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Large-time behavior of the motion of a viscous heat-conducting one-dimensional gas coupled to radiation

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Abstract We study the large-time behavior of the solution of an initial-boundary value problem for the equations of 1D motions of a compressible viscous heat-conducting gas coupled to radiation through a radiative transfer equation. Assuming suitable hypotheses on the transport coefficients and adapted boundary conditions, we prove that the unique strong solution of this problem converges toward a well-determined equilibrium state at exponential rate.

Keywords Compressible · Viscous · Heat-conducting fluids · One-dimensional symmetry · Radiative transfer

Mathematics Subject Classification (2000) 35Q30 · 76N10

1 Introduction

We consider the asymptotic behavior of the compressible Navier–Stokes system when radiation, traveling at the velocity of light c, is present with coupling terms between matter and radiation, which appears naturally in various astrophysical contexts [23] and in high-temperature plasma physics [32]. These couplings, introducing momentum and energy sources, depend on the radiative intensity I driven by the so called radiative transfer integro-differential equation introduced and discussed by Chandrasekhar in [4].

Supposing that the matter is in local thermodynamical equilibrium, the coupled system for the density $\rho(x,t)$, velocity $\vec{u}(x,t)$, temperature $\theta(x,t)$, and radiative intensity $I(x,t,\vec{\Omega},\nu)$ reads [28,30]

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$$
\begin{cases}
\partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0, \\
\partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) = -\nabla \cdot \overset{\Rightarrow}{\Pi} - \vec{S}_F, \\
\partial_t (\rho \varepsilon) + \nabla \cdot (\rho \varepsilon \vec{u}) = -\nabla \vec{q} - \overset{\Rightarrow}{D} : \overset{\Rightarrow}{\Pi} - S_E, \\
\dfrac{1}{c} \dfrac{\partial}{\partial t} I \left(r, t, \vec{\Omega}, \nu\right) + \vec{\Omega} \cdot \nabla I \left(r, t, \vec{\Omega}, \nu\right) = S_t \left(r, t, \vec{\Omega}, \nu\right),
\end{cases}
\tag{1}
$$

for $(x, t, \vec{\Omega}, \nu) \in \mathbb{R}^3 \times [0, T] \times \mathbb{S}^2 \times \mathbb{R}$, where $\vec{\Omega}$ and ν are the angular variable and the frequency of the radiation, and where $\overset{\Rightarrow}{\Pi}$ is the stress tensor for matter, ε is the internal energy, $\vec{q}$ is the thermal heat flux, and $\vec{S_F}$, S_E, and S_t are the radiative coupling terms.

The foundations of the previous system have been extensively described by Pomraning [30] and Mihalas and Weibel-Mihalas [28] in the full framework of special relativity (oversimplified in our present considerations), and the system (1) has been recently investigated (in the inviscid case) by Lowrie et al. in [27], Buet and Després [3] with a special attention to asymptotic regimes and by Dubroca and Feugeas in [7], Lin in [25] and Lin et al. in [26] for various numerical aspects. Concerning the existence of solutions, a proof of local-in-time existence and blow-up of solutions (in the inviscid case) has been recently proposed by Zhong and Jiang [33] (see also the recent papers by Jiang and Wang [20,21] for a 1D related "Euler-Boltzmann" model), moreover a simplified version of the system has been investigated by Golse and Perthame [14].

In [8–10], we derived and studied the one-dimensional version of (1), which rewrites as follows

$$
\begin{cases}
\rho_\tau + (\rho v)_y = 0, \\
(\rho v)_\tau + (\rho v^2)_y + p_y = \left(\mu v_y\right)_y - (S_F)_R, \\
\left[\rho \left(e + \dfrac{1}{2} v^2\right)\right]_\tau + \left[\rho v \left(e + \dfrac{1}{2} v^2\right) + pv - \kappa \theta_y - \mu v v_y\right]_y = -(S_E)_R, \\
\dfrac{1}{c} I_\tau + \omega I_y = S,
\end{cases}
\tag{2}
$$

in the domain $\mathcal{O} \times \mathbb{R}_+$ with $\mathcal{O} := (0, L)$, where the density ρ, the velocity v, the temperature θ depend on the coordinates (y, τ). The radiative intensity $I = I(y, \tau, \nu, \omega)$ depends also on two extra variables: the radiation frequency $\nu \in \mathbb{R}_+$ and the angular variable $\omega \in S^1 := [-1, 1]$ (let us stress that here S^1 is *not* the unit circle). The state functions are the pressure p, the internal energy e, the heat conductivity κ and the viscosity coefficient μ. The thermal flux is $q = -\kappa \theta_y$

In the standard radiative transfer equation, the source term is

$$
S(y, \tau, \nu, \omega) := S_{a,e}(y, \tau, \nu, \omega) + S_s(y, \tau, \nu, \omega), \tag{3}
$$

where the absorption–emission term is

$$
S_{a,e}(y, \tau, \nu, \omega) = \sigma_a(\nu, \omega; \rho, \theta) \, [\mathcal{B}(\nu; \theta) - I(y, \tau, \nu, \omega)], \tag{4}
$$

and the scattering term is

$$
S_s(y, \tau, \nu, \omega) = \sigma_s(\nu; \rho, \theta) \, \left[\tilde{I}(y, \tau, \nu, \theta) - I(y, \tau, \nu, \omega)\right], \tag{5}
$$

where $\tilde{I}(y, \tau, \nu) := \frac{1}{2} \int_{-1}^{1} I(y, \tau, \nu, \omega) \, d\omega$ and $\mathcal{B}$ is a function of temperature and frequency describing the equilibrium state.

Typically, taking

$$\mathcal{B}(\nu;\theta) = 2h\nu^3 c^{-2}\,\mathcal{P}\left(\frac{\nu}{\theta}\right), \tag{6}$$

with $\mathcal{P}(u) := (e^{\frac{h}{k_B}\,u} - 1)^{-1}$, where k_B is the Boltzmann's constant and h is the Planck's constant, corresponds to the Planck's equilibrium distribution of photons in a cavity at temperature θ (black body).

Moreover, we suppose that $\sigma_a(\nu,\omega;\rho,\theta)$ and $\sigma_s(\nu;\rho,\theta)$ are positive functions. An example of σ_a is the Kramer's formula $\sigma_a(\nu,\theta) = \frac{C(\theta)}{\nu^3}\,(1 - e^{-\frac{h\nu}{k_B\theta}})$, where C is a positive function.

Defining the radiative energy

$$E_R := \frac{1}{c}\int_{-1}^{1}\int_{0}^{\infty} I(y,\tau,\nu,\omega)\,d\nu\,d\omega,$$

the radiative flux

$$F_R := \int_{-1}^{1}\int_{0}^{\infty} \omega I(y,\tau,\nu,\omega)\,d\nu\,d\omega,$$

and the radiative pressure

$$P_R := \frac{1}{c}\int_{-1}^{1}\int_{0}^{\infty} \omega^2 I(y,\tau,\nu,\omega)\,d\nu\,d\omega,$$

one can define in turn the radiative energy source

$$(S_E)_R := \int_{-1}^{1}\int_{0}^{\infty} S(y,\tau,\nu,\omega)\,d\nu\,d\omega,$$

and the radiative force

$$(S_F)_R := \frac{1}{c}\int_{-1}^{1}\int_{0}^{\infty} \omega S(y,\tau,\nu,\omega)\,d\nu\,d\omega.$$

We associate to (2) the initial and boundary conditions

$$\begin{cases} v|_{y=0} = v|_{y=L} = 0, \\ \theta|_{y=0} = \theta_0, \quad q|_{y=L} = 0, \end{cases} \tag{7}$$

for a given temperature $\theta_0 > 0$, and transparent boundary conditions for the radiative intensity (see [6] and [13])

$$\begin{cases} I|_{y=0} = I_b \quad \text{for } \omega \in (0,1) \\ I|_{y=L} = I_b \quad \text{for } \omega \in (-1,0), \end{cases} \tag{8}$$

for $t > 0$, and initial conditions

$$\rho|_{t=0} = \rho^0(y), \quad v|_{t=0} = v^0(y), \quad \theta|_{t=0} = \theta^0(y), \quad \text{on } \Omega. \tag{9}$$

and

$$I|_{t=0} = I^0(y, \nu, \omega) \quad \text{on } \mathcal{O} \times \mathbb{R}_+ \times S^1. \tag{10}$$

Finally, we assume that state functions e, p and κ (resp. σ_a and σ_s) are C^2 (resp. C^0) functions of their arguments for $0 < \rho < \infty$ and $0 \leq \theta < \infty$.

In (8), the function $I_b(\omega, \nu)$ is supposed to be integrable on $S^1 \times [0, T]$ and will be properly chosen below.

In [9], we considered the Lagrangian version of the previous model with transparent conditions for I (i.e., $I_b = 0$), given by the coupled system

$$\begin{cases} \eta_t = v_x, \\ v_t = \sigma_x - \eta (S_F)_R, \\ \left(e + \frac{1}{2} v^2\right)_t = (\sigma v - q)_x - \eta (S_E)_R, \\ I_t + \eta^{-1}(c\omega - v) I_x = cS, \end{cases} \tag{11}$$

in the domain $Q := \Omega \times \mathbf{R}^+$ with $\Omega := (0, M)$ (M is the total mass of matter), where the specific volume η (with $\eta := \frac{1}{\rho}$), the velocity v, the temperature θ and the radiative intensity I depend on the lagrangian mass coordinates (x, t) and also on the radiation frequency $\nu \in \mathbb{R}_+$ and the angular variable $\omega \in S^1 := [-1, 1]$.

We also denote by $\sigma := -p + \mu \, \frac{v_x}{\eta}$ the stress and by $q := -\kappa \, \frac{\theta_x}{\eta}$ the heat flux, and the source term in the last equation is

$$\begin{aligned} S(x, t, \nu, \omega) = {} & \sigma_a(\nu, \omega; \eta, \theta)\,[B(\nu, \omega; v, \theta) - I(x, t; \nu, \omega)] \\ & + \sigma_s(\nu; \eta, \theta)\, \left[\tilde{I}(x, t, \nu) - I(x, t, \nu, \omega)\right], \end{aligned} \tag{12}$$

with $\tilde{I}(x, t, \nu) := \frac{1}{2} \int_{-1}^{1} I(x, t, \nu, \omega)\, d\omega$.

In this expression, we consider, as explained in [9], a (phenomenological) relativistic modification of the equilibrium distribution $\mathcal{B}$, substituting in (6) the argument of $\mathcal{P}$ by $\frac{\nu_0}{\theta}$ with $\nu_0 = \left(1 - \frac{\omega v}{c}\right) \nu$ (just notice that $\nu_0 \sim \nu$ when $\frac{v}{c} \ll 1$), and we denote by $B(\nu, \omega; v, \theta)$ the renormalized function. We also note $B_0(\nu; \theta) := B(\omega, \nu; 0, \theta)$ the associated unrenormalized function.

The lagrangian radiative energy is

$$E_R := \frac{1}{c} \int_{-1}^{1} \int_{0}^{\infty} I(x, t, \nu, \omega)\, d\nu\, d\omega, \tag{13}$$

the radiative flux

$$F_R := \int_{-1}^{1} \int_{0}^{\infty} \omega I(x, t, \nu, \omega)\, d\nu\, d\omega, \tag{14}$$

and the radiative pressure

$$P_R := \frac{1}{c} \int_{-1}^{1} \int_{0}^{\infty} \omega^2 I(x, t, \nu, \omega)\, d\nu\, d\omega. \tag{15}$$

The radiative energy source in the right-hand side of $(11)_3$ is then

$$(S_E)_R := \int_{-1}^{1}\int_{0}^{\infty} S(x,t,\nu,\omega)\,d\nu\,d\omega, \tag{16}$$

and the radiative force source in the right-hand side of $(11)_2$ is

$$(S_F)_R := \frac{1}{c}\int_{-1}^{1}\int_{0}^{\infty} \omega S(x,t,\nu,\omega)\,d\nu\,d\omega. \tag{17}$$

Now, from $(11)_4$ and the definitions (13)–(17), one derives the equations

$$(\eta I)_t + ((c\omega - v)I)_x = c\eta S. \tag{18}$$

and after integrating in frequency and angular variables

$$\begin{cases} (\eta E_R)_t + (F_R - vE_R)_x = \eta\,(S_E)_R\,, \\ (\eta F_R)_t + (P_R - vF_R)_x = \eta\,(S_F)_R\,. \end{cases} \tag{19}$$

Dirichlet-Neumann boundary conditions for the fluid unknowns are

$$\begin{cases} v|_{x=0} = v|_{x=M} = 0, \\ \theta|_{x=0} = \theta_0, \quad q|_{x=M} = 0, \end{cases} \tag{20}$$

and transparent boundary conditions for the radiative intensity (see [6])

$$\begin{cases} I|_{x=0} = 0 \quad \text{for } \omega \in (0,1) \\ I|_{x=M} = 0 \quad \text{for } \omega \in (-1,0), \end{cases} \tag{21}$$

for $t > 0$, and initial conditions

$$\eta|_{t=0} = \eta^0(x), \quad v|_{t=0} = v^0(x), \quad \theta|_{t=0} = \theta^0(x), \quad \text{on} \quad \Omega. \tag{22}$$

and

$$I|_{t=0} = I^0(x,\nu,\omega) \quad \text{on} \quad \Omega \times \mathbb{R}_+ \times S^1. \tag{23}$$

Recall that pressure and energy of the matter are related by the thermodynamical relation

$$e_\eta(\eta,\theta) = -p(\eta,\theta) + \theta p_\theta(\eta,\theta). \tag{24}$$

We denote by Ξ and Ξ' the auxiliary functions

$$\Xi(\nu,\omega;\eta,\theta) := (\eta\sigma_a(\nu,\omega;\eta,\theta))^{1/2},$$

and

$$\Xi'(\nu,\omega;\eta,v,\theta) := B(\nu,\omega;v,\theta)\,\Xi(\nu,\omega;\eta,\theta),$$

and we assume that state functions e, p and κ (resp. σ_a and σ_s) are C^2 (resp. C^0) functions of their arguments for $0 < \eta < \infty$ and $0 \le \theta < \infty$, and, for any $\underline{\eta} \ge 0$, we suppose the following growth conditions for $\eta \ge \underline{\eta}$ and $\theta \ge 0$

$$
\begin{cases}
e(\eta, 0) \geq 0, \quad c_1(1+\theta^r) \leq e_\theta(\eta,\theta) \leq C_1(\underline{\eta})(1+\theta^r), \\
-c_2\eta^{-2}(1+\theta^{1+r}) \leq p_\eta(\eta,\theta) \leq -C_2\eta^{-2}(1+\theta^{1+r}), \\
|p_\theta(\eta,\theta)| \leq C_3(\underline{\eta})\eta^{-1}(1+\theta^r), \\
\eta p(\eta,\theta) \leq C_4(1+\theta^{1+r}), \\
c_5(\underline{\eta})(1+\theta^{1+r}) \leq p(\eta,\theta) \leq C_5(\underline{\eta})(1+\theta^{1+r}), \\
c_6(1+\theta^q) \leq \kappa(\eta,\theta) \leq C_6(\underline{\eta})(1+\theta^q), \\
|\kappa_\eta(\eta,\theta)| + |\kappa_{\eta\eta}(\eta,\theta)| \leq C_7(\underline{\eta})(1+\theta^q), \\
\left|\Xi(\nu,\omega;X,Z) - \Xi(\nu,\omega;X',Z')\right| B_0(\nu;Z') \\
\leq C_8|Z^\alpha - Z'^\alpha| f(\nu) \quad \text{for } X,X',Y,Y',Z,Z' \geq 0, \\
\left|\Xi'(\nu,\omega;X,Y,Z) - \Xi'(\nu,\omega;X',Y',Z')\right| \\
\leq C_9|Z^\alpha - Z'^\alpha| g(\nu) \quad \text{for} \quad X,X',Y,Y',Z,Z' \geq 0, \\
\eta\sigma_a(\nu,\omega;\eta,\theta) \leq C_{10}h(\nu), \\
\left(\left|(\sigma_a)_\eta\right| + \left|(\sigma_a)_\theta\right|\right)(1+B+|B_\theta|+|B_\nu|) \leq C_{11}j(\nu), \\
\eta\sigma_s(\nu;\eta,\theta) \leq C_{12}k(\nu), \\
\left(\left|(\sigma_s)_\eta\right| + \left|(\sigma_s)_\theta\right|\right)(1+B+|B_\theta|) \leq C_{13}\ell(\nu),
\end{cases}
\tag{25}
$$

where the numbers $c_j, C_j, j = 1, \ldots, 13$ are positive constants and the functions f, g, h, k, ℓ, m are such that

$$f, g \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+),$$

and

$$h, j, k, \ell \in L^{1+\gamma}(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+),$$

for any arbitrary, small $\gamma > 0$.

Concerning the viscosity, we suppose that it does not depend on temperature and that

$$0 < \mu_0 \leq \mu(\eta) \leq \mu_1, \tag{26}$$

for some positive constants μ_0 and μ_1.

Remark 1 1. The importance of relative growth of the exponents $r \geq 0$ and $q \geq 0$ has been the subject of a number of works in the context of real gas flows. For simplicity, we assume here that

$$r \in [0,1], \quad q \geq r+1,$$

but one can check that our results also hold in more general situations (see the book of Qin [31] for a general presentation).
We also suppose that

$$0 < \alpha < \frac{1}{2}(q + r + 1).$$

2. The growth hypotheses for Ξ and Ξ' mimic the behavior of the Kramer's absorption coefficient and the Planck's function (see above after formula (6)) with a $C(\theta) \sim \theta^{\alpha-1}$ (see [30] for information concerning transport coefficients, and [33] for integrated growth hypotheses of the same type).
3. The assumption $\sigma_s = \sigma_s(\nu)$ (independent of ω) is crucial in our arguments.

We consider smooth solutions of system (11) and denote by $C^{\beta}(\Omega)$ and $C^{\beta,\frac{\beta}{2}}(Q_T)$ for $0 \le \beta \le 1$ and $T > 0$, the usual and anisotropic Hölder spaces, where $Q_T := \Omega \times (0, T)$ (see [1] for complete definitions).

In the following, we use the following notation for the integrated radiative intensity

$$\mathcal{I}(x,t) := \int_0^{\infty} \int_{S^1} I(x,t;\omega,\nu)\, d\omega\, d\nu.$$

In [9], we proved the following existence result

Theorem 1 *Suppose that the initial data satisfy*

$$\left(\eta^0, \eta^0_x, v^0, v^0_x, v^0_{xx}, \theta^0, \theta^0_x, \theta^0_{xx}, \mathcal{I}^0, \mathcal{I}^0_x\right) \in \left(C^{\alpha}(\Omega)\right)^{10},$$

that $I_b \equiv 0$ and that T is an arbitrary positive number.
Let $\eta^0 > 0$ and $\theta^0 > 0$ for any $x \in \Omega$, and assume that

$$v^0(0) = v^0(M) = 0,$$
$$\theta^0_x(0) = 0, \quad \theta^0_x(M) = 0,$$

and

$$I^0(0;\omega,\nu) = I^0(M;\omega,\nu) = 0 \quad \text{for } (\omega,\nu) \in S^1 \times \mathbb{R}_+.$$

Then, system (11) with boundary conditions

$$\begin{cases} v|_{x=0} = v|_{x=M} = 0, \\ q|_{x=0} = 0, \quad q|_{x=M} = 0, \end{cases} \tag{27}$$

and (21) together with initial conditions (22)–(23) possesses a unique global solution $(\eta, v, \theta, \mathcal{I})$ such that $\eta > 0$ and $\theta > 0$ for $(x,t) \in \overline{\Omega} \times [0,T]$, and such that

$$(\eta, \eta_x, v, v_x, v_{xx}, \theta, \theta_x, \theta_{xx}, \mathcal{I}, \mathcal{I}_x) \in \left(C^{\alpha,\frac{\alpha}{2}}(Q_T)\right)^{10},$$

and

$$(\eta_{tt}, v_{xt}, \theta_{xt}) \in (L^2(Q_T))^3.$$

We also realized that the previous homogeneous problem did not admit any stationary solution, when absorption–emission term is present (in contrast, see [10] for the pure scattering case), which raised the problem of large-time behavior for its time-dependent solution.

The absence of stationary solution for the problem (11), (20)–(23) is clearly due to the boundary condition for I: the exterior of Ω plays the role of vacuum and transparent boundary conditions cannot be satisfied by the equilibrium solution of the radiative transfer equation (Planck's distribution). A way out is precisely to modify the homogeneous boundary condition by a suitable boundary (source) term I_b. One guesses that, in order to accommodate the presence of this "external vacuum," the requested boundary contribution must exactly be the radiative intensity corresponding to the static solution (if any)

$$(\eta_\infty, v_\infty = 0, \theta_\infty, I_\infty),$$

of the system (11), corresponding to $S \equiv 0$.

In fact, we have

Lemma 1 *The unique stationary solution* $(\eta_\infty(x), v_\infty(x), \theta_\infty(x), I_\infty(x))$, *of the problem* (11) *satisfying the system*

$$\begin{cases} P_x = -\eta_\infty (\mathcal{S}_F)_R, \\ Q_x = -\eta_\infty (\mathcal{S}_E)_R, \\ \omega\,(I_\infty)_x = \eta_\infty \mathcal{S}, \end{cases} \tag{28}$$

where $P = p(\eta_\infty, \theta_\infty)$, $Q = q(\eta_\infty, \theta_\infty)$,

$$\mathcal{S} := \sigma_a(\nu, \omega; \eta_\infty, \theta_\infty)\,(B_\infty - I_\infty) + \sigma_s(\nu; \eta_\infty, \theta_\infty)\left(\tilde{I}_\infty - I_\infty\right),$$

with $B_\infty = B(\nu, \omega; \eta_\infty, v_\infty, \theta_\infty, I_\infty)$, *and*

$$(\mathcal{S}_E)_R = \int_{S^1}\int_0^\infty \mathcal{S}\, d\nu\, d\omega, \quad (\mathcal{S}_F)_R = \int_{S^1}\int_0^\infty \omega\mathcal{S}\, d\nu\, d\omega,$$

with boundary conditions

$$\begin{cases} v_\infty|_{x=0} = v_\infty|_{x=M} = 0, \\ \theta_\infty|_{x=0} = \theta_0, \quad (\theta_\infty)_x|_{x=M} = 0, \end{cases} \tag{29}$$

and

$$\begin{cases} I_\infty|_{x=0} = I_b \quad \text{for } \omega \in (0,1), \\ I_\infty|_{x=M} = I_b \quad \text{for } \omega \in (-1,0), \end{cases} \tag{30}$$

for $t > 0$, is given by the formulas

$$\begin{cases} \eta_\infty(x) = \eta_0 := \frac{1}{M} \int\limits_\Omega \eta^0(x)\, dx, \\ \\ v_\infty(x) = 0, \\ \\ \theta_\infty(x) = \theta_0, \\ \\ I_\infty(x; \nu) = I_b(\nu), \end{cases} \tag{31}$$

provided that

$$I_b(\nu) \equiv B_\infty(\nu).$$

The proof is straightforward and we omit it.

Let us comment about boundary conditions for temperature. In fact, keeping Neumann boundary conditions $(27)_2$ for temperature on both ends would lead to complications in computing the equilibrium solution, due to the nonlinear dependence of the Planck's function $B(\nu; \theta)$ with respect to temperature. As we focus here on the radiative boundary conditions, we avoid this difficulty by putting the mixed condition $(29)_2$ in place of $(27)_2$.

In the sequel, we will suppose that $I_b(\nu) = B_\infty(\nu)$, for any $\omega \in S^1$. Then, our main results read

Theorem 2 *Suppose that the initial data satisfy*

$$\left(\eta^0, \eta^0_x, v^0, v^0_x, v^0_{xx}, \theta^0, \theta^0_x, \theta^0_{xx}, \mathcal{I}^0, \mathcal{I}^0_x\right) \in \left(C^\alpha(\Omega)\right)^{10},$$

and that T is an arbitrary positive number.

Let $\eta^0 > 0$ and $\theta^0 > 0$ for any $x \in \Omega$, and assume the compatibility conditions

$$v^0(0) = v^0(M) = 0,$$
$$\theta^0(0) = \theta_0, \quad \theta^0_x(M) = 0,$$

and

$$I^0(0; \omega, \nu) = I^0(M; \omega, \nu) = I_b(\nu) \quad \textit{for } (\omega, \nu) \in S^1 \times \mathbb{R}_+.$$

Then system (11) with boundary conditions (20)

and

$$\begin{cases} I|_{x=0} = I_b(\nu) & \text{for } \omega \in (0, 1), \\ \\ I|_{x=M} = I_b(\nu) & \text{for } \omega \in (-1, 0), \end{cases} \tag{32}$$

where I_b is fixed as in Lemma 1, together with initial conditions (22–23) possesses a unique global solution $(\eta, v, \theta, \mathcal{I})$ such that $\eta > 0$ and $\theta > 0$ for $(x, t) \in \overline{\Omega} \times [0, T]$, and such that

$$(\eta, \eta_x, v, v_x, v_{xx}, \theta, \theta_x, \theta_{xx}, \mathcal{I}, \mathcal{I}_x) \in \left(C^{\alpha, \frac{\alpha}{2}}(Q_T)\right)^{10},$$

and

$$(\eta_{tt}, v_{xt}, \theta_{xt}) \in (L^2(Q_T))^3.$$

Theorem 3 *Let the assumptions of Theorem 2 hold. The solution described in Theorem 2 converges to the constant state*

$$(\eta_\infty, v_\infty = 0, \theta_\infty, I_\infty),$$

given by Lemma 1.

The decay takes place in $H^1(\Omega)$ for the fluid variables η, v and θ, and in $L^2(\Omega)$ for the radiative intensity $\mathcal{I}$.

Moreover, there exist two positive numbers T_∞ and γ such that

$$\|\eta - \eta_\infty\|_{L^2(\Omega)} + \|\theta - \theta_0\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} + \|\mathcal{I} - \mathcal{I}_\infty\|_{L^2(\Omega)} \leq K e^{-\gamma t}, \tag{33}$$

for $t \geq T_\infty$.

Finally, for the decoupled system, we have the simple improvement

Proposition 1 *Suppose that system (11) is decoupled i.e.,*

$$\sigma_a \equiv 0 \quad \text{and} \quad \sigma_s \equiv 0.$$

If the conditions of Theorem 3 hold, then the solution described in Theorem 2 decays as previously to the constant state $(\eta_\infty, v_\infty = 0, \theta_\infty, I_\infty)$ given by Lemma 1.

Moreover, there exist two positive numbers T'_∞ and γ' such that

$$\|\eta - \eta_\infty\|_{H^1(\Omega)} + \|\theta - \theta_0\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)} + \|\mathcal{I} - \mathcal{I}_\infty\|_{L^2(\Omega)} \leq K e^{-\gamma' t}, \tag{34}$$

for $t \geq T'_\infty$.

One first observes that Theorem 2 is a direct extension of Theorem 1. In fact, one checks that modifying the boundary conditions does not essentially modify the proof of Theorem 1 in [9], so we only sketch its main steps in Appendix.

To achieve the proof of Theorem 3 (and of Proposition 1), we need to get suitable time-independent estimates, which constitutes the main part of this article, and we adapt to the radiative case the general strategy of Jiang in [18].

Remark 2 1. Let us recall that the investigation of existence and asymptotic for 1D viscous heat-conducting (nonradiative) flows for compressible media goes back to the pioneer work of Antonsev-Kazhikov-Monakov [1] and has been largely extended to real gases by a number of authors (see among related works: Kawohl [22], Dafermos-Hsiao [5], Jiang [17–19], and also Hsiao [15], Hsiao-Jiang [16] and Qin [31] for recent presentations in the heat conductive case). The full theory in 3D case can be found in the work of Feireisl (see [12]).

2. Just mention that when local thermodynamical equilibrium for matter *and* radiation is almost achieved, a proper scaling in the Chapman–Enskog expansion (see [3]) leads to the so called equilibrium diffusion limit, decoupled from the radiative transfer equation for I, given by

$$\begin{cases} \eta_t = v_x, \\ v_t = \left(-(p + p_r) + \frac{\mu}{\eta}\, v_x\right)_x, \\ \left(e + e_r + \frac{1}{2}\, v^2\right)_t = (\sigma v - (q + q_r))_x, \end{cases} \tag{35}$$

with effective state functions $p_r = \frac{a}{3}\,\theta^4$, $e_r = a\eta\theta^4$ and $q_r = -\kappa_r(\theta)\,\frac{\theta_x}{\eta}$. For this model, one can prove an exponential decay for (η, v, θ) of the type described in Theorem 3, see [11].

2 Time-independent a priori estimates

Let T be an arbitrary positive number and let us denote by K, K_j $j = 1, 2, \ldots$ various positive constants which do not depend on T, but only on the physical constants of the problem.

We first get usual mass-energy estimates

Lemma 2 *Under the following condition on the data*

$$\|v^0\|_{L^2(\Omega)} + \|\eta^0\|_{L^1(\Omega)} + \|\theta^0\|_{(L^1 \cap L^{r+1})(\Omega)} + \|I^0\|_{L^1(\Omega\times\mathbb{R}_+\times S^1)} \leq N, \tag{36}$$

there exists a positive constant $K = K(N)$ such that

1. *the mass conservation*

$$\int_\Omega \eta\, dx = \int_\Omega \eta^0\, dx, \tag{37}$$

2. *the energy–entropy inequality*

$$\int_\Omega \left[e + \frac{1}{2}v^2 + \frac{1}{c}\,\eta E_R\right] dx + \int_{Q_t} \left(\frac{\kappa(\eta,\theta)}{\eta\theta^2}\,\theta_x^2 + \frac{\mu(\eta)}{\eta\theta}\,v_x^2\right) dx\, ds$$
$$\leq \int_\Omega \left[e^0 + \frac{1}{2}(v^0)^2 + \frac{1}{c}\,\eta E_R^0\right] dx, \tag{38}$$

   where $E_R^0(x) = \int_0^\infty \int_{S^1} \omega I^0(x,\nu,\omega)\, d\nu\, d\omega$

3. *the estimates*

$$\|\eta\|_{L^\infty(0,T;L^1(\Omega))} + \|v\|_{L^\infty(0,T;L^2(\Omega))} + \|\theta\|_{L^\infty(0,T;L^\delta(\Omega))} \leq K, \tag{39}$$

   for any $1 \leq \delta \leq r+1$, and

$$\|\eta E_R\|_{L^\infty(0,T;L^1(\Omega))} \leq K, \tag{40}$$

4. *the condition*

$$\theta(x,t) > 0 \quad \text{for any} \quad (x,t) \in Q_T, \tag{41}$$

hold.

Proof 1. Integrating the first Eq. (11) and using boundary conditions give (37).

2. Total entropy $s = s_m + s_R$ is the sum of the entropy of matter s_m and entropy of radiation s_R, and the second principle of thermodynamics tells us that $\theta(s_m)_t = e_t + p\eta_t$, so using (11), one finds

$$(s_m)_t = \left(\frac{\kappa\theta_x}{\eta\theta}\right)_x + \frac{\mu v_x^2}{\eta\theta} + \frac{\kappa\theta_x^2}{\eta\theta^2} - \frac{\eta}{\theta}\,(S_E)_R + \frac{\eta}{\theta}\,v(S_F)_R. \tag{42}$$

   From statistical mechanics, the entropy per mode of a boson gas is $k_B[(n+1)\log(n+1) - n\log n]$, where n is the occupation number related to I by

$$n = n(I) := \frac{c^2}{2h}\,\frac{I}{\nu^3}.$$

Multiplying by the number of modes, we find the entropy per mass unit

$$s_R = \eta \int_0^\infty \int_{S^1} \frac{2k_B \nu^2}{c^3} \left[(n+1)\log(n+1) - n\log n\right] d\nu\, d\omega.$$

Using the last Eq. (11), observing that for any regular function $n \to \chi(n)$ one has the identity

$$(\eta\chi)_t + [(c\omega - v)\chi]_x = \frac{c^3}{2h\nu^3}\, \chi' \eta S,$$

and choosing $\chi(n) = (n+1)\log(n+1) - n\log n$, we get after a direct computation

$$(s_R)_t + \left[\int_0^\infty \int_{S^1} \frac{2k_B \nu^2}{c^3} (c\omega - v) \left[(n+1)\log(n+1) - n\log n\right] d\nu\, d\omega\right]_x$$
$$= \eta \int_0^\infty \int_{S^1} \frac{k_B}{h\nu} \log\frac{n+1}{n}\, S\, d\nu\, d\omega =: Q_R. \tag{43}$$

Decomposing

$$\eta \int_0^\infty \int_{S^1} \frac{k_B}{h\nu} \log\frac{n+1}{n}\, S\, d\nu\, d\omega$$
$$= \eta \int_0^\infty \int_{S^1} \frac{k_B}{h\nu} \log\frac{n+1}{n}\, \sigma_a(B - I)\, d\nu\, d\omega$$
$$+ \eta \int_0^\infty \int_{S^1} \frac{k_B}{h\nu} \log\frac{n+1}{n}\, \sigma_s(\tilde I - I)\, d\nu\, d\omega,$$

and checking the identity

$$\log\frac{n(B)+1}{n(B)} = \left(1 - \frac{\omega v}{c}\right) \frac{h\nu}{k_B\theta},$$

the right-hand side of (43) reads

$$Q_R = \eta \int_0^\infty \int_{S^1} \frac{k_B}{h\nu} \left[\log\frac{n(I)+1}{n(I)} - \log\frac{n(B)+1}{n(B)}\right] \sigma_a(B - I)\, d\nu\, d\omega$$
$$+ \frac{\eta}{\theta}\, (S_E)_R - \frac{\eta}{\theta}\, v\, (S_F)_R$$
$$+ \eta \int_0^\infty \int_{S^1} \frac{k_B}{h\nu} \left[\log\frac{n(I)+1}{n(I)} - \log\frac{n(\tilde I)+1}{n(\tilde I)}\right] \sigma_s(\tilde I - I)\, d\nu\, d\omega.$$

As $u \to \log\frac{u+1}{u}$ is decreasing for $u > 0$, the first and last terms are positive. So, using the isotropy of scattering in the lagrangian coordinates (see [3] for a general derivation, and also [9]), we get finally

$$
\begin{aligned}
(s_R)_t = & -\left[\int_0^\infty \int_{S^1} \frac{2k_B \nu^2}{c^3}(c\omega - v)\,[(n+1)\log(n+1) - n\log n]\,d\nu\,d\omega\right]_x \\
& -\eta \int_0^\infty \int_{S^1} \frac{k_B}{h\nu}\left[\log\frac{n(I)+1}{n(I)} - \log\frac{n(\tilde{I})+1}{n(\tilde{I})}\right]\sigma_s(\tilde{I} - I)\,d\nu\,d\omega \\
& -\eta \int_0^\infty \int_{S^1} \frac{k_B}{h\nu}\left[\log\frac{n(I)+1}{n(I)} - \log\frac{n(B)+1}{n(B)}\right]\sigma_a(B - I)\,d\nu\,d\omega \\
& +\frac{\eta}{\theta}\,(S_E)_R - \frac{\eta}{\theta}\,v\,(S_F)_R\,.
\end{aligned} \tag{44}
$$

Using the technique of [17] and defining the free energy $\psi := e - \theta s_m$ of the fluid, with $\psi_\theta = -s_m$ and $\psi_\eta = -p$, let us introduce the auxiliary function

$$
\mathcal{E}(\eta,\theta) := \psi(\eta,\theta) - \psi(\eta_0,\theta_0) - (\eta-\eta_0)\psi_\eta(\eta_0,\theta_0) - (\theta-\theta_0)\psi_\theta(\eta,\theta) - \theta_0 s_R.
$$

Using (44), we compute

$$
\begin{aligned}
& \left(\mathcal{E} + \frac{1}{2}\,v^2 + \eta E_R\right)_t + \theta_0\left(\frac{\mu v_x^2}{\eta\theta} + \frac{\kappa\theta_x^2}{\eta\theta^2}\right) \\
& +\theta_0\eta \int_0^\infty \int_{S^1} \frac{k_B}{h\nu}\left[\log\frac{n(I)+1}{n(I)} - \log\frac{n(\tilde{I})+1}{n(\tilde{I})}\right]\sigma_s(I - \tilde{I})\,d\nu\,d\omega \\
& +\theta_0\eta \int_0^\infty \int_{S^1} \frac{k_B}{h\nu}\left[\log\frac{n(I)+1}{n(I)} - \log\frac{n(B)+1}{n(B)}\right]\sigma_a(I - B)\,d\nu\,d\omega \\
= & \left[\sigma v + p(\eta_0,\theta_0)v - \left(1 - \frac{\theta_0}{\theta}\right)q - F_R + vE_R \right. \\
& \left. + \int_0^\infty \int_{S^1} \frac{2k_B \nu^2}{c^3}(c\omega - v)\,[(n+1)\log(n+1) - n\log n]\,d\nu\,d\omega\right]_x.
\end{aligned} \tag{45}
$$

Integrating on Q_t and using (38) and (20), the contribution of the first three boundary term is zero. Moreover using (21) to compute the contribution of the radiative terms boundary terms, we have the final equality

$$
\begin{aligned}
&\int_\Omega \left(\mathcal{E}+\frac{1}{2}\,v^2+\eta E_R\right)\,dx+\theta_0\int_{Q_t}\left(\frac{\mu v_x^2}{\eta\theta}+\frac{\kappa\theta_x^2}{\eta\theta^2}\right)\,dx\,ds\\
&\quad+\theta_0\int_{Q_t}\eta\int_0^\infty\int_{S^1}\frac{k_B}{h\nu}\left[\log\frac{n(I)+1}{n(I)}-\log\frac{n(\tilde{I})+1}{n(\tilde{I})}\right]\sigma_s(I-\tilde{I})\,d\nu\,d\omega\,dx\,ds\\
&\quad+\theta_0\int_{Q_t}\eta\int_0^\infty\int_{S^1}\frac{k_B}{h\nu}\left[\log\frac{n(I)+1}{n(I)}-\log\frac{n(B)+1}{n(B)}\right]\sigma_a(I-B)\,d\nu\,d\omega\,dx\,ds\\
&\quad+\int_0^t\int_0^\infty\int_0^1\omega I(M,s;\omega,\nu)\,d\nu\,d\omega\,ds-\int_0^t\int_0^\infty\int_{-1}^0\omega I(0,s;\omega,\nu)\,d\nu\,d\omega\,ds\\
&\quad+\theta_0\int_0^t\int_0^\infty\int_0^1\frac{2k_B\nu^2}{c^2}\,\omega\,[(n+1)\log(n+1)-n\log n](M,s;\omega,\nu)\,d\nu\,d\omega\,ds\\
&\quad-\theta_0\int_0^t\int_0^\infty\int_{-1}^0\frac{2k_B\nu^2}{c^2}\,\omega\,[(n+1)\log(n+1)-n\log n](0,s;\omega,\nu)\,d\nu\,d\omega\,ds\\
&=\int_\Omega\left(\mathcal{E}^0+\frac{1}{2}\,{v^0}^2+\eta^0E_R^0\right)\,dx=:\mathcal{E}_0.
\end{aligned}
\tag{46}
$$

Now, we argue in the same way as [17] noting that, by using Taylor formula, for any $\eta>0$

$$
\begin{aligned}
&\mathcal{E}(\eta,\theta)-\psi(\eta,\theta)+\psi(\eta,\theta_0)+(\theta-\theta_0)\psi_\theta(\eta,\theta)-\theta_0 s_R\\
&\quad=\psi(\eta,\theta_0)-\psi(\eta_0,\theta_0)-(\eta-\eta_0)\psi_\eta(\eta_0,\theta_0)\geq 0,
\end{aligned}
$$

and that

$$
\psi(\eta,\theta)=\psi(\eta,\theta_0)-(\theta-\theta_0)\psi_\theta(\eta,\theta_0)+\int_{\theta_0}^{\theta}(\theta-\alpha)\psi_{\theta\theta}(\eta,\alpha)\,d\alpha.
$$

Using $\psi_{\theta\theta}=-\theta^{-1}e_\theta$ and estimates (25), we find

$$
\begin{aligned}
&\psi(\eta,\theta)-\psi(\eta,\theta_0)-(\theta-\theta_0)\psi_\theta(\eta,\theta)\\
&\quad\geq c_1(\theta-\theta_0)^2\times\int_0^1\frac{\theta_0+[\theta_0+s(\theta-\theta_0)]^r}{\theta_0+s(\theta-\theta_0)}\,(1-s)\,ds\\
&\quad\geq c_1\left(\frac{\theta}{\theta_0}-\log\frac{\theta}{\theta_0}-1\right)+\Psi_r(\theta),
\end{aligned}
\tag{47}
$$

where

$$
\Psi_r(z)=\left|\begin{array}{l}
c_1(z-\log z-1)\quad\text{for } r=0,\\[2ex]
\frac{1}{1+r}\,z^{1+r}+\left(\frac{1}{1+r}-z^r\right)\quad\text{for } r>0,
\end{array}\right.
$$

so

$$\mathcal{E}(\eta,\theta) - \theta_0 s_R \geq c_1\,(\theta - \log\theta - 1).$$

Now, one checks by elementary computations that $\eta E_R - \theta_0 s_R \geq K$, so we deduce that

$$\mathcal{E}(\eta,\theta) + \eta E_R \geq \frac{c_1}{2}\,\theta + \frac{1}{2(1+r)}\,\theta^{1+r} - K,$$

and we conclude that (38) holds.

3. Estimates (39) and (40) follow directly from (37) and (38).
4. Using (25), the positivity of $\theta(x,t)$ follows from that of $\theta^0(x)$ after the maximum principle applied to the third Eq. (11) □

Lemma 3 *Any solution of the integro-differential problem*

$$\begin{cases} \dfrac{\partial}{\partial t}\,[\eta I(x,t;\nu,\omega)] + \dfrac{\partial}{\partial x}\,[(c\omega - v)\,I(x,t;\nu,\omega)] \\ = c\eta\sigma_a(\nu,\omega;\eta,\theta)\,[B(\nu,\omega;v,\theta) - I(x,t;\nu,\omega)] \\ \quad + c\eta\sigma_s(\nu,\eta;\theta)\left[\tilde{I}(x,t;\nu) - I(x,t;\nu,\omega)\right] \quad \text{on} \quad \Omega\times[0,T]\times\mathbb{R}_+\times S^1, \\ I(0;\nu,\omega) = I_b \quad \text{for } \omega\in(0,1), \\ I(M;\nu,\omega) = I_b \quad \text{for } \omega\in(-1,0), \\ I(x,0;\nu,\omega) = I^0(x;\nu,\omega) \quad \text{on } \Omega\times\mathbb{R}_+\times S^1 \end{cases} \tag{48}$$

satisfies the following bounds

$$\max_{[0,T]}\int_\Omega\int_0^\infty\int_{S^1}\eta I^2(x,t;\nu,\omega)\,d\omega\,d\nu\,dx \leq K, \tag{49}$$

$$\int_{Q_T}\int_0^\infty\int_{S^1}\eta\sigma_a(\eta,\theta;\nu,\omega)\,(I(x,t;\nu,\omega) - I_b(\nu))^2\;d\omega\,d\nu\,dx\,dt \leq K, \tag{50}$$

$$\int_{Q_T}\int_0^\infty\int_{S^1}\eta\sigma_s(\eta,\theta;\nu)\left(\tilde{I}(x,t;\nu) - I(x,t;\nu,\omega)\right)^2\;d\omega\,d\nu\,dx\,dt \leq K. \tag{51}$$

$$\int_0^T\int_0^\infty\int_{S^1}\omega\,(I(M,t;\nu,\omega) - I_b(\nu,\omega))^2\;d\omega\,d\nu\,dt \leq K, \tag{52}$$

$$\int_0^T\int_0^\infty\int_{S^1}\omega\,(I(0,t;\nu,\omega) - I_b(\nu,\omega))^2\;d\omega\,d\nu\,dt \leq K, \tag{53}$$

$$\|\eta\,(S_E)_R\|_{L^2(Q_T)} \leq K, \tag{54}$$

$$\|\eta\,(S_F)_R\|_{L^2(Q_T)} \leq K. \tag{55}$$

Proof 1. Setting $J := I - I_b$ and observing that $\tilde{I}_b = I_b$, $(48)_1$ rewrites

$$(\eta J)_t + [(c\omega - v)J]_x + \eta\sigma_a J = \eta\sigma_a(B - B_\infty) + \eta\sigma_s(\tilde{J} - J).$$

Multiplying by J we get

$$\frac{1}{2}\,(\eta J^2)_t + \frac{1}{2}\,[(c\omega - v)J^2]_x + \eta\sigma_a J^2 + \eta\sigma_s(\tilde{J} - J)^2$$
$$+\eta\sigma_s(\tilde{J} - J)^2 \le \frac{1}{2}\eta\sigma_a(B - B_\infty)^2 + \frac{1}{2}\eta\sigma_a J^2 + \eta\sigma_s\tilde{J}(\tilde{J} - J).$$

Integrating on $\Omega \times S^1$ and using boundary conditions and Cauchy–Schwarz inequality, we get

$$\frac{1}{2}\,\frac{d}{dt}\int_\Omega\int_{S^1} \eta J^2\,dx\,d\omega + \frac{c}{2}\int_{S^1} \omega J^2(M, t; \nu, \omega)\,d\omega$$
$$-\frac{c}{2}\int_{S^1} \omega I^2(0, t; \nu, \omega)\,d\omega + \frac{1}{2}\int_\Omega\int_{S^1} \eta\sigma_a J^2\,dx\,d\omega$$
$$+\int_\Omega\int_{S^1} \eta\sigma_s(\tilde{J} - J)^2\,dx\,d\omega \le \int_\Omega\int_{S^1} \eta\sigma_a(B - B_\infty)^2\,dx\,d\omega.$$

Integrating on time and frequency and using (25), we have

$$\frac{1}{2}\int_\Omega\int_0^\infty\int_{S^1} \eta J^2\,d\omega\,d\nu\,dx - \frac{1}{2}\int_\Omega\int_0^\infty\int_{S^1} \eta {J^0}^2\,d\omega\,d\nu\,dx$$
$$+\frac{c}{2}\int_0^T\int_0^\infty\int_0^1 \omega J^2(M, t; \nu, \omega)\,d\omega\,d\nu\,dt - \frac{c}{2}\int_0^T\int_0^\infty\int_{-1}^0 \omega J^2(0, t; \nu, \omega)\,d\omega\,d\nu\,dt$$
$$+\frac{1}{2}\int_{Q_T}\int_0^\infty\int_{S^1} \eta\sigma_a J^2\,dx\,d\omega\,d\nu\,dx\,dt + \int_{Q_T}\int_0^\infty\int_{S^1} \eta\sigma_s(\tilde{J} - J)^2\,dx\,d\omega\,d\nu\,dt$$
$$\le \int_{Q_T}\int_0^\infty\int_{S^1} |\Xi(\nu, \omega; \eta, \theta) - \Xi(\nu, \omega; \eta_0, \theta_0)|\,B_0(\nu; \theta_0)\,d\omega\,d\nu\,dx\,ds$$
$$+\int_{Q_T}\int_0^\infty\int_{S^1} \left|\Xi'(\nu, \omega; \eta, v, \theta) - \Xi(\nu, \omega; \eta_0, 0, \theta_0)\right|\,d\omega\,d\nu\,dx\,ds$$
$$\le d\int_{Q_T} (\theta^\alpha - \theta_0^\alpha)^2\,dx\,dt,$$

where $d := C_8^2\|f\|^2_{L^2(\mathbb{R}_+)} + C_9^2\|g\|^2_{L^2(\mathbb{R}_+)}$.
To bound the integral in the right-hand side, we observe that, for any $\lambda > 0$

$$\left(\theta^\lambda(x, t) - {\theta_0}^\lambda\right)^2 \le KV(t)\int_\Omega \theta^{2\lambda - q}\,dx, \tag{56}$$

where $t \to V(t) := \int_\Omega \frac{\kappa}{\eta\theta^2}\, \theta_x^2\, dx \in L^1(0,T)$, after Lemma 2. In particular

$$\int_0^t \left(\theta^\lambda(x,t) - {\theta_0}^\lambda\right)^2\, ds \le K,$$

for any $2\lambda \le q$. So choosing $\lambda = \alpha/2$, we find that $U \le K$, and we get (49), (50) and (51) together with (52) and (53).

2. Decomposing the radiative source as

$$S = \eta\sigma_a(B - B_\infty) + \eta\sigma_a(I_b - I) + \eta\sigma_s(\tilde{I} - I),$$

using Cauchy–Schwarz inequality together with (57) and the previous bounds (50) and (51), inequalities (54) and (55) follow □

Lemma 4 *Under the previous condition on the data* (36), *there exist positive constants* $\underline{\eta}$ *and* $\overline{\eta}$ *independent of* T *such that*

$$\underline{\eta} \le \eta(x,t) \le \overline{\eta} \quad \text{for } (t,x) \in Q_T. \tag{57}$$

Proof As we follow the line of the proof of Jiang [18], we only sketch the necessary modifications involving essentially the source $(S_F)_R$ and the variable viscosity.

Introducing the strictly increasing function $s \to \mathcal{M}(s) := \int_1^s \frac{\mu(\xi)}{\xi}\, d\xi$, one observes that $\mathcal{M}$ maps $(0, \inf_\Omega \eta^0]$ onto $(-\infty, 0)$.

If $\phi(x,t) := \int_0^t \sigma\, ds + \int_0^x v^0\, dy - \int_0^t \int_0^x \eta(S_F)_R\, dy\, ds$, then ϕ satisfies the equations $\phi_x = v$ and $\phi_t = \frac{\mu(\eta)}{\eta}\, v_x - p - \int_0^x \eta(S_F)_R\, dy$. Multiplying the last equation by η, we find that

$$(\eta\phi)_t = (v\phi)_x + \mu\phi_{xx} - p\eta - v^2 - \eta \int_0^x \eta(S_F)_R\, dy.$$

Integrating on Q_t, and using boundary conditions we find

$$\begin{aligned} \int_\Omega \phi\eta\, dx &= \int_\Omega \phi^0\eta^0\, dx \\ &\quad - \int_{Q_t} \left(p\eta + v^2\right)\, dx\, ds - \int_0^t \int_\Omega \eta \int_0^x \eta(S_F)_R\, dy\, dx\, ds. \end{aligned} \tag{58}$$

Using (37) and a standard argument of [1], there exists a point $X(t) \in \Omega$ such that $\phi(X(t),t) = \frac{1}{R} \int_\Omega \phi\eta\, dx$ with $R := \int_\Omega \eta^0\, dx$. Then, after the definition of ϕ and (58), we find

$$\int_0^t \sigma(X(t),t)\,ds + \int_0^{X(t)} v^0 dy - \int_0^t \int_0^{X(t)} \eta(S_F)_R\,dy\,ds$$
$$= \frac{1}{R}\left\{\int_\Omega \eta^0(x)\int_0^x v^0(y)dy\,dx - \int_{Q_t}\left(p\eta + v^2\right)\,dx\,ds\right.$$
$$\left. - \int_0^t \int_\Omega \eta \int_0^x \eta(S_F)_R\,dy\,dx\,ds\right\}. \tag{59}$$

Now rewriting the second Eq. (11) as $\mathcal{M}_{xt} = v_t + p_x + \eta\,(S_F)_R$ and integrating it first on $[0,t]$ then on $[X(t),x]$, we find

$$\mathcal{M}(x,t) - \mathcal{M}(X(t),t) - \mathcal{M}^0(x) + \mathcal{M}^0(X(t))$$
$$= \int_{X(t)}^x \left(v(y,t) - v^0(y)\right)\,dy + \int_0^t p(x,s)\,ds - \int_0^t p(X(t),s)\,ds$$
$$+ \int_{X(t)}^x \int_0^t \eta(S_F)_R\,ds\,dy.$$

After the definition of $\mathcal{M}$

$$\int_0^t \sigma(X(t),s)\,ds = -\int_0^t p(X(t),s)\,ds + \mathcal{M}(X(t),t) - \mathcal{M}^0(X(t)),$$

so, we get

$$\mathcal{M}(\eta(x,t)) = \mathcal{M}(\eta^0(x)) + \int_0^t p\,ds + \int_{X(t)}^x \left(v(y,t) - v^0(y)\right)\,dy$$
$$+ \int_0^t \sigma(X(t),s)\,ds + \int_{X(t)}^x \int_0^t \eta\,(S_F)_R\,ds\,dx, \tag{60}$$

and using (59), we obtain

$$\mathcal{M}(\eta(x,t)) = \mathcal{M}(\eta^0(x)) + \int_0^t p\,ds - \frac{1}{R}\int_0^t \int_\Omega \eta \int_0^x \eta\,(S_F)_R\,dy\,dx\,ds$$
$$+ \int_{X(t)}^x \left(v(y,t) - v^0(y)\right)\,dy - \int_0^{X(t)} v^0 dy$$

$$+\frac{1}{R}\int_{\Omega}\eta^0(x)\int_0^x v^0(y)dy\,dx+\int_0^t\int_0^x \eta(S_F)_R\,dy\,ds$$
$$-\frac{1}{R}\int_0^t\int_{\Omega}(v^2+p\eta)\,dx\,ds.$$

Integrating by parts in the second integral in the right-hand side, we get

$$\begin{aligned}
&\int_0^t\int_{\Omega}\eta\int_0^x \eta\,(S_F)_R\;dy\,dx\,ds\\
&=\int_0^t\left[\left(\int_x^M \eta\,dy\right)\left(\int_0^x \eta(S_F)_R\,dy\right)\Bigg|_0^M+\int_0^t\int_{\Omega}\left(\int_x^M \eta\,dy\right)\eta\,(S_F)_R\;dx\right]ds\\
&=\int_0^t\int_{\Omega}\left(\int_x^M \eta\,dy\right)\eta\,(S_F)_R\;dx\,ds.
\end{aligned}$$

So, we get

$$\mathcal{M}(\eta(x,t))=\int_0^t p^*\,ds+\Psi(x,t), \tag{61}$$

where

$$p^*(x,t):=p(x,t)+f(t),$$

with

$$f(t):=-\frac{1}{R}\int_0^t\int_{\Omega}\eta\int_0^x \eta\,(S_F)_R\;dy\,dx\,ds,$$

and

$$\begin{aligned}
\Psi(x,t):=\mathcal{M}(\eta^0(x))+\int_{X(t)}^x \big(v(y,t)-v^0(y)\big)\;dy-\int_0^{X(t)} v^0dy\\
+\frac{1}{R}\int_{\Omega}\eta^0(x)\int_0^x v^0(y)dy\,dx-\frac{1}{R}\int_0^t\int_{\Omega}(v^2+p\eta)\,dx\,ds=:\sum_{j=1}^5\Psi_j.
\end{aligned}$$

Integrating (26), we find $\mu_0\log\eta\le\mathcal{M}(\eta)\le\mu_1\log\eta$, then after a standard computation we get from (61) the inequalities

$$\eta(x,t)\le\left\{\exp\left(\frac{1}{\mu_1}\;\Psi(x,t)\right)\left[1+\frac{1}{\mu_1}\int_0^t(p^*\eta)(x,s)\exp\left(-\frac{1}{\mu_1}\;\Psi(x,s)\right)ds\right]\right\}^{\frac{\mu_1}{\mu_0}}, \tag{62}$$

and

$$\eta(x,t) \geq \left\{ \exp\left(\frac{1}{\mu_0}\,\Psi(x,t)\right)\left[1+\frac{1}{\mu_0}\int_0^t (p^*\eta)(x,s)\exp\left(-\frac{1}{\mu_0}\,\Psi(x,s)\right)ds\right]\right\}^{\frac{\mu_0}{\mu_1}}, \tag{63}$$

so we are led to bound the right (resp. left)-hand side in (62) (resp. (63)).

One first easily check by using conditions on initial data, (38) and Lemma 3 that

$$K^{-1} \leq \exp\left(\frac{1}{\mu_0}\sum_{j=1}^{4}\Psi_j(x,t)\right) \leq K.$$

Then, we get

$$\eta(x,t) \leq K\left[\int_0^t (p\eta)(x,s)\exp\left(-\frac{1}{\mu_1}\int_s^t\int_\Omega \left\{v^2+p\eta\right\}\,dx\,d\tau\right)\,ds\right]^{\frac{\mu_1}{\mu_0}}, \tag{64}$$

and

$$\eta(x,t) \geq K^{-1}\left[\int_0^t (p\eta)(x,s)\exp\left(-\frac{1}{\mu_1}\int_s^t\int_\Omega \left\{v^2+p\eta\right\}\,dx\,d\tau\right)\,ds\right]^{\frac{\mu_0}{\mu_1}} \tag{65}$$

In (64) we have, after (25)

$$\exp\left(-\frac{1}{\mu_1}\int_s^t\int_\Omega (v^2+p\eta)\,dy\,d\tau\right) \leq e^{-Mc_4(t-s)}.$$

Then,

$$\eta(x,t) \leq K_1\left[1+\frac{C_4}{\mu_1}\int_0^t (1+\theta^{1+r}+|f(t)|)\,e^{-K(t-s)}ds\right]^{\frac{\mu_1}{\mu_0}}. \tag{66}$$

Now, using Cauchy–Schwarz inequality and Lemma 2, we get that $\theta^{1+r} \leq K(1+V(t))$, where $V \in L^1(0,T)$; moreover, $f \in L^2(0,T)$ after Lemma 3 so using these properties into (66), we get clearly that $\eta(x,t) \leq \overline{\eta}$, for a positive constant $\overline{\eta}$ independent of T.

The lower bound in (65) is obtained in the same way (as in [18]) and we skip the proof. □

Lemma 5

$$K(1 - V(t)) \leq \theta^{2\lambda}(x,t) \leq K(1 + V(t)), \tag{67}$$

where $V(t) := \int_\Omega \frac{1+\theta^q}{\theta^2}\, \theta_x^2\, dx$, *for any* $\lambda \leq \frac{q+r+1}{2}$.

Proof Just use the inequality $\theta^\lambda(x,t) \leq K + K\int_\Omega \theta^{\lambda-1}|\theta_x|\, dx$ together with (38) and Lemma 4 □

Lemma 6

$$\int_\Omega \eta_x^2\, dx + \int_{Q_t} v_x^2\, dx\, ds + \int_{Q_t} (1+\theta^{1+r})\eta_x^2\, dx\, ds \leq K. \tag{68}$$

Proof 1. Multiplying the second Eq. (11) by v and integrating by parts on Q_t for any $t \in [0, T]$, we get

$$\begin{aligned} \int_\Omega v^2\, dx + \int_{Q_t} \frac{\mu}{\eta} v_x^2\, dx\, ds &= \int_\Omega (v^0)^2\, dx + \int_{Q_t} p_x v\, dx\, ds \\ &\quad - \int_{Q_t} \eta v (S_F)_R\, dx\, ds. \end{aligned} \tag{69}$$

Using Cauchy–Schwarz inequality in the right-hand side, the last term in the right-hand side is bounded as follows

$$\left| \int_{Q_t} \eta v (S_F)_R\, dx\, ds \right| \leq \frac{1}{2} \int_{Q_t} \eta v\, dx\, ds + \frac{1}{2} \int_{Q_t} [\eta (S_F)_R]^2\, dx\, ds,$$

where the last term is bounded using Lemma 3 and the first one is estimated by using

$$\begin{aligned} \int_{Q_t} v^2\, dx\, ds &\leq K \int_0^t \max_\Omega v^2\, ds \leq K \int_0^t \left(\int_\Omega |v_x|\, dx \right)^2 ds \\ &\leq K \int_0^t \int_\Omega \frac{v_x^2}{\eta\theta}\, dx\, ds \leq K, \end{aligned}$$

after (38) and Lemma 4.

Using this in (69), we find

$$\int_\Omega v^2\,dx+\int_{Q_t}\frac{\mu}{\eta}\,v_x^2\,dx\,ds\le K+\int_{Q_t}|p_xv|\,dx\,ds$$

$$\le K+\int_{Q_t}(1+\theta^{1+r})|\eta_xv|\,dx\,ds+\int_{Q_t}(1+\theta^r)|\theta_xv|\,dx\,ds$$

$$\le K_\varepsilon+\varepsilon\int_{Q_t}(1+\theta^{1+r})\eta_x^2\,dx\,ds+\int_0^t\max_\Omega v^2\int_\Omega(1+\theta^{1+r})\,dx\,ds$$

$$+K\int_{Q_t}\frac{(1+\theta^r)\theta_x^2}{\eta\theta^2}\,dx\,ds.$$

So using (57), we get

$$\int_\Omega v^2\,dx+\int_{Q_t}v_x^2\,dx\,ds\le K_\varepsilon+\varepsilon\int_{Q_t}(1+\theta^{1+r})\eta_x^2\,dx\,ds. \tag{70}$$

2. Multiplying the second Eq. (11) by $\mathcal{M}_x$ and integrating by parts on Q_t for any $t\in[0,T]$, we get

$$\frac{1}{2}\int_\Omega\mathcal{M}_x^2\,dx-\frac{1}{2}\int_\Omega{\mathcal{M}^0}_x^2\,dx\le\int_{Q_t}v_t\mathcal{M}_x\,dx-K\int_{Q_t}(1+\theta^{1+r})\eta_x^2\,dx\,ds$$

$$+K\int_{Q_t}(1+\theta^r)|\eta_x\theta_x|\,dx\,ds+\int_{Q_t}\eta(S_F)_R\mathcal{M}_x\,dx\,ds=:A+B+C+D. \tag{71}$$

After integrating by parts, the first term in the right-hand side reads

$$A=\int_\Omega v\mathcal{M}_x\,dx-\int_\Omega v^0{\mathcal{M}^0}_x\,dx+\int_{Q_t}v_x\mathcal{M}_t\,dx\,ds.$$

So,

$$A\le K+\varepsilon\int_\Omega\mathcal{M}_x^2\,dx+K\int_\Omega v_x^2\,dx\,ds.$$

Using Cauchy–Schwarz inequality

$$B+C\le K_\varepsilon+\varepsilon\int_{Q_t}(1+\theta^{1+r})\eta_x^2\,dx\,ds.$$

Finally,

$$D\le\varepsilon\int_{Q_t}\mathcal{M}_x^2\,dx\,ds+K_\varepsilon\int_{Q_t}[\eta(S_F)_R]^2\,dx\,ds,$$

where the last term is bounded after Lemma 3.

Plugging all of these estimates into (71) and using (57), we have

$$\int_\Omega \eta_x^2\,dx + \int_{Q_t} (1+\theta^{1+r})\eta_x^2\,dx\,ds \le K_\varepsilon + K\int_{Q_t} v_x^2\,dx\,ds. \tag{72}$$

Finally, multiplying (72) by 2ε, adding to (70) and choosing $\varepsilon < \frac{1}{4K}$, we recover the estimate (68) □

Lemma 7

$$\int_\Omega v_x^2\,dx + \int_0^t \max_\Omega v_x^2(\cdot,s)\,ds + \int_{Q_t} v_{xx}^2\,dx\,ds \le K. \tag{73}$$

Proof 1. Let us define the auxiliary function F by

$$F(\xi) := \int_{\theta_0}^{\xi} e_\theta(\eta,\zeta)\,d\zeta,$$

for any $\xi > 0$.
After (25), one checks that $F(\xi) \le K|\xi-\theta_0|(1+\xi^r)$.
Moreover, $(F(\theta))_t = e_t - e_\eta v_x$ and $(F(\theta))_x = e_\theta\theta_x$.
As the third Eq. (11) gives the following equation for the internal energy

$$e_t = -pv_x + \frac{\mu}{\eta}\,v_x^2 - q_x + \eta v(S_F)_R - \eta(S_E)_R,$$

multiplying this equation by $F(\theta)$ and integrating by parts on Q_t one gets

$$\int_\Omega F(\theta)e\,dx + \int_{Q_t} e_\theta\,\frac{\kappa}{\eta}\,\theta_x^2\,dx\,ds \le \int_{Q_t} (ee_\eta + pF(\theta))|v_x|\,dx\,ds$$
$$+\int_{Q_t} |F(\theta)|\frac{\mu}{\eta}\,v_x^2\,dx\,ds + \int_{Q_t} \eta F(\theta)[|v(S_F)_R| + |(S_E)_R|]\,dx\,ds.$$

Then, using (25) and Lemma 4

$$\int_\Omega (\theta^2+\theta^{r+2})\,dx + \int_{Q_t} (1+\theta^{q+r})\theta_x^2\,dx\,ds$$
$$\le K_\varepsilon + \varepsilon\int_{Q_t} v_{xx}^2\,dx\,ds + \int_0^t V(s)\int_\Omega \theta^{2r+2}\,dx$$
$$+\int_{Q_t} F(\theta)\eta\,(\,|v(S_F)_R| + |(S_E)_R|)\,dx\,ds,$$

for a $\varepsilon > 0$ and a positive $V \in L^1(0,T)$.

In the same way, the last integral in the right-hand side can be bounded by using Cauchy–Schwarz inequality, Lemma 2 and Lemma 3. We get

$$\int_{Q_t} F(\theta)\eta\,(\,|v(S_F)_R| + |(S_E)_R|)\;dx\,ds$$
$$\le K + K\int_{Q_t} v^2(1+\theta^{2r+2})\,dx\,ds + K\int_{Q_t}(\theta-\theta_0)^2(1+\theta^{2r})\,dx\,ds$$
$$\le K\int_0^t W(s)\int_\Omega \theta^{2r+2}\,dx\,ds,$$

for a positive $W \in L^1(0,T)$.

Finally taking ε_1 small enough, we get

$$\int_\Omega(\theta^2+\theta^{2r+2})\,dx + \int_{Q_t}(1+\theta^{q+r})\theta_x^2\,dx\,ds$$
$$\le K + \epsilon_1\varepsilon\int_{Q_t} v_{xx}^2\,dx\,ds + K\int_0^t V(s)\int_\Omega \theta^{2r+2}\,dx\,ds. \tag{74}$$

2. Multiplying the second Eq. (11) by $-v_{xx}$ and integrating by parts on Ω, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_x^2\,dx + \int_\Omega \frac{\mu}{\eta}\,v_{xx}^2\,dx = \int_\Omega v_{xx}p_x\,dx - \int_\Omega\left(\frac{\mu}{\eta}\right)_x v_x v_{xx}\,dx + \int_\Omega v_{xx}\eta(S_F)_R\,dx.$$

Integrating on $[0,t]$ and using (26), we find

$$\frac{1}{2}\int_\Omega v_x^2\,dx + \frac{\mu_1}{\overline{\eta}}\int_{Q_t} v_{xx}^2\,dx\,ds \le \int_{Q_t}|v_{xx}(p_\eta\eta_x + p_\theta\theta_x)|\,dx\,ds$$
$$+\left(\frac{\mu_1}{\overline{\eta}^2}+\frac{\mu_2}{\overline{\eta}}\right)\int_{Q_t}|\eta_x v_x v_{xx}|\,dx\,ds + \int_{Q_t}|v_{xx}\eta(S_F)_R|\,dx\,ds$$
$$\le \varepsilon\int_{Q_t} v_{xx}^2\,dx\,ds + K_\varepsilon\int_{Q_t} v_x^2\eta_x^2\,dx\,ds$$
$$+K_\varepsilon\int_{Q_t}(1+\theta^{q+r})\theta_x^2\,dx\,ds + K_\varepsilon\int_{Q_t}(1+\theta^{2+2r})\eta_x^2\,dx\,ds.$$

In order to bound the second term in the right-hand side we remark that, using Lemma 6

$$\max_\Omega v_x^2(\cdot,t) \le K\int_\Omega v_x^2\,dx + \varepsilon\int_\Omega v_{xx}^2\,dx \le K + \varepsilon\int_\Omega v_{xx}^2\,dx.$$

To bound the last term in the right-hand side, we apply (56) and we have, using Lemma 6

$$\int_{Q_t}(1+\theta^{2+2r})\eta_x^2\,dx\,ds \le K + \int_{Q_t}(\theta^{1+r}-\theta^*)^2\eta_x^2\,dx\,ds \le K.$$

Finally, we get for ε small enough

$$\int_{\Omega} v_x^2\,dx + \int_{Q_t} v_{xx}^2\,dx\,ds \le K + K\int_{Q_t}(1+\theta^{q+r})\theta_x^2\,dx\,ds. \tag{75}$$

Now adding (75) to (74) and applying Gronwall's Lemma gives (73) □

Lemma 8 *Let us introduce the two quantities*

$$Y(t) := \int_{\Omega}\left(1+\theta^{2q}\right)\theta_x^2\,dx, \quad X(t) := \int_{Q_t}\left(1+\theta^{q+r}\right)\theta_t^2\,dx\,ds.$$

The following estimates hold

$$X(t) + Y(t) \le K, \tag{76}$$

and

$$\max_{Q_t}\theta \le K. \tag{77}$$

Proof Observing that

$$\theta^{q+r+2} - {\theta_0}^{q+r+2} \le (q+r+2)\int_{\Omega}\theta^{q+r+1}\,|\theta_x|\,dx,$$

and using Lemma 2 and Cauchy–Schwarz inequality, we get

$$\max_{Q_t}\theta \le K(1+Y^{\frac{1}{2q+r+3}}). \tag{78}$$

From (11), the equation for the internal energy reads

$$e_\theta\theta_t + \theta p_\theta v_x - \frac{\mu}{\eta}\,v_x^2 = \left(\frac{\kappa\theta_x}{\eta}\right)_x - \eta(S_E)_R.$$

Defining the auxiliary function $K(\eta,\theta) := \int_0^\theta \frac{\kappa(\eta,u)}{u}\,du$, multiplying the previous equation by K_t and integrating by parts, we get

$$\int_{Q_t}\left(e_\theta\theta_t + \theta p_\theta v_x - \frac{\mu}{\eta}\,v_x^2 + \eta(S_E)_R\right)K_s\,dx\,ds + \int_{Q_t}\left(\frac{\kappa\theta_x}{\eta}\right)K_{sx}\,dx\,ds = 0. \tag{79}$$

Observing that $K_t = K_\eta v_x + \frac{\kappa}{\eta}\,\theta_t$, $K_{xt} = \left(\frac{\kappa\theta_x}{\eta}\right)_t + K_{\eta\eta}v_x\eta_x + \left(\frac{\kappa}{\eta}\right)_\eta\eta_x\theta_t$ and that after (25) $|K_\eta| + |K_{\eta\eta}| \le C(1+\theta^{q+1})$, we can estimate all the contributions in (79).

After (25), we have the lower bound

$$\int_{Q_s}\kappa e_\theta\theta_s^2\,dx\,ds \ge \frac{c_6c_1}{\overline{\eta}}\,X(t),$$

Using (25) and Lemma 4

$$\left|\int_{Q_t} e_\theta \theta_s K_\eta v_x \, dx \, ds\right| \leq K \int_{Q_t} (1+\theta)^{q+r+1} |\theta_s v_x| \, dx \, ds$$
$$\leq \frac{c_6 c_1}{8\overline{\eta}} X(t) + K(1 + \max_{Q_t} \theta^{q+r+2}).$$

Arguing similarly

$$\left|\int_{Q_t} \left(\theta p_\theta v_x - \frac{\mu}{\eta} v_x^2\right) K_s \, dx \, ds\right|$$
$$\leq K \int_{Q_t} (1+\theta)^{q+r+2} v_x^2 \, dx \, ds + K \int_{Q_t} (1+\theta)^{q+1} |v_x|^3 \, dx \, ds$$
$$+K \int_{Q_t} (1+\theta)^{q+r+1} |v_x \theta_s| \, dx \, ds + K \int_{Q_t} (1+\theta)^q v_x^2 |\theta_s| \, dx \, ds$$
$$\leq \frac{c_6 c_1}{8\overline{\eta}} X(t) + K(1 + \max_{Q_t} \theta^{q+r+2}).$$

Using (25), we have

$$\left|\int_{Q_T} \frac{\kappa \theta_x}{\eta} \left(\frac{\kappa \theta_x}{\eta}\right)_s dx \, ds\right| \geq \frac{c_6^2}{2\overline{\eta}^2} Y(t) - K.$$

$$\left|\int_{Q_T} \frac{\kappa \theta_x}{\eta} (K_\eta v_{xx} + K_{\eta\eta} v_x \eta_x) \, dx \, ds\right| \leq K \int_{Q_T} (1+\theta)^{2q+1} |\theta_x| (|v_{xx}| + |v_x \eta_x|) \, dx \, ds$$
$$\leq K \left(\int_{Q_T} (1+\theta)^{4q+2} \theta_x^2| \, dx \, ds\right)^{1/2} \leq K \left(1 + \max_{Q_t} \theta^{1+\frac{3q}{2}}\right).$$

Using (25), we have also

$$\left|\int_{Q_T} \frac{\kappa \theta_x}{\eta} \left(\frac{\kappa}{\eta}\right)_\eta \eta_x \theta_t \, dx \, ds\right| \leq \frac{c_6 c_1}{8\overline{\eta}} X(t) + K \int_{Q_t} \left[\frac{\kappa \theta_x}{\eta}\right]^2 (1 + \theta^{q-r}) \eta_x^2 \, dx \, ds$$
$$\leq K + \frac{c_6 c_1}{8\overline{\eta}} X(t) + K \left(1 + \max_{Q_t} \theta^{2q-2r}\right)$$
$$+K \left(1 + \max_{Q_t} \theta^{q-r}\right) \times \int_{Q_T} \left|\frac{\kappa \theta_x}{\eta}\right| \left|\left[\frac{\kappa \theta_x}{\eta}\right]_x\right| dx \, ds.$$

But the last integral is estimated by

$$\int_{Q_T} \left|\frac{\kappa\theta_x}{\eta}\right| \left|\left[\frac{\kappa\theta_x}{\eta}\right]_x\right| \, dx\, ds.$$

$$\leq \left(\int_{Q_T} (1+\theta^{q-r}) \left[\frac{\kappa\theta_x}{\eta}\right]_x^2 \, dx\, ds\right)^{1/2}$$

$$\leq K \left(\int_{Q_T} (1+\theta^{q-r})\theta_s^2 + (1+\theta^{q+r+2})v_x^2 + (1+\theta^{q-r})v_x^4 \, dx\, ds\right)^{1/2}$$

$$\leq K X(t) + K\left(1 + \max_{Q_t} \theta^{\frac{q+r+2}{2}}\right),$$

so finally,

$$\left|\int_{Q_T} \frac{\kappa\theta_x}{\eta} \left(\frac{\kappa}{\eta}\right)_\eta \eta_x\theta_t \, dx\, ds\right| \leq \frac{c_6 c_1}{4\overline{\eta}} X(t) + K\left(1 + \max_{Q_t}\theta^{2q+1}\right),$$

Let us estimate the last contribution in the left-hand side of (79).

$$\left|\int_{Q_T} \eta\, (S_E)_R\, K_t \, dx\, ds\right| \leq \int_{Q_T} \left(\int_0^\infty \int_{S^1} \eta\sigma_a B \, d\nu\, d\omega\right) |K_t| \, dx\, ds$$

$$+ \int_{Q_T} \left(\int_0^\infty \int_{S^1} \eta\sigma_a I \, d\nu\, d\omega\right) |K_t| \, dx\, ds$$

$$+ \int_{Q_T} \left(\int_0^\infty \int_{S^1} \eta\sigma_s |\tilde{I} - I| \, d\nu\, d\omega\right) |K_t| \, dx\, ds =: P + Q + R.$$

After (25) and Lemma 3,

$$P \leq C \int_{Q_T} |K_t| \left(1+\theta^\alpha\right) \, dx\, ds$$

$$\leq C \int_{Q_T} \left(1+\theta^{q+\alpha+1}\right) |v_x| \, dx\, ds + C \int_{Q_T} \left(1+\theta^{q+\alpha}\right) |\theta_t| \, dx\, ds =: A + B.$$

Using Cauchy–Schwarz inequality and Lemma 5, we have

$$A \leq K \max_{Q_t}\theta^{r+2} + K \int_{Q_T} \left(1+\theta^{q+2\alpha-r}\right) \, dx\, ds \leq K\left(1 + \max_{Q_t}\theta^{r+2}\right),$$

and

$$B \leq \frac{c_6 c_1}{8\overline{\eta}} X(t) + C \int_{Q_T} \left(1+\theta^{q+r}\right) \, dx\, ds \leq \frac{c_6 c_1}{8\overline{\eta}} X(t) + K.$$

Using (25), Lemma 3 and Cauchy–Schwarz inequality, we have

$$R \le K \int_{Q_T} \int_0^\infty \int_{S^1} \left[|(\tilde{I} - I) K_\eta v_x + (\tilde{I} - I) \frac{\kappa}{\eta} \theta_t| \right] d\nu \, d\omega \, dx \, ds$$

$$\le K \int_{Q_T} \int_0^\infty \int_{S^1} \eta \sigma_s |\tilde{I} - I|^2 \, d\nu \, d\omega \, dx \, ds + K \int_{Q_T} \left(1 + \theta^{2q+2}\right) v_x^2 \, dx \, ds$$

$$+ K \int_{Q_T} \left(1 + \theta^q\right) |\tilde{I} - I| \, |\theta_t| \, dx \, ds$$

$$\le K + \int_{Q_T} \left(1 + \theta^{2q+2}\right) v_x^2 \, dx \, ds + \frac{c_6 c_1}{8\overline{\eta}} X(t) + \int_{Q_T} \int_0^\infty \int_{S^1} \left(1 + \theta^{q-r}\right) (\tilde{I} - I)^2 d\nu \, d\omega \, dx \, ds$$

$$\le K + K \max_{Q_t} \theta^{q+2} + \frac{c_6 c_1}{8\overline{\eta}} X(t) + C \max_{Q_t} \theta^{q-r}.$$

Using the same technique, we get also

$$Q \le K + K \max_{Q_t} \theta^{q+2} + \frac{c_6 c_1}{8\overline{\eta}} X(t) + C \max_{Q_t} \theta^{q-r}.$$

Plugging all the previous estimates into (79), we get

$$\frac{c_6 c_1}{2\overline{\eta}} \, X(t) + \frac{c_6^2}{2\overline{\eta}^2} \, Y(t) \le K \left(1 + \max_{Q_t} \theta^{2q+1}\right).$$

Using (78), we end with

$$\frac{c_6 c_1}{2\overline{\eta}} \, X(t) + \frac{c_6^2}{2\overline{\eta}^2} \, Y(t) \le K \left(1 + Y^{\frac{2q+1}{2q+r+3}}\right),$$

which ends the proof. □

Corollary 1 *The quantities*

$$\int_{Q_t} \theta_{xx}^2 \, dx \, ds, \quad \int_{Q_t} v_s^2 \, dx \, ds, \quad \int_{Q_t} \theta_s^2 \, dx \, ds, \tag{80}$$

are bounded independently of time.

Proof The first bound is a consequence of the following inequality (itself following from the third Eq. (11))

$$\theta_{xx}^2 \le K[\theta_t^2 + v_x^2 + v_x^4 + \eta_x^2 \theta_x^2 + \theta_x^4].$$

We know from Lemmas that $v_x \in L^2(Q_T)$ and $\theta_t \in L^2(Q_T)$. Moreover, as

$$\int_{Q_t} v_x^4 \, dx \, ds \le \int_0^t \max_\Omega v_x^2 \int_\Omega v_x^2 \, dx \, ds \le K,$$

after Lemma 7,

$$\int_{Q_t} \theta_x^4 \, dx \, ds \le \int_0^t \max_{\Omega} \theta_x^2 \int_{\Omega} \theta_x^2 \, dx \, ds \le K,$$

after Lemma 8 and

$$\int_{Q_t} \eta_x^2 \theta_x^2 \, dx \, ds \le \int_0^t \max_{\Omega} \theta_x^2 \int_{\Omega} \eta_x^2 \, dx \, ds \le K,$$

after Lemma 6, bound (80) follows. □

3 Proofs of Theorem 3 and Proposition 1

1. Applying the elementary fact [2] that if, for a $1 \le p < \infty$, the function u is in $W^{1,p}(\mathbb{R}_+)$ then $\lim_{t\to\infty} u(t) = 0$, to the quantities $\|\eta - \eta_\infty\|_{H^1(\Omega)}$, $\|v\|_{H^1(\Omega)}$, $\|\theta - \theta_\infty\|_{H^1(\Omega)}$ and $\|\mathcal{I} - \mathcal{I}_b\|_{L^2(\Omega)}$, one has first to check that

$$\int_0^\infty \left[\left| \frac{d}{dt} \int_{\Omega} \eta_x^2 \, dx \right| + \left| \frac{d}{dt} \int_{\Omega} v_x^2 \, dx \right| + \left| \frac{d}{dt} \int_{\Omega} \theta_x^2 \, dx \right| \right] dt \le K,$$

which follows from the fact that η_t, v_t, θ_t, η_{xx}, v_{xx} and θ_{xx} are in $L^2(\Omega)$ after the results of Sect. 2, and

$$\int_0^\infty \left| \frac{d}{dt} \int_{\Omega} \mathcal{I}^2 \, dx \right| dt \le K,$$

which follows from Lemmas 3.
This proves that

$$\lim_{t\to\infty} \left(\|\eta_x\|^2 + \|\theta_x\|^2 + \|v_x\|^2 \right) = 0.$$

So after the mass conservation and Poincaré's inequality,

$$\lim_{t\to\infty} \left(\|\eta - \eta_0\|^2 + \|v\|^2 + \|\theta - \theta_0\|^2 \right) = 0,$$

which gives the requested H^1-decay for (η, v, θ), and L^2-decay for $\mathcal{I}$, for large t.
2. The exponential decay is finally obtained by applying the method of [29].
Let us define the modified energy of the matter

$$E(\eta, v, \theta) := \frac{1}{2} v^2 + \psi(\eta, \theta) - \psi(\eta_0, \theta_0) - (\eta - \eta_0)\psi_\eta(\eta_0, \theta_0) - (\theta - \theta_0)\psi_\theta(\eta, \theta),$$

where ψ is the free energy, and the modified energy–entropy of the radiation

$$\mathbf{e} := \eta E_R(I) - \theta_0 s_R(\eta, I) - \eta E_R(I_0) + \theta_0 s_R(\eta_0, I_0).$$

Introduce the set

$$\mathcal{O}_{k_1,k_2} := \left\{ \eta, \theta \ : \ \log \left| \frac{\eta}{\eta_0} \right| < k_1, \quad \log \left| \frac{\theta}{\theta_0} \right| < k_2 \right\}.$$

We have the following two-sided inequalities for the energy and "reduced" production of radiative entropy

Lemma 9 1. *There exists $a > 0$ such that* $\forall(\eta,\theta) \in \mathcal{O}(k_1,k_2)$,

$$\frac{1}{2}\,v^2 + a^{-1}\left(|\eta-\eta_0|^2 + |\theta-\theta_0|^2\right) \le E$$

$$\le \frac{1}{2}\,v^2 + a\left(|\eta-\eta_0|^2 + |\theta-\theta_0|^2\right) \tag{81}$$

where the parameter a depends on k_1 and k_2.

2. *There exists $b > 0$ such that* $\forall I > 0, \theta > 0$

$$b^{-1}\int_0^\infty\int_{S^1} |I-I_b|^2\,d\omega\,d\nu \le \mathbf{e} \le b\int_0^\infty\int_{S^1} |I-I_b|^2\,d\omega\,d\nu. \tag{82}$$

3. *There exists $d > 0$ such that* $\forall I > 0, \theta > 0$

$$d^{-1}\left(|\theta^\alpha-\theta_0^\alpha|^2 + \int_0^\infty\int_{S^1}\left\{|I-I_b|^2 + |\tilde{I}-I|^2\right\}\,d\omega\,d\nu\right)$$

$$\le \mathcal{Q} \le d\left(|\theta^\alpha-\theta_0^\alpha|^2 + \int_0^\infty\int_{S^1}\left\{|I-I_b|^2 + |\tilde{I}-I|^2\right\}\,d\omega\,d\nu\right) \tag{83}$$

with

$$\mathcal{Q} := \theta_0\eta\int_0^\infty\int_{S^1}\frac{k_B}{h\nu}\left[\log\frac{n(I)+1}{n(I)} - \log\frac{n(\tilde{I})+1}{n(\tilde{I})}\right]\sigma_s(I-\tilde{I})\,d\nu\,d\omega$$

$$+\theta_0\eta\int_0^\infty\int_{S^1}\frac{k_B}{h\nu}\left[\log\frac{n(I)+1}{n(I)} - \log\frac{n(B)+1}{n(B)}\right]\sigma_a(I-B)\,d\nu\,d\omega.$$

Proof The first inequality (81) is a slight modification of the result of Okada and Kawashima (see Lemma 3.1 of [29]) and the second and third inequalities (82) and (83) follow after an elementary analysis of the integrands in $\mathbf{e}$ and $\mathcal{Q}$ □

Now, we rewrite Eq. (45) as

$$(E+\mathbf{e})_t + \frac{\theta_0}{\theta}\left(\frac{\mu v_x^2}{\eta} + \frac{\kappa\theta_x^2}{\eta\theta}\right) + \mathcal{Q}$$

$$= \Bigg[(p(1,\theta_0)-p(\eta,\theta))\,v + \frac{\mu}{\eta}\,vv_x + \left(1-\frac{\theta_0}{\theta}\right)\frac{\kappa}{\eta}\,\theta_x - F_R + vE_R$$

$$-\int_0^\infty\int_{S^1}\frac{2k_B\nu^2}{c^3}(c\omega-v)\left[(n+1)\log(n+1)-n\log n\right]d\nu\,d\omega\Bigg]_x. \tag{84}$$

On the other hand, multiplying the second Eq. (11) by $\mathcal{M}_x$, (recall that $\mathcal{M}(s) := \int_1^s \frac{\mu(\xi)}{\xi}\, d\xi$), we get

$$\begin{aligned}&\left(\frac{1}{2}\mathcal{M}_x^2 - \mathcal{M}_x v\right)_t - \frac{\mu}{\eta}\, p_\eta \eta_x^2\\&\quad = \frac{\mu}{\eta}\, v_x^2 - \mathcal{M}_x p_\theta \theta_x - \left(\frac{\mu}{\eta}\, v v_x\right)_x + \mathcal{M}_x \eta (S_F)_R.\end{aligned} \tag{85}$$

After the proof of Lemma 3, we have finally

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega\int_0^\infty\int_{S^1} \eta J^2\, d\omega\, d\nu\, dx + \frac{c}{2}\int_0^\infty\int_{S^1} \omega J^2(M,t;\nu,\omega)\, d\omega\, d\nu\\&\quad -\frac{c}{2}\int_0^\infty\int_{S^1} \omega I^2(0,t;\nu,\omega)\, d\omega\, d\nu + \frac{1}{2}\int_\Omega\int_0^\infty\int_{S^1} \eta\sigma_a J^2\, d\omega\, d\nu\, dx\\&\quad + \int_\Omega\int_0^\infty\int_{S^1} \eta\sigma_s (\tilde{I} - I)^2\, d\omega\, d\nu\, dx\\&\quad \le w \int_\Omega\int_0^\infty\int_{S^1} \eta\sigma_a (\theta^\alpha - \theta_0^\alpha)^2\, d\omega\, d\nu\, dx,\end{aligned} \tag{86}$$

where $w := C_8^2 \|f\|^2_{L^2(\mathbb{R}_+)} + C_9^2 \|g\|^2_{L^2(\mathbb{R}_+)}$.
Multiplying (84) by $e^{\beta_1 t}$ then (85) by $\beta_2 e^{\beta_1 t}$ with $\beta_{1,2} > 0$ and adding the resulting identities, we get

$$\begin{aligned}&\frac{d}{dt}\, e^{\beta_1 t}\left\{E + \mathbf{e} + \beta_2\left(\frac{1}{2}\mathcal{M}_x^2 - \mathcal{M}_x v\right)\right\}\\&\quad + e^{\beta_1 t}\left\{\frac{\theta_0}{\theta}\left(\frac{\mu v_x^2}{\eta} + \frac{\kappa\theta_x^2}{\eta\theta}\right) + \mathcal{Q} + \beta_2\left(-\frac{\mu}{\eta}\, p_\eta \eta_x^2 - \frac{\mu}{\eta}\, v_x^2 + \mathcal{M}_x p_\theta \theta_x - \mathcal{M}_x \eta (S_F)_R\right)\right\}\\&= \beta_1 e^{\beta_1 t}\left\{E + \mathbf{e} + \beta_2\left(\frac{1}{2}\mathcal{M}_x^2 - \mathcal{M}_x v\right)\right\}\\&\quad + e^{\beta_1 t}\Bigg[(p(\eta_0,\theta_0) - p(\eta,\theta))\, v + (1-\beta_2)\frac{\mu}{\eta}\, v v_x + \left(1 - \frac{\theta_0}{\theta}\right)\frac{\kappa}{\eta}\,\theta_x - F_R + v E_R\\&\quad - \int_0^\infty\int_{S^1} \frac{2k_B \nu^2}{c^3}(c\omega - v)\,[(n+1)\log(n+1) - n\log n]\, d\nu\, d\omega\Bigg]_x.\end{aligned} \tag{87}$$

Multiplying (86) by $\beta_3 e^{\beta_1 t}$ with $\beta_3 > 0$, integrating on $(0,t)$ and using (25), we get

$$e^{\beta_1 t}\int_\Omega\int_0^\infty\int_{S^1}\frac{1}{2}\,\beta_3\eta(I-I_b)^2\,d\nu\,d\omega\,dx$$
$$+\frac{1}{2}\int_0^t e^{\beta_1 s}\int_\Omega\int_0^\infty\int_{S^1}\beta_3\eta\sigma_a(I-I_b)^2\,d\nu\,d\omega\,dx$$
$$+\int_0^t e^{\beta_1 s}\int_\Omega\int_0^\infty\int_{S^1}\beta_3\eta\sigma_s(\tilde{I}-I)^2\,d\nu\,d\omega\,dx\,ds$$
$$\leq K+\int_0^t e^{\beta_1 s}\int_\Omega\int_0^\infty\int_{S^1}\frac{1}{2}\,\beta_1\beta_3\eta\sigma_a(I-I_b)^2\,d\nu\,d\omega\,dx\,ds$$
$$+\int_0^t e^{\beta_1 s}\int_\Omega\beta_3 w|\theta^\alpha-\theta_0^\alpha|^2\,dx\,ds. \tag{88}$$

Integrating now (87) on $(0,t)$, adding to (88) and using Lemma 9, we get

$$e^{\beta_1 t}\int_\Omega\left\{a^{-1}\left(|\eta-\eta_0|^2+|\theta-\theta_0|^2\right)+\frac{1}{2}\,v^2+\frac{1}{2}\,\beta_2\mathcal{M}_x^2\right.$$
$$\left.+\left(b^{-1}+\frac{1}{2}\,\beta_3\underline{\eta}\right)\int_0^\infty\int_{S^1}|I-I_b|^2\,d\nu\,d\omega\right\}\,dx$$
$$+\int_0^t e^{\beta_1 s}\int_\Omega d^{-1}\left\{\int_0^\infty\int_{S^1}\left\{|I-I_b|^2+|\tilde{I}-I|^2\right\}\,d\nu\,d\omega+|\theta^\alpha-\theta_0^\alpha|^2\right\}\,dx\,ds$$
$$+\int_0^t e^{\beta_1 s}\int_\Omega\left\{a_1v_x^2+a_2\theta_x^2+a_3\mathcal{M}_x^2\right\}\,dx\,ds+\frac{1}{2}\int_0^t e^{\beta_1 s}\int_\Omega\int_0^\infty\int_{S^1}\beta_3\eta\sigma_a(I-I_b)^2$$
$$\times d\nu\,d\omega\,dx\,ds$$
$$+\int_0^t e^{\beta_1 s}\int_\Omega\int_0^\infty\int_{S^1}\beta_3\eta\sigma_s(\tilde{I}-I)^2\,d\nu\,d\omega\,dx\,ds$$
$$\leq K+e^{\beta_1 t}\int_\Omega\beta_2\mathcal{M}_x v\,dx$$
$$+\int_0^t e^{\beta_1 s}\int_\Omega\left\{\beta_1\left[a\left(|\eta-\eta_0|^2+|\theta-\theta_0|^2\right)+\frac{1}{2}\,v^2+b\int_0^\infty\int_{S^1}|I-I_b|^2\,d\nu\,d\omega\right]\right.$$
$$+\beta_2\left[\mathcal{M}_x\eta(S_F)_R-\mathcal{M}_x p_\theta\theta_x\right]+\beta_1\beta_2\left[\frac{1}{2}\,\mathcal{M}_x^2-\mathcal{M}_x v\right]+\beta_2\frac{\mu}{\eta}\,v_x^2$$

$$\left. +\beta_3 w|\theta^\alpha - \theta_0^\alpha|^2 \right\} dx\, ds$$

$$+\int_0^t e^{\beta_1 s} \int_\Omega \int_0^\infty \int_{S^1} \frac{1}{2}\, \beta_1\beta_3\eta\sigma_a(I - I_b)^2\, d\nu\, d\omega\, dx\, ds =: \mathbf{R}, \tag{89}$$

with the constants $a_1 = \frac{\theta_0\mu_0}{\overline{\theta}\overline{\eta}}$, $a_2 = \frac{c_6\theta_0(1+\underline{\theta}^{1+r})\mu_0}{\overline{\theta}\overline{\eta}}$ and $a_3 = \frac{C_2\underline{\eta}(1+\underline{\theta}^{1+r})\mu_0}{\overline{\eta}^2}$. The right-hand side is estimated by using Cauchy–Schwarz inequality.

$$\begin{aligned}
|\mathbf{R}| \le\; & K + e^{\beta_1 t} \int_\Omega \frac{1}{2}\, \beta_2 \left(\varepsilon_1 \mathcal{M}_x^2 + \frac{1}{\varepsilon_1}\, v^2\right) dx \\
& + \int_0^t e^{\beta_1 s} \int_\Omega \left\{ a\beta_1 \left(|\eta - \eta_0|^2 + |\theta - \theta_0|^2\right) + \frac{1}{2}\, \beta_1 v^2 \right. \\
& \left. + \beta_1 b \int_0^\infty \int_{S^1} |I - I_b|^2\, d\nu\, d\omega \right\} dx\, ds \\
& + \int_0^t e^{\beta_1 s} \int_\Omega \frac{1}{2}\, \beta_2^2 \left(\varepsilon_2 \mathcal{M}_x^2 + \frac{1}{\varepsilon_2}\, [\eta(S_F)_R]^2\right) dx\, ds \\
& + \int_0^t e^{\beta_1 s} \int_\Omega \frac{1}{2}\, \beta_2^2 \left(\varepsilon_3 \mathcal{M}_x^2 + \frac{1}{\varepsilon_3}\, p_\theta^2\theta_x^2\right) dx\, ds \\
& + \int_0^t e^{\beta_1 s} \int_\Omega \frac{1}{2}\, \beta_1^2\beta_2^2 \left(\varepsilon_4 \mathcal{M}_x^2 + \frac{1}{\varepsilon_4}\, v^2\right) dx\, ds \\
& + \int_0^t e^{\beta_1 s} \int_\Omega \beta_2 \frac{\mu}{\eta}\, v_x^2 + \beta_3 w|\theta^\alpha - \theta_0^\alpha|^2\, dx\, ds \\
& + \int_0^t e^{\beta_1 s} \int_\Omega \int_0^\infty \int_{S^1} \frac{1}{2}\, \beta_1\beta_3\eta\sigma_a(I - I_b)^2\, d\nu\, d\omega\, dx\, ds.
\end{aligned} \tag{90}$$

Exploiting the structure of $\eta(S_F)_R$ and using (25), we get then

$$\begin{aligned}
|\mathbf{R}| \le\; & K + e^{\beta_1 t} \int_\Omega \frac{1}{2}\, \beta_2 \left(\varepsilon_1 \mathcal{M}_x^2 + \frac{1}{\varepsilon_1}\, v^2\right) dx \\
& + \int_0^t e^{\beta_1 s} \int_\Omega \beta_2 a_4\, v_x^2\, dx\, ds
\end{aligned}$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega a\beta_1\int_0^\infty\int_{S^1}|I-I_b|^2\,d\nu\,d\omega\,dx\,ds$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega \frac{1}{2}\,(a_5\beta_1+\varepsilon_2\beta_2+\varepsilon_3\beta_2+\beta_1\beta_2\varepsilon_4)\,\mathcal{M}_x^2\,dx\,ds$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega \beta_2^2\frac{1}{\varepsilon_3}a_6\theta_x^2\,dx\,ds$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega \left(d\beta_3+a_7\frac{\beta_2}{\varepsilon_2}\right)|\theta^\alpha-\theta_0^\alpha|^2\,dx\,ds.$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega\int_0^\infty\int_{S^1} a_8\frac{\beta_2}{\varepsilon_2}\,\eta\sigma_a(I-I_b)^2\,d\nu\,d\omega\,dx\,ds$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega\int_0^\infty\int_{S^1} a_9\frac{\beta_2}{\varepsilon_2}\,\eta\sigma_s(\tilde{I}-I)^2\,d\nu\,d\omega\,dx\,ds$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega\int_0^\infty\int_{S^1} \frac{1}{2}\,\beta_1\beta_3\eta\sigma_a(I-I_b)^2\,d\nu\,d\omega\,dx\,ds,$$

for any $\varepsilon_{1,2,3} > 0$, for the positive constants $a_4 = \frac{\mu_1}{\underline{\eta}}$, $a_5 = \frac{\overline{\eta}a}{\mu_0}$, $a_6 = \frac{C_3(1+\overline{\theta}^r)}{2\underline{\eta}}$, $a_7 = C_8C_{10}\|f\|_{L^1(\mathbb{R}_+)}$, and $a_8 = a_9 = C_{10}\|f\|_{L^1(\mathbb{R}_+)}$.
One sees that, in order to absorb all of the terms in the right-hand side of (90) by the the left-hand side of (89), parameters $\varepsilon_{1,2,3}$ and $\beta_{1,2,3}$ have to satisfy the constraints

$$\begin{cases} \beta_2 < \varepsilon_1 < 1, \\ \beta_2 < \frac{a_1}{a_4}, \\ \beta_1 < \frac{1}{ad}, \\ \beta_2 < \frac{a_2}{a_6}\,\varepsilon_3, \\ d\beta_3\varepsilon_2+\beta_2 a_7 < d^{-1}\varepsilon_2, \\ \beta_2 a_8 < \beta_3\varepsilon_2, \end{cases}$$

where the a_j $j = \overline{1,9}$ are positive number depending only on the physical constants $(M, c, h, \theta_0, \overline{\eta}, \underline{\eta}, \overline{\theta}, \underline{\theta})$ and those appearing in (25). An elementary analysis of this system of algebraic inequalities shows that, taking for example $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1/2$, it admits nontrivial solutions $(\beta_1, \beta_2, \beta_3)$ in a neighborhood of $(\beta_1, \beta_2, \beta_3) = (0, 0, 0)$

Then, we end with the estimate

$$e^{\beta_1 t} \int_\Omega \left\{ \left(|\eta - \eta_0|^2 + |\theta - \theta_0|^2\right) + \frac{1}{2}\, v^2 + \eta_x^2 + \int_0^\infty \int_{S^1} |I - I_b|^2 \, d\nu \, d\omega \right\} dx$$

$$+ \int_0^t e^{\beta_1 s} \int_\Omega \left\{ v_x^2 + \theta_x^2 + \eta_x^2 \right\} dx \, ds \le K, \tag{91}$$

which gives the exponential decay of Theorem 3, for $\gamma = \beta_1$.

3. When the system is decoupled, the decay of the fluid variables is improved and we can use almost verbatim the argument of [29]: one multiplies the second Eq. (11) by $-e^{\delta t} v_{xx}$ and the third Eq. (11) by $-e^{\delta t}\theta_{xx}$. Then integrating on Q_t, one obtain

$$e^{\delta t} \int_\Omega \left(\theta_x^2 + v_x^2\right) dx + \int_0^t e^{\delta s} \int_\Omega \left(v_{xx}^2 + \theta_{xx}^2\right) dx \, ds \le K,$$

which gives, using also (91) and taking $\gamma' = \min\{\gamma, \delta\}$, the H^1 exponential decay described in Proposition 1.

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Appendix

In all this Appendix, we denote by C various positive constants, possibly depending on T.

The proof of Theorem 2 relies on a priori estimates allowing to apply a fixed-point theorem in the same conditions as Theorem 1 in [9]. Then, it is sufficient to show the following

Theorem 4 *Let* $(\eta, v, \theta, \mathcal{I})$ *be a smooth solution of* (11) (20) (21) (22) (23) *described in Theorem* 2.

The functions $\eta, \eta_x, \eta_t, \eta_{xt}, v, v_x, v_t, v_{xx}, \theta, \theta_x, \theta_t, \theta_{xx}, \mathcal{I}, \mathcal{I}_x$ *all belong to* $C^{\alpha,\frac{\alpha}{2}}(Q_T)$ *and there is a* $C > 0$ *such that*

$$\|\eta, \eta_x, \eta_t, \eta_{xt}, v, v_x, v_t, v_{xx}, \theta, \theta_x, \theta_t, \theta_{xx}, \mathcal{I}, \mathcal{I}_x\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le C,$$

where C *depends on* T*, an the parameters of the system, on the size of the initial data* $\|\eta^0, \eta_x^0, v^0, v_x^0, \theta^0, \theta_x^0, \mathcal{I}^0, \mathcal{I}_x^0\|_{C^\alpha(\Omega)}$ *and on* $\inf_\Omega \eta^0$.

Moreover,

$$0 < \underline{\eta} \le \eta \le \overline{\eta}, \quad 0 < \underline{\theta} \le \theta \le \overline{\theta},$$

where the bounds also depend on T*, the parameters of the system, the initial data* $\|\eta^0, \eta_x^0, v^0, v_x^0, \theta^0, \theta_x^0, \mathcal{I}^0, \mathcal{I}_x^0\|_{C^\alpha(\Omega)}$ *and where* $\underline{\theta}$ *depends on* $\inf_\Omega \theta^0$.

Proof 1. After the a priori estimates in Sect. 2, one checks that all the quantities

$$\int_{Q_T} v_x^2 \, dx \, dt, \quad \int_{Q_T} \theta_x^2 \, dx \, dt, \quad \int_{Q_T} v_t^2 \, dx \, dt, \quad \int_{Q_T} \theta_t^2 \, dx \, dt, \tag{92}$$

are bounded.

2. As the boundary conditions for I are only shifted by the positive time-independent quantity I_b, the same proof as in [9] gives the bounds

$$\max_{[0,T]} \int_\Omega \int_0^\infty \int_{S^1} I_t^2 \, d\omega \, d\nu \, dx \leq C, \tag{93}$$

$$\max_{[0,T]} \int_\Omega \int_0^\infty \int_{S^1} I_x^2 \, d\omega \, d\nu \, dx \leq C. \tag{94}$$

3. The following bounds hold

$$\max_{[0,T]} \int_\Omega v_t^2 \, dx + \int_{Q_T} v_{xt}^2 \, dx \, dt \leq C, \tag{95}$$

$$\max_{[0,T]} \int_\Omega v_{xx}^2 \, dx \leq C, \tag{96}$$

$$\max_{[0,T]} \int_\Omega \eta_x^2 \, dx \leq C. \tag{97}$$

Formally derivating the second Eq. (11) with respect to t, multiplying by v_t, integrating by parts and using (25), we find first

$$\begin{aligned} &\frac{1}{2}\int_\Omega v_t^2 \, dx + \frac{\mu_0}{2\overline{\eta}} \int_{Q_T} v_{xt}^2 \, dx \, dt \\ &\leq C + C\int_{Q_T} \left(1+\theta^{2r+2}\right) v_x^2 \, dx \, dt + C\int_{Q_T} \left(1+\theta^{2r+2}\right) v_x^2 \, dx \, dt \\ &\quad + C\int_{Q_T} v_x^4 \, dx \, dt + \varepsilon \max_{[0,T]} \int_\Omega v_t^2 \, dx \, dt + C\int_{Q_T} ((S_F)_R)_t^2 \, dx \, dt. \end{aligned}$$

As $|((S_F)_R)_t| \leq C + C\,[(1+\theta^\alpha)|\theta_t| + |v_x| + |I_t|]$, we get

$$\frac{1}{2}\int_\Omega v_t^2 \, dx + \frac{\mu_0}{2\overline{\eta}} \int_{Q_T} v_{xt}^2 \, dx \, dt \leq C + \varepsilon \max_{[0,T]} \int_\Omega v_t^2 \, dx \, dt \leq C,$$

for ε small enough, which proves (95).
From the second Eq. (11)

$$v_{xx} = \frac{\eta}{\mu}\left[v_t + p_x - \left(\frac{\mu}{\eta}\right)_\eta \eta_x v_x + \eta (S_F)_R \right],$$

then we get

$$\int_\Omega v_{xx}^2 \, dx \leq C + C\int_\Omega \left[v_t^2 + \eta_x^2 + \theta_x^2 + \eta_x^2 v_x^2 \right] dx,$$

which implies (96), after (95)

Finally, using the first Eq. (11), one gets

$$\int_\Omega \eta_x^2\,dx \le C + C\int_\Omega v_{xx}^2\,dx\,dt \le C,$$

after (96) □

4. Under the previous condition on the data, applying the maximum principle to the (parabolic) energy equation, there exists positive constant $\overline{\theta}$ and $\underline{\theta}$ depending on T and N such that

$$0 < \underline{\theta} \le \theta(x,t) \le \overline{\theta} \quad \text{for } (t,x) \in Q_T. \tag{98}$$

5. After the a priori estimates in Sect. 2, all the quantities

$$\max_{Q_T}|v_x|, \quad \max_{[0,T]}\int_\Omega v_x^2\,dx, \quad \int_{Q_T} v_x^4\,dx, \quad \max_{[0,T]}\int_\Omega v_t^2\,dx, \quad \int_{Q_T} v_{xt}^2\,dx\,dt,$$

are bounded.

6. The following estimate holds

$$\int_{Q_T}\theta_x^4\,dx\,dt \le C. \tag{99}$$

We first observe that

$$\int_{Q_T}\theta_x^4\,dx\,dt \le C\int_0^t \max_\Omega \theta_x^2\,ds \tag{100}$$

so, in order to prove (99), it is sufficient to bound the right-hand side.
First multiplying the equation of the internal energy by $\frac{\eta}{\kappa}\,\theta_t$ and integrating on Q_t, we get

$$\int_{Q_t}\frac{\eta e_\theta}{\kappa}\,\theta_t^2\,dx\,dt - \int_{Q_t}\frac{\eta p}{\kappa}\,\theta_t v_x\,dx\,dt + \int_{Q_t}\frac{\eta\theta p_\theta}{\kappa}\,\theta_t v_x\,dx\,dt - \int_{Q_t}\frac{\mu}{\kappa}\,\theta_t\,v_x^2\,dx\,dt$$
$$= \int_{Q_t}\frac{\eta}{\kappa}\,\theta_t\left(\frac{\kappa\theta_x}{\eta}\right)_x\,dx\,dt - \int_{Q_t}\frac{\eta^2}{\kappa}\,\theta_t\left[(S_E)_R - v\,(S_F)_R\right]\,dx\,dt.$$

The first term in the right-hand side rewrites

$$\int_{Q_t}\frac{\eta}{\kappa}\,\theta_t\left(\frac{\kappa\theta_x}{\eta}\right)_x\,dx\,dt = \int_{Q_t}\frac{\kappa_\theta}{\kappa}\,\theta_t\theta_x^2\,dx\,dt + \int_{Q_t}\left(\frac{\kappa_\eta}{\kappa} - \frac{1}{\kappa}\right)\theta_t\theta_x\eta_x\,dx\,dt$$
$$+ \int_{Q_t}\theta_t\theta_{xx}\,dx\,dt.$$

Integrating by parts, we get

$$\int_{Q_t} \frac{\eta e_\theta}{\kappa}\, \theta_t^2 \, dx\, dt + \frac{1}{2}\int_{\Omega} \theta_x^2 \, dx\, dt$$
$$\leq C + C\int_{Q_t} \left\{ |\theta_t v_x| + |\theta_t| v_x^2 \right.$$
$$\left. + |\theta_t|\theta_x^2 + |\theta_t \theta_x \eta_x| + |\theta_t\, (S_E)_R\,| \right\} \, dx\, dt.$$

So, for any $\varepsilon > 0$

$$\int_{Q_t} \frac{\eta e_\theta}{\kappa}\, \theta_t^2 \, dx\, dt + \frac{1}{2}\int_{\Omega} \theta_x^2 \, dx\, dt$$
$$\leq C + \varepsilon \int_{Q_t} \theta_t^2 \, dx\, dt + C\int_0^t \max_{\Omega} \theta_x^2 \, ds + \int_0^t \max_{\Omega} \theta_x^2 \int_{\Omega} \eta_x^2 \, dx\, ds.$$

Finally,

$$\int_{Q_t} \theta_t^2 \, dx\, dt + \int_{\Omega} \theta_x^2 \, dx\, dt \leq C + C\int_0^t \max_{\Omega} \theta_x^2 \, ds. \tag{101}$$

Multiplying now the equation of the internal energy by $\frac{(x-M)\kappa}{\eta}\, \theta_x$ and integrating on Ω, we get

$$\int_{\Omega} \frac{(x-M)\kappa e_\theta}{\eta}\, \theta_t \theta_x \, dx - \int_{\Omega} \frac{(x-M)\kappa}{\eta}\, p\theta_x v_x \, dx$$
$$+ \int_{\Omega} \frac{(x-M)\kappa}{\eta}\, \theta p_\theta \theta_x v_x \, dx - \int_{\Omega} \frac{(x-M)\mu\kappa}{\eta^2} \theta_x\, v_x^2 \, dx$$
$$= \int_{\Omega} (x-M)\, \frac{\kappa\theta_x}{\eta} \left(\frac{\kappa\theta_x}{\eta}\right)_x \, dx - \int_{\Omega} (x-M)\kappa\, \theta_x \left[(S_E)_R - v\, (S_F)_R \right] \, dx.$$

Then integrating in t and using boundary conditions, we have the estimate

$$\frac{1}{2}\int_0^t \left(\frac{\kappa\theta_x}{\eta}\right)^2 (0,s)\, ds \leq C + C\int_{Q_t} \left\{ \theta_x^2 + \theta_t^2 + v_x^2 + v_x^4 \right\} \, dx\, dt.$$

So, we end with

$$\int_0^t \left(\frac{\kappa\theta_x}{\eta}\right)^2 (0,s)\, ds \leq C. \tag{102}$$

Multiplying now the same equation of the internal energy by $\frac{\kappa}{\eta}\,\theta_x$ and integrating on $[0, x]$, we get

$$\int_0^x \frac{\kappa e_\theta}{\eta}\,\theta_t\theta_y\,dy - \int_0^x \frac{\kappa}{\eta}\,p\theta_y v_y\,dy + \int_0^x \frac{\kappa}{\eta}\,\theta p_\theta \theta_y v_y\,dy - \int_0^x \frac{\mu\kappa}{\eta^2}\,\theta_y\,v_y^2\,dy$$
$$= \int_0^x \frac{\kappa\theta_y}{\eta}\left(\frac{\kappa\theta_y}{\eta}\right)_y dy - \int_0^x \kappa\,\theta_y\left[(S_E)_R - v\,(S_F)_R\right] dy.$$

Then, integrating in t and using boundary conditions, we have the estimate

$$\frac{1}{2}\int_0^t \left(\frac{\kappa\theta_x}{\eta}\right)^2 (x,s)\,ds \le \frac{1}{2}\int_0^t \left(\frac{\kappa\theta_x}{\eta}\right)^2 (0,s)\,ds$$
$$+C + C\int_{Q_t}\left\{\theta_x^2 + \theta_t^2 + v_x^2 + v_x^4\right\}\,dx\,dt.$$

So after (102), we end with

$$\int_0^t \left(\frac{\kappa\theta_x}{\eta}\right)^2 (x,s)\,ds \le C, \tag{103}$$

which gives (99).

7. All the quantities

$$\max_{[0,T]}\int_\Omega \theta_x^2\,dx,\quad \max_{[0,T]}\int_\Omega \theta_{xx}^2\,dx,\quad \int_{Q_T}\theta_{xt}^2\,dx\,dt, \tag{104}$$

are bounded.
In fact, derivating formally the internal energy equation with respect to t, multiplying by $e_\theta\theta_t$ and using integration by parts on Q_T, we get

$$\frac{1}{2}\int_\Omega (e_\theta\theta_t)^2\,(x,t)\,dx - \frac{1}{2}\int_\Omega (e_\theta\theta_t)^2\,(x,0)\,dx + \int_{Q_T} p_\theta v_x e_\theta\theta_t^2\,dx\,dt$$
$$+ \int_{Q_T}\theta p_{\theta\theta} v_x e_\theta \theta_t^2\,dx\,dt + \int_{Q_T}\theta p_{\theta\eta} v_x^2 e_\theta\theta_t\,dx\,dt + \int_{Q_T}\theta p_\theta v_{xt} e_\theta\theta_t\,dx\,dt$$
$$- \int_{Q_T}\left[\left(\frac{\mu(\eta)}{\eta}\right)_\eta v_x^3 + 2\frac{\mu(\eta)}{\eta}\,v_x v_{xt}\right] e_\theta\theta_t\,dx\,dt$$
$$= -\int_{Q_T}\frac{\kappa}{\eta} e_\theta\theta_{tx}^2\,dx\,dt - \int_{Q_T}\left[\left(\frac{\kappa}{\eta}\right)_\eta v_x\theta_x + \frac{\kappa_\theta}{\eta}\,\theta_t\theta_x\right](e_\theta\theta_t)_x\,dx\,dt$$
$$- \int_{Q_T}\theta_x\left(e_{\theta\eta}\eta_x + e_{\theta\theta}\theta_x\right)\,dx\,dt - \int_{Q_T}\eta\,[(S_E)_R]_t\;e_\theta\theta_t\,dx\,dt - \int_{Q_T} v_x (S_E)_R\,e_\theta\theta_t\,dx\,dt.$$

After [5] (see the proof of Lemma 3.6), we get

$$\frac{1}{2}\int_{\Omega}(e_\theta\theta_t)^2(x,t)\,dx+\int_{Q_T}\frac{\kappa}{\eta}e_\theta\theta_{tx}^2\,dx\,dt$$
$$\leq C-\int_{Q_T}[(S_E)_R]_t\ e_\theta\theta_t\,dx\,dt-\int_{Q_T}v_x(S_E)_R\,e_\theta\theta_t\,dx\,dt. \qquad (105)$$

As the two integrals in the right-hand side are bounded after Lemma 3 and estimate (99) we obtain the first two estimates (104).

From the internal energy equation

$$\frac{\kappa}{\eta}\,\theta_{xx}=\left(\frac{\kappa-\eta\kappa_\eta}{\eta^2}\right)\eta_x\theta_x-\frac{\kappa_\theta}{\eta}\,\theta_x^2+e_\theta\theta_t+\theta p_\theta v_x-\frac{\mu}{\eta}\,v_x^2+\eta\,(S_E)_R\,,$$

then

$$|\theta_{xx}|\leq C\left(|\eta_x\theta_x|+\theta_x^2+|\theta_t|+|v_x|+v_x^2+|\,(S_E)_R\,|\right),$$

where all of the terms in the right-hand side are in $L^2(\Omega)$ after previous a priori estimates, which proves the last bound (104).

8. As $\max_{Q_T}|v_x|$ is bounded, we have

$$\left|\eta(x,t)-\eta(x,t')\right|\leq|t-t'|^{1/2}\left(\int_0^T v_x^2\,dt\right)^{1/2}\leq C|t-t'|^{1/2}.$$

We have also

$$\left|\eta(x,t)-\eta(x',t)\right|\leq C|x-x'|^{1/2}\left(1+\int_{\Omega}\eta_x^2\,dx\right)\leq C|x-x'|^{1/2},$$

so we find that $\eta\in C^{1/2,1/4}(Q_T)$.

After (104), we have

$$\left|\theta(x,t)-\theta(x,t')\right|\leq|t-t'|^{1/2}\left(\int_0^T\theta_t^2\,dt\right)^{1/2}\leq C|t-t'|^{1/2}.$$

We see also that

$$\left|\theta(x,t)-\theta(x',t)\right|\leq C|x-x'|^{1/2}\left(T\cdot\max_{[0,T]}\int_{\Omega}\theta_t^2\,dx+\int_0^T\int_{\Omega}\theta_{xt}^2\,dx\right)\leq C|x-x'|^{1/2},$$

so we find that $\theta\in C^{1/2,1/4}(Q_T)$. We have also

$$\left|\theta_x(x,t)-\theta_x(x',t)\right|\leq|x-x'|^{1/2}\left(\int_{\Omega}\theta_{xx}^2\,dt\right)^{1/2}\leq|x-x'|^{1/2},$$

we conclude, by using an interpolation argument of [24], that $\theta_x\in C^{1/3,1/6}(Q_T)$.

The same arguments holding true verbatim for v and v_x, we have that $v,v_x\in C^{1/3,1/6}(Q_T)$.

Let us note $\mathcal{I}(x,t):=\int_0^\infty\int_{S^1}I(x,t;\omega,\nu)\,d\omega\,d\nu$.

As $\max_{[0,T]} \|\mathcal{I}_t\|_{L^2(\Omega)} \leq C$, after Lemma 3, it follows that

$$|\mathcal{I}(x,t) - \mathcal{I}(x',t)| \leq \int_{x'}^{x} |I_y|\, dy \leq C|x - x'|^{1/2}.$$

As $\max_{[0,T]} \|\mathcal{I}_x\|_{L^2(\Omega)} \leq C$, also after Lemma 3, it also follows that

$$|\mathcal{I}(x,t) - \mathcal{I}(x,t')| \leq \int_{t'}^{t} |I_s|\, ds \leq C|t - t'|^{1/2}.$$

Then we conclude in particular that $\mathcal{I} \in C^{1/3,1/6}(Q_T)$, which ends the proof. □

References

1. Antontsev, S.N., Kazhikhov, A.V., Monakhov, V.N.: Boundary Value Problems in Mechanics of Nonhomogeneous Fluids. North-Holland, Amsterdam, New-York, Oxford, Tokyo (1990)
2. Brézis, H.: Analyse Fonctionnelle. Masson, Paris (1983)
3. Buet, C., Després, B.: Asymptotic analysis of fluid models for the coupling of radiation and hydrodynamics. J. Quant. Spectrosc. Radiat. Transf. **85**, 385–418 (2004)
4. Chandrasekhar, S.: Radiative Transfer. Dover Publications Inc., New York (1960)
5. Dafermos, C.M., Hsiao, L.: Global smooth thermomechanical processes in one-dimensional nonlinear thermoviscoelasticity. Nonlinear Anal. **6**, 435–454 (1982)
6. Dautray, R., Lions, J.L.: Analyse Mathématique et Calcul Numërique Pour les Sciences et les Techniques Tome 3. Masson, Paris, New York, Barcelone, Milan, Mexico, Sao Paulo (1985)
7. Dubroca, B., Feugeas, J.-L.: Etude théorique et numérique d'une hiérarchie de modèles aux moments pour le transfert radiatif. C. R. Acad. Sci. Paris **329**(Série I), 915–920 (1999)
8. Ducomet, B., Nečasová, Š.: Global existence of solutions for the one-dimensional motions of a compressible gas with radiation: An "infrarelativistic model". Nonlinear Anal. **72**, 3258–3274 (2010)
9. Ducomet, B., Nečasová, Š.: Global weak solutions to the 1D compressible Navier–Stokes equations with radiation. Commun. Math. Anal. **8**(3), 23–65 (2010)
10. Ducomet, B., Nečasová, Š.: Asymptotic behavior of the motion of a viscous heat-conducting one-dimensional gas with radiation: the pure scattering case. (in preparation)
11. Ducomet, B., Zlotnik, A.: Lyapunov functional method for 1D radiative and reactive viscous gas dynamics. Arch. Ration. Mech. Anal. **177**, 185–229 (2005)
12. Feireisl, E.: Dynamics of Viscous Compressible Fluids. Oxford University Press, Oxford (2003)
13. Golse, F., Allaire, G.: Transport et diffusion. Cours à l'Ecole Polytechnique, MAP 567 (2009)
14. Golse, F., Perthame, B.: Generalized solutions of the radiative transfer equations in a singular case. Commun. Math. Phys. **106**, 211–239 (1986)
15. Hsiao, L.: Quasilinear Hyperbolic Systems and Dissipative Mechanisms. World Scientific, Singapore, New Jersey, London, Hong Kong (1997)
16. Hsiao, L., Jiang, S.: Nonlinear parabolic equations equations and systems. In: Dafermos, C.M., Feireisl, E. (eds.) Handbook of Differential Equations, vol. 1, Elsevier, North Holland, Amsterdam, Boston, Heidelberg (2004)
17. Jiang, S.: On initial-boundary value problems for a viscous heat-conducting one-dimensional gas. J. Differ. Equ. **110**(2), 157–181 (1994)
18. Jiang, S.: On the asymptotic behavior of the motion of a viscous heat-conducting one-dimensional real gas. Math. Z. **216**, 317–336 (1994)
19. Jiang, S.: Global smooth solutions to the equations of a viscous heat-conducting one-dimensional gas with density-dependent viscosity. Math. Nachr. **190**, 169–183 (1998)
20. Jiang, P., Wang, D.: Formation of Singularities of Solutions to the Three-Dimensional Euler-Boltzmann Equations in Radiation Hydrodynamics. Preprint, March 11 (2009)
21. Jiang P., Wang D.: Global Weak Solutions to the Euler-Boltzmann Equations in Radiation Hydrodynamics. Preprint, June 27 (2009)
22. Kawohl, B.: Global existence of large solutions to initial-boundary value problems for a viscous heat-conducting, one-dimensional real gas. J. Differ. Equ. **58**(1), 76–103 (1985)

23. Kippenhahn, R., Weigert, A.: Stellar Structure and Evolution. Springer, Berlin, Heidelberg (1994)
24. Ladyženskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N.: Linear and Quasilinear Equations of Parabolic Type. AMS Providence, Rhode Island (1968)
25. Lin, C.: Mathematical Analysis of Radiative Transfer Models. PhD Thesis (2007)
26. Lin, C., Coulombel, J.-F., Goudon, T.: Shock profiles for nonequilibrium radiating gases. Phys. D **218**, 83–94 (2006)
27. Lowrie, R.B., Morel, J.E., Hittinger, J.A.: The coupling of radiation and hydrodynamics. Astrophys. J. **521**, 432–450 (1999)
28. Mihalas, D., Weibel-Mihalas, B.: Foundations of Radiation Hydrodynamics. Oxford University Press, New York (1984)
29. Okada, M., Kawashima, S.: On the equations of one-dimensional motion of compressible viscous fluids. J. Math. Kyoto Univ. **23**, 55–71 (1983)
30. Pomraning, G.C.: Radiation Hydrodynamics. Dover Publications Inc., Mineola, New York (2005)
31. Qin, Y.: Nonlinear Parabolic-Hyperbolic Coupled Systems and Their Attractors. Birkhäuser, Basel, Boston, Berlin (2008)
32. Zel'dovich, Y.B., Raiser, Y.P.: Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena. Dover Publications Inc., Mineola, New York (2002)
33. Zhong, X., Jiang, S.: Local existence and finite-time blow-up in multidimensional radiation hydrodynamics. J. Math. Fluid Mech. **9**, 543–564 (2007)