is a continuous and piecewise differentiable map $s : [0, 1] \to U$ such that $\dot{s}(t) \in \mathbf{Y}_{s(t)} M$, except for the finite number of points where $\dot{s}(t)$ does not exist.

For an open set U of M and a point $p \in U$, the *Sussmann leaf* of $\mathfrak{Y}(M)$ in U through p is the set

$$S(\mathfrak{Y}(M), U, p) = \left\{ q \in U \;\middle|\; \begin{array}{c} \exists \text{ a } \mathfrak{Y}(M)\text{-path } s : [0,1] \to U, \\ \text{with } s(0) = p, \ s(1) = q \end{array} \right\}. \tag{1.2}$$

The leaf $S(\mathfrak{Y}(M), U, p)$ is a smooth submanifold of U (see [10]).

Definition 1.2 We say that $\mathfrak{Z}(M)$ satisfies the *S-condition* at $p \in M$ if, for every open neighborhood U of p in M, the Sussmann leaf $S(\mathbb{H}_{\mathfrak{Z}}(M), U, p)$ is an open neighborhood of p in M.

We say that $\mathfrak{Z}(M)$ satisfies the *H-condition* at $p \in M$ if $\mathbb{H}_{\mathfrak{Z}}(M)$ satisfies the Hörmander condition at p. This means that, denoting by

$$\tilde{\mathbb{H}}_{\mathfrak{Z}}(M) = \mathbb{H}_{\mathfrak{Z}}(M) + [\mathbb{H}_{\mathfrak{Z}}(M), \mathbb{H}_{\mathfrak{Z}}(M)] + [\mathbb{H}_{\mathfrak{Z}}(M), [\mathbb{H}_{\mathfrak{Z}}(M), \mathbb{H}_{\mathfrak{Z}}(M)]] + \cdots$$

the Lie subalgebra of $\mathfrak{X}(M)$ generated by $\mathbb{H}_{\mathfrak{Z}}(M)$,

$$\{X_p \mid X \in \tilde{\mathbb{H}}_{\mathfrak{Z}}(M)\} = T_p M.$$

The validity of the H-condition at $p \in M$ implies the validity of the S-condition at $p \in M$.

Definition 1.3 We say that $\mathfrak{Z}(M)$ is Lipschitz-hypoelliptic if, for every open subset U of M, all weak L^2_{loc}-solutions of (1.1) on U are Lipschitz continuous.

This is the case, for instance, when the system $\mathfrak{Z}(M)$ satisfies the weak pseudo-concavity conditions of [2]. These conditions lead to the validity of some subelliptic estimates, while there are examples of hypoelliptic and non subelliptic systems with a big loss of derivatives (see [8]).

Theorem 1.4 *Assume that $\mathfrak{Z}(M)$ is Lipschitz-hypoelliptic and satisfies the S-condition on U. If $u \in \mathcal{O}_{\mathfrak{Z}}(U) \cap \mathcal{C}^0(U)$ and $|u|$ has a local maximum at a point $p_0 \in U$, then u is constant on a neighborhood of p_0 in U.*

Proof By the Banach-Schauder open mapping theorem, our assumption implies that for every pair U, U' of open subsets of M, with $U' \Subset U$, we have, for some positive constant C,

$$\|u\|_{W^{1,\infty}(U')} \leq C\|u\|_{L^2(U)}, \quad \forall u \in \mathcal{O}_{\mathfrak{Z}}(U) \cap L^2(U). \tag{1.3}$$

Let now $u \in \mathcal{O}_{\mathfrak{Z}}(U) \cap \mathcal{C}^0(U)$ and assume that $|u|$ has a local maximum at a point $p_0 \in U$. We can as well assume that U is relatively compact in M, that u is defined and continuous on a neighborhood of $\bar{U}$, and that $|u(p_0)| \geq |u(p)|$ for all $p \in U$. If $u(p_0) = 0$, there is nothing to prove. Consider then the case where $u(p_0) \neq 0$. Multiplying u by a complex constant, we can also take $u(p_0) = 1$. Fix any connected open neighborhood U' of p_0 in U, with $U' \Subset U$. By (1.3), $\{u^k|_{U'}\}_{k \in \mathbb{N}}$ is a sequence bounded in $W^{1,\infty}(U')$. By the Ascoli-Arzelà compactness theorem, there is a subsequence $\{u^{k_\nu}\}$ which converges, uniformly on $\bar{U}'$, to a continuous function u_∞ on U'. Let $F = \{p \in U' \mid |u(p)| = 1\}$. We have $u_\infty \in \mathcal{O}_{\mathfrak{Z}}(U')$ and $|u_\infty(p)| = 1$ for $p \in F$, $u_\infty(p) = 0$ for all $p \in U' \cap \complement F$. Since $\{|u(p)| \mid p \in U'\}$ is connected, it follows that $|u(p)| = 1$ for all $p \in U'$. Let $U'' = \{p \in U' \mid \operatorname{Re} u(p) > 0\}$. This is an open neighborhood of p_0 in U. Then, we can define $w(p) = i \log(u(p))$ on U'' in such a way that $w(p_0) = 0$. The function w belongs to $\mathcal{O}_{\mathfrak{Z}}(U'')$, is continuous, and takes real values. Then, it is Lipschitz continuous and satisfies also $\bar{Z}w = 0$ for all $Z \in \mathfrak{Z}(M)$, hence $Xw = 0$ for all $X \in \mathbb{H}_{\mathfrak{Z}}(M)$. Having assumed that $\mathfrak{Z}(M)$ satisfies the S-condition, it follows that w, and hence also u, is constant on U''. □

Corollary 1.5 *Assume that U is a relatively compact open subset of M, that $\mathfrak{Z}(M)$ is Lipschitz-hypoelliptic and satisfies the S-condition on U. Then*

$$|u(p)| \leq \max_{q \in \partial U} |u(q)|, \quad \forall p \in U, \quad \forall u \in \mathcal{O}_{\mathfrak{Z}}(U) \cap \mathcal{C}^0(\bar{U}). \tag{1.4}$$

Corollary 1.6 *Assume that U is a relatively compact open subset of M, that $\mathfrak{Z}(M)$ is Lipschitz-hypoelliptic and satisfies the S-condition on U. Then, there are no nonzero $u \in \mathcal{O}_{\mathfrak{Z}}(M)$ having compact support in U.*

Proof This follows indeed from (1.3). □

1.2 The global maximum modulus principle

The local maximum modulus principle, together with a weak unique continuation result for funtions in $\mathcal{O}_{\mathfrak{Z}}(U)$, yields a global maximum modulus principle.

Theorem 1.7 *Let U be a connected open set of M, and assume that $\mathfrak{Z}(M)$ is Lipschitz-hypoelliptic on U. If weak unique continuation holds true for $\mathcal{O}_{\mathfrak{Z}}(M)$ on U, then any function $u \in \mathcal{O}_{\mathfrak{Z}}(U)$ such that $|u|$ has a local maximum in U is constant in U.*

Proof Let $u \in \mathcal{O}_{\mathfrak{Z}}(U)$, and assume that $|u|$ has a local maximum at $p_0 \in U$. Then, by Theorem 1.4, u is constant in an open neighborhood V of p_0 in U. By weak unique continuation, $u(p) - u(p_0)$ is then equal to zero on U. □

2 Weak unique continuation

In this section, we shall discuss weak continuation for general systems of complex vector fields. For the convenience of the reader, we begin by stating the trapping lemma, that will be a fundamental tool in our approach. Although Propositions 2.4 and 2.5 in the following may be considered standard, we found it convenient to provide the reader with their statement and short proofs, to emphasize the fact that a different approach is needed to prove unique continuation in the more general framework of Theorem 2.6. Although this theorem is a consequence of Theorem 2.12, we found it cleaner to distinguish the two statements.

2.1 The trapping lemma

Let M be a smooth manifold, $T^*M \xrightarrow{\pi} M$ its cotangent bundle. Given a closed subset F of M, the set $N_e(F)$ of its exterior normals consists of all $\nu \in T^*M$ such that $\nu \neq 0$, $p = \pi(\nu) \in F$, and there exists a $\mathcal{C}^2$ real valued function $\phi : M \to \mathbb{R}$ satisfying:

$$d\phi(p) = \nu, \quad \phi(q) \leq \phi(p) \qquad \forall q \in F.$$

The main properties of $N_e(F)$ are collected in the following:

Proposition 2.1 *Let F be any closed subset of M. Then:*

$\pi(N_e(F)) \subset \partial F$ and is dense in ∂F.
Let X be a Lipshitz-continuous real vector field in M. If $\nu(X) \leq 0$ for all $\nu \in N_e(F)$, then F contains all integral curves of X issuing from a point of F.

For the proof, see Proposition 8.5.8 and Theorem 8.5.11 in [7].
From Proposition 2.1 one obtains

Proposition 2.2 *Let $\mathfrak{Y}(M)$ be a system of real vector fields on M and let F be a closed subset of M. Then, the following are equivalent:*

(i) $p \in F$ *and* $U^{open} \ni p \implies S(\mathfrak{Y}(M), U, p) \subset F$,

(ii) $\nu(Y) = 0,$ *for all* $\nu \in N_e(F),$ *and all* $Y \in \mathfrak{Y}(M)$.

2.2 Real analytic systems

First, we consider the case of a real analytic system, where we can use Holmgren's uniqueness theorem and the trapping lemma, to generalize Zachmanoglou's uniqueness theorem (see [13, 14]) to the case of overdetermined systems.

Definition 2.3 We say that $\mathfrak{Z}(M)$ is real analytic at $p_0 \in M$ if there is a coordinate patch U centered at p_0 such that $\mathfrak{Z}(U)$ is generated by complex vector fields $Z_1, \ldots, Z_m \in \mathfrak{Z}(M)$ that are real analytic on a neighborhood of p_0.

Conditions S and H are equivalent at a point p_0 of M where $\mathfrak{Z}(M)$ is real analytic (see [9]).

Proposition 2.4 *Let U be a connected open subset of M. Assume that $\mathfrak{Z}(M)$ is real analytic at p and has the H-property at all points of U. Then, any $u \in \mathcal{O}_{\mathfrak{Z}}(U)$ which vanishes on a non empty open subset of U is identically zero on U.*

Proof Let $u \in \mathcal{O}_{\mathfrak{Z}}(U)$, and $F = \operatorname{supp} u \subset U$. Assume by contradiction that $\emptyset \neq F \neq U$. By the trapping lemma and the assumption that $\mathfrak{Z}(M)$ satisfies the H-condition on U, there are points p_0 of $\partial F \cap U$ with an exterior normal ν_{p_0} which is non characteristic for $\mathfrak{Z}(M)$. Consider a smooth hypersurface S through p_0 with normal ν_{p_0} at p_0 and $F \cap S = \{p_0\}$, and a $Z \in \mathfrak{Z}(M)$ that is real analytic on a neighborhood of p_0 in U, which satisfies $\nu_{p_0}(\operatorname{Re} Z) \neq 0$. Then, u solves the Cauchy problem

$$\begin{cases} Zu = 0 & \text{on } U, \\ u = 0 & \text{on } S. \end{cases}$$

Applying the Holmgren uniqueness theorem, we obtain that $u = 0$ on a neighborhood of p_0, and this gives a contradiction. □

2.3 Embedded CR manifolds

Here, we consider the case where M is a CR manifold and $\mathfrak{Z}(M)$ is the system of its smooth $(0, 1)$-vector fields.

Proposition 2.5 *Let M be a CR submanifold of a complex manifold N, and let $\mathfrak{Z}(M)$ be the space of smooth complex tangent vector fields of type* $(0, 1)$ *on M. If M is minimal in the sense of* [11] *and* [12], *i.e. if $\mathfrak{Z}(M)$ has the S-property, then $\mathcal{O}_{\mathfrak{Z}}(M)$ satisfies the weak unique continuation principle.*

Proof Indeed, by [12], for every connected open subset U of M, there is a connected wedge $\tilde{U}$ in N, with edge U, such that every continuous $u \in \mathcal{O}_{\mathfrak{Z}}(U)$ extends to a continuous function $\tilde{u}$ on $\tilde{U}$, that is holomorphic in the interior of $\tilde{U}$. If u vanishes on a nonempty open subset V of U, then $\tilde{u}$ vanishes on the corresponding wedge $\tilde{V}$ and hence in $\tilde{U}$, by the unique continuation theorem for holomorphic functions. Hence, also the boundary value u of $\tilde{u}$ vanishes on U. □

2.4 S-condition for $\Theta_{\mathfrak{Z}}(M)$

Following [2], we associate to $\mathfrak{Z}(M)$ the new system of complex vector fields

$$\Theta_{\mathfrak{Z}}(M) = \left\{ Z \in \mathfrak{Z}(M) \,\middle|\, \begin{matrix} \exists r \geq 0,\ \exists Z_1, \ldots, Z_r \in \mathfrak{Z}(M),\ \text{s.t.} \\ i[Z, \bar{Z}] + i \sum_{j=1}^{r} [Z_j, \bar{Z}_j] \in \mathbb{H}_{\mathfrak{Z}}(M) \end{matrix} \right\}. \tag{2.1}$$

The result of [3] and [5, Theorem 5.1] can be generalized to:

Theorem 2.6 *Let U be a connected open subset of M, and assume that $\Theta_{\mathfrak{Z}}(M)$ has the S-property on U. Then, if $u \in L^2_{loc}(U)$ satisfies*

$$\begin{cases} \textit{for every } Z \in \mathfrak{Z}(M), \ Zu \in L^2_{loc}(U), \\ \textit{there exists } \kappa_Z \in L^\infty_{loc}(U) \textit{ such that} \\ |Zu(p)| \leq \kappa_Z(p)|u(p)| \quad \textit{a.e. on } U, \end{cases} \tag{2.2}$$

and u is zero a.e. on a non empty open subset of U, then $u = 0$ a.e. on U.

If $u \in L^2_{loc}(U)$ is a solution of (2.2), we can apply Proposition 2.2 to $F = \operatorname{supp} u$. The assumption that $\Theta_{\mathfrak{Z}}(M)$ has the S-property on U implies that either $\partial F \cap U = \emptyset$, or else there is $\nu \in N_e(\operatorname{supp} u)$ and $Z \in \Theta_{\mathfrak{Z}}(M)$ with $\nu(Z) \neq 0$. Thus, Theorem 2.6 will follow from

Proposition 2.7 *Let U be an open subset of M, and U^- an open subset of U such that $\partial U^- \cap U$ is smooth and*

$$\forall \nu \in N_e(\bar{U}^-), \quad \exists Z \in \Theta_{\mathfrak{Z}}(M), \ \textit{with } \nu(Z) \neq 0.$$

Then, every $u \in L^2_{loc}(U)$ which satisfies (2.2) and vanishes on U^- is zero a.e. on a neighborhood of $\bar{U}^-$ in U.

After introducing a Riemannian metric on M, the proof of Proposition 2.7 reduces to the following Carleman type estimate:

Proposition 2.8 *Let U be a relatively compact open subset of M. Let ϕ be a real valued smooth function on U, and $p_0 \in U$ a point where $\phi(p_0) = 0$ and $(Z\phi)(p_0) \neq 0$ for some $Z \in \Theta_{\mathfrak{Z}}(M)$. Then, we can find a neighborhood U_0 of p_0 in U, $L_1, \ldots, L_n \in \mathfrak{Z}(M)$, and constants $A > 0$, $c > 0$ and $\tau_0 > 0$ such that*

$$\begin{gathered} \tau \, \|u \exp(\tau\psi_A)\|_0^2 \leq c \sum_{i=1}^{n} \|(L_i u) \exp(\tau\psi_A)\|_0^2, \\ \textit{where } \psi_A = \phi + A\phi^2, \quad \forall u \in \mathcal{C}_0^\infty(U_0), \ \forall \tau \geq \tau_0. \end{gathered} \tag{2.3}$$

Proof Having fixed $u \in \mathcal{C}_0^\infty(U)$, we set $v_\tau = u \cdot \exp(\tau\psi_A)$, so that

$$\|(Zu)\exp(\tau\psi_A)\|_0 = \|Zv_\tau - \tau v_\tau Z\psi_A\|_0.$$

Let $L_1 \in \Theta_{\mathfrak{Z}}(M)$ be such that $(L_1\phi)(p_0) \neq 0$, and let $L_2, \ldots, L_n \in \mathfrak{Z}(M)$ be such that

$$i \sum_{j=1}^{n-1} [L_j, \bar{L}_j] = \operatorname{Re} L_n.$$

It suffices to prove that there is an open neighborhood U_0 of p_0 in U and constants $A > 0$, $c > 0$, $\tau_0 > 0$ such that

$$c \sum_{j=1}^{n} \|L_i v - \tau v L_i \psi_A\|_0^2 \geq \tau \|v\|_0^2, \quad \forall v \in \mathcal{C}_0^\infty(U_0), \ \forall \tau > \tau_0. \tag{2.4}$$

Using integration by parts, we obtain

$$\begin{aligned} \sum_{j=1}^{n-1} \|L_i v - \tau v L_i \psi_A\|_0^2 &= \sum_{j=1}^{n-1} \|L_i^* v - \tau v \bar{L}_i \psi_A\|_0^2 \\ &\quad + \int \sum_{j=1}^{n-1} [L_i^*, L_i] v \cdot v d\lambda + 2\tau \operatorname{Re} \int |v|^2 \sum_{j=1}^{n-1} L_i \bar{L}_i \psi_A d\lambda. \end{aligned}$$

The second term in the right hand side can be estimated by

$$c_1\|L_n v - \tau v L_n \psi_A\|_0 \cdot \|v\|_0 + c_2(U_0)\tau\|v\|_0^2,$$

where c_1 is a constant that can be taken independent of A and τ, provided the support of v is contained in a fixed relatively compact open neighborhood of p_0 in U; if $\mathrm{supp}(v) \subset U_0$, the constant $c_2(U_0)$ is the supremum of $(1 + 2A\phi(p))|L_n\phi(p)|$ on U_0.

The third term in the right hand side is bounded from below by

$$c_3(U_0)\tau\|v\|_0^2,$$

where

$$c_3(U_0) = \inf_{p\in U_0}\left(A\sum_{j=1}^{n-1}|L_j\phi(p)|^2 - c_4(1 + A|\phi(p)|)\sum_{j=1}^{n-1}(|L_i\phi(p)| + |L_i\bar{L}_i\phi(p)|)\right).$$

By taking U_0 sufficiently small, so that $|\phi(p)| \ll 1$ on U_0, and taking A sufficiently large, we obtain with some constants $c_5 > 0$, $c_6 > 0$,

$$\sum_{j=1}^{n}\|L_j v - \tau v L_i\psi_A\|_0^2 \geq (c_5\,\tau - c_6)\,\|v\|_0^2,\ \forall v \in \mathcal{C}_0^\infty(U_0),$$

and the Carleman estimate (2.4) follows for $\tau > \tau_0$, provided we take $c_5\tau_0 > c_6$. □

Proof of Proposition 2.7 Fix a point $p_0 \in \partial U^- \cap U$ and a defining function ρ for U^- in a neighborhood $V \subset U$ of p_0: $\rho \in \mathcal{C}^2(V, \mathbb{R})$, $U^- \cap V = \{p \in V \mid \rho(p) < 0\}$. By our assumption, there is $Z \in \Theta_3(M)$ such that $Z\rho(p_0) \neq 0$. After shrinking, we can assume that $Z\rho(p) \neq 0$ for all $p \in V$. Take for V a coordinate patch, with coordinates $x \in \mathbb{R}^m$ vanishing at p_0, and set $\phi(p) = \rho(p) - C\cdot|x(p)|^2$ with C sufficiently large, so that $\phi(p) < -1$ outside a compact neighborhood of p_0 in V. By Proposition 2.8, there are an open neighborhood U_0 of p_0 in V, and constants $A > 0$, $c > 0$ and $\tau_0 > 0$ such that (2.3) is valid for the weight function ϕ. Fix another function $\nu : \mathbb{R} \to \mathbb{R}$ with:

$$\begin{cases} 0 \leq \nu(\theta) \leq 1 & \forall\theta \in \mathbb{R}, \\ \nu(\theta) = 1 & \text{if} \quad \theta > 1, \\ \nu(\theta) = 0 & \text{if} \quad \theta < -1. \end{cases}$$

Given a solution $u \in L^2_{loc}(U)$ of (2.2) vanishing on U^-, for real $\delta > 0$, we consider the function $u_\delta = u \cdot \nu(\phi/\delta)$. Its support is contained in $\{\phi(p) \geq -\delta\} \cap \complement U^-$ and therefore is compact and contained in U_0 if $\delta > 0$ is sufficiently small, say $\delta < \delta_0$. The estimate (2.3) is valid for u_δ when $\delta < \delta_0$ by Friedrichs extension theorem (cf. [4]). For a fixed $0 < \delta < \delta_0$ and $\psi_A = \phi + A\phi^2$, we obtain

$$\tau\|u_\delta \cdot \exp(\tau\psi_A)\|^2 \leq c\sum_{j=1}^{n}\|\exp(\tau\psi_A)\,L_j u_\delta)\|^2.$$

For $\lambda = \delta + A\delta^2$, we obtain:

$$\begin{aligned}
&\tau \| \exp(\tau\psi_A - \lambda) \cdot u\|^2_{\phi\geq\delta} \\
&\quad\leq c\left(\sum_{j=1}^{n} \|exp(\tau\psi_A - \lambda)L_j u\|_{\phi\geq\delta} + \sum_{j=1}^{n} \| \exp(\tau\psi_A - \lambda)L_j u_\delta\|^2_{\phi\leq\delta}\right) \\
&\quad\leq \text{constant}\| \exp(\tau\psi_A - \lambda) \cdot u\|^2_{\phi\geq\delta} + c\sum_{j=1}^{n} \| \exp(\tau\psi_A - \lambda)L_j u_\delta)\|^2_{\phi\leq\delta}.
\end{aligned}$$

This gives:

$$(\tau - \text{constant})\|u\|^2_{\phi\geq\delta} \leq c\, \|\overline{\partial}_M(f_\delta)\|^2_{\phi\leq\delta}$$

for all $\tau \geq \tau_0$ and hence $u = 0$ a.e. for $\phi \geq \delta$, showing that u vanishes on a neighborhood of p_0.

The proof is complete. □

Example 2.9 Let $N = \{\ell_1 \subset \ell_3 \subset \mathbb{C}^6\}$ be the complex flag manifold of complex lines and 3-planes in $\mathbb{C}^6$. We denote by ℓ_i a $\mathbb{C}$-linear subspace of dimension i. Fix a Hermitian symmetric form h with signature (2, 4) and let M be the minimal orbit in N of the group **SU**(2, 4) of $\mathbb{C}$-linear transformations of $\mathbb{C}^6$ leaving h invariant. This is a compact smooth real submanifold of M, consisting of the pairs $\ell_1 \subset \ell_2$ with rank $h|_{\ell_1} = 0$, rank $h|_{\ell_3} = 1$. M has a natural structure of a generic real analytic CR submanifold of N, with CR dimension 5 and CR codimension 6. Let $\mathfrak{Z}(M)$ be the system of (0, 1)-vector fields tangent to M. Then, $\mathfrak{Z}(M)$ is a distribution of complex vector fields of rank 5, and $\Theta_3(M)$ a distribution of complex vector fields of rank 4, that satisfies the H-condition, and hence the S condition. In particular, Propositions 1.7 and 2.6 apply.

Example 2.10 Fix a Hermitian symmetric form h on $\mathbb{C}^7$, with signature (2, 5). Let M be the manifold consisting of the flags $\ell_1 \subset \ell_3 \subset \ell_4 \subset \mathbb{C}^7$ such that $h|_{\ell_1}$, $h|_{\ell_3}$, and $h|_{\ell_4}$ have ranks 0, 1, and 2, respectively. As a submanifold of the complex flag manifold $N = \{\ell_1 \subset \ell_3 \subset \ell_4 \subset \mathbb{C}^4\}$, M has the structure of a generic real analytic CR submanifold of CR dimension 7 and CR codimension 10. Let $\mathfrak{Z}(M)$ be the set of smooth complex vector fields of type (0, 1), tangent to M. It is a distribution of complex vector fields of rank 7. Then, $\Theta_3(M)$ is a distribution of complex vector fields of rank 6, and $\mathfrak{Z}(M)$ is generated by $\Theta_3(M)$ as a Lie subalgebra of $\mathfrak{X}^{\mathbb{C}}(M)$. Conditions H and S are then satisfied, and, in particular, Propositions 1.7 and 2.6 apply. (For other similar examples, we refer to [1]).

2.5 Dropping the S-condition for $\Theta_3(M)$

Definition 2.11 We denote by $\mathbb{Y}_3(M)$ the Lie subalgebra of $\mathfrak{X}(M)$ generated by $\{\text{Re}\, Z \mid Z \in \Theta_3(M)\}$.

We have

Theorem 2.12 *Let U be an open set of M and assume that $u \in L^2_{loc}(U)$ satisfies (2.2). Then,* supp(u) *is foliated by the Sussmann leaves of $\mathbb{Y}_3(M)$ in U.*

Proof Assume that $F = \text{supp}(u) \neq U$. By using the Carleman estimate of Proposition 2.7, we can prove that $\nu(Z) = 0$ for all $Z \in \Theta_3(M)$. By Proposition 2.2, this implies that a Sussmann leaf of $\mathbb{Y}_3(M)$ in U, which has a point in F is contained completely in F. □

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