Annali di Matematica (2012) 191:431–457
DOI 10.1007/s10231-011-0189-y

Self-adjoint curl operators

Ralf Hiptmair · Peter Robert Kotiuga · Sébastien Tordeux

Received: 10 July 2009 / Accepted: 1 February 2011 / Published online: 25 February 2011

Abstract We study the exterior derivative as a symmetric unbounded operator on square integrable 1-forms on a 3D bounded domain D. We aim to identify boundary conditions that render this operator self-adjoint. By the symplectic version of the Glazman-Krein-Naimark theorem, this amounts to identifying complete Lagrangian subspaces of the trace space of $\boldsymbol{H}(\mathbf{curl}, D)$ equipped with a symplectic pairing arising from the $\wedge$-product of 1-forms on ∂D. Substantially generalizing earlier results, we characterize Lagrangian subspaces associated with closed and co-closed traces. In the case of non-trivial topology of the domain, different contributions from co-homology spaces also distinguish different self-adjoint extensions. Finally, all self-adjoint extensions discussed in the paper are shown to possess a discrete point spectrum, and their relationship with **curl curl**-operators is discussed.

Keywords **curl** operators · Self-adjoint extension · Complex symplectic space · Glazman-Krein-Naimark theorem · Co-homology spaces · Spectral properties of **curl**

Mathematics Subject Classification (2000) 47F05 · 46N20

R. Hiptmair (✉)
Seminar for Applied Mathematics, ETH Zurich,
8092 Zurich, Switzerland
e-mail: hiptmair@sam.math.ethz.ch

P. R. Kotiuga
Department of Electrical and Computer Engineering,
Boston University, Boston, MA, USA
e-mail: prk@bu.edu

S. Tordeux
Institut de Mathématiques de Toulouse, INSA-Toulouse,
Toulouse, France
e-mail: sebastien.tordeux@insa-toulouse.fr

List of symbols

$C^\infty(D)$	Infinite differentiable functions on D, $\mathbf{C}^\infty(D) = (C^\infty(D))^3$
$C_0^\infty(D)$	Compactly supported functions in $C^\infty(D)$, $\mathbf{C}_0^\infty(D) = (C_0^\infty(D))^3$
curl_∂	Scalar valued surface rotation
d	Exterior derivative of differential forms
∂M	Boundary of M
$\mathcal{D}(\mathsf{T})$	Domain of definition of the linear operator T
D	Bounded (open) Lipschitz domain in affine space $\mathbb{R}^3$
D'	Closure of the complement of D, $D' := \mathbb{R}^3 \backslash \bar{D}$
div_∂	Surface divergence
$\mathbf{grad}_\partial$	Surface gradient
$\boldsymbol{H}(\mathbf{curl}, D)$	Real Hilbert space $\{\mathbf{v} \in \boldsymbol{L}^2(D) : \mathbf{curl}\,\mathbf{v} \in \boldsymbol{L}^2(D)\}$ with graph norm
$\boldsymbol{H}_0(\mathbf{curl}, D)$	Closure of $\mathbf{C}_0^\infty(D)$ in $\boldsymbol{H}(\mathbf{curl}, D)$
$H^{\frac{1}{2}}(\partial D)$	Trace space of $H^1(D) := \{u \in L^2(D) : \nabla u \in L^2(D)\}$
$\mathbf{H}_{\mathbf{t}}^{-\frac{1}{2}}(\mathrm{curl}_\partial, \partial D)$	Tangential traces of vector fields in $\boldsymbol{H}(\mathbf{curl}, D)$
$\mathbf{H}_{\mathbf{t}}^s(\partial D), \mathbf{L}_{\mathbf{t}}^2(\partial D)$	Tangential trace spaces
$H^{\frac{3}{2}}(\partial D)$	See (5.3)
$\mathcal{HF}^k(\mathsf{d}, D)$	Square integrable k-forms with square integrable exterior derivative
$\mathcal{HF}_0^k(\mathsf{d}, D)$	Completion of compactly supported k-forms in $\mathcal{HF}^k(\mathsf{d}, D)$
$\mathcal{HF}^{-\frac{1}{2},k}(\mathsf{d}, \partial D)$	Trace space of $\mathcal{HF}^k(\mathsf{d}, D)$
$\mathcal{HZ}^{-\frac{1}{2},k}(\partial D)$	Closed k-forms in $\mathcal{HF}^{-\frac{1}{2},1}(\mathsf{d}, \partial D)$, see for instance (6.1)
$\mathcal{HF}^{\frac{3}{2},0}(\partial D)$	See (5.5)
$\mathcal{H}^1(\partial D)$	Co-homology space of harmonic 1-forms on ∂D
i^*	Natural trace operator for differential forms
$L^2(D)$	Real square integrable functions on D, $\boldsymbol{L}^2(D) = (L^2(D))^3$
$L^\sharp$	Symplectic orthogonal of subspace L of a symplectic space
$L^2(\Lambda^k(M))$	Hilbert space of square integrable k-forms on M
$\mathbf{n}$	Exterior unit normal vector field on ∂D
S_i, S_i'	Inside and outside cuts of D, see Sect. 6.3
$\mathcal{N}(\mathsf{T})$	Kernel (null space) of linear operator T
$\mathcal{R}(\mathsf{T})$	Range space of a linear operator T
T, T^*	An (unbounded) linear operator and its adjoint
$\mathsf{T}_{min}, \mathsf{T}_s, \mathsf{T}_{max}$	Min, self-adj. and max closures of a symmetric operator T
$\mathbf{u}, \mathbf{v}, \ldots$	Vector fields on a three-dimensional domain or elements of trace space of vector proxies
γ_t, γ_n	Tangential and normal boundary traces of a vector field
$\omega, \eta, \ldots$	Differential forms
$\omega^0, \omega^\perp$	Components of the Hodge decomposition of $\omega \in \mathcal{HF}^{-\frac{1}{2},1}(\mathsf{d}, \partial D)$
$\langle\cdot\rangle$	(Relative) Homology class of a cycle
$\wedge$	Exterior product of differential forms
$\star\ (\star_g)$	Hodge operator (induced by metric g)
$\cdot$	Euclidean inner product in $\mathbb{R}^3$
$\times$	Cross product of vectors $\in \mathbb{R}^3$
$(\cdot, \cdot)$	Inner product: for $\omega \in L^2(\Lambda^k(M))$, $(\omega, \omega)_{k,M} = \int_M \omega \wedge \star\omega$
$\Vert\cdot\Vert$	Norm: for $\omega \in L^2(\Lambda^k(M))$, $\Vert\omega\Vert_{k,M}^2 := (\omega, \omega)_{k,M}$
$[\cdot, \cdot]$	Symplectic pairing: for 1-forms on 2-manifold M, $[\omega, \eta]_M = \int_M \omega \wedge \eta$
$[\cdot]_\Gamma$	Jump of trace of a function across 2-manifold Γ

1 Introduction

The **curl** operator is pervasive in field models, in particular in electromagnetics, but hardly ever occurs in isolation. Most often, we encounter a **curl curl** operator, and its properties are starkly different from those of the **curl** alone. We devote the final section of this article to investigation of their relationship.

The notable exception, starring a sovereign **curl**, is the question of stable force-free magnetic fields in plasma physics [3,13,24]. They are solutions of the eigenvalue problem

$$\alpha \in \mathbb{R}\backslash\{0\} : \mathbf{curl}\,\mathbf{H} = \alpha \mathbf{H}, \tag{1.1}$$

posed on a suitable domain, see [25,12,21,32]. A solution theory for (1.1) must scrutinize the spectral properties of the **curl** operator. The mature theory of unbounded operators in Hilbert spaces is a powerful tool. For the **curl** operator, this approach was pioneered by R. Picard [30,33,34], see also [40].

The main thrust of research was to convert **curl** into a self-adjoint operator by a suitable choice of domains of definition. This is suggested by the following Green's formula for the **curl** operator:

$$\int_D \mathbf{curl}\,\mathbf{u} \cdot \mathbf{v} - \mathbf{curl}\,\mathbf{v} \cdot \mathbf{u}\,\mathrm{d}\mathbf{x} = \int_{\partial D} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{n}\,\mathrm{d}S, \tag{1.2}$$

for any domain $D \subset \mathbb{R}^3$ with sufficiently regular boundary ∂D and $\mathbf{u}, \mathbf{v} \in C^1(\overline{D})$. This reveals that the **curl** operator is truly symmetric, for instance, when acting on vector fields with vanishing tangential components on ∂D.

On bounded domains D, several instances of what qualifies as a self-adjoint **curl** operators were found. Invariably, their domains were defined through judiciously chosen boundary conditions. It also became clear that the topological properties of D have to be taken into account carefully, see [33, Thm. 2.4] and [40, Sect. 4].

In this paper, we carry these developments further with quite a few novel twists: we try to give a rather systematic treatment of different options to obtain self-adjoint **curl** operators. It is known that the **curl** operator is an incarnation of the exterior derivative of 1-forms. Thus, to elucidate structure, we will mainly adopt the perspective of differential forms.

Further, we base our considerations on recent discoveries linking symplectic algebra and self-adjoint extensions of symmetric operators, see [17] for a survey. In the context of ordinary differential equations, this connection was intensively studied by Markus and Everitt during the past few years [15]. They also extended their investigations to partial differential operators like Δ [16]. We are going to apply these powerful tools to the special case of **curl** operators. Here, the crucial symplectic space is a Hilbert space of 1-forms on ∂D equipped with the pairing

$$[\omega, \eta]_{\partial D} := \int_{\partial D} \omega \wedge \eta.$$

We find out that it is the Hodge decomposition of the trace space for 1-forms on D that allows a classification of self-adjoint extensions of **curl**: the main distinction is between boundary conditions that impose closed and co-closed traces. Moreover, further constraints are necessary in the form of vanishing circulation along certain fundamental cycles of ∂D. This emerges from an analysis of the space of harmonic 1-forms on ∂D as a finite-dimensional symplectic space. For all these self-adjoint **curl** operators, we show that they possess a complete orthonormal system of eigenfunctions.

The plan of the article is as follows: the next section reviews the connection between vector analysis and differential forms in 2D and 3D with an emphasis on coordinate-free aspects that are best described in terms of differential forms. Then, in the third section, we summarize how self-adjoint extensions of a symmetric operator can be identified with complete Lagrangian subspaces of a symplectic form. The fourth section specializes these results to the **curl** operator. It is proved that every self-adjoint **curl** operator can be associated with a boundary condition defined by a Lagrangian subspace. The differential form point of view reveals the invariance of the underlying symplectic form under pullback. The following section describes important complete Lagrangian subspaces spawned by the Hodge decomposition of 1-forms on surfaces. The role of co-homology spaces comes under scrutiny. In the sixth section, we elaborate on concrete boundary conditions for self-adjoint **curl** operators induced by the complete Lagrangian subspaces discussed before. A practical prescription is presented for handling global aspects of boundary conditions imposed by co-homology groups. The two final sections study the spectral properties of the classes of self-adjoint **curl** operators examined before and explore their relationships with **curl curl** operators. We show that the computation of eigenvectors of **curl** operators usually cannot rely on the computation of eigenvectors of self-adjoint **curl curl** operators.

2 The curl operator and differential forms

In classical vector analysis, the operator **curl** is introduced as first-order partial differential operator acting on vector fields with three components. Thus, given a domain $D \subset \mathbb{R}^3$, we may formally consider $\mathbf{curl} : \mathbf{C}_0^\infty(D) \mapsto \mathbf{C}_0^\infty(D)$ as an unbounded operator on $\boldsymbol{L}^2(D)$. Integration by parts according to (1.2) shows that this basic **curl** operator is symmetric, hence closable [39, Ch. 5]. Its closure is given by the *minimal curl operator*

$$\mathbf{curl}_{\min} : \boldsymbol{H}_0(\mathbf{curl}, D) \mapsto \boldsymbol{L}^2(D). \tag{2.1}$$

Its adjoint is the *maximal curl operator*, see [33, Sect. 0],

$$\mathbf{curl}_{\max} := \mathbf{curl}_{\min}^* : \boldsymbol{H}(\mathbf{curl}, D) \mapsto \boldsymbol{L}^2(D). \tag{2.2}$$

Note that $\mathbf{curl}_{\max}$ is no longer symmetric, and neither operator is self-adjoint. This motivates the search for self-adjoint extensions $\mathbf{curl}_s : \mathcal{D}(\mathbf{curl}_s) \subset \boldsymbol{L}^2(D) \mapsto \boldsymbol{L}^2(D)$ of $\mathbf{curl}_{\min}$. If they exist, they will satisfy, *c.f.* [17, Example 1.13],

$$\mathbf{curl}_{\min} \subset \mathbf{curl}_s \subset \mathbf{curl}_{\max}\,. \tag{2.3}$$

Remark 1 The classical route in the study of self-adjoint extensions of symmetric operators is via the famous Stone-von Neumann extension theory, see [39, Ch. 6]. It suggests that, after complexification, we examine the deficiency spaces ($\mathcal{N}$ stands for the null space of a linear operator) $N^\pm := \mathcal{N}(\mathbf{curl}_{\max} \pm \imath \cdot \mathsf{Id}) \subset \mathcal{D}(\mathbf{curl}_{\max})$. As $\dim N^\pm = \infty$, the spaces $N^\pm$ reveal little about the structure governing self-adjoint extensions of **curl**. Yet, the relationship of **curl** and differential forms suggests that there is rich structure underlying self-adjoint extensions of $\mathbf{curl}_{\min}$.

The **curl** operator owes its significance to its close link with the exterior derivative operator in the calculus of differential forms. Let us write $\Lambda^k(M)$ for the space of ("sufficiently smooth") k-forms on the manifold M with boundary ∂M. On any piecewise smooth orientable k-dimensional sub-manifold of M, we can evaluate the *integral* $\int_\Sigma \omega$ of a k-form ω over a k-dimensional sub-manifold Σ of M. Also recall the concepts of trace $i^* : \Lambda^l(M) \mapsto$

Table 1 The standard choice of vector proxy u, $\mathbf{u}$ for a differential form ω in $\mathbb{R}^3$

Differential form ω	Related function u or vector field $\mathbf{u}$
$\boldsymbol{x} \mapsto \omega(\boldsymbol{x})$	$u(\boldsymbol{x}) := \omega(\boldsymbol{x})$
$\boldsymbol{x} \mapsto \{\mathbf{v} \mapsto \omega(\boldsymbol{x})(\mathbf{v})\}$	$\mathbf{u}(\boldsymbol{x}) \cdot \mathbf{v} := \omega(\boldsymbol{x})(\mathbf{v})$
$\boldsymbol{x} \mapsto \{(\mathbf{v}_1, \mathbf{v}_2) \mapsto \omega(\boldsymbol{x})(\mathbf{v}_1, \mathbf{v}_2)\}$	$\mathbf{u}(\boldsymbol{x}) \cdot (\mathbf{v}_1 \times \mathbf{v}_2) := \omega(\boldsymbol{x})(\mathbf{v}_1, \mathbf{v}_2)$
$\boldsymbol{x} \mapsto \{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \mapsto \omega(\boldsymbol{x})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\}$	$u(\mathbf{x}) \det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) := \omega(\boldsymbol{x})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$

Here, $\cdot$ denotes the Euclidean inner product of vectors in $\mathbb{R}^3$, whereas $\times$ designates the cross product

$\Lambda^l(\partial M)$, exterior product ($\wedge$-product) $\wedge : \Lambda^k(M) \times \Lambda^j(M) \mapsto \Lambda^{k+j}(M)$, exterior derivative $\mathsf{d} : \Lambda^k(M) \to \Lambda^{k+1}(M)$, Stokes' theorem $\int_\Sigma \mathsf{d}\omega = \int_{\partial\Sigma} \omega$, $\omega \in \Lambda^{k-1}(M)$, and the product rule, which gives rise to the most general integration by parts formula. For more details refer to the large body of introductory literature on differential forms, e.g. [11].

A metric g defined on the manifold M permits us to introduce the *Hodge operator* $\star_g : \Lambda^k(M) \mapsto \Lambda^{m-k}(M)$. It gives rise to the inner product on $\Lambda^k(M)$

$$(\omega, \eta)_{k,M} := \int_M \omega \wedge *_g \eta, \quad \omega, \eta \in \Lambda^k(M). \tag{2.4}$$

Thus, we obtain an L^2-type norm $\|\cdot\|$ on $\Lambda^k(M)$. Completion of smooth k-forms with respect to this norm yields the Hilbert space $L^2(\Lambda^k(M))$ of square integrable (w.r.t. g) k-forms on M. Its elements are equivalence classes of k-forms defined almost everywhere on M. Since Lipschitz manifolds possess a tangent space almost everywhere, for them $L^2(\Lambda^k(M))$ remains meaningful. As straightforward is the introduction of "Sobolev spaces" of differential forms, see [2, Sect. 1],

$$H\!F^k(\mathsf{d}, M) := \{\omega \in L^2(\Lambda^k(M)) : \; \mathsf{d}\omega \in L^2(\Lambda^{k+1}(M))\}, \tag{2.5}$$

which are Hilbert spaces with the graph norm. The completion of the subset of smooth k-forms with compact support in $H\!F^k(\mathsf{d}, M)$ is denoted by $H\!F^k_0(\mathsf{d}, D)$.

Let us zero in on the three-dimensional "manifold" D. Choosing bases for the spaces of alternating k-multilinear forms, differential k-forms can be identified with vector fields with $\binom{3}{k}$ components, their so-called vector proxies [2, Sect. 1]. The usual association of "Euclidean vector proxies" in three-dimensional space is summarized in Table 1. The terminology honors the fact that the Hodge operators $\star : \Lambda^1(D) \mapsto \Lambda^2(D)$ and $\star : \Lambda^0(D) \mapsto \Lambda^3(D)$ connected with the Euclidean metric of 3-space leave the vector proxies invariant (this is not true in 2D since $\star^2 = -1$ on 1-forms). In addition, the exterior product of forms is converted into the pointwise Euclidean inner product of vector fields. Thus, the inner product $(\cdot, \cdot)_{k,D}$ of k-forms on D becomes the conventional $L^2(D)$ inner product of the vector proxies. Further, the spaces $H\!F^k(\mathsf{d}, D)$ boil down to the standard Sobolev spaces $H^1(D)$ (for $k = 0$), $\boldsymbol{H}(\mathbf{curl}, D)$ (for $k = 1$), $\boldsymbol{H}(\mathrm{div}, D)$ (for $k = 2$), and $L^2(D)$ (for $k = 3$).

Using Euclidean vector proxies, the **curl** operator turns out to be an incarnation of the *exterior derivative* for 1-forms. Please note that since the Hodge operator is invisible on the vector proxy side, **curl** can as well stand for the operator

$$\mathbf{curl} \quad \longleftrightarrow \quad \star\mathsf{d} : \Lambda^1(D) \mapsto \Lambda^1(D), \tag{2.6}$$

which is naturally viewed as an unbounded operator on $L^2(\Lambda^1(D))$. Thus, (2.6) puts the formal **curl** operator introduced above in the framework of differential forms on D. Translated into the language of differential forms, the Green's formula (1.2) can be stated as

Table 2 Euclidean vector proxies for differential forms on ∂D

Differential forms	Related function u or vector field $\mathbf{u}$
$\boldsymbol{x} \mapsto \omega(\boldsymbol{x})$	$u(\boldsymbol{x}) := \omega(\boldsymbol{x})$
$\boldsymbol{x} \mapsto \{\mathbf{v} \mapsto \omega(\boldsymbol{x})(\mathbf{v})\}$	$\mathbf{u}(\boldsymbol{x}) \cdot \mathbf{v} := \omega(\boldsymbol{x})(\mathbf{v})$
$\boldsymbol{x} \mapsto \{(\mathbf{v}_1, \mathbf{v}_2) \mapsto \omega(\boldsymbol{x})(\mathbf{v}_1, \mathbf{v}_2)\}$	$u(\boldsymbol{x})\ \det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{n}(\boldsymbol{x})) := \omega(\boldsymbol{x})(\mathbf{v}_1, \mathbf{v}_2)$

Note that the test vectors $\mathbf{v}$, $\mathbf{v}_1$, $\mathbf{v}_2$ have to be chosen from the tangent space $T_{\boldsymbol{x}}(\partial D)$

$$(\star\, \mathsf{d}\,\omega, \eta)_{1,D} - (\omega, \star\, \mathsf{d}\,\eta)_{1,D} = \int_{\partial D} i^*\omega \wedge i^*\eta, \quad \omega, \eta \in \mathit{HF}^1(\mathsf{d}, D). \tag{2.7}$$

A metric on $\mathbb{R}^3$ induces a metric on the embedded 2-dimensional manifold ∂D. Thus, the Euclidean inner product on local tangent spaces becomes a meaningful concept, and Euclidean vector proxies for k-forms on ∂D, $k = 0, 1, 2$, can be defined as in Table 1, see Table 2.

This choice of vector proxies leads to convenient vector analytic expressions for the trace operator i^*:

$$\begin{cases} \omega \in \Lambda^0(D): & i^*\omega \longleftrightarrow \gamma u(\boldsymbol{x}) := u(\boldsymbol{x}), & u : D \mapsto \mathbb{R}, \\ \omega \in \Lambda^1(D): & i^*\omega \longleftrightarrow \gamma_t \mathbf{u}(\boldsymbol{x}) := \mathbf{u}(\boldsymbol{x}) - (\mathbf{u}(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}))\mathbf{n}(\boldsymbol{x}), & \mathbf{u} : D \mapsto \mathbb{R}^3, \\ \omega \in \Lambda^2(D): & i^*\omega \longleftrightarrow \gamma_n \mathbf{u}(\boldsymbol{x}) := \mathbf{u}(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}), & \mathbf{u} : D \mapsto \mathbb{R}^3, \\ \omega \in \Lambda^3(D): & i^*\omega \longleftrightarrow 0, & \end{cases}$$

where $\boldsymbol{x} \in \partial D$.

Remark 2 Vector proxies offer an isomorphic model for the calculus of differential forms. However, one must be aware that the choice of bases, and therefore the description of a differential form by a vector proxy, is essentially arbitrary. In particular, a change of metric of space suggests a different choice of vector proxies for which the Hodge operators reduce to the identity. Thus, metric and topological aspects are hard to disentangle from a vector analysis point of view. This made us prefer the differential forms point of view in the remainder of the article.

3 Self-adjoint extensions and Lagrangian subspaces

Symplectic geometry, see [26, Ch. 2], offers an abstract framework to deal with self-adjoint extensions of symmetric operators in Hilbert spaces. Here, we briefly recall how one can associate a Lagrangian subspace of the Gelfand-Robbin quotient to any self-adjoint extension. The reader can refer to [16,17] for a more detailed treatment.

Let H be a real Hilbert space and T a closed symmetric linear operator with dense domain $\mathcal{D}(\mathsf{T}) \subset H$. We denote by T^* its adjoint. Let us first recall, see [39], that each self-adjoint extension T^s of T is a restriction of T^*, which is classically written as

$$\mathsf{T} \subset \mathsf{T}^s \subset \mathsf{T}^*. \tag{3.1}$$

Next, introduce a degenerate symplectic pairing on $\mathcal{D}(\mathsf{T}^*)$ by

$$[\cdot, \cdot] : \mathcal{D}(\mathsf{T}^*) \times \mathcal{D}(\mathsf{T}^*) \longrightarrow \mathbb{R} \quad \text{such that } [u, v] = (\mathsf{T}^* u, v) - (u, \mathsf{T}^* v). \tag{3.2}$$

From the definition of T^*, the symmetry of T, and the fact $\mathsf{T}^{**} = \mathsf{T}$, we infer that, see [15, Appendix],

$$\begin{cases} [u + u_0, v + v_0] = [u, v], \ \forall u_0, v_0 \in \mathcal{D}(\mathsf{T}), \ \forall u, v \in \mathcal{D}(\mathsf{T}^*), \\ u \in \mathcal{D}(\mathsf{T}^*), \ [u, v] = 0, \quad \forall v \in \mathcal{D}(\mathsf{T}^*), \quad \Longrightarrow \quad u \in \mathcal{D}(\mathsf{T}). \end{cases} \tag{3.3}$$

As a consequence, we obtain a symplectic quotient space, see Appendix of [15]. In Salamon et al. [36], it is called the Gelfand–Robbin quotient.

Lemma 3.1 *The quotient space $S = (\mathcal{D}(\mathsf{T}^*)/\mathcal{D}(\mathsf{T}), [\cdot, \cdot])$ is a symplectic space.*

The graph norm on $\mathcal{D}(\mathsf{T}^*)$ induces a quotient norm on S, and due to (3.3), the symplectic pairing $[\cdot, \cdot]$ is continuous with respect to this norm

$$|[u, v]|^2 \leq \left(\|u\|^2 + \left\|\mathsf{T}^* u\right\|^2\right) \cdot \left(\|v\|^2 + \left\|\mathsf{T}^* v\right\|^2\right) \quad \forall u \in \mathcal{D}(\mathsf{T}^*), \ v \in \mathcal{D}(\mathsf{T}^*),$$

Let $L \oplus \mathcal{D}(\mathsf{T})$ denote the preimage of L under the quotient map $\mathcal{D}(\mathsf{T}^*) \mapsto S$.

Corollary 3.2 *The symplectic orthogonal complement*[1] *$V^\sharp$ of any subspace V of S is closed (in the quotient space topology).*

Any linear subspace L of S defines an extension T_L of T through

$$\mathsf{T} \subset \mathsf{T}_L := \mathsf{T}^*{}_{|L \oplus \mathcal{D}(\mathsf{T})} \subset \mathsf{T}^*. \tag{3.4}$$

This relationship allows to characterize self-adjoint extensions of T by means of the symplectic properties of the associated subspaces L. This statement is made precise in the Glazman-Krein-Naimark Theorem, see Theorem 1 of [15, Appendix].

Theorem 3.3 (Glazman-Krein-Naimark Theorem symplectic version) *The mapping $L \mapsto \mathsf{T}_L$ is a bijection between the space of complete Lagrangian subspaces of S and the space of self-adjoint extensions of T. The inverse mapping is given by*

$$L = \mathcal{D}(\mathsf{T}_L)/\mathcal{D}(\mathsf{T}). \tag{3.5}$$

4 Symplectic space for curl

The unbounded **curl** operators introduced in Sect. 2 (resorting to the vector proxy point of view) fit the framework of the preceding section, and Theorem 3.3 can be applied. To begin with, from (2.1) and (2.2), we arrive at the quotient space

$$S_{\mathbf{curl}} := \boldsymbol{H}(\mathbf{curl}, D)/\boldsymbol{H}_0(\mathbf{curl}, D). \tag{4.1}$$

By (1.2), it can be equipped with a symplectic pairing that can formally be written as

$$[\mathbf{u}, \mathbf{v}]_{\partial D} := \int_{\partial D} (\mathbf{u}(\boldsymbol{y}) \times \mathbf{v}(\boldsymbol{y})) \cdot \mathbf{n}(\boldsymbol{y}) \, \mathrm{d}S(\boldsymbol{y}), \tag{4.2}$$

[1] Let L be a linear subspace of the symplectic space S

(1) The *symplectic orthogonal complement* of L is $L^\sharp = \{u \in S : \ [u, L] = 0\}$;
(2) L is a *Lagrangian* subspace, if $L \subset L^\sharp$ i. e. $[u, v] = 0$ for all u and v in L;
(3) A *Lagrangian* subspace L is complete, if $L^\sharp = L$.

One can refer to [26, Chap. 2] for an introduction to symplectic geometry.

for any representatives of the equivalence classes of $S_{\mathbf{curl}}$. From (4.1), it is immediate that $S_{\mathbf{curl}}$ is algebraically and topologically isomorphic to the natural trace space of $\boldsymbol{H}(\mathbf{curl}, D)$.

This trace space is well understood, see the seminal work of Paquet [29], [8,9,7,10] for the extension to generic Lipschitz domains, and [38] for a presentation in the context of differential forms. To begin with, the topology of $S_{\mathbf{curl}}$ is intrinsic, that is, with $D' := \mathbb{R}^3 \backslash \overline{D}$, the norm of

$$S^c_{\mathbf{curl}} := \boldsymbol{H}(\mathbf{curl}, D')/\boldsymbol{H}_0(\mathbf{curl}, D') \tag{4.3}$$

is equivalent to that of $S_{\mathbf{curl}}$; both spaces are isomorphic algebraically and topologically. This can be proved appealing to an extension theorem for $\boldsymbol{H}(\mathbf{curl}, D)$. The trace space also allows a characterization via surface differential operators. It relies on the space $\mathbf{H}_t^{\frac{1}{2}}(\partial D)$ of tangential surface traces of vector fields in $(H^1(D))^3$ and its dual $\mathbf{H}_t^{-\frac{1}{2}}(\partial D) := (\mathbf{H}_t^{\frac{1}{2}}(\partial D))'$. Then, one finds that, algebraically and topologically, $S_{\mathbf{curl}}$ is isomorphic to

$$S_{\mathbf{curl}} \cong \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\partial, \partial D) := \{\mathbf{v} \in \mathbf{H}_t^{-\frac{1}{2}}(\partial D) :\ \mathrm{curl}_\partial\, \mathbf{v} \in H^{-\frac{1}{2}}(\partial D)\}. \tag{4.4}$$

The intricate details and the proper definition of curl_∂ can be found in [10].

When we adopt the perspective of differential forms, the domain of $\mathbf{curl}_{\max}$ is the Sobolev space $H\!F^1(\mathsf{d}, D)$ of 1-forms. Thus, $S_{\mathbf{curl}}$ has to be viewed as a trace space of 1-forms, that is, a space of 1-forms (more precisely, 1-currents) on ∂D. In analogy to (2.5) and (4.4), we adopt the notation

$$S_{\mathbf{curl}} \cong H\!F^{-\frac{1}{2},1}(\mathsf{d}, \partial D). \tag{4.5}$$

The corresponding symbol for the trace space of $H\!F^0(\mathsf{d}, D)$ will be $H\!F^{-\frac{1}{2},0}(\mathsf{d}, \partial D)$ (and not $H\!F^{\frac{1}{2},0}(\mathsf{d}, D)$ as readers accustomed to the conventions used with Sobolev spaces might expect).

In light of (2.7), the symplectic pairing on $H\!F^{-\frac{1}{2},1}(\mathsf{d}, \partial D)$ can be expressed as

$$[\omega, \eta]_{\partial D} := \int\limits_{\partial D} \omega \wedge \eta, \quad \omega, \eta \in H\!F^{-\frac{1}{2},1}(\mathsf{d}, \partial D). \tag{4.6}$$

Whenever $H\!F^{-\frac{1}{2},1}(\mathsf{d}, \partial D)$ is treated as a real symplectic space, the pairing (4.6) is assumed.

The most important observation about (4.6) is that the pairing $[\cdot, \cdot]$ is *invariant under pullback*. Indeed, let us introduce a bi-Lipschitz homeomorphism Φ from the closure of D to the closure of another domain $\widehat{D}$ in $\mathbb{R}^3$. The pullback[2] of a form ω is denoted by $\Phi^\star\omega$,

[2] From the integral perspective, the transformation (pullback) $\Phi^*\omega$ of a k-form under a sufficiently smooth mapping $\Phi : \widehat{M} \mapsto M$ appears natural: $\Phi^*\omega$ is a k-form on $\widehat{M}$ that fulfills

$$\int\limits_{\widehat{\Sigma}} \Phi^*\omega = \int\limits_{\Sigma} \omega$$

for all k-dimensional orientable sub-manifolds $\widehat{\Sigma}$ of $\widehat{M}$. We remark that pullbacks commute with the exterior derivative and exterior product

$$\mathsf{d}\,\Phi^*\omega = \Phi^*\,\mathsf{d}\,\omega \quad \text{and} \quad \Phi^*(\omega \wedge \eta) = \Phi^*\omega \wedge \Phi^*\eta.$$

see [18, Sect. 3.3]. Let $\widehat{\omega}, \widehat{\eta} \in \mathit{HF}^{-\frac{1}{2},1}(\mathsf{d}, \partial\widehat{D})$ and $\omega = \Phi^*\widehat{\omega}$ and $\eta = \Phi^*\widehat{\eta}$, We have

$$[\omega, \eta]_{\partial D} = \int\limits_{\partial D} \Phi^*\widehat{\omega} \wedge \Phi^*\widehat{\eta} = \int\limits_{\partial D} \Phi^*(\widehat{\omega} \wedge \widehat{\eta}) = \int\limits_{\partial \widehat{D}} \widehat{\omega} \wedge \widehat{\eta} = [\widehat{\omega}, \widehat{\eta}]_{\partial\widehat{D}}\,.$$

Consequently, determining all the complete Lagrangian subspaces of $(\mathit{HF}^{-\frac{1}{2},1}(\mathsf{d}, \partial\widehat{D}),$ $[\cdot,\cdot]_{\partial D})$ is completely equivalent to determine all the Lagrangian subspaces of $(\mathit{HF}^{-\frac{1}{2},1}(\mathsf{d}, \widehat{D}),$ $[\cdot,\cdot]_{\partial\widehat{D}})$, with $\widehat{D}$ a topologically equivalent domain to D: the pullback operator will map bijectively between both sets. Tersely speaking, the set of complete Lagrangian subspaces is invariant under topology-preserving invertible maps. This is obvious from the perspective of differential forms.

Now we can specialize Theorem 3.3 to the **curl** operator. To emphasize topological aspects, we formulate it in terms of 1-forms:

Theorem 4.1 (GKN-theorem for curl, version for 1-forms) *The mapping that associates with $L \subset \mathit{HF}^{-\frac{1}{2},1}(\mathsf{d}, \partial D)$ the $\star\,\mathsf{d}$ operator with domain*

$$\{\eta \in \mathit{HF}^1(\mathsf{d}, D) :\ i^*\eta \in L\}$$

is a bijection between the set of complete Lagrangian subspaces of $\mathit{HF}^{-\frac{1}{2},1}(\mathsf{d}, \partial D)$ and the self-adjoint extensions of **curl** *defined on $\mathit{HF}^1_0(\mathsf{d}, D)$.*

We point out that the constraint $i^*\eta \in L$ on traces amounts to imposing linear *boundary conditions*. In other words, the above theorems tell us that self-adjoint extensions of $\mathbf{curl}_{\min}$ will be characterized by demanding particular boundary conditions for their argument vector fields, *cf.* [33].

5 Hodge theory and consequences

Now we study particular subspaces of the trace space $\mathit{HF}^{-\frac{1}{2},1}(\mathsf{d}, \partial D)$. We will take for granted a *generic* metric on ∂D, which has no relationship, whatsoever, with the Euclidean metric on D. It induces a Hodge operator $\star : \Lambda^1(\partial D) \mapsto \Lambda^1(\partial D)$, which will be used throughout this and the next section.

5.1 The Hodge decomposition

Let us first recall the well-known Hodge decomposition of spaces of square-integrable differential 1-forms on ∂D. We refer to [38, Thm. 9], [27, Lemma 2.1], and, for a more general exposition, to [28].

Lemma 5.1 *We have the following decomposition, which is orthogonal w.r.t. the inner product of $L^2(\Lambda^1(\partial D))$ ("$\star$"-orthogonal"):*

$$L^2(\Lambda^1(\partial D) = \mathsf{d}\,\mathit{HF}^0(\mathsf{d}, \partial D) \oplus \star\,\mathsf{d}\,\mathit{HF}^0(\mathsf{d}, \partial D) \oplus \mathcal{H}^1(\partial D).$$

Here, $\mathcal{H}^1(\partial D)$ designates the finite-dimensional space of harmonic 1-forms on ∂D:

$$\mathcal{H}^1(\partial D) := \{\omega \in L^2(\Lambda^1(\partial D)) :\ \mathsf{d}\,\omega = 0 \quad \text{and} \quad \mathsf{d}\star\omega = 0\}. \tag{5.1}$$

In terms of Euclidean vector proxies, the space $L^2(\Lambda^1(\partial D)$ is modeled by the space $\mathbf{L}^2_t(\partial D)$ of square integrable tangential vector fields on ∂D. Then, in the special case of $\star$ connected with the induced Euclidean metric on ∂D, the decomposition of Lemma 5.1 has a vector proxy incarnation as $\mathbf{L}_t^2(\partial D)$-orthogonal splitting

$$\begin{aligned}\mathbf{L}_t^2(\partial D) &= \mathbf{grad}_\partial\, H^1(\partial D) \oplus_{L^2} \mathbf{curl}_\partial H^1(\partial D) \oplus_{L^2} \mathcal{H}^1(\partial D),\\ \mathcal{H}^1(\partial D) &:= \{\mathbf{v} \in \mathbf{L}_t^2(\partial D) :\ \mathbf{curl}_\partial \mathbf{v} = 0 \quad \text{and} \quad \mathrm{div}_\partial \mathbf{v} = 0\}.\end{aligned}$$

The Hodge decomposition can be extended to $\mathcal{H\!F}^{-\frac{1}{2},1}(\mathsf{d}, \partial D)$ on Lipschitz domains, as was demonstrated in [10, Sect. 5] and [6]. There, the authors showed that with a suitable extension of the surface differential operators

$$\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\partial, \partial D) = \mathbf{grad}_\partial\, H^{\frac{1}{2}}(\partial D) \oplus_{L^2} \mathbf{curl}_\partial H^{\frac{3}{2}}(\partial D) \oplus_{L^2} \mathcal{H}^1(\partial D), \tag{5.2}$$

where, formally,

$$H^{\frac{3}{2}}(\partial D) := \Delta_{\partial D}^{-1} H_*^{-\frac{1}{2}}(\partial D), \quad H_*^{-\frac{1}{2}}(\partial D) := \{v \in H^{-\frac{1}{2}}(\partial D) : \int_{\partial D_i} v \,\mathrm{d}S = 0\}. \tag{5.3}$$

with ∂D_i the connected components of ∂D. For C^1-boundaries, the space $H^{\frac{3}{2}}(\partial D)$ agrees with the trace space of $H^2(D)$. Using the techniques of [10], the result (5.2) can be generalized in the calculus of differential forms, see [38, Thm. 10]:

Theorem 5.2 (Hodge decomposition of trace space) *We have the following $\star$-orthogonal decomposition*

$$\mathcal{H\!F}^{-\frac{1}{2},1}(\mathsf{d}, \partial D) = \mathsf{d}\, \mathcal{H\!F}^{-\frac{1}{2},0}(\mathsf{d}, \partial D) \oplus \star\, \mathsf{d}\, \mathcal{H\!F}^{\frac{3}{2},0}(\partial D) \oplus \mathcal{H}^1(\partial D), \tag{5.4}$$

with

$$\mathcal{H\!F}^{\frac{3}{2},0}(\partial D) := \Delta_{\partial D}^{-1} \left\{\varphi \in \mathcal{H\!F}^{-\frac{1}{2},2}(\partial D) \ : \ \langle \varphi, \mathbf{1}\rangle_{\partial D_i} = 0\right\}, \tag{5.5}$$

with ∂D_i the connected components of ∂D and the Hodge–Laplacian $\Delta_{\partial D} = -\mathsf{d}\star\mathsf{d} :$ $\Lambda^0(\partial D) \mapsto \Lambda^2(\partial D)$.

As $\mathsf{d}^2 = 0$, the first subspace in the decomposition (5.4) of Theorem 5.2 comprises only closed 1-forms, that is, $\mathsf{d}\left(\mathsf{d}\, \mathcal{H\!F}^{-\frac{1}{2},0}(\mathsf{d}, \partial D)\right) = 0$. The second subspace contains only so-called co-closed 1-forms in the kernel of $\mathsf{d}\star$, since $\mathsf{d}\star\left(\star\,\mathsf{d}\, \mathcal{H\!F}^{\frac{3}{2},0}(\partial D)\right) = \mathsf{d}^2\, \mathcal{H\!F}^{\frac{3}{2},0}(\partial D) =$ 0. Again, we point out that the Hodge decomposition (5.4) hinges on the choice of the Hodge operator $\star$. Consequently, it depends on the underlying metric on ∂D.

5.2 Lagrangian properties of the Hodge decomposition

The subspaces occurring in the Hodge decomposition of Theorem 5.2 can be used as building blocks for (complete) Lagrangian subspaces of $\mathcal{H\!F}^{-\frac{1}{2},1}(\mathsf{d}, \partial D)$.

Proposition 5.3 *The linear space $\mathsf{d}\, \mathcal{H\!F}^{-\frac{1}{2},0}(\mathsf{d}, \partial D)$ is a Lagrangian subspace of the symplectic space $(\mathcal{H\!F}^{-\frac{1}{2},1}(\mathsf{d}, \partial D), [\cdot,\cdot])$.*

Proof We have to show that

$$\left[\mathsf{d}\,\omega, \mathsf{d}\,\eta\right]_{\partial D} = 0 \quad \forall \omega, \eta \in \mathcal{H\!F}^{-\frac{1}{2},0}(\mathsf{d}, \partial D). \tag{5.6}$$

By density, we need merely consider ω, η in $HF^0(\mathsf{d}, \partial D)$. In this case, it is immediate from Stokes' Theorem (∂D has no boundary)

$$[\mathsf{d}\,\omega, \mathsf{d}\,\eta]_{\partial D} = \int_{\partial D} \mathsf{d}\,\omega \wedge \mathsf{d}\,\eta = \int_{\partial D} \omega \wedge \mathsf{d}^2\,\eta = 0. \tag{5.7}$$

□

Proposition 5.4 *The linear space* $\star\,\mathsf{d}\,HF^{\frac{3}{2},0}(\partial D)$ *is a Lagrangian subspace of* $(HF^{-\frac{1}{2},1}(\mathsf{d}, \partial D), [\cdot,\cdot])$.

Proof The proof is the same as above, except that one has to use that $\star$ is an isometry with respect to the inner product $(\cdot,\cdot)_{1,\partial\Omega}$ induced by it (note that $\star\star = -1$ for 1-forms on ∂D):

$$\begin{aligned}[\star\,\mathsf{d}\,\omega, \star\,\mathsf{d}\,\eta]_{\partial D} &= \int_{\partial D} \star\,\mathsf{d}\,\omega \wedge \star\,\mathsf{d}\,\eta = \int_{\partial D} \mathsf{d}\,\omega \wedge \star^2\,\mathsf{d}\,\eta \\ &= -\int_{\partial D} \mathsf{d}\,\omega \wedge \mathsf{d}\,\eta = -\int_{\partial D} \omega \wedge \mathsf{d}^2\,\eta = 0. \end{aligned} \tag{5.8}$$

□

In a similar way, we prove the next proposition.

Proposition 5.5 *The space of harmonic 1-forms* $\mathcal{H}^1(\partial D)$ *is symplectically orthogonal to* $\mathsf{d}\,HF^{-\frac{1}{2},0}(\mathsf{d}, \partial D)$ *and* $\star\,\mathsf{d}\,HF^{\frac{3}{2},0}(\partial D)$.

The Hodge decomposition of Theorem 5.2 offers a tool for the evaluation of the symplectic pairing $[\cdot,\cdot]_{\partial D}$. Below, we tag the three components of the Hodge decomposition of Theorem 5.2 by subscripts 0, $\perp$, and $\mathcal{H}$: for $\omega, \eta \in HF^{-\frac{1}{2},1}(\mathsf{d}, \partial D)$ we thus express (5.4) as

$$\omega = \mathsf{d}\,\omega_0 + \star\,\mathsf{d}\,\omega_\perp + \omega_{\mathcal{H}} \quad \text{and} \quad \eta = \mathsf{d}\,\eta_0 + \star\,\mathsf{d}\,\eta_\perp + \eta_{\mathcal{H}}. \tag{5.9}$$

Note that the forms ω_0 and $\omega_\perp$ are not unique since the kernels of d and $\star\,\mathsf{d}$ are not empty (they contain the piecewise constants on connected components of ∂D).

Taking into account the symplectic orthogonalities stated in Propositions 5.3, 5.4, and 5.5, we see that all the following terms vanish

$$\begin{cases} [\mathsf{d}\,\omega_0, \mathsf{d}\,\eta_0]_{\partial D} &= [\star\,\mathsf{d}\,\omega_\perp, \star\,\mathsf{d}\,\eta_\perp]_{\partial D} = [\mathsf{d}\,\omega_0, \eta_{\mathcal{H}}]_{\partial D} &= 0, \\ [\star\,\mathsf{d}\,\omega_\perp, \eta_{\mathcal{H}}]_{\partial D} &= [\omega_{\mathcal{H}}, \mathsf{d}\,\eta_0]_{\partial D} \qquad = [\omega_{\mathcal{H}}, \star\,\mathsf{d}\,\eta_\perp]_{\partial D} &= 0. \end{cases}$$

Hence, we can compute the symplectic pairing on $HF^{-\frac{1}{2},1}(\mathsf{d}, \partial D)$ according to

$$[\omega, \eta]_{\partial D} = [\mathsf{d}\,\omega_0, \star\,\mathsf{d}\,\eta_\perp]_{\partial D} + [\star\,\mathsf{d}\,\omega_\perp, \mathsf{d}\,\eta_0]_{\partial D} + [\omega_{\mathcal{H}}, \eta_{\mathcal{H}}]_{\partial D}, \tag{5.10}$$

which extends [10, Lemma 5.6].

5.3 The symplectic space $\mathcal{H}^1(\partial D)$

Let us recall that the space of harmonic 1-forms on ∂D (a 2-dimensional compact C^∞-manifold without boundary) is a finite-dimensional linear space with

$$\dim(\mathcal{H}^1(\partial D)) = 2g, \tag{5.11}$$

with g the genus of the boundary. The genus is also the first Betti number of D. We refer to Theorem 5.1, Proposition 5.3.1 of [5], and Theorem 7.4.3 of [28].

Since the set of harmonic vector fields is stable with respect to the Hodge operator,

$$\begin{cases} \eta \in \mathcal{H}^1(\partial D) \Longrightarrow \eta \in L^2(\Lambda^1(\partial D)),\ \mathsf{d}\,\eta = 0,\ \mathsf{d}\star\eta = 0 \\ \qquad\qquad\qquad \Longrightarrow \star\eta \in L^2(\Lambda^1(\partial D)),\ \mathsf{d}\star(\star\eta) = 0,\ \mathsf{d}(\star\eta) = 0 \\ \qquad\qquad\qquad \Longrightarrow \star\eta \in \mathcal{H}^1(\partial D), \end{cases} \tag{5.12}$$

we find that the pairing $[\cdot,\cdot]_{\partial D}$ is non-degenerate on $\mathcal{H}^1(\partial D)$:

$$\left([\omega_{\mathcal{H}}, \eta_{\mathcal{H}}]_{\partial D} = 0, \quad \forall \eta_{\mathcal{H}} \in \mathcal{H}^1(D)\right) \Longrightarrow [\omega_{\mathcal{H}}, \star\omega_{\mathcal{H}}]_{\partial D} = (\omega_H, \omega_H)_{1,\partial D} = 0. \tag{5.13}$$

Lemma 5.6 *The space of harmonic 1-forms $\mathcal{H}^1(\partial D)$ is a symplectic space with finite dimension when equipped with the symplectic pairing $[\cdot,\cdot]_{\partial D}$. It is a finite-dimensional symplectic subspace of $(H\!F^{-\frac{1}{2},1}(\mathsf{d}, \partial D), [\cdot,\cdot])$.*

6 Some examples of self-adjoint curl operators

Starting from the Hodge decomposition of Theorem 5.2, we now identify important classes of self-adjoint extensions of **curl**. We rely on a generic Riemannian metric on ∂D and the associated Hodge operator $\star$.

6.1 Self-adjoint **curl** associated with closed traces

In this section, we aim to characterize the complete Lagrangian subspaces L of $(H\!F^{-\frac{1}{2},1}(\mathsf{d}, \partial D), [\cdot,\cdot]_{\partial D})$, which contain only closed forms:

$$L \subset H\!Z^{-\frac{1}{2},1}(\partial D) := \{\eta \in H\!F^{-\frac{1}{2},1}(\mathsf{d}, \partial D) :\ \mathsf{d}\,\eta = 0\}. \tag{6.1}$$

Hodge theory (see Theorem 5.2) provides the tools to study these Lagrangian subspaces, since we have the following result:

Lemma 6.1 *The set of closed 1-forms in $H\!F^{-\frac{1}{2},1}(\mathsf{d}, \partial D)$ admits the following direct $\star$-orthogonal decomposition*

$$H\!Z^{-\frac{1}{2},1}(\partial D) = \mathsf{d}\, H\!F^{-\frac{1}{2},0}(\mathsf{d}, \partial D) \oplus \mathcal{H}^1(\partial D). \tag{6.2}$$

Proof For $\omega \in H\!Z^{-\frac{1}{2},1}(\partial D)$, $\star\,\mathsf{d}\,\omega_\perp$ from (5.9) satisfies $\mathsf{d}(\star\,\mathsf{d}\,\omega_\perp) = 0$, $\mathsf{d}\star(\star\,\mathsf{d}\,\omega_\perp) = 0$, $(\star\,\mathsf{d}\,\omega_\perp)_{\mathcal{H}} = 0$, which implies that $\star\,\mathsf{d}\,\omega_\perp = 0$ and yields the assertion of the lemma. □

The next result highlights the limited leeway in choosing Lagrangian subspaces included in $H\!Z^{-\frac{1}{2},1}(\partial D)$.

Lemma 6.2 *The symplectic orthogonal of the space $H\!Z^{-\frac{1}{2},1}(\partial D)$ is the space $\mathsf{d}\, H\!F^{-\frac{1}{2},0}(\mathsf{d}, \partial D)$ of exact 1-forms.*

Proof The symplectic orthogonal of $H\!Z^{-\frac{1}{2},1}(\partial D)$ is defined as the set

$$\{\omega \in H\!F^{-\frac{1}{2},1}(\mathsf{d}, \partial D) :\ [\omega, \eta]_{\partial D} = 0, \quad \forall \eta \in H\!Z^{-\frac{1}{2},1}(\partial D)\}. \tag{6.3}$$

Using Theorem 5.2 for $\omega = \mathsf{d}\,\omega_0 + \star\mathsf{d}\,\omega_\perp + \omega_{\mathcal{H}}$ and Lemma 6.1 for $\eta = \mathsf{d}\,\eta_0 + \eta_{\mathcal{H}}$, we conclude from (5.10):

$$[\omega, \eta]_{\partial D} = [\star\mathsf{d}\,\omega_\perp, \mathsf{d}\,\eta_0]_{\partial D} + [\omega_{\mathcal{H}}, \eta_{\mathcal{H}}]_{\partial D}. \tag{6.4}$$

When $\eta = \star\omega_{\mathcal{H}} = \eta_{\mathcal{H}}$ (here we use the stability of $\mathcal{H}^1(\partial D)$ with respect to the Hodge operator), this implies

$$[\omega, \star\omega_{\mathcal{H}}]_{\partial D} = [\omega_{\mathcal{H}}, \star\omega_{\mathcal{H}}]_{\partial D} = \int_{\partial D} \omega_{\mathcal{H}} \wedge \star\omega_{\mathcal{H}} = 0 \quad \Longrightarrow \omega_{\mathcal{H}} = 0, \tag{6.5}$$

and, for $\eta = \mathsf{d}\,\eta_0$ with $\eta_0 = \omega_\perp \in \mathit{HF}^{3/2,0}(\partial D)$

$$[\omega, \mathsf{d}\,\omega_\perp]_{\partial D} = [\star\mathsf{d}\,\omega_\perp, \mathsf{d}\,\omega_\perp]_{\partial D} = -\int_{\partial D} \mathsf{d}\,\omega_\perp \wedge \star\mathsf{d}\,\omega_\perp \quad \Longrightarrow \quad \mathsf{d}\,\omega_\perp = 0. \tag{6.6}$$

Hence, we have $\omega = \mathsf{d}\,\omega_0$ (and $\omega_{\mathcal{H}} = 0$). The converse inclusion holds due to (6.4). □

Lemma 6.2 combined with the splitting (6.4) also means that we can evaluate $[\cdot,\cdot]_{\partial D}$ on $\mathit{HZ}^{-\frac{1}{2},1}(\partial D)$ according to

$$\boxed{[\omega, \eta]_{\partial D} = [\omega_{\mathcal{H}}, \eta_{\mathcal{H}}]_{\partial D}, \quad \forall \omega,\ \eta \in \mathit{HZ}^{-\frac{1}{2},1}(\partial D)}, \tag{6.7}$$

that is, the pairing $[\cdot,\cdot]$ on $\mathit{HZ}^{-\frac{1}{2},1}(\partial D)$ depends only on the harmonic components.

This means that all complete Lagrangian subspaces L of $\mathit{HF}^{-\frac{1}{2},1}(\mathsf{d}, \partial D)$ contained in $\mathit{HZ}^{-\frac{1}{2},1}(\partial D)$ are related to complete Lagrangian subspaces $L_{\mathcal{H}}$ of $\mathcal{H}^1(\partial D)$ by

$$\boxed{L = \mathsf{d}\,\mathit{HF}^{-\frac{1}{2},0}(\mathsf{d}, \partial D) \oplus L_{\mathcal{H}}}. \tag{6.8}$$

From (6.7), we directly infer the following lemma:

Lemma 6.3 *There is a one-to-one correspondence between the complete Lagrangian subspaces L of the symplectic space $(\mathit{HF}^{-\frac{1}{2},1}(\mathsf{d}, \partial D), [\cdot,\cdot]_{\partial D})$ satisfying*

$$L \subset \mathit{HZ}^{-\frac{1}{2},1}(\partial D) \tag{6.9}$$

and the complete Lagrangian subspaces $L_{\mathcal{H}}$ of $\mathcal{H}^1(\partial D)$. The bijection is given by (6.8).

Via Theorem 4.1, Lemma 6.3 leads to the characterization of the self-adjoint **curl** operators whose domains contain only functions with closed traces.

Theorem 6.4 *There is a one-to-one correspondence between the set of all self-adjoint* **curl** *operators* $\mathbf{curl}_S$ *satisfying*

$$\mathcal{D}(\mathbf{curl}_S) \subset \left\{\omega \in \mathit{HF}^1(\mathsf{d}, D) : \ i^*\omega \in \mathit{HZ}^{-\frac{1}{2},1}(\partial D)\right\} \tag{6.10}$$

and the set of complete Lagrangian subspaces $L_{\mathcal{H}}$ of $\mathcal{H}^1(\partial D)$. They are related according to

$$\mathcal{D}(\mathbf{curl}_S) = \left\{\omega \in \mathit{HF}^1(\mathsf{d}, D) : \ i^*\omega \in \mathsf{d}\,\mathit{HF}^{-\frac{1}{2},0}(\mathsf{d}, \partial D) \oplus L_{\mathcal{H}}\right\}. \tag{6.11}$$

Obviously, the constraint $i^*\omega \in \mathsf{d}\,\mathit{HF}^{-\frac{1}{2},0}(\mathsf{d}, \partial D) \oplus L_{\mathcal{H}}$ is a boundary condition, *cf.* the discussion in Sect. 4.

Remark 3 Now, assume the domain D to feature trivial topology, that is, the genus of D is zero, and the space of harmonic forms is trivial. Theorem 6.4 reveals that there is only one self-adjoint **curl** with domain containing only forms with closed traces

$$\mathcal{D}(\mathbf{curl}_S) = \left\{\omega \in HF^1(\mathsf{d}, D) : \ \mathsf{d}\,(i^*\omega) = 0\right\}. \tag{6.12}$$

In terms of Euclidean vector proxies, this leads to the self-adjoint **curl** operator with domain

$$\mathcal{D}(\mathbf{curl}_S) = \{\mathbf{u} \in \boldsymbol{H}(\mathbf{curl}, D) : \ \mathbf{curl}(\mathbf{u}) \cdot \mathbf{n} = 0 \text{ on } \partial D\}, \tag{6.13}$$

which has been investigated in [34,40]. In case D has non-trivial topology, then $\dim(\mathcal{H}^1(\partial D)) = 2g \neq 0$, and one has to examine the complete Lagrangian subspaces of $\mathcal{H}^1(\partial D)$, which is postponed to Sect. 6.3.

6.2 Self-adjoint **curl** based on co-closed traces

In this section, we seek to characterize those Lagrangian subspaces L of the symplectic space $(HF^{-\frac{1}{2},1}(\mathsf{d}, \partial D), [\cdot,\cdot])$ that contain only co-closed forms, i.e.,

$$L \subset \left\{\omega \in HF^{-1/2,1}(\mathsf{d}, \partial D) : \ \mathsf{d} \star \omega = 0\right\}. \tag{6.14}$$

The developments are parallel to those of the previous section, because, as is illustrated by (5.10), from a symplectic point of view, the subspaces of closed and co-closed 1-forms occurring in the Hodge decomposition of Theorem 5.2 are symmetric. The next two lemmas are counterparts of Lemmas 6.1 and 6.2 with similar proofs.

Lemma 6.5 *The subspace of co-closed 1-forms of $HF^{-1/2,1}(d, \partial D)$ admits the following $\star$-orthogonal decomposition*

$$\left\{\omega \in HF^{-1/2,1}(d, \partial D) : \ \mathsf{d} \star \omega = 0\right\} = \star\, d\, HF^{3/2,0}(\partial D) \oplus \mathcal{H}^1(\partial D). \tag{6.15}$$

Lemma 6.6 *The symplectic orthogonal of the subspace of co-closed forms of $HF^{-1/2,1}(d, \partial D)$ is $\star\, d\, HF^{3/2,0}(\partial D)$.*

Again, we observe that when restricted to the space of co-closed forms, the bilinear pairing $[\cdot,\cdot]_{\partial D}$ becomes degenerate. On the other hand, also on the subset of co-closed forms, one can evaluate $[\cdot,\cdot]_{\partial D}$ through a formula analogous to (6.7):

$$\boxed{[\omega, \eta]_{\partial D} = [\omega_{\mathcal{H}}, \eta_{\mathcal{H}}]_{\partial D}, \quad \forall \omega,\ \eta \in \left\{\kappa \in HF^{-1/2,1}(\mathsf{d}, \partial D) : \ \mathsf{d}\star\kappa = 0\right\}}, \tag{6.16}$$

which implies a result analogous to Lemma 6.3.

Lemma 6.7 *The complete Lagrangian subspaces L of $(HF^{-1/2,1}(d, \partial D), [\cdot,\cdot]_{\partial D})$ containing only co-closed forms are one-to-one related to the complete Lagrangian subspaces $L_{\mathcal{H}}$ of $(\mathcal{H}^1(\partial D), [\cdot,\cdot]_{\partial D})$ by*

$$\boxed{L = \star\, d\, HF^{3/2,0}(\partial D) \oplus L_{\mathcal{H}}}. \tag{6.17}$$

Theorem 4.1 and Lemma 6.7 lead to the characterization of the self-adjoint **curl** operators based on co-closed forms:

Theorem 6.8 *There is a one to one correspondence between the set of all self-adjoint operators $\mathbf{curl}_S$ satisfying*

$$\mathcal{D}(\mathbf{curl}_S) \subset \left\{\omega \in HF^1(d, \partial D) : \ d\star(i^*\omega) = 0\right\} \tag{6.18}$$

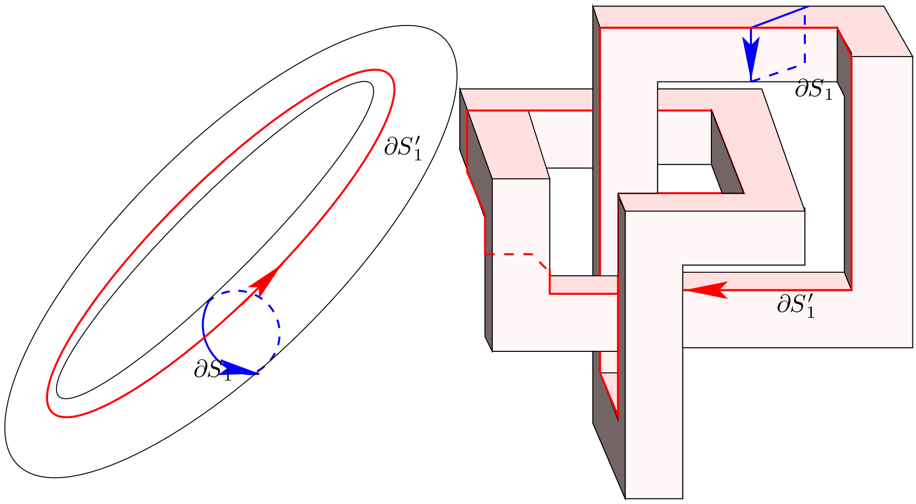

Fig. 1 Boundaries of cuts ∂S_1 and $\partial S_1'$ for the torus (*left*) and trefoil knot (*right*), $g = 1$ in each case

and the set of complete Lagrangian subspaces $L_{\mathcal{H}}$ *of* $\mathcal{H}^1(\partial D)$ *equipped with* $[\cdot,\cdot]_{\partial D}$*. The underlying bijection is*

$$\mathcal{D}(\mathbf{curl}_S) = \left\{\omega \in HF^1(\mathsf{d}, D) : \ i^*\omega \in \star\, \mathsf{d}\, HF^{3/2,0}(\partial D) \oplus L_{\mathcal{H}}\right\}. \tag{6.19}$$

Remark 4 Let D be a domain with trivial topology, so that $\mathcal{H}^1(\partial D) = \{0\}$. Then, there is only one self-adjoint **curl**-operator with domain containing only forms whose traces are co-closed

$$\mathcal{D}(\mathbf{curl}_S) = \left\{\omega \in HF^0(\mathsf{d}, \Omega) : \ \mathsf{d} \star (i^*\omega) = 0\right\}. \tag{6.20}$$

In terms of Euclidean vector proxies and $\star$ arising from the Euclidean metric on ∂D, we obtain the self-adjoint **curl** operator with domain

$$\mathcal{D}(\mathbf{curl}_S) = \{\mathbf{u} \in \boldsymbol{H}(\mathbf{curl}, D) : \ \mathrm{div}_\partial(\gamma_t(\mathbf{u})) = 0 \quad \text{on } \partial D\}. \tag{6.21}$$

On the contrary, if D has non-trivial topology, then one has to identify the complete Lagrangian subspaces of $\mathcal{H}^1(\partial D)$. This is the topic of the next section.

6.3 Complete Lagrangian subspaces of $\mathcal{H}^1(\partial D)$

The goal is to give a rather concrete description of the boundary conditions implied by (6.11) and (6.19). Concepts and results from (algebraic) topology and homology theory as introduced in [5,23] will be pivotal. In particular, we rely on the existence of $2g$ compact orientable 2-manifolds, so-called "*cuts*" [22,19,14],

$$S_i \subset \overline{D}, \quad S_i' \subset \overline{D'}\ (D' := \mathbb{R}^3 \setminus \overline{D}), \qquad 1 \leqslant i \leqslant g, \tag{6.22}$$

with boundaries contained in ∂D, such that $\overline{D}\setminus S_i$ or $\mathbb{R}^3\setminus(D \cup S_i')$, respectively, are still connected, see Fig. 1

In [23], it was established that the set of cuts $\{\langle S_i\rangle\}_{i=1}^g \cup \{\langle S_i'\rangle\}_{i=1}^g$ can be chosen so that they are "dual to each other". Here, this duality is expressed through the intersection numbers of their boundaries (with induced orientation), see [18, Sect. 6.4] and, in particular, Chapter 5 of [19].

Lemma 6.9 *The set of cuts* $\{\langle S_i\rangle\}_{i=1}^{g} \cup \{\langle S_i'\rangle\}_{i=1}^{g}$ *can be chosen such that they satisfy*

$$\begin{cases} \mathrm{Int}(\langle \partial S_i\rangle, \langle \partial S_j'\rangle) = \delta_{i,j}, \\ \mathrm{Int}(\langle \partial S_i'\rangle, \langle \partial S_j\rangle) = -\delta_{i,j}, \end{cases} \quad 1 \le i, j \le g, \tag{6.23}$$

where $\mathrm{Int}(\gamma_1, \gamma_2)$ *designates the intersection number*[3] *of two curves on* ∂D.

These dual pairs of cuts are a tool for constructing a particular basis of $\mathcal{H}^1(\partial D)$, see [23]:

Lemma 6.10 *The harmonic 1-forms* $\kappa_1, \dots, \kappa_g, \kappa_1', \dots, \kappa_g' \in \mathcal{H}^1(\partial D)$ *uniquely defined by*

$$\begin{aligned} &\int_{\partial S_j} \kappa_i = \delta_{ij}, \quad \int_{\partial S_j'} \kappa_i = 0, \\ &\int_{\partial S_i} \kappa_j' = 0, \quad \int_{\partial S_i'} \kappa_j' = \delta_{ij}, \qquad 1 \le i, j \le g, \end{aligned} \tag{6.24}$$

form a basis of $\mathcal{H}^1(\partial D)$.

Proof Linear independence of $\{\kappa_1, \dots, \kappa_g, \kappa_1', \dots, \kappa_g'\}$ is immediate from (6.24). Hence, a counting argument confirms the basis property.

Uniqueness: if there was another basis complying with (6.24), the differences of the basis forms would harmonic 1-forms with vanishing integral over *any* closed curve (cycle) in ∂D. Those must vanish identically. □

These basis 1-forms are traces of closed 1-forms on D and D', respectively. For instance, κ_i can be obtained as trace of the *piecewise* exterior derivative of a 0-form (a scalar function) on $D' \setminus S_i'$ that has a jump of height 1 across S_i', see [1, Sect. 3] and [23]. An analogous statement holds for κ_i' with S_i' replaced with S_i. More precisely, one has for $1 \le i \le g$

$$\exists \psi_i \in HZ^1(D') := \{\omega \in HF^1(\mathsf{d}, D') : \mathsf{d}\,\omega = 0\} \quad \kappa_i = i_{out}^* \psi_i, \tag{6.25}$$

$$\exists \psi_i' \in HZ^1(D) := \{\omega \in HF^1(\mathsf{d}, D) : \mathsf{d}\,\omega = 0\} \quad \kappa_i' = i_{in}^* \psi_i', \tag{6.26}$$

where i_{in} and i_{out} stand for the traces onto ∂D from D (*in*side) and D' (*out*side), respectively.

Lemma 6.11 *For* $1 \leqslant m, n \leqslant g$*, we have*

$$\int_{\partial D} \kappa_m \wedge \kappa_n = 0, \quad \int_{\partial D} \kappa_m' \wedge \kappa_n' = 0. \tag{6.27}$$

Proof Using (6.25) and (6.26), we rewrite the integral as one over D, as the following calculation shows

$$\begin{cases} \displaystyle\int_{\partial D} i_{out}^*(\psi_m) \wedge i_{out}^*(\psi_n) = \int_{\partial D} i_{out}^*(\psi_m \wedge \psi_n) = \int_D \mathsf{d}(\psi_m \wedge \psi_n) \\ \displaystyle\qquad\qquad = \int_D (\mathsf{d}\,\psi_m) \wedge \psi_n - \psi_m \wedge (\mathsf{d}\,\psi_n) = 0. \end{cases}$$

The other relationship follows from an analogous calculation with primed entities. □

[3] The reader may resort to the geometric intuition that the intersection number $\mathrm{Int}(\gamma_1, \gamma_2)$ counts the number of times γ_1 crosses γ_2 from "left to right" minus the number of times it crosses from "right to left". Thus, the intersection numbers $\mathrm{Int}(\partial S_1, \partial S_1')$ in Fig. 1 are equal to -1 is each case.

Lemma 6.12 *For* $1 \leqslant i,\ j \leqslant g$, *we have*

$$\int\limits_{\partial D} \kappa_i \wedge \kappa_j' = \delta_{i,j}. \tag{6.28}$$

Proof Following [23] or [1, Sect. 3], let us represent the 1-forms ψ_i, ψ_j' from (6.25) and (6.26) by means of the local exterior derivative of 0-forms (scalar functions) φ_i, φ_j', which jump across S_i' and S_j, respectively.

$$\psi_i = \mathsf{d}\,\varphi_i \quad \text{in } D'\backslash S_i', \qquad \psi_j' = \mathsf{d}\,\varphi_j' \quad \text{in } D\backslash S_j, \tag{6.29}$$

$$[\varphi_i]_{S_i'} = 1, \quad \left[\varphi_j'\right]_{S_j} = 1, \tag{6.30}$$

with $[\cdot]_\Gamma$ denoting the (signed) jump across the oriented surface Γ. Taking into account that the exterior derivative commutes with the trace, we find

$$\kappa_i \wedge \kappa_j' = i_{out}\,\mathsf{d}\,\varphi_i \wedge i_{in}\,\mathsf{d}\,\varphi_j' = \mathsf{d}\,i_{out}\varphi_i \wedge \mathsf{d}\,i_{in}\varphi_j' \quad \text{on } \partial D\backslash(\partial S_i' \cup \partial S_j). \tag{6.31}$$

Then, applying integration by parts, we infer (one has to take care of the orientation)

$$\int\limits_{\partial D} \kappa_i \wedge \kappa_j' = \int\limits_{\partial S_i'} [i_{out}^* \varphi_i \wedge i_{in}^*\,\mathsf{d}\,\varphi_j]_{\partial S_i'} + \int\limits_{\partial S_j} \underbrace{\left[i_{out}^*\varphi_i \wedge i_{in}^*\,\mathsf{d}\,\varphi_j'\right]_{\partial S_j}}_{=0}$$

$$= \underbrace{[i_{out}^*\varphi_i]_{\partial S_i'}}_{=1} \wedge \underbrace{\int\limits_{\partial S_i'} \kappa_j'}_{=\delta_{ij}} = \delta_{ij}.$$

by the property (6.30) of φ_i and Lemma 6.10. □

Owing to Lemma 6.11 and Lemma 6.12, the symplectic pairing $[\cdot,\cdot]_{\partial D}$ has the matrix representation

$$\begin{bmatrix} \mathbf{0}_{g\times g} & \mathbf{I}_{g\times g} \\ -\mathbf{I}_{g\times g} & \mathbf{0}_{g\times g} \end{bmatrix} \in \mathbb{R}^{2g,2g}, \tag{6.32}$$

with respect to the basis

$$\left(\{\kappa_i\}_{i\in I} \cup \{-\kappa_i'\}_{i\in I'}\right) \cup \left(\{-\kappa_i'\}_{i\in I} \cup \{\kappa_i\}_{i\in I'}\right) \tag{6.33}$$

of $\mathcal{H}^1(\partial D)$. A basis of a finite-dimensional symplectic space with this property is called a canonical symplectic basis [15, Sect. 1].

Lemma 6.13 *The set* $\{\kappa_i, \kappa_i'\}_{i=1}^g$ *is a canonical symplectic basis of the* $2g$*-dimensional symplectic space* $(\mathcal{H}^1(\partial D), [\cdot,\cdot])$.

Given a canonical symplectic basis, we can appeal to abstract results [15, Ex. 2] to build complete Lagrangian subspaces: for any canonical symplectic basis $\{u_i\}_{i=1}^{2g}$ of a $2g$-dimensional symplectic space and mapping $\sigma : \{1,\ldots,g\} \mapsto \{0,1\}$ the span of $\{u_{i+\sigma(i)g}\}_{i=1}^g$ is a complete Lagrangian subspace.

Thus, starting from a partition

$$I \cup I' = \{1,\ldots,g\}, \quad I \cap I' = \emptyset. \tag{6.34}$$

we can construct a complete Lagrangian subspace of $(\mathcal{H}^1(\partial D), [\cdot,\cdot])$

$$\operatorname{span}\left\{\{\kappa_i\}_{i\in I}\cup\{\kappa_i'\}_{i\in I'}\right\} = \left\{\omega\in\mathcal{H}^1(\partial D):\ \int_{\partial S_i}\omega = \int_{\partial S_j'}\omega = 0\quad\forall i\in I',\ j\in I\right\}. \tag{6.35}$$

By Theorems 6.4 and 6.8, $L_{\mathcal{H}}$ induces self-adjoint **curl** operators. Denoting i_{in}^* by i^* in what follows and invoking (6.2), for those specimens with closed traces, we find the domains

$$\begin{aligned}\mathcal{D}(\mathbf{curl}_s) &= \left\{\omega\in HF^1(\mathsf{d},D):\ i^*\omega\in \mathsf{d}\,HZ^{-\frac{1}{2},0}(\partial D)+\mathcal{H}^1(\partial D),\right.\\ &\qquad\left.\int_{\partial S_i}\omega=\int_{\partial S_j'}\omega=0\quad\forall i\in I',\ j\in I\right\}\\ &=\left\{\omega\in HF^1(\mathsf{d},D):\ \mathsf{d}\,i^*\omega=0,\ \int_{\partial S_i}\omega=\int_{\partial S_j'}\omega=0\quad\forall i\in I',\ j\in I\right\}.\end{aligned} \tag{6.36}$$

We point out that $\mathcal{D}(\mathbf{curl}_s)$ is well defined, since integration over boundaries of cuts provides bounded functionals on $\{\omega\in HF^1(\mathsf{d},D):\ i^*\omega\in HZ^{-\frac{1}{2},0}(\partial D)\}$.

A key observation with (6.36) is that replacing any boundary of a cut with a closed oriented curve (cycle) that is homologous to it in ∂D does not change the space at all. More precisely, let $C_i\subset\partial D$, $i\in I'$, $C_j'\subset\partial D$, $j\in I$ be cycles that form bases of the homology spaces, that is, of subspaces of the $2g$-dimensional homology space $\mathbb{H}_1(\partial D,\mathbb{R})$, that are generated by the cycles ∂S_i, $i\in I'$, and $\partial S_j'$, $j\in I$, respectively. These cycles allow an equivalent definition of the domain from (6.36):

$$\mathcal{D}(\mathbf{curl}_s) = \left\{\omega\in HF^1(\mathsf{d},D):\ \mathsf{d}\,i^*\omega=0,\ \int_{C_i}\omega=\int_{C_j'}\omega=0\quad\forall i\in I',\ j\in I\right\}. \tag{6.37}$$

We point out that the choice $I=\emptyset$ together with closed trace is the one proposed in [40] to obtain a self-adjoint **curl**.

Homeomorphisms map homologous cycles to homologous ones, while the pullback preserves closedness of a form. This implies that the boundary conditions inherent in (6.37) are *invariant under pullback* in the sense explained earlier in Sect. 4.

Relying on (6.15), (6.17), and Theorem 6.8, we can pursue the same considerations for self-adjoint **curl** operators characterized by co-closed traces. They lead to the domains

$$\mathcal{D}(\mathbf{curl}_s) := \left\{ \omega \in HF^1(\mathsf{d}, D) : \; \mathsf{d}\star(i^*\omega) = 0, \; \int\limits_{\partial S_i} \omega = 0, \; i \in I', \; \int\limits_{\partial S'_j} \omega = 0, \; j \in I \right\}, \tag{6.38}$$

In stark contrast to the case of closed traces, these domains will depend on the concrete choice of the cycles ∂S_i, $\partial S'_j$! Since they also depend on $\star$, they will fail to be invariant under pullback; though the formulas (6.8) and (6.17) enjoy a striking symmetry they result in boundary conditions of a completely different nature.

7 Spectral properties

Having constructed self-adjoint versions of the **curl** operator, we go on to verify whether their essential spectrum is confined to 0 and their eigenfunctions can form a complete orthonormal system in $\boldsymbol{L}^2(D)$. These are common important features of self-adjoint partial differential operators.

The following compact embedding result is instrumental in investigating the spectrum of $\mathbf{curl}_s$. Related results can be found in [37] and [31].

Theorem 7.1 **(Compact embedding)** *The spaces, endowed with the $HF^1(\mathsf{d}, D)$-norm,*

$$X_0 := \{\omega \in HF^1(\mathsf{d}, D) : \; \mathsf{d}\star\omega = 0, \; i^*(\star\omega) = 0\} \quad \textit{and}$$
$$X^\perp := \quad \{\omega \in HF^1(\mathsf{d}, D) : \; \mathsf{d}\star\omega = 0, \; \mathsf{d}\star(i^*\omega) = 0\}$$

are compactly embedded into $L^2(\Lambda^1(D))$.

Remark 5 In terms of Euclidean vector proxies and Euclidean Hodge operator on ∂D, these spaces read

$$X_0 = \{\mathbf{v} \in \boldsymbol{H}(\mathbf{curl}, D) : \; \operatorname{div} \mathbf{v} = 0, \; \gamma_n \mathbf{u} = 0\},$$
$$X^\perp = \{\mathbf{v} \in \boldsymbol{H}(\mathbf{curl}, D) : \; \operatorname{div} \mathbf{v} = 0, \; \operatorname{div}_\partial(\gamma_t \mathbf{u}) = 0\}$$

where the constraint $\operatorname{div}_\partial(\gamma_t \mathbf{u}) = 0$ should be read as "orthogonality" to $\mathbf{grad}_\partial H^{\frac{1}{2}}(\partial D)$ in the sense of the Hodge decomposition.

Proof *(of Thm. 7.1)* The proof will be given for $X^\perp$ only. The simpler case of X_0 draws on the same ideas. We are using vector proxy notation, because the proof takes us beyond the calculus of differential forms. Note that the inner product chosen for the vector proxies does not affect the statement of the theorem.

A key tool is the so-called regular decomposition theorem that was discovered in [4], consult [20, Sect. 2.4] for a comprehensive presentation including proofs. It asserts that there is $C > 0$ depending only on D such that for all $\mathbf{u} \in \boldsymbol{H}(\mathbf{curl}, D)$, there are functions $\boldsymbol{\Phi} \in (H^1(D))^3$, $\varphi \in H^1(D)$, with

$$\mathbf{u} = \boldsymbol{\Phi} + \mathbf{grad}\,\varphi, \quad \|\boldsymbol{\Phi}\|_{H^1(D)} + |\varphi|_{H^1(D)} \leq C\,\|\mathbf{u}\|_{\boldsymbol{H}(\mathbf{curl}, D)}\,. \tag{7.1}$$

Let $(\mathbf{u}_n)_{n\in\mathbb{N}}$ be a bounded sequence in $X^\perp$ that is

$$\operatorname{div}\mathbf{u}_n = 0 \quad \text{in } D \quad \text{and} \quad \operatorname{div}_\partial(\gamma_t\mathbf{u}_n) = 0 \quad \text{on } \partial D, \tag{7.2}$$
$$\exists C > 0: \quad \|\mathbf{u}_n\|_{L^2(D)} + \|\mathbf{curl}\,\mathbf{u}_n\|_{L^2(D)} \leq C. \tag{7.3}$$

Write $\mathbf{u}_n = \boldsymbol{\Phi}_n + \mathbf{grad}\,\varphi_n$ for the regular decomposition according to (7.1). Thus, $(\boldsymbol{\Phi}_n)_{n\in\mathbb{N}}$ is bounded in $(H^1(D))^3$ and, by Rellich's theorem, will possess a sub-sequence that converges in $\boldsymbol{L}^2(D)$. We pick the corresponding sub-sequence of $(\mathbf{u}_n)_{n\in\mathbb{N}}$ without changing the notation.

Further,

$$\operatorname{div}\mathbf{u}_n = 0 \Rightarrow -\Delta\varphi_n = \operatorname{div}\boldsymbol{\Phi}_n \quad (\text{bounded in } L^2(D)), \tag{7.4}$$
$$\operatorname{div}_\partial(\gamma_t\mathbf{u}) = 0 \Rightarrow -\Delta_{\partial D}(\gamma\varphi_n) = \operatorname{div}_\partial(\gamma_t\boldsymbol{\Phi}_n) \quad (\text{bounded in } H^{-\frac{1}{2}}(\partial D)). \tag{7.5}$$

We conclude that $(\gamma\varphi_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\partial D)$ and, hence, has a convergent sub-sequence in $H^{\frac{1}{2}}(\partial D)$ (for which we still use the same notation). The harmonic extensions $\widetilde{\varphi}_n$ of $\gamma\varphi_n$ will converge in $H^1(D)$.

Finally, the solutions $\widehat{\varphi}_n \in H^1(D)$ of the boundary value problems

$$-\Delta\widehat{\varphi}_n = \operatorname{div}\boldsymbol{\Phi}_n \quad \text{in } D, \quad \widehat{\varphi}_n = 0 \quad \text{on } \partial D, \tag{7.6}$$

will possess a sub-sequence that converges in $H^1(D)$, as $(-\Delta_{\mathrm{Dir}})^{-1}L^2(D)$ is compactly embedded in $H^1(D)$. Since $\varphi_n = \widetilde{\varphi}_n + \widehat{\varphi}_n$, this provides convergence of a subsequence of $(\boldsymbol{\Phi}_n + \mathbf{grad}\,\varphi_n)_{n\in\mathbb{N}}$ in $\boldsymbol{L}^2(D)$. □

Let $\mathbf{curl}_s : \mathcal{D}_s \subset L^2(\Lambda^1(D)) \mapsto L^2(\Lambda^1(D))$ be one of the self-adjoint realizations of **curl** discussed in the previous section. Recall that we pursued two constructions based on closed and co-closed traces, respectively[4].

Remark 6 Even if the domain $\mathcal{D}_s$ of the self-adjoint $\mathbf{curl}_s$ is known only up to the contribution of a Lagrangian subspace of $L_{\mathcal{H}}$, we can already single out special subspaces of $\mathcal{D}_s$:

(1) For the **curl** operators based on closed traces, see Sect. 6.1, in particular Thm. 6.4, we find

$$\mathsf{d}\,HF^0(\mathsf{d}, D) \subset \mathcal{D}_s. \tag{7.7}$$

Indeed, for $\omega \in \mathsf{d}\,HF^0(\mathsf{d}, D)$, there exists $\eta \in HF^0(\mathsf{d}, D)$ with $\omega = \mathsf{d}\,\eta$. Due to the trace theorem, $i^*\eta$ belongs to $HF^{-\frac{1}{2}}(\mathsf{d}, \partial D)$. Consequently, it follows from the relation $i^*\,\mathsf{d} = \mathsf{d}\,i^*$ that $i^*\omega = \mathsf{d}\,i^*\eta$ belongs to $\mathsf{d}\,HF^{-\frac{1}{2}}(\mathsf{d}, \partial D)$. We conclude using (6.11).

(2) For the **curl** operators based on co-closed traces introduced in Sect. 6.2, it follows that

$$\mathsf{d}\,HF_0^0(\mathsf{d}, D) \subset \mathcal{D}_s. \tag{7.8}$$

This is immediate from the fact that

$$\eta \in HF_0^0(\mathsf{d}, D) \quad \text{and} \quad \omega = \mathsf{d}\,\eta \text{ implies } i^*\omega = \mathsf{d}\,i^*\eta = 0, \tag{7.9}$$

which means that ω belongs to $\mathcal{D}_s$, see (6.19).

[4] In the continuation, $\mathbf{curl}_s$ is a generic notation for a self-adjoint realization of **curl**. This notation will be used indiscriminately for the closed or co-closed case. It will be clear from the context to which operator it refers.

In the sequel, the kernel of $\mathbf{curl}_s$ will be required. We recall that

$$\mathcal{N}(\mathbf{curl}_s) = \mathcal{D}_s \cap \mathcal{N}(\mathbf{curl}_{\max})$$

is a closed subspace of $L^2(\Lambda^1(D))$. Moreover, since $\mathsf{d}^2 = 0$ and due to (7.7) and (7.8), one has

$$\mathsf{d}\, \mathit{HF}^0(\mathsf{d}, D) \subset \mathcal{N}(\mathbf{curl}_s) \quad \text{in the closed case,} \tag{7.10}$$

$$\mathsf{d}\, \mathit{HF}_0^0(\mathsf{d}, D) \subset \mathcal{N}(\mathbf{curl}_s) \quad \text{in the co-closed case.} \tag{7.11}$$

Lemma 7.2 *The operator* $\mathbf{curl}_s$ *is bounded from below on* $\mathcal{D}_s \cap \mathcal{N}(\mathbf{curl}_s)^{\perp}$*:*

$$\exists C = C(D): \quad \|\omega\| \leq C\, \|\mathbf{curl}_s\, \omega\| \quad \forall \omega \in \mathcal{D}_s \cap \mathcal{N}(\mathbf{curl}_s)^{\perp}.$$

Proof The indirect proof will be elaborated for the case of co-closed traces only. The same approach will work for closed traces.

We assume that there is a sequence $(\omega_n)_{n\in\mathbb{N}} \subset \mathcal{D}_s \cap \mathcal{N}(\mathbf{curl}_s)^{\perp}$ such that

$$\|\omega_n\| = 1, \quad \|\mathbf{curl}\, \omega_n\| \leq n^{-1} \quad \forall n \in \mathbb{N}. \tag{7.12}$$

Since $\omega_n \in \mathcal{N}(\mathbf{curl}_s)^{\perp}$, the inclusion (7.11) implies that $d^*\omega_n = 0$. As a consequence of (7.12), $(\omega_n)_{n\in\mathbb{N}}$ is a bounded sequence in $X^{\perp}$. Theorem 7.1 tells us that it will possess a subsequence that converges in $L^2(\Lambda^1(D))$, again we call it $(\omega_n)_{n\in\mathbb{N}}$. Thanks to (7.12), it will converge in the graph norm on $\mathcal{D}_s$, and the non-zero limit will belong to $\mathcal{N}(\mathbf{curl}_s) \cap \mathcal{N}(\mathbf{curl}_s)^{\perp} = \{0\}$. This contradicts $\|\omega_n\| = 1$. □

From Lemma 7.2, we conclude that the range space $\mathcal{R}(\mathbf{curl}_s)$ is a closed subspace of $L^2(\Lambda^1(D))$, which means

$$\mathcal{R}(\mathbf{curl}_s) = \mathcal{N}(\mathbf{curl}_s)^{\perp}. \tag{7.13}$$

Thus, we are led to consider the symmetric, bijective operator

$$\mathsf{C} := \mathbf{curl}_s : \mathcal{D}_s \cap \mathcal{N}(\mathbf{curl}_s)^{\perp} \subset \mathcal{N}(\mathbf{curl}_s)^{\perp} \mapsto \mathcal{N}(\mathbf{curl}_s)^{\perp}. \tag{7.14}$$

It is an isomorphism, when $\mathcal{D}_s \cap \mathcal{N}(\mathbf{curl}_s)^{\perp}$ is equipped with the graph norm, and $\mathcal{N}(\mathbf{curl}_s)^{\perp}$ with the $L^2(\Lambda^1(D))$-norm. Its inverse C^{-1} is a bounded, self-adjoint operator.

Theorem 7.3 *The operator* $\mathbf{curl}_s$ *has a pure point spectrum with* ∞ *as sole accumulation point. It possesses a complete* L^2*-orthonormal system of eigenfunctions.*

Proof The inverse operator

$$\mathsf{C}^{-1} : \mathcal{N}(\mathbf{curl}_s)^{\perp} \mapsto \mathcal{D}_s \cap \mathcal{N}(\mathbf{curl}_s)^{\perp} \tag{7.15}$$

is even *compact* as a mapping $L^2(\Lambda^1(D)) \mapsto L^2(\Lambda^1(D))$. Indeed, due to (7.10) and (7.11), the range of C^{-1} satisfies

$$\mathcal{D}_s \cap \mathcal{N}(\mathbf{curl}_s)^{\perp} \subset X_0 \text{ in the closed case,} \tag{7.16}$$

$$\mathcal{D}_s \cap \mathcal{N}(\mathbf{curl}_s)^{\perp} \subset X^{\perp} \text{ in the co-closed case.} \tag{7.17}$$

By Theorem 7.1, the compactness follows. Riesz-Schauder theory [41, Sect. X.5] tells us that, except for 0, its spectrum will be a pure (discrete) point spectrum with zero as accumulation point, and it will possess a complete orthonormal system of eigenfunctions.

The formula, see [39, Thm. 5.10],

$$\lambda^{-1} - \mathsf{C}^{-1} = \lambda^{-1}(\mathsf{C} - \lambda)\mathsf{C}^{-1} \tag{7.18}$$

shows that for $\lambda \neq 0$,

- $\lambda^{-1} - \mathsf{C}^{-1}$ bijective $\quad\Rightarrow\quad \mathsf{C} - \lambda$ bijective,
- $\mathcal{N}(\lambda^{-1} - \mathsf{C}^{-1}) = \mathcal{N}(\mathsf{C} - \lambda)$.

Thus, $\sigma(\mathsf{C}) = (\sigma(\mathsf{C}^{-1})\backslash\{0\})^{-1}$, and the eigenfunctions are the same. □

8 curl and curl curl

8.1 Self-adjoint **curl curl** operators

In the context of electromagnetism, we mainly encounter the self-adjoint operator **curl curl**. Now, we explore its relationship with the **curl** operators discussed before. The Euclidean metric on D and the associated Hodge operator $\star$ will be taken for granted.

Definition 8.1 A linear operator $\mathsf{S} : \mathcal{D}(\mathsf{S}) \subset L^2(\Lambda^1(D)) \mapsto L^2(\Lambda^1(D))$ is a **curl curl** operator, if and only if S is a closed extension of the operator $\star\mathsf{d}\star\mathsf{d}$ defined for smooth compactly supported 1-forms.

Two important extensions of the **curl curl** operator are the maximal and the minimal extensions:

Lemma 8.2 *The domain of the minimal closed extension* $(\mathbf{curl\,curl})_{\min}$ *of the* **curl curl** *operator is*

$$\mathcal{D}_{\min} = \left\{\omega \in HF_0^1(\mathsf{d}, D) : \ \star\mathsf{d}\omega \in HF_0^1(\mathsf{d}, D)\right\} \tag{8.1}$$

or, equivalently, in terms of Euclidean vector proxies

$$\begin{aligned}\mathcal{D}_{\min} = \big\{\mathbf{u} \in \boldsymbol{L}^2(D) : \ \mathbf{curl}\,\mathbf{u} \in \boldsymbol{L}^2(D), \mathbf{curl\,curl}\,\mathbf{u} \in \boldsymbol{L}^2(D),\\ \gamma_t(\mathbf{u}) = 0, \ \textit{and}\ \gamma_t(\mathbf{curl}\,(\mathbf{u})) = 0 \ \ \textit{on}\ \partial D\big\}.\end{aligned}$$

The adjoint of $(\mathbf{curl\,curl})_{\min}$ *is the maximal closed extension* $(\mathbf{curl\,curl})_{\max}$. *It is an extension of the* **curl curl** *operator with domain*

$$\mathcal{D}_{\max} = \mathcal{D}_1 \oplus \mathcal{D}_2, \tag{8.2}$$

with

$$\mathcal{D}_1 = \left\{\omega \in HF_0^1(\mathsf{d}, D) : \ \star\mathsf{d}\omega \in HF^1(\mathsf{d}, D)\right\}, \tag{8.3}$$

$$\mathcal{D}_2 = \left\{\omega \in L^2(\Lambda^1(D)) : \ \mathsf{d}\star\mathsf{d}\omega = 0\right\}. \tag{8.4}$$

Proof The domain $\mathcal{D}_{\min}$ of the minimal closure is straightforward. We recall the definition of the domain of the adjoint T^* of an operator $\mathsf{T} : \mathcal{D}(\mathsf{T}) \subset H \mapsto H$

$$\mathcal{D}(\mathsf{T}^*) = \{u \in H : \ \exists C_u > 0 : \quad (u, \mathsf{T}v)_H \leqslant C_u\, \|v\|_H \quad \forall v \in \mathcal{D}(\mathsf{T})\}. \tag{8.5}$$

Let $\mathcal{D}_{\max}$ stand for the domain of the adjoint of the minimal **curl curl** operator. First, we show that

$$\mathcal{D}_1 \oplus \mathcal{D}_2 \subset \mathcal{D}_{\max}. \tag{8.6}$$

Let us consider $\omega \in \mathcal{D}_1$ and $\eta \in \mathcal{D}_{\min}$. By integration by parts and the isometry properties of $\star$, we get

$$\int_D \omega \wedge \mathsf{d}\star\mathsf{d}\,\eta = \int_D \mathsf{d}\star\mathsf{d}\,\omega \wedge \eta \leq \|\mathsf{d}\star\mathsf{d}\,\omega\|\,\|\eta\|\,. \tag{8.7}$$

This involves $\mathcal{D}_1 \subset \mathcal{D}_{\max}$.

Now we consider $\omega \in \mathcal{D}_2$. The relation $\mathsf{d}\star\mathsf{d}\,\omega = 0$ has to be understood as

$$\int_D \mathsf{d}\star\mathsf{d}\,\omega \wedge \eta = 0 \quad \forall \eta \in \Lambda^1(D) \text{ smooth, compactly supported.} \tag{8.8}$$

As the smooth compactly supported 1-forms are dense in $\mathcal{D}_{\min}$ with respect to the topology induced by the norm

$$\|\omega\| + \|\mathbf{curl}(\omega)\| + \|\mathbf{curl}(\mathbf{curl}(\omega))\|\,, \tag{8.9}$$

it follows that

$$\int_D \omega \wedge \mathsf{d}\star\mathsf{d}\,\eta = 0 \quad \forall \eta \in \mathcal{D}_{\min}, \tag{8.10}$$

and, finally, $\mathcal{D}_2 \subset \mathcal{D}_{\max}$. This confirms (8.6).

Next, we prove

$$\mathcal{D}_{\max} \subset \mathcal{D}_1 \oplus \mathcal{D}_2. \tag{8.11}$$

Pick, $\omega \in \mathcal{D}_{\max}$. There exists $\varphi \in L^2(\Lambda^1(D))$ such that

$$\int_D \omega \wedge \mathsf{d}\star\mathsf{d}\,\eta; = \int_D \varphi \wedge \star\eta \quad \forall \eta \in \mathcal{D}_{\min}. \tag{8.12}$$

Since $\mathsf{d}^*\varphi = 0$ (pick $\eta = \mathsf{d}\,\nu$ in (8.12)), and $\int_D \varphi \wedge \star\eta_{\mathcal{H}} = 0$ for $\eta_{\mathcal{H}} \in \mathcal{H}^1(D)$, there exists $\omega_1 \in HF^1(\mathsf{d}, D)$ satisfying

$$\begin{cases} \star\mathsf{d}\star\mathsf{d}\,\omega_1 = \varphi & \text{in } D, \\ i^*\omega_1 = 0 & \text{on } \partial D. \end{cases} \tag{8.13}$$

Note that this ω_1 belongs to $\mathcal{D}_1$. Then, $\omega_2 = \omega - \omega_1$ satisfies

$$\int_D (\omega - \omega_1) \wedge \mathsf{d}\star\mathsf{d}\,\eta = 0 \quad \forall \eta \in \mathcal{D}_{\min} \quad \Longrightarrow \quad \mathsf{d}\star\mathsf{d}\,\omega_2 = 0. \tag{8.14}$$

It follows that $\omega_2 \in \mathcal{D}_2$. Since $\omega = \omega_1 + \omega_2$, we have proven (8.11). □

Remark 7 The last lemma gives a nice example for

$$(\mathsf{T}^2)^* \neq (\mathsf{T}^*)^2.$$

Indeed, the minimal extension of the formal **curl curl** boils down to the squared minimal **curl** operator $\mathbf{curl}_{\min}$ with domain $HF_0^1(\mathsf{d}, D)$

$$(\mathbf{curl}\,\mathbf{curl})_{\min} = \mathbf{curl}_{\min}\ \mathbf{curl}_{\min}$$

The adjoint of $\mathbf{curl}_{\min}$ is the $\mathbf{curl}_{\max}$ operator with domain $HF^1(\mathsf{d}, D)$, but

$$(\mathbf{curl}\,\mathbf{curl})_{\max} \neq \mathbf{curl}_{\max}\ \mathbf{curl}_{\max}\,.$$

To identify self-adjoint **curl curl** operators, we could also rely on the toolkit of symplectic algebra, using the metric-dependent symplectic pairing

$$[\omega, \eta] = \int_D \mathsf{d} \star \mathsf{d}\, \omega \wedge \eta - \int_D \omega \wedge \mathsf{d} \star \mathsf{d}\, \eta. \tag{8.15}$$

As before, complete Lagrangian subspaces will give us self-adjoint extensions of $(\mathbf{curl\,curl})_{\min}$ that are restrictions of $(\mathbf{curl\,curl})_{\max}$. However, we will not pursue this further.

There are two classical self-adjoint **curl curl** operators that play a central role in electromagnetic boundary value problems. Their domains are

$$\mathcal{D}((\mathbf{curl\,curl})_{\mathrm{Dir}}) = \left\{\omega \in H\!F_0^1(\mathsf{d}, D) : \star \mathsf{d}\, \omega \in H\!F^1(\mathsf{d}, D)\right\}, \tag{8.16}$$

$$\mathcal{D}((\mathbf{curl\,curl})_{\mathrm{Neu}}) = \left\{\omega \in H\!F^1(\mathsf{d}, D) : \star \mathsf{d}\, \omega \in H\!F_0^1(\mathsf{d}, D)\right\}. \tag{8.17}$$

Both can be written as the product of a **curl** operator and its adjoint:

$$(\mathbf{curl\,curl})_{\mathrm{Dir}} = \mathbf{curl}_{\max}\, \mathbf{curl}_{\min}, \quad (\mathbf{curl\,curl})_{\mathrm{Neu}} = \mathbf{curl}_{\min}\, \mathbf{curl}_{\max}\,. \tag{8.18}$$

Less familiar self-adjoint **curl curl** operators will emerge from taking the square of a self-adjoint **curl** operator as introduced in Sect. 6.

8.2 Square roots of **curl curl** operators

It is natural to ask whether any self-adjoint **curl curl** operator can be obtained as the square of a self-adjoint **curl**. We start with reviewing the abstract theory of square roots of operators, see [39, Sect. 7.3].

Let S be a positive (unbounded) self-adjoint operator on the Hilbert space H. We recall from [39, Thm. 7.20] that there exists a unique self-adjoint positive (unbounded) operator R satisfying

$$\mathsf{S} = \mathsf{R}^2, \text{ i.e. } \mathsf{D}(\mathsf{S}) = \mathsf{D}(\mathsf{R}^2) := \{u \in \mathsf{D}(\mathsf{R}) / \mathsf{R}u \in \mathsf{D}(\mathsf{R})\} \text{ and } \mathsf{S}u = \mathsf{R}^2 u \text{ if } u \in \mathsf{D}(\mathsf{S}). \tag{8.19}$$

Lemma 8.3 (domain of square roots) *Let* R_1 *and* R_2 *be two closed densely defined unbounded operators on* H *with domains* $\mathsf{D}(\mathsf{R}_1)$, $\mathsf{D}(\mathsf{R}_2) \subset \mathsf{H}$.

If $\mathsf{R}_1^* \mathsf{R}_1 = \mathsf{R}_2^* \mathsf{R}_2$, *that is,*

$$\mathsf{D}(\mathsf{R}_1^* \mathsf{R}_1) = \mathsf{D}(\mathsf{R}_2^* \mathsf{R}_2) \;\; \textit{and} \;\; \forall u \in \mathsf{D}(\mathsf{R}_1^* \mathsf{R}_1),\; \mathsf{R}_1^* \mathsf{R}_1 u = \mathsf{R}_2^* \mathsf{R}_2 u,$$

then $\mathsf{D}(\mathsf{R}_1) = \mathsf{D}(\mathsf{R}_2)$.

Proof For $i = 1, 2$, $\mathsf{D}(\mathsf{R}_i)$ equipped with the scalar product $(u, v)_i = (u, v)_{\mathsf{H}} + (\mathsf{R}_i u, \mathsf{R}_i v)_{\mathsf{H}}$ is a Hilbert space.

Let us first prove that $\mathsf{D}(\mathsf{R}_i^* \mathsf{R}_i)$ is dense in $\mathsf{D}(\mathsf{R}_i)$ with respect to $(\cdot, \cdot)_i$. We consider $u \in \mathsf{D}(\mathsf{R}_i^* \mathsf{R}_i)^{\perp}$

$$\forall v \in \mathsf{D}(\mathsf{R}_i^* \mathsf{R}_i), \quad 0 = (u, v)_i = (u, v)_{\mathsf{H}} + (\mathsf{R}_i u, \mathsf{R}_i v)_{\mathsf{H}} = (u, v + \mathsf{R}_i^* \mathsf{R}_i v)_{\mathsf{H}} \tag{8.20}$$

As $\mathsf{Id} + \mathsf{R}_i^* \mathsf{R}_i$ is surjective from $\mathsf{D}(\mathsf{R}_i^* \mathsf{R}_i)$ to H, see [35, Theorem 13.31], u is equal to zero.

Hence, the spaces $\mathsf{D}(\mathsf{R}_1)$ and $\mathsf{D}(\mathsf{R}_2)$ share the dense subspace $\mathsf{D}(\mathsf{R}_1^*\,\mathsf{R}_1) = \mathsf{D}(\mathsf{R}_2^*\,\mathsf{R}_2)$. Moreover, their scalar products coincide on this subset:

$$\begin{aligned}(u, v)_{\mathsf{H}} + (\mathsf{R}_1 u, \mathsf{R}_1 v)_{\mathsf{H}} &= (u, v + \mathsf{R}_1^*\,\mathsf{R}_1 v)_{\mathsf{H}} = (u, v + \mathsf{R}_2^*\,\mathsf{R}_2 v)_{\mathsf{H}} \\ &= (u, v)_{\mathsf{H}} + (\mathsf{R}_2 u, \mathsf{R}_2 v)_{\mathsf{H}}.\end{aligned}$$

We conclude using Cauchy sequences. □

Surprisingly, the simple self-adjoint operator $(\mathbf{curl}\,\mathbf{curl})_{\mathrm{Dir}}$ does not have a square root that is a self-adjoint **curl**:

Lemma 8.4 *The* **curl curl** *operator* $\mathbf{curl}_{\max}\,\mathbf{curl}_{\min}$ *does not have a square root that is a self-adjoint* **curl**.

Proof Let us suppose that $\mathsf{T} = \mathbf{curl}_{\max}\,\mathbf{curl}_{\min}$ admits a **curl** self-adjoint square root S which implies that

$$\mathbf{curl}_{\max}\,\mathbf{curl}_{\max}^* = \mathbf{curl}_{\max}\,\mathbf{curl}_{\min} = \mathsf{T} = \mathsf{S}^2 = \mathsf{S}\,\mathsf{S}^*. \tag{8.21}$$

since $\mathbf{curl}_{\max}$ and $\mathbf{curl}_{\min}$ are adjoint and S is self-adjoint. Due to lemma 8.3, we have $\mathsf{D}(\mathbf{curl}_{\max}) = \mathsf{D}(\mathsf{S})$ and therefore

$$\mathsf{S} = \mathbf{curl}_{\max} \tag{8.22}$$

since S and $\mathbf{curl}_{\max}$ are both **curl** operators. Clearly, this is not possible since $\mathbf{curl}_{\max}$ is not self-adjoint. □

Remark 8 We remark that the same arguments apply to the operator $(\mathbf{curl}\,\mathbf{curl})_{\mathrm{Neu}}$.

8.3 **curl curl** $\neq$ **curl** $\mathbf{curl}^*$ is possible

Finally, we would like to show that not all the self-adjoint **curl curl** operators are of the form $\mathsf{R}\,\mathsf{R}^*$ with R a **curl** operator.

Following an idea of Everitt and Markus—a similar construction or the Laplacian is introduced in [17]— we consider the self-adjoint **curl curl** operator

$$\mathsf{T}^0 : \mathcal{D}(\mathsf{T}^0) \subset \boldsymbol{L}^2(D) \longmapsto \boldsymbol{L}^2(D), \qquad \mathbf{u} \longmapsto \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} \tag{8.23}$$

with domain

$$\mathcal{D}(\mathsf{T}^0) = \mathcal{D}_{\min} \oplus \mathcal{D}_2, \tag{8.24}$$

where $\mathcal{D}_{\min}$ and $\mathcal{D}_2$ are defined in (8.1) and (8.4).

Proposition 8.5 *There exists no* **curl** *operator* R *such that*

$$\mathsf{T}^0 = \mathsf{R}\,\mathsf{R}^*. \tag{8.25}$$

Proof Suppose that there exists a **curl** operator R satisfying (8.25). By definition of the composition of operators, one has

$$\mathcal{D}(\mathsf{T}^0) = \left\{u \in \mathsf{D}(\mathsf{R}^*) : \ \mathsf{R}u \in \mathsf{D}(\mathsf{R})\right\}.$$

Hence, this implies

$$\mathcal{D}_2 \subset \mathcal{D}(\mathsf{T}^0) \subset \mathsf{D}(\mathsf{R}^*) \subset \mathcal{HF}^1(\mathsf{d}, D).$$

This is not possible since $\mathcal{D}_2$ is not a subspace $\mathcal{HF}^1(\mathsf{d}, D)$.

This can be illustrated by means of vector proxies and in the case of the unit sphere D. Consider the function

$$\mathbf{u}(r, \theta, z) = \left(\sum_{n=1}^{+\infty} r^n \ \sin n\theta \right) \mathbf{e}_z,$$

given the cylindrical coordinates. The **curl** and **curl curl** of **u** are

$$\mathbf{curl}\,\mathbf{u} = \left(\sum_{n=1}^{+\infty} n\ r^{n-1} \ \cos n\theta \right) \mathbf{e}_r - \left(\sum_{n=1}^{+\infty} n\ r^{n-1} \ \sin n\theta \right) \mathbf{e}_\theta,$$
$$\mathbf{curl}\,\mathbf{curl}\ \mathbf{u} = 0.$$

Direct computation leads to

$$\|\mathbf{u}\|^2 < +\infty \quad \text{and} \quad \|\mathbf{curl}\,\mathbf{u}\| = +\infty.$$

Hence, this $\mathbf{u}$ satisfies $\mathbf{u} \in \mathcal{D}_2$ but $\mathbf{u} \notin \boldsymbol{H}(\mathbf{curl}, D)$. □

Remark 9 In the same way, we show that there exists no **curl** operators R_1 and R_2 satisfying

$$\mathsf{T}^0 = \mathsf{R}_1\ \mathsf{R}_2. \tag{8.26}$$

References

1. Amrouche, C., Bernardi, C., Dauge, M., Girault, V.: Vector potentials in three–dimensional nonsmooth domains. Math. Meth. Appl. Sci **21**, 823–864 (1998)
2. Arnold, D., Falk, R., Winther, R.: Finite element exterior calculus, homological techniques, and applications. Acta Numerica **15**, 1–155 (2006)
3. Arnold, V., Khesin, B.: Topological Methods in Hydrodynamics, vol. 125 of Applied Mathematical Sciences. Springer, New York (1998)
4. Birman, M., Solomyak, M.: L_2-theory of the Maxwell operator in arbitrary domains. Russian Math. Surv. **42**, 75–96 (1987)
5. Bott, R., Tu, L.: Differential Forms in Algebraic Topology. Springer, New York (1982)
6. Buffa, A.: Hodge decompositions on the boundary of a polyhedron: the multiconnected case. Math. Mod. Meth. Appl. Sci. **11**, 1491–1504 (2001)
7. Buffa, A: Traces theorems on non-smooth boundaries for functional spaces related to Maxwell equations: An overwiew. In: Carstensen, C., Funken, S., Hackbusch, W., Hoppe, R., Monk, P. (eds.) Cmputational Electromagnetics, vol. 28 of Lecture Notes in Computational Science and Engineering., pp. 23–34. Springer, Berlin (2003)
8. Buffa, A., Ciarlet, P.: On traces for functional spaces related to Maxwell's equations. Part I: an integration by parts formula in Lipschitz polyhedra. Math. Meth. Appl. Sci. **24**, 9–30 (2001)
9. Buffa, A., Ciarlet, P.: On traces for functional spaces related to Maxwell's equations. Part II: Hodge decompositions on the boundary of Lipschitz polyhedra and applications. Math. Meth. Appl. Sci. **24**, 31–48 (2001)
10. Buffa, A., Costabel, M., Sheen, D.: On traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains. J. Math. Anal. Appl **276**, 845–867 (2002)
11. Cartan, H.: Differentialformen. Bibliographisches Institut, Zürich (1974)
12. Chandrasekhar, S., Kendall, P.: On force-free magnetic fields. Astrophys. J. **126**, 457–460 (1957)
13. Crager, J., Kotiuga, P.: Cuts for the magnetic scalar potential in knotted geometries and force-free magnetic fields. IEEE Trans. Magn. **38**, 1309–1312 (2002)
14. Dłotko, P., Specogna, R., Trevisan, F.: Automatic generation of cuts on large-sized meshes for the T-Ω geometric eddy-current formulation. Comput. Methods Appl. Mech. Eng. **198**, 3765–3781 (2009)
15. Everitt, W., Markus, L.: Complex symplectic geometry with applications to ordinary differential equations. Trans. Am. Math. Soc. **351**, 4905–4945 (1999)

16. Everitt, W., Markus, L.: Elliptic Partial Differential Operators and Symplectic Algebra, no. 770 in Memoirs of the American Mathematical Society. American Mathematical Society, Providence (2003)
17. Everitt, W., Markus, L.: Complex symplectic spaces and boundary value problems Bull. A. Math. Soc **42**, 461–500 (2005)
18. Flanders, H.: Differential Forms with Applications to the Physical Sciences. Academic Press, New York (1963)
19. Gross, P., Kotiuga, P.: Electromagnetic Theory and Computation: A Topological Approach, vol. 48 of Mathematical Sciences Research Institute Publications. Cambridge University Press, Cambridge, UK (2004)
20. Hiptmair, R.: Finite elements in computational electromagnetism. Acta Numerica **11**, 237–339 (2002)
21. Jette, A.: Force-free magnetic fields in resistive magnetohydrostatics. J. Math. Anal. Appl. **29**, 109–122 (1970)
22. Kotiuga, P.: On making cuts for magnetic scalar potentials in multiply connected regions. J. Appl. Phys. **61**, 3916–3918 (1987)
23. Kotiuga, P.: Topological duality in three-dimensional eddy-current problems and its role in computer-aided problem formulation. J. Appl. Phys. **9**, 4717–4719 (1990)
24. Kotiuga, P.: Topology-based inequalities and inverse problems for near force-free magnetic fields. IEEE. Trans. Magn. **40**, 1108–1111 (2004)
25. Lundquist, S.: Magneto-hydrostatic fields. Ark. Fysik **2**, 361–365 (1950)
26. McDuff, D., Salamon, D.: Introduction to Symplectic Topology, Oxford Mathematical Monographs. Oxford University Press, Oxford, UK (1995)
27. McLeod, K., Picard, R.: A compact imbedding result on Lipschitz manifolds. Math. Annalen **290**, 491–508 (1991)
28. Morrey, C.: Multiple integrals in the calculus of variations. vol. 130 of Grundlehren der mathematischen Wissenschaften. Springer, New York (1966)
29. Paquet, L.: Problemes mixtes pour le systeme de Maxwell. Ann. Fac. Sci. Toulouse, **4**, 103–141 (1982)
30. Picard, R.: Ein Randwertproblem in der Theorie kraftfreier Magnetfelder. Z. Angew. Math. Phys. **27**, 169–180 (1976)
31. Picard, R.: An elementary proof for a compact imbedding result in generalized electromagnetic theory. Math. Z. **187**, 151–161 (1984)
32. Picard, R.: Über kraftfreie Magnetfelder. Wissenschaftliche Zeitschrift der Technischen Universität Dresden **45**, 14–17 (1996)
33. Picard, R.: On a selfadjoint realization of curl and some of its applications. Riceche di Matematica **XLVII**, 153–180 (1998)
34. Picard, R.: On a selfadjoint realization of curl in exterior domains. Mathematische Zeitschrift **229**, 319–338 (1998)
35. Rudin, W.: Functional Analysis 1st ed. McGraw-Hill, NY (1973)
36. Salamon, D., Wehrheim, K.: Instanton Floer homology with Lagrangian boundary conditions. Geom. Topol. **12**, 747–918 (2008)
37. Weber, C.: A local compactness theorem for Maxwell's equations. Math. Meth. Appl. Sci. **2**, 12–25 (1980)
38. Weck, N.: Traces of differential forms on Lipschitz boundaries, Analysis. Int. Math. J. Anal. Appl. **24**, 147–169 (2004)
39. Weidmann, J.: Linear Operators in Hilbert spaces, vol. 68 of Graduate Texts in Mathematics. Springer, New York (1980)
40. Yoshida, Z., Giga, Y.: Remarks on spectra of operator rot. Math. Z. **204**, 235–245 (1990)
41. Yosida, K.: Functional Analysis, Classics in Mathematics 6th ed. Springer, Berlin (1980)