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An extension of coherent sheaves defined outside holomorphically convex compact sets

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Abstract We show that a coherent analytic sheaf $\mathcal{F}$ with prof $\mathcal{F} \geq 2$ defined outside a holomorphically convex compact set K in a 1-convex space X admits a coherent extension to the whole space X if, and only if, the canonical topology on $H^1(X \setminus K, \mathcal{F})$ is separated.

Keywords Coherent sheaf · Coherent extension · Holomorphically convex compact set · 1-convex space · Remmert reduction

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1 Introduction

Let X be a Stein space and $K \subset X$ a Stein compact set, i.e., K admits a neighborhood system of Stein open sets.

A theorem due to Bănică [4] states that, for any coherent analytic sheaf $\widehat{\mathcal{F}}$ on X and any positive integer q, the canonical topology (defined via the Čech cohomology) on $H^q(X \setminus K, \widehat{\mathcal{F}})$ is separated.

On the other hand, it is shown in [5] that if $\mathcal{F}$ is a coherent analytic sheaf on $X \setminus K$ and prof $\mathcal{F} \geq 3$, then $\mathcal{F}$ admits a coherent extension to X, namely there is a coherent analytic sheaf $\widehat{\mathcal{F}}$ on X such that $\widehat{\mathcal{F}}|_{X \setminus K} = \mathcal{F}$ (equality means $\mathcal{O}_{X \setminus K}$-module isomorphism). If, moreover, K is holomorphically convex, then $\mathcal{F} = \mathcal{F}^{[1]}$ is sufficient for a coherent extension, see [3]; this will be improved in the subsequent Proposition 1. The gap condition is equivalent to saying that prof $\mathcal{F} \geq 2$ and the set $\{x \in X \setminus K\,;\ \text{prof}\, \mathcal{F}_x = 2\}$ is discrete in $X \setminus K$.

In this circle of ideas, we prove:

Theorem 1 *Let X be a 1-convex space and $K \subset X$ a holomorphically convex compact set. Let $\mathcal{F}$ be a coherent analytic sheaf on $X \setminus K$ with prof $\mathcal{F} \geq 2$. Then $\mathcal{F}$ admits a coherent extension $\widehat{\mathcal{F}}$ to X if, and only if, $H^1(X \setminus K, \mathcal{F})$ is separated.*

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Remark 1 If X is Stein, then the extension $\widehat{\mathcal{F}}$ can be chosen such that $\operatorname{prof} \widehat{\mathcal{F}} \geq 2$; moreover such an $\widehat{\mathcal{F}}$ is unique (up to an isomorphism). However, the unicity fails in the 1-convex case. For instance, if $\pi : X \longrightarrow \mathbb{C}^2$ is the blowing-up of the origin in $\mathbb{C}^2$, then X is 1-convex and its exceptional set S is a rational curve; its canonically associated invertible sheaf is holomorphically trivial on $X \setminus S$. Therefore, $\mathcal{O}_{X \setminus S}$ admits two non isomorphic coherent extensions. Note also that, if $\iota : X \setminus S \hookrightarrow X$ is the canonical inclusion, then $\iota_\star(\mathcal{O}_{X \setminus S})$ is not coherent.

Corollary 1 *Let $K \subset \mathbb{C}^2$ be a polynomially convex set and $\mathcal{L}$ an invertible sheaf on $\mathbb{C}^2 \setminus K$. Then $H^1(\mathbb{C}^2 \setminus K, \mathcal{L})$ is separated if, and only if, $\mathcal{L}$ is the trivial invertible sheaf.*

Remark 2 This corollary shows that condition $\operatorname{prof} \mathcal{F} \geq 2$ alone in Theorem 1 does not guarantee the coherent extension. (Take $K = \{0\}$ and $\mathcal{L}$ an invertible sheaf of over $\mathbb{C}^2 \setminus \{0\}$ that is not holomorphically trivial. If $\mathcal{L}$ would extend coherently, then the extension can be chosen to be an invertible sheaf over $\mathbb{C}^2$ which would be trivial. See [10].)

Proposition 1 *Let X be a complex space, $K \subset X$ a Stein compact set and $\mathcal{F}$ a coherent sheaf on $X \setminus K$ such that $\mathcal{F} = \mathcal{F}^{[1]}$. Then there exists a coherent sheaf $\widehat{\mathcal{F}}$ on X that extends $\mathcal{F}$, i.e. $\widehat{\mathcal{F}}|_{X \setminus K} = \mathcal{F}$.*

Corollary 2 *Let X be a Stein space, $K \subset X$ a Stein compact set and $\mathcal{F}$ a coherent analytic sheaf on $X \setminus K$ such that $\mathcal{F} = \mathcal{F}^{[1]}$. Then $H^1(X \setminus K, \mathcal{F})$ is separated.*

In order to put our results in a larger context, we note that one recurring theme in Complex Analysis is "Hartogs type extension theorems." Specifically, let X be a complex space, $S \subset X$ a closed subset and $\mathcal{F}$ a coherent analytic sheaf on $X \setminus S$. Find reasonable conditions such that $\mathcal{F}$ admits a coherent extension to the whole space X. In particular, if $\iota : X \setminus S \longrightarrow X$ is the inclusion map, the sheaf $\iota_\star(\mathcal{F})$ is an analytic extension and one looks for conditions such that $\iota_\star(\mathcal{F})$ is coherent.

The found necessary conditions are *local* and stated *in terms of the absolute or relative gap sheaves* and require either (i) that S is analytic [7,10,12–14] or (ii) that S is a holomorphically convex compact set (or, more generally, a Stein compact set) (as in [5]), or (iii) that S is the complement of an open set fulfilling certain generalized pseudoconvexity at the boundary (see [11] and [12] for more details).

The extension stated above in Proposition 1 is complementary to results around 1970 and, perhaps, it has been essentially known, but we did not found an appropriate reference. In the same vein (see [11]), an extension is done for K a closed set of a complex space X admitting a smooth proper function $\varphi : X \longrightarrow (0, \infty)$ that is q-convex on X (the normalization is such that 1-*convex* $\equiv$ *strictly plurisubharmonic*), $K = \{x \in X\,;\ \varphi(x) \leq c\}$ for some $c > 0$ and $\mathcal{F} = \mathcal{F}^{[q]}$. However, our proposition is not a consequence of this result for $q = 1$ because a Stein compact set does not necessarily have a Stein open neighborhood with respect to which it becomes holomorphically convex. A straightforward example in $\mathbb{C}$ is given by the Stein compact set

$$K = \{0\} \cup \bigcup_{n \geq 1} \partial\Delta(1/n),$$

where for $r > 0$ we set $\Delta(r) := \{z \in \mathbb{C}\,;\ |z| < r\}$. (Use the subsequent Lemma 1 and the maximum principle for subbarmonic functions.)

2 Preliminaries

Throughout this paper, complex spaces, whose structural sheaves might have nilpotents, are such that their underlying topology admits a countable base of open sets.

Let $X = (X, \mathcal{O}_X)$ be a complex space and $\mathcal{F}$ a coherent sheaf on X. For each point $x \in X$ there exists an holomorphic embedding $\iota : U \longrightarrow \widehat{U} \subset \mathbb{C}^{m(x)}$ of an open neighborhood $U \ni x$ into the Zariski tangent space $\mathbb{C}^{m(x)}$ of X at x. Let $\widehat{\mathcal{F}}$ be the trivial extension of $\iota_\star(\mathcal{F}|_U)$; it is a coherent sheaf on $\widehat{U}$. Let

$$0 \longrightarrow \mathcal{O}^{p_d} \longrightarrow \mathcal{O}^{p_{d-1}} \longrightarrow \cdots \longrightarrow \mathcal{O}^{p_0} \longrightarrow \widehat{\mathcal{F}} \longrightarrow 0$$

be a resolution of $\widehat{\mathcal{F}}$ on a neighborhood of $\iota(x)$ of minimal length. It can be shown that $d \leq m(x)$ and the number $\operatorname{prof} \mathcal{F}_x := m(x) - d$ does not depend on the embedding ι. If $\mathcal{F}_x = 0$, then we set $\operatorname{prof} \mathcal{F}_x = \infty$. We let $\operatorname{prof}_X \mathcal{F} := \inf_{x \in X} \operatorname{prof} \mathcal{F}_x$; if X is clearly understood from the context, we write $\operatorname{prof} \mathcal{F}$ instead of $\operatorname{prof}_X \mathcal{F}$.

(Note that $\operatorname{prof} \mathcal{F}$ can be larger than $\operatorname{prof} \mathcal{O}_X$. Take X the image of the holomorphic mapping $h : \mathbb{C}^2 \longrightarrow \mathbb{C}^4$, $(z, w) \mapsto (z^2, z^3, w, zw)$; X is an analytic subset of $\mathbb{C}^4$ of dimension 2, it has only one singularity at the origin and $X \setminus \{0\}$ is connected (so that X is irreducible). The map h is the normalization of X, $\operatorname{prof} \mathcal{O}_X = 1$ and $\operatorname{prof} \widetilde{\mathcal{O}}_X = 2$, where $\widetilde{\mathcal{O}}_X$ is the coherent sheaf of germs of weakly holomorphic functions in X.)

For a non-negative integer q the set $S_q(\mathcal{F}) := \{x \in X ;\ \operatorname{prof} \mathcal{F}_x \leq q\}$ is analytic in X of dimension $\leq q$; these are called the singular sets of $\mathcal{F}$.

Also the *qth-absolute gap sheaf* of $\mathcal{F}$, denoted by $\mathcal{F}^{[q]}$, is the canonical sheaf associated to the presheaf which to an open subset U of X associated $\lim \Gamma(U \setminus A, \mathcal{F})$, where in the inductive limit A runs over all analytic subsets of U of dimension $\leq q$, and with the natural restrictions mappings. One has a canonical morphism $\mathcal{F} \longrightarrow \mathcal{F}^{[q]}$. This is an isomorphism, and in that case, we write $\mathcal{F} = \mathcal{F}^{[q]}$ if, and only if,

$$\dim S_{k+2}(\mathcal{F}) \leq k \quad \text{for } k = -1, 0, \ldots, q-1.$$

Thus, $\mathcal{F} = \mathcal{F}^{[1]}$ means precisely that $\operatorname{prof} \mathcal{F} \geq 2$ and $\{x \in X ;\ \operatorname{prof} \mathcal{F}_x = 2\}$ is a discrete set; *a fortiori* $\mathcal{F} = \mathcal{F}^{[1]}$ whenever $\operatorname{prof} \mathcal{F} \geq 3$.

From ([5], pp. 356 and 357), we quote the following two propositions:

Proposition 2 *Let X a Stein space and $K \subset X$ a Stein compact set. Let $\mathcal{F}$ be a coherent sheaf on X with $\operatorname{prof} \mathcal{F} \geq 2$. Then for every coherent sheaf $\mathcal{G}$ on X the natural map*

$$\operatorname{Hom}_{\mathcal{O}}(\mathcal{G}, \mathcal{F}) \longrightarrow \operatorname{Hom}_{\mathcal{O}}(\mathcal{G}|_{X \setminus K}, \mathcal{F}|_{X \setminus K})$$

is bijective.

Remark 3 The proposition fails if X is 1-convex and $K \subset X$ is holomorphically convex; see Remark 1.

Proposition 3 *Let X be a complex space and $\Lambda \subset X$ a discrete subset. Let $\mathcal{E}$ be a coherent sheaf on $X \setminus \Lambda$ with* $\operatorname{prof} \mathcal{E} \geq 2$. *If $\mathcal{E}$ admits a coherent extension to X, then $\iota_\star(\mathcal{E})$ is coherent on X and* $\operatorname{prof} \iota_\star(\mathcal{E}) \geq 2$.

(Here $\iota : X \setminus A \longrightarrow X$ is the inclusion map.)

3 Holomorphic convexity in 1-convex spaces

Here, we recall that a complex space X is said to be 1-convex if it satisfies one of the following four equivalent conditions, see [9]:

- There exists a continuous function $\varphi : X \longrightarrow \mathbb{R}$ such that φ is exhaustive, i.e., for every $c \in \mathbb{R}$ the set $\{x \in X\,; \varphi(x) < c\}$ is relatively compact in X and φ strictly plurisubharmonic outside a compact subset of X.
- The space X is *cohomologically 1-convex*, that is, for every coherent analytic sheaf $\mathcal{F}$ on X, the cohomology groups $H^q(X, \mathcal{F})$, $q = 1, 2, \ldots$, have finite dimension (as complex vector spaces);
- The space X is holomorphically convex and admits a maximally compact analytic set S, called the *exceptional set*.
- The space X is a *proper modification of a Stein space at a finite number of points*, i.e., there is a Stein space Y, a proper holomorphic map $\rho : X \longrightarrow Y$ with $\rho_\star(\mathcal{O}_X) \simeq \mathcal{O}_Y$ (in particular ρ is surjective and has connected fibers) and a finite set $B \subset Y$ such that ρ induces a biholomorphism between $X \setminus \rho^{-1}(B)$ and $Y \setminus B$.

The map ρ is called the *Remmert's reduction* of X. The exceptional set of X is $S = \rho^{-1}(B)$.

A compact set $K \subset X$ is "saturated" with respect to ρ, which means that $K = \rho^{-1}(\rho(K))$, if, and only if, every irreducible component of S meeting K lies entirely in K. For instance, any holomorphically convex compact set in X is saturated.

Notice that Stein spaces are considered as 1-convex with empty exceptional set.

The following result, which in particular shows that 1-convexity is stable under normalization, can be immediately deduced from [16].

Proposition 4 *Let $\pi : X \longrightarrow Y$ be a holomorphic map of complex spaces that is finite and surjective. Then X is 1-convex if, and only if, Y is 1-convex.*

Lemma 1 *Let K be a holomorphically convex compact set in a 1-convex space X with exceptional set S. Then there is a C^∞-smooth, proper function $\varphi : X \longrightarrow [0, \infty)$ such that $K = \{\varphi = 0\}$ and φ is strictly plurisubharmonic on $X \setminus (K \cup S)$.*

Proof Observe that φ as above results immediately plurisubharmonic on X.

Let $\rho : X \longrightarrow Y$ be the Remmert's reduction. Then $L = \rho(K)$ is holomorphically convex in Y and $\rho(S)$ is a finite set. Therefore, it will be enough to produce φ when X is Stein and $S = \emptyset$. To this purpose, we let $\psi : X \longrightarrow [0, \infty)$ be a C^∞-smooth strictly plurisubharmonic proper function. Let $r > \max_K \psi$. Since K is holomorphically convex, there is a sequence of holomorphic functions $\{f_n\}_n$ on X such that $|f_n| \leq 1$ on K for all n and for any point $x_0 \in X \setminus K$ there is an index n_0 with $|f_{n_0}(x_0)| \geq \sqrt{1+r}$. Select $\rho : [0, \infty) \longrightarrow [0, \infty)$ be smooth of class C^∞ and convex such that $\{\rho = 0\} = [0, 1 + r]$ and ρ be strictly increasing on $[1 + r, \infty)$. Then, we define $\varphi : X \longrightarrow [0, \infty)$ by setting

$$\varphi(x) := \sum \epsilon_n \rho(|f_n(x)|^2 + \psi(x)), \quad x \in X,$$

where $\{\epsilon_\nu\}_\nu$ is a sequence of positive numbers that decreases fast enough to zero. This φ has the required properties. □

Lemma 2 *Let X be a 1-convex space and $K \subset X$ a compact set. Let $A \subset X$ be a compact analytic set that does not meet K. Then K is holomorphically convex if, and only if, $K \cup A$ is holomorphically convex.*

Proof First notice the following fact. Let Y be a Stein space, $L \subset X$ a compact set and F a finite set of points in $Y \setminus K$. Then L and $L \cup F$ are simultaneously holomorphically convex or not. (If L is holomorphically convex, and $y_0 \in Y \setminus (L \cup F)$, then there is f and g holomorphic functions on Y such that $|f(y_0)| > \|f\|_L$ and $\{g = 0\} = F$. It follows that $F := f^N g$ for N positive integer large enough is such that $|F(y_0)| > \|F\|_{L \cup F}$. For the other implication, we choose $\psi : Y \longrightarrow [0, \infty)$ that is proper, smooth of class C^∞, plurisubharmonic on Y and strictly plurisubharmonic on $Y \setminus (L \cup F)$ and such that $\{\psi = 0\} = L \cup F$. It follows that the union of the connected components of $\{\psi < \epsilon\}$ ($\epsilon > 0$) meeting K form a Runge neighborhoods system for L so that L follows holomorphically convex.)

Now let $\rho : X \longrightarrow Y$ be the Remmert's reduction and S the exceptional set of X. Since a compact set $T \subset X$ is holomorphically convex in X if, and only if, T is saturated and $\rho(T)$ is holomorphically convex in Y, the proof of the lemma follows easily. □

Proposition 5 *Let X be a Stein space, $K \subset X$ a holomorphically convex set and $\mathcal{F}$ a coherent analytic sheaf on $X \setminus K$.*

(a) *If* prof $\mathcal{F} \geq 3$, *then $H^1(X \setminus K, \mathcal{F})$ has finite dimension.*
(b) *Let $L \subset X$ be another holomorphically convex compact set, $K \subset L$. If* prof $\mathcal{F} \geq 2$, *then the restriction map $\Gamma(X \setminus K, \mathcal{F}) \longrightarrow \Gamma(X \setminus L, \mathcal{F})$ is bijective.*

Proof This is only a reformulation of some results from [2]. First select $\varphi : X \longrightarrow [0, \infty)$ that is proper, smooth of class C^∞, plurisubharmonic on X and strictly plurisubharmonic on $Y \setminus K$ and such that $K = \{\varphi = 0\}$. Let $0 < a < b$ and $D := \{a < \varphi < b\}$, which is relatively compact in $X \setminus K$. The bumping technique gives that the restriction map $H^1(X \setminus K, \mathcal{F}) \longrightarrow H^1(D, \mathcal{F})$ is bijective. So one concludes by using the classical finiteness lemma.

To verify the second statement, we deal first with the surjectivity. Let $\sigma \in \Gamma(X \setminus L, \mathcal{F})$. Let also $\psi : Y \longrightarrow [0, \infty)$ be proper, smooth of class C^∞, plurisubharmonic on Y and strictly plurisubharmonic on $Y \setminus L$ and such that $L = \{\psi = 0\}$. Then for $c > 0$ large enough, the restriction of σ to $\{c < \varphi\}$ extends to $\widehat{\sigma} \in \Gamma(X \setminus K, \mathcal{F})$. Then $\widehat{\sigma}|_{X \setminus L} - \sigma$ vanishes on the set $\{c_1 < \psi\}$ for $c_1 > 0$ sufficiently large such that the set $\{c_1 < \psi\}$ is contained in $\{c < \varphi\}$. Then it vanishes on $X \setminus L$. The injectivity of the said restriction is similar so it is omitted. □

For the sake of completeness we mention (cf. Proposition 2)

Proposition 6 *Let $\pi : X \longrightarrow Y$ be a finite holomorphic surjection map between 1-convex spaces X and Y. Let $K \subset Y$ be a compact set. Then $\pi^{-1}(K)$ is holomorphically convex if, and only if, K is holomorphically convex.*

Toward the proof we prepare:

Lemma 3 *Let Z be a 1-convex space and $K \subset Z$ a compact set. Then K is holomorphically convex if, and only if, for any coherent analytic sheaf $\mathcal{F}$ on Z, the restriction map*

$$\Gamma(Z, \mathcal{F}) \longrightarrow \Gamma(K, \mathcal{F})$$

has dense image.[1]

[1] For a compact set K in a complex space Z and $\mathcal{F}$ a coherent analytic sheaf on Z, then $\Gamma(K, \mathcal{F})$ is the inductive limit of $\Gamma(U_\nu, \mathcal{F})$ where (U_ν) forms a neighborhood system of open sets of K and has a structure of LF topological vector space that is separated as the continuous map

$$\Gamma(X, \mathcal{F}) \longrightarrow \prod_{x \in K} \prod_{\nu \geq 0} \mathcal{F}_x / m_x^\nu \mathcal{F}_x$$

is injective according to Krull's theorem.

As a matter of fact, it is enough to take $\mathcal{F}$ only coherent ideal subsheaves of $\mathcal{O}_X$ (or more simply ideal sheaves $\mathcal{I}_a$, $a \in X$).

Proof Indeed, for the "if" part, we consider $\mathcal{F}$ the ideal sheaf defined by some point x_0 outside K. For the "only if", let $\rho : X \longrightarrow Y$ be the Remmert's reduction. Since $\rho(K)$ is holomorphically convex in Y, thanks to Grauert's coherence theorem $\rho_\star(\mathcal{F})$ is coherent on Y and since $\Gamma(Y, \pi_\star\mathcal{F}) = \Gamma(X, \mathcal{F})$ and $\Gamma(\rho(K), \pi_\star\mathcal{F}) = \Gamma(K, \mathcal{F})$ the lemma results easily. □

Lemma 4 *Let $\pi : X \longrightarrow Y$ be a finite holomorphic surjection map between normal 1-convex spaces X and Y. Let $K \subset Y$ be a compact set. Then the holomorphically convex hull of $\pi^{-1}(K)$ equals $\pi^{-1}(\widehat{K})$.*

Proof A sketch of the proof is as follows. First there is no loss in generality to assume that X and Y are connected so that there is a nowhere dense analytic set $B \subset Y$ such that $A := \pi^{-1}(B)$ is nowhere dense in X and π induces an holomorphic covering map between $X \setminus A$ and $Y \setminus B$, say with n sheets. Also, we may take $K = \overline{U}$, where $U \subset Y$ is open so that the closure of $K \setminus B$ equals K and, consequently, for any holomorphic function g on Y, $\sup_K |g| = \sup_{K \setminus B} |g|$.

Now, any holomorphic function f on X satisfies a polynomial equation of the form

$$f^n + \sum_{\nu=1}^{n} (a_\nu \circ \pi) f^{n-\nu} = 0,$$

where $a_1, \ldots, a_n$ are holomorphic on Y. In fact, on $Y \setminus B$, one has:

$$a_\nu(y) = \sum_{1 \le i_1 < \cdots < i_\nu \le n} f(x_{i_1}) \cdots f(x_{i_\nu}),$$

where $\pi^{-1}(y) = \{x_1, \ldots, x_n\}$. Thus, for all $x \in X$, if $y = \pi(x)$, then

$$|f(x)| \le \max(1, |a_1(y)| + \cdots + |a_n(y)|).$$

Then, we conclude in a standard manner. □

Proof of Proposition 6 Let $n = \dim(X) = \dim(Y)$. By Lemma 4 and straightforward arguments, we reduce ourselves to show that holomorphic convexity of $\pi^{-1}(K)$ implies that of K when $\pi : X \longrightarrow Y$ is the normalization map of Y and assuming the proposition holds true for complex spaces of dimension $\le n - 1$.

Let S be the exceptional set of Y. Then $\pi^{-1}(S)$ is the exceptional set of X. Thanks to Lemma 6, we may assume that S lies in K. Now, we follow the technique of Narasimhan for the Stein setting [8]. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a coherent ideal sheaf. We want to check that $\Gamma(Y, \mathcal{I}) \longrightarrow \Gamma(K, \mathcal{I})$ has dense image.

Let $\mathcal{A}$ be the subsheaf of $\mathcal{O}_Y$ given as the sheaf of universal denominators of $\pi_\star(\mathcal{O}_X)$, which is the coherent sheaf of weakly holomorphic functions on Y in $\mathcal{O}_Y$. Thus, $\mathcal{A} \cdot \pi_\star(\mathcal{O}_X) \subset \mathcal{O}_Y$. Let $\mathcal{B} = \pi_\star(\widetilde{\mathcal{B}})$, where $\widetilde{\mathcal{B}} = \pi^\star(\mathcal{A} \cdot \mathcal{I}) \cdot \mathcal{O}_X$. Thus, $\mathcal{B}$ is a coherent subsheaf of $\mathcal{I}$ and $\mathcal{I}/\mathcal{B}$ has the support of dimension $\le n - 1$. Furthermore, $H^1(Y, \mathcal{B}) \longrightarrow H^1(K, \mathcal{B})$ is an isomorphism (because π is finite and $H^1(Y, \mathcal{B}) = H^1(X, \widetilde{\mathcal{B}})$, the last being isomorphic due to 1-convexity of X to $H^1(\pi^{-1}(K), \widetilde{\mathcal{B}}) = H^1(K, \mathcal{B})$). From the commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(Y,\mathcal{B}) & \longrightarrow & \Gamma(Y,\mathcal{I}) & \longrightarrow & \Gamma(Y,\mathcal{I}/\mathcal{B}) & \longrightarrow & H^1(Y,\mathcal{B}) \\ & & \downarrow u & & \downarrow v & & \downarrow w & & \downarrow \theta \\ 0 & \longrightarrow & \Gamma(K,\mathcal{B}) & \longrightarrow & \Gamma(K,\mathcal{I}) & \longrightarrow & \Gamma(K,\mathcal{I}/\mathcal{B}) & \longrightarrow & H^1(K,\mathcal{B}) \end{array}$$

since u has dense image (because $\Gamma(Y, \mathcal{B}) = \Gamma(X, \widetilde{\mathcal{B}}), \Gamma(K, \mathcal{B}) = \Gamma(\pi^{-1}(K), \widetilde{\mathcal{B}})$ and $\pi^{-1}(K)$ is holomorphicaly convex in X), w has dense image by the induction hypothesis, we conclude easily by diagramm chasing the density of v, so that K results holomorphically convex in Y from Lemma 3, whence the proposition. □

4 Decoding separatedness

Below, we give a key fact encapsuled in the separation assumption, namely:

Proposition 7 *Let X be a Stein space and $K \subset X$ a holomorphically convex compact set. Let $\mathcal{F}$ be a coherent sheaf on $X \setminus K$ such that $H^1(X \setminus K, \mathcal{F})$ is separated.*

Then $\mathcal{F}$ satisfies Theorem A, that is, for every $x \in X \setminus K$, the sections of $\Gamma(X \setminus K, \mathcal{F})$ generates $\mathcal{F}_x$ over $\mathcal{O}_{X,x}$.

For the proof of this, we first prepare a few lemmata. From [6], we deduce in a straightforward way:

Lemma 5 *Let Z be complex space that is exhausted by an increasing sequence of open sets $\{Z_n\}_n$ and let $\mathcal{F}$ be a coherent analytic sheaf on Z. Suppose that for some integer $q \geq 1$ the following conditions are satisfied:*

(a) *$H^q(Z, \mathcal{F})$ is separated.*
(b) *Each restriction $H^q(Z_{n+1}, \mathcal{F}) \longrightarrow H^q(Z_n, \mathcal{F})$ is surjective and induces a bijection between the associated separated spaces.*

Then, for each $n = 1, 2, \ldots$, the topology on $H^q(Z_n, \mathcal{F})$ is separated.

The following statement is easy and is left to the reader.

Lemma 6 *Let Z be a complex space and $\mathcal{F}$ a coherent analytic sheaf on Z such that for some integer $q \geq 1$ the topology on $H^q(Z, \mathcal{F})$ is separated. Let $\mathcal{I}$ be a coherent ideal subsheaf of $\mathcal{O}_Z$ such that* $\mathrm{Supp}(\mathcal{O}_Z/\mathcal{I})$ *is a finite set. Then $H^q(Z, \mathcal{I}\mathcal{F})$ is separated.*

Lemma 7 *Let Z be a complex space which is the union of two open sets Y and U such that the pair $(Y \cap U, U)$ is Runge.*[2] *Then for each coherent analytic sheaf $\mathcal{F}$ on Z the restriction $H^1(Z, \mathcal{F}) \longrightarrow H^1(Y, \mathcal{F})$ induces a bijection between the associated separated parts. Besides, if $H^1(Z, \mathcal{F})$ is separated, then the mapping $H^0(Z, \mathcal{F}) \longrightarrow H^0(Y, \mathcal{F})$ has dense image and $H^1(Z, \mathcal{F}) \simeq H^1(Y, \mathcal{F})$.*

Proof Consider the exact portion of the Mayer-Vietoris sequence with coefficients in $\mathcal{F}$ (which we omit for practical purposes) associated to $Z = U \cup Y$,

$$H^0(Z) \longrightarrow H^0(Y) \oplus H^0(U) \longrightarrow H^0(Y \cap U) \longrightarrow H^1(Z) \longrightarrow H^1(Y) \longrightarrow 0,$$

[2] This means that U and $Y \cap U$ are Stein and $\mathcal{O}(U) \longrightarrow \mathcal{O}(Y \cap U)$ has dense range.

where we used Theorem B for vanishing of cohomology of coherent sheaves on Stein spaces. It is known that in the above diagram the canonical maps are continuous for the natural topologies.

Let $\mathcal{W} = \{W_m\}_{m=0,1,\dots}$ be a Stein open covering of Z with $W_0 = U$ and $W_m \subset Y$ for $m > 0$. Let $\mathcal{V} = \{V_m\}_m$, where $V_m := U_m \cap Y$ for $m \geq 0$. Clearly, $\mathcal{V}$ is a Stein covering of Y.

Then, since $W_k \cap W_m = V_k \cap V_m$ for $k \neq m$ and $\mathcal{F}(U) \longrightarrow \mathcal{F}(U \cap Y)$ has dense image, it results that $C^i(\mathcal{W}, \mathcal{F}) = C^i(\mathcal{V}, \mathcal{F})$ for $i > 0$ and the canonical map $C^0(\mathcal{W}, \mathcal{F}) \longrightarrow C^0(\mathcal{V}, \mathcal{F})$ has dense image. The lemma follows readily using the Čech definition of cohomology with alternate cycles.

Now, the additional statement results in the following way. Because the restriction map $H^0(U, \mathcal{F}) \longrightarrow H^0(Y \cap U, \mathcal{F})$ has dense image, it follows that the natural map $u : H^0(Y, \mathcal{F}) \oplus H^0(U, \mathcal{F}) \longrightarrow H^0(Y \cap U, \mathcal{F})$ has dense image, too. But $\operatorname{Im} u$ is the kernel of the continuous map $H^0(Y \cap U, \mathcal{F}) \longrightarrow H^1(Z, \mathcal{F})$ which is closed since $\{0\}$ is closed in $H^1(Z, \mathcal{F})$. Therefore, the map u is surjective and the proof finishes easily by diagram chasing from the following simple fact.

Let

$$0 \longrightarrow E' \xrightarrow{u} E_1 \oplus E_2 \xrightarrow{v} E'' \longrightarrow 0$$

be an exact sequence of Fréchet spaces where $u = (u_1, u_2)$, $u_1 : E' \longrightarrow E_1$, $u_2 : E' \longrightarrow E_2$, and $v = v_1 - v_2$ where $v_1 : E_1 \longrightarrow E''$, $v_2 : E_2 \longrightarrow E''$ are all continuous linear mappings. Then v_2 has dense range if, and only if, u_1 has dense range, too. □

Putting these together, we obtain in a standard way a "bumping lemma":

Proposition 8 *Let Z be a complex space and $\mathcal{F}$ a coherent sheaf on Z. Assume that Z is exhausted by an increasing sequence $\{Z_n\}_n$ of open sets such that $Z_{n+1} = Z_n \cup U_{n+1}$ and each pair $(U_{n+1}, Z_n \cap U_{n+1})$ is Runge. Then the following statements hold true:*

(a) *Each restriction $H^1(Z, \mathcal{F}) \longrightarrow H^1(Z_n, \mathcal{F})$ induces a bijection between their separated spaces.*
(b) *If $H^1(Z, \mathcal{F})$ is separated, then $H^1(Z_n, \mathcal{F})$ is separated for all n and each restriction $H^1(Z, \mathcal{F}) \longrightarrow H^1(Z_n, \mathcal{F})$ is bijective. Moreover, each mapping $H^0(Z, \mathcal{F}) \longrightarrow H^0(Z_n, \mathcal{F})$ has dense image.*

Proof of Proposition 7 The assertion results immediately from Nakayama's lemma and the following more general fact that will be proved subsequently:

($\star$) *For any coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that $\Lambda := \operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$ is discrete and does not meet K, the restriction map*

$$H^0(X \setminus K, \mathcal{F}) \longrightarrow H^0(X \setminus K, \mathcal{F}/\mathcal{I}\mathcal{F})$$

is surjective.

(For instance, one may take $\mathcal{I}$ be defined by a suitable chosen sequence $\{x_k\}$ of X.)

To start the proof, let φ be as in Lemma 1 (with $S = \emptyset$) and set $c_0 := \inf_\Lambda \varphi > 0$. Let $\{X_n\}$ be an exhaustion of X by increasing open subsets obtained by the bumping method in [2] such that $X_0 = \{\varphi < c\}$ with $c \in (0, c_0)$. For the sake of simplicity, let us adopt the following *ad-hoc* notation: For a subset T of X containing K, we denote by T' the set $T \setminus K$.

These $\{X'_n\}$ fulfill the hypothesis of Proposition 8 corresponding to X'. Besides, it is easily seen that, for all n, $\Lambda \cap X_n$ is a finite set of points (possibly the empty set).

By Proposition 8, for any n, $H^1(X'_n, \mathcal{F})$ is separated, the restriction $H^1(X'_{n+1}, \mathcal{F}) \longrightarrow H^1(X'_n, \mathcal{F})$ is bijective and $H^0(X'_{n+1}, \mathcal{F}) \longrightarrow H^0(X'_n, \mathcal{F})$ have dense images. Therefore,

$$H^1(X', \mathcal{F}) \longrightarrow H^1(X'_0, \mathcal{F})$$

is bijective.

From Lemma 6 all cohomological vector spaces $H^1(X'_n, \mathcal{IF})$ are separated. Thanks to Lemma 7, the restrictions $H^1(X'_{n+1}, \mathcal{IF}) \longrightarrow H^1(X'_n, \mathcal{IF})$ are bijective and $H^0(X'_{n+1}, \mathcal{IF}) \longrightarrow H^0(X'_n, \mathcal{IF})$ have dense images. Thus,

$$H^1(X', \mathcal{IF}) \longrightarrow H^1(X'_0, \mathcal{IF})$$

is also bijective. Consider now the following canonical commutative diagram

$$\begin{array}{ccccccc} H^0(X', \mathcal{F}) & \longrightarrow & H^0(X', \mathcal{F}/\mathcal{IF}) & \longrightarrow & H^1(X', \mathcal{IF}) & \xrightarrow{t} & H^1(X', \mathcal{F}) \\ & & & & \downarrow u & & \downarrow v \\ & & & & H^1(X'_0, \mathcal{IF}) & \xrightarrow{w} & H^1(X'_0, \mathcal{F}) \end{array}$$

where the mappings u and v are bijective by the above discussion. Because w is obviously bijective, t follows bijective, too. Hence, the restriction $H^0(X \backslash K, \mathcal{F}) \longrightarrow H^0(X \backslash K, \mathcal{F}/\mathcal{IF})$ is surjective, whence the proof of the proposition. □

5 Complements and some examples

Let X be a complex space. Let $A \subset X$ be a closed set. We say that A is *pseudoconcave* at a point $x_0 \in A$ if either x_0 is an interior point of A or else x_0 is a boundary point of A and there is a non empty open neighborhood U of x_0 such that $U \setminus A$ is Stein. Therefore, the set A^o of pseudoconcave points of A is open in A.

For instance, the compact set $K := \partial\Delta \times \overline{\Delta}$ in $\mathbb{C}^\star \times \mathbb{C}$, which is holomorphically convex in $\mathbb{C}^\star \times \mathbb{C}$, is pseudoconcave at every point of $\partial\Delta \times \Delta$. (Here, Δ is the open unit disk in $\mathbb{C}$.)

Lemma 8 *Let A be a complex hypersurface in a complex space X. Then A^o is dense in A.*

Proof The question being local, there is no loss in generality to assume that X is Stein. Thus, there is a holomorphic function f on X that vanishes on A and its zero set $Z_f := \{f = 0\}$ is a hypersurface in X. Therefore, A is a union of some irreducible components of Z_f; let B be the union of the remaining ones. Thus, B is closed in X and $A \setminus B$ is dense in A.

Now, for any point $x_0 \in A \setminus B$ and every Stein open neighborhood V of x_0 such that $V \cap B = \emptyset$, $V \setminus A$ is Stein as it equals $V \setminus \{f = 0\}$, whence $A \setminus B \subset A^o$. The proof follows. □

Proposition 9 *Let X be a complex space and $A \subset X$ a closed set. Let $\iota : X \setminus A \hookrightarrow X$ be the inclusion map. Then $\iota_\star(\mathcal{O}_{X\setminus A})$ is not of finite type at any pseudoconcave boundary point of A.*

Proof Since "pseudoconcavity" is an open property as well as "being of finite type", assume, in order to reach a contradiction that there is a point $a \in \partial A$ such that A is pseudoconcave at a and $\mathcal{H} := \iota_\star(\mathcal{O}_{X\setminus A})$ is of finite type at a. Thus, there is a Stein open neighborhood W of

a such that $W \setminus A$ is Stein and sections $\sigma_1, \ldots, \sigma_k \in \Gamma(W, \mathcal{H})$ whose germs at a generates $\mathcal{H}_a$. Let $\{x_\nu\}_\nu$ be a sequence of points in $W \setminus A$ converging to a. Because $W \setminus A$ is Stein we may choose $\sigma \in \mathcal{O}(W \setminus A)$ such that, for all ν one has:

$$|\sigma(x_\nu)| > \nu(1 + \max(|\sigma_1(x_\nu)|, \ldots, |\sigma_k(x_\nu)|)).$$

On the other hand, on a suitable open neighborhood V of a in W, there are $f_1, \ldots, f_k \in \mathcal{O}(V)$, which might be assumed bounded in modulus, say by some $C > 0$, such that $\sigma = f_1\sigma_1 + \cdots f_k\sigma_k$ on $V \setminus A$. Therefore, for ν sufficiently large one has $1 \leq kC/\nu$ which is absurd! □

Corollary 3 *Let X be a complex space and A a hypersurface in X. Let $\iota : X \setminus A \hookrightarrow X$ be the inclusion map. Then $\iota_\star(\mathcal{O}_{X\setminus A})$ is not of finite type at any point of A, a fortiori, $\iota_\star(\mathcal{O}_{X\setminus A})$ is not coherent.*

Proof This follows immediately from the above lemma and proposition. □

Proposition 10 *Let X be a Stein space and $K \subset X$ a holomorphically convex compact set. Let $A \subset X \setminus K$ be a discrete set and $\mathcal{I}_A$ its ideal sheaf, which is coherent on $X \setminus K$. Then $H^1(X \setminus K, \mathcal{I}_A)$ is separated, if, and only if, A has no accumulation point in K.*

Proof Let $\varphi : X \longrightarrow [0, \infty)$ be smooth, proper such that $\{\varphi = 0\} = K$ and φ is strictly plurisubharmonic on $X \setminus K$. It is straightforward to find a sequence $\{\epsilon_\nu\}_\nu$ that strictly decreases to 0 such that $A \cap \{\varphi = \epsilon_\nu\} = \emptyset$. Thus, setting $X_\nu := \{\varphi > \epsilon_\nu\}$ it follows that each $H^1(X_\nu, \mathcal{I}_A)$ is separated. Since $\{X_\nu\}_\nu$ increases to X, granting [6], $H^1(X, \mathcal{I}_A)$ is separated if, and only if, the system $\{H^0(X_\nu, \mathcal{I}_A)\}_\nu$ satisfies a condition of Mittag–Leffler type and this is readily shown to be equivalent to the fact that A has no accumulation point in K. □

Let, we also note the following statement due to Markoe.

Proposition 11 *Let X be a complex space and $\mathcal{F}$ a coherent analytic sheaf on X such that $H^1(X, \mathcal{F})$ is separated and $H^0(X, \mathcal{F})$ has finite dimension. Then, for any coherent subsheaf $\mathcal{F}'$ of $\mathcal{F}$, $H^1(X, \mathcal{F}')$ is separated.*

The proof results using a Stein open covering of X and the simple functional analysis fact saying that if $u : E \longrightarrow F$ is a continuous surjective morphism of Fréchet spaces, then, for every closed subspace $E' \subset E$, $u(E')$ is closed in F if, and only if $E' + \operatorname{Ker} u$ is closed in E.

For instance, this can be applied for $X = Y \setminus K$, where Y is a compact (connected) normal complex space of dimension ≥ 2 and $K = \{\psi \leq 0\}$, where $\psi : W \longrightarrow \mathbb{R}$ (W an open neighborhood of K) is continuous on W and strictly plurisubharmonic on $W \setminus K$. Then, for any coherent subsheaf $\mathcal{I}$ of $\mathcal{O}_X$, $H^1(X, \mathcal{I})$ is separated. In particular, for any coherent subsheaf $\mathcal{I}$ of $\mathcal{O}_M$, where M is the complex manifold of regular points of Y, $H^1(M, \mathcal{I})$ is separated.

Corollary 4 *Let A be a complex submanifold of $\mathbb{P}^n$ of pure codimension ≥ 2. Let $\mathcal{F}$ be a locally free sheaf on $\mathbb{P}^n \setminus A$ of finite rank. Then, for every coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$, $H^1(\mathbb{P}^n \setminus A, \mathcal{F}')$ is separated.*

Corollary 5 *Let X be a pseudoconcave space and $\mathcal{F}$ a torsion free-coherent sheaf on X such that $H^1(X, \mathcal{F}')$ is separated. Then, for every coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$, $\mathcal{H}^\infty(\mathcal{X}, \mathcal{F}')$ is separated.*

Note. Recall that a complex space X is said to be *pseudoconcave* in the sense of Andreotti [1] if there is a relatively compact open subset Y of X such that every point $x_0 \in \partial Y$ admits a neighborhood system of open sets $\{U_\nu\}_\nu$ such that x_0 is an interior point of the holomorphically convex hull of $U_\nu \cap Y$ with respect to $\mathcal{O}(U_\nu)$. It is shown in [1] that if X is pseudoconcave and $\mathcal{F}$ a torsion-free coherent sheaf on X, then $H^0(X, \mathcal{F})$ has finite dimension.

6 The proofs of theorem 1 and proposition 1

Proof of Theorem 1, the Stein case Let K be a holomorphically convex compact set in a Stein space X and $\mathcal{F}$ a coherent analytic sheaf on $X \setminus K$ such that $\operatorname{prof} \mathcal{F} \geq 2$ and $H^1(X \setminus K, \mathcal{F})$ is separated. Since the extension is a question around K, there is no loss in generality to consider X embedded as a closed analytic subset of some complex number space $\mathbb{C}^n$ $(n \geq 3)$. Let $\iota : X \hookrightarrow \mathbb{C}^n$ be the analytic embedding. Clearly, $\iota_\star(\mathcal{F})$ is coherent on $\mathbb{C}^n$, $\operatorname{prof} \iota_\star\mathcal{F} = \operatorname{prof} \mathcal{F}$, K stays holomorphically convex in $\mathbb{C}^n$, and $H^\bullet(\mathbb{C}^n \setminus K, \iota_\star\mathcal{F}) = H^\bullet(X \setminus K, \mathcal{F})$. Therefore, there is no loss in generality to take $X = \mathbb{C}^n$ with $n \geq 3$. From Proposition 7, $\mathcal{F}$ satisfies Theorem A on $X \setminus K$.

Let $\varphi : X \longrightarrow [0, \infty)$ proper, smooth, plurisubharmonic on X and strictly plurisubharmonic on $X \setminus K$ such that $\{\varphi = 0\} = K$. Let $L = \{\varphi \leq \epsilon\}$ $(\epsilon > 0)$ be a compact neighborhood of K and $c > 0$ such that $L \subset D := \{\varphi < c\}$. Then $D \setminus L$ is relatively compact in $X \setminus K$ and there are sections $s_1, \ldots, s_q \in \Gamma(X \setminus K, \mathcal{F})$ that generates $\mathcal{F}$ on $D \setminus L$; in other words, we have a morphisms of sheaves $\mathcal{O}_X^q \longrightarrow \mathcal{F}$ that is surjective on $D \setminus L$. Let $\mathcal{G}$ be the kernel of the above $\mathcal{O}_{D\setminus L}$-morphism. One has $\operatorname{prof}(\mathcal{G}) \geq 3$. (For the exact sequence, $0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}^q \longrightarrow \mathcal{F} \longrightarrow 0$ implies that for every $x \in D \setminus L$ either $\mathcal{G}_x$ is free so that $\operatorname{prof} \mathcal{G}_x = n$, or $\operatorname{prof} \mathcal{G}_x = 1 + \operatorname{prof} \mathcal{F}_x$. In any case, we deduce $\operatorname{prof} \mathcal{G} \geq 3$.)

Because $H^1(D \setminus L, \mathcal{G})$ has finite dimension from Proposition 5, the exact sequence

$$H^0(D \setminus L, \mathcal{O}^q) \longrightarrow H^0(D \setminus L, \mathcal{F}) \longrightarrow H^1(D \setminus L, \mathcal{G}) \longrightarrow H^1(D \setminus L, \mathcal{O}^q)$$

and $H^1(D \setminus L, \mathcal{O}^q) = 0$ implies that $H^0(D \setminus L, \mathcal{F})$ regarded as $H^0(D \setminus L, \mathcal{O})$-module is of finite type. But since $H^0(D \setminus K, \mathcal{F}) \simeq H^0(D \setminus L, \mathcal{F})$ and $H^0(D \setminus K, \mathcal{O}) \simeq H^0(D \setminus L, \mathcal{O})$ there are sections $\sigma_1, \ldots, \sigma_M \in \Gamma(D \setminus K, \mathcal{F})$ which generates $H^0(D \setminus K, \mathcal{F})$ as an $H^0(D \setminus K, \mathcal{O})$-module. Because Theorem A is true for $\mathcal{F}$ on $D \setminus K$, we get a surjective morphism $\mu : \mathcal{O}^M \longrightarrow \mathcal{F}$ on $D \setminus K$. For $\mathcal{K}$, the kernel of μ, we apply again the above discussion (possibly, we shrink D a little bit, as a matter of fact, one may choose a new D given as $\{\varphi < c'\}$ with $c' < c$ close to c. Thus, we obtain an exact sequence

$$\mathcal{O}^N \longrightarrow \mathcal{O}^M \longrightarrow \mathcal{F} \longrightarrow 0$$

on $D \setminus K$. The matrix giving the $\mathcal{O}$-morphism $\mathcal{O}^N \longrightarrow \mathcal{O}^M$ has as entries holomorphic functions on $D \setminus K$ so that they extend holomorphically D and therefore it gives a morphism $\mathcal{O}^N \longrightarrow \mathcal{O}^M$ over D. Its cokernel is the desired extension $\mathcal{H}$ to D. Furthermore, since the set $\{x \in D\, ;\ \operatorname{prof} \mathcal{H}_x \leq 1\}$ is analytic in D and contained in K, thus compact, it is a finite set, say Λ. Finally, we glue $\iota_\star(\mathcal{H}|_{D\setminus\Lambda})$ with $\mathcal{F}$ on $D \setminus K$ and get a coherent extension $\widehat{\mathcal{F}}$ of $\mathcal{F}$ with $\operatorname{prof} \widehat{\mathcal{F}} \geq 2$. □

Remark 4 From Proposition 2, it results that there is a unique (up to an isomorphism) coherent extension $\widehat{\mathcal{F}}$ of $\mathcal{F}$ with $\operatorname{prof} \widehat{\mathcal{F}} \geq 2$. Furthermore, in the same vein, we can show the following:

Let X a Stein space, $K \subset X$ a Stein compact set and $\mathcal{F}$ a coherent sheaf on $X \setminus K$ with $\operatorname{prof} \mathcal{F} \geq 2$ such that, for every Stein open neighborhood W of K, $H^1(W \setminus K, \mathcal{F})$ is separated. Then $\mathcal{F}$ admits a coherent extension to X.

Proof of Theorem 1, the 1-convex case First, we show the "if" part, which amounts to complement [4] by proving:

Proposition 12 *Let X be a 1-convex space and $K \subset X$ a holomorphically convex compact set. Then for any coherent sheaf $\mathcal{H}$ on X, the cohomology groups $H^i(X \setminus K, \mathcal{H})$, $i \geq 0$, are separated.*

Indeed, since $\Gamma(X \setminus K, \mathcal{H})$ is separated, it remains to check the assertion for $H^i(X \setminus K, \mathcal{H})$ with $i \geq 1$. Let S be the exceptional set of X and decompose S as union $S' \cup S''$ into disjoint analytic sets such that $S'' \subset K$ and $S' \cap K = \emptyset$. From a standard long exact cohomology sequence, we retain the exact portion

$$H^i(X, \mathcal{H}) \longrightarrow H^i(X \setminus K, \mathcal{H}) \longrightarrow H^{i+1}_K(X, \mathcal{H}) \longrightarrow H^{i+1}(X, \mathcal{H}) \qquad (\ddagger)$$

in which the extremes are complex vector spaces of finite dimension.

Let $\rho : X \longrightarrow Y$ be the Remmert's reduction. Then $L := \rho(K)$ is holomorphically convex in Y. Consider L defined by a function ψ as in Lemma 1 and take $V = \{\psi < \epsilon\}$ for $\epsilon > 0$ small enough such that $V \cap \rho(S') = \emptyset$. Let $U := \rho^{-1}(V)$. Now write the exact sequence as in ($\ddagger$) this time for U; it follows using excision and 1-convexity that the canonical restriction map

$$H^i(X \setminus K, \mathcal{H}) \longrightarrow H^i(U \setminus K, \mathcal{H}),$$

which is linear and continuous, has finite dimensional kernel and cokernel. On the other hand, as $\rho_\star(\mathcal{H})$ is coherent on Y, thanks to Bănică [4] and the isomorphism $H^i(U \setminus K, \mathcal{H}) \longrightarrow H^i(V \setminus L, \rho_\star(\mathcal{H}))$ it follows that $H^i(X \setminus K, \mathcal{H})$ is Fréchet by using the following simple functional analysis fact:

Let $u : E \longrightarrow F$ be a continuous linear map between QF-spaces such that $\operatorname{Ker} u$ and $\operatorname{Coker} u$ are of finite dimension. Then E is separated provided that F is separated.

Lemma 9 *Let $\pi : X \longrightarrow Y$ be a holomorphic map of complex spaces X and Y that contracts an analytic subset A of X onto a Stein analytic subset B of Y (e.g. B a finite set of points). Let $\mathcal{F}$ be a coherent analytic sheaf on X. Then the natural map $H^1(Y, \pi_\star(\mathcal{F})) \longrightarrow H^1(X, \mathcal{F})$ is injective. Moreover, if $H^1(X, \mathcal{F})$ is separated, then $H^1(Y, \pi_\star(\mathcal{F}))$ is separated, too.*

Proof Let V be a Stein open neighborhood of B and $W := \pi^{-1}(V)$. The Mayer–Vietoris sequence induces a commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^0(Y \setminus B) \oplus H^0(V) & \longrightarrow & H^0(V \setminus A) & \longrightarrow & H^1(Y) & \longrightarrow & H^1(Y \setminus B) \oplus \{0\} \\
\downarrow u & & \downarrow v & & \downarrow w & & \downarrow \delta \\
H^0(X \setminus A) \oplus H^0(W) & \longrightarrow & H^0(W \setminus A) & \longrightarrow & H^1(X) & \longrightarrow & H^1(X \setminus A) \oplus H^1(W)
\end{array}$$

where, the first row has coefficients in $\pi_\star \mathcal{F}$ and the second in $\mathcal{F}$. Since u is surjective and v and δ are injective, the five lemma shows that w is injective. Finally, as w is continuous, the additional statements results, too. $\square$

To conclude the proof of Theorem 1 in the 1-convex case, keeping the notations from above, because $L := \rho(K)$ is holomorphically convex in Y and granting Lemma 9, using that $H^1(X \setminus K, \mathcal{F})$ is separated, it follows that $H^1(Y \setminus L, \rho_\star(\mathcal{F}))$ is separated, too. Consider L defined by a function ψ as in Lemma 1 and take $V = \{\psi < \epsilon\}$ for $\epsilon > 0$ small enough such that $V \cap \rho(S') = \emptyset$. It follows now that $H^1(V \setminus L, \rho_\star(\mathcal{F}))$ is separated and prof $\rho_\star(\mathcal{F}|_{V\setminus L}) \geq 2$ so that there is a coherent extension $\mathcal{B}$ on V of $\rho_\star(\mathcal{F}|_{V\setminus L})$. Then $\rho^\star(\mathcal{B})$ can be glued with $\mathcal{F}$ on $\rho^{-1}(V \setminus K)$ to produced the desired coherent extension of $\mathcal{F}$. □

Proof of Proposition 1 We follow an idea from Bănică and Stănaşilă [5]. Let $\Sigma := \{x \in X \setminus K\,;\ \text{prof}\,\mathcal{F}_x = 2\}$. This set is discrete in $X \setminus K$ and prof $\mathcal{F} \geq 2$ on $X \setminus K$. Then one produces in a standard way a sequence $\{\Omega_\nu\}_\nu$ of Stein open subsets of X and compact neighborhoods K_ν of K in Ω_ν with the following properties:

(1) $\Omega_{\nu+1} \Subset \Omega_\nu$ and $\{\Omega\}_\nu$ decreases to K,
(2) K_ν is holomorphically convex in Ω_ν and
(3) $(\Omega_\nu \setminus K_\nu) \cap \Sigma = \emptyset$.

It follows that there is a coherent sheaf $\mathcal{F}_\nu$ on X such that

$$\mathcal{F}_\nu|_{X\setminus K_\nu} \simeq \mathcal{F}|_{X\setminus K_\nu} \qquad (\star)$$

and prof $\mathcal{F}_\nu \geq 2$. For each couple (i, j) and any holomorphically convex compact set L in X containing $\widehat{K}_i \cup \widehat{K}_j$, we obtain from the above fact an isomorphism

$$\xi_i^{-1}\xi_j : \mathcal{F}_j|_{X\setminus L} \simeq \mathcal{F}_i|_{X\setminus L}$$

which, thanks to Proposition 2, induces an isomorphism

$$\xi_{ij} : \mathcal{F}_j \simeq \mathcal{F}_i.$$

Because the bijections in ($\star$) are functorial, it is easily shown that ξ_{ij} does not depend on L and for each i and each triple (i, j, k), we have the relations $\xi_{ii} = \text{id}$ and $\xi_{ij}\xi_{jk}\xi_{ki} = \text{id}$. We, then, obtain the searched sheaf $\widetilde{\mathcal{F}}$. □

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