Annali di Matematica (2012) 191:205–217
DOI 10.1007/s10231-010-0179-5

On the asymptotic behavior of solutions of Emden–Fowler equations on time scales

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Received: 30 April 2010 / Accepted: 26 November 2010 / Published online: 14 December 2010

Abstract Consider the Emden-Fowler dynamic equation

$$x^{\Delta\Delta}(t) + p(t)x^{\alpha}(t) = 0, \quad \alpha > 0, \tag{0.1}$$

where $p \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, α is the quotient of odd positive integers, and $\mathbb{T}$ denotes a time scale which is unbounded above and satisfies an additional condition (C) given below. We prove that if $\int_{t_0}^{\infty} t^{\alpha}|p(t)|\Delta t < \infty$ (and when $\alpha = 1$ we also assume $\lim_{t\to\infty} tp(t)\mu(t) = 0$), then (0.1) has a solution $x(t)$ with the property that

$$\lim_{t\to\infty} \frac{x(t)}{t} = A \neq 0.$$

Keywords Asymptotic behavior · Emden-Fowler equation · Generalized Gronwall's Inequality

Mathematics Subject Classification (2010) 34K11 · 39A10 · 39A99 · 34N05

1 Introduction

Consider the second-order Emden–Fowler dynamic equation

$$x^{\Delta\Delta}(t) + p(t)x^{\alpha}(t) = 0, \tag{1.1}$$

Jia Baoguo is supported by the National Natural Science Foundation of China (No. 10971232).

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where $p : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ is rd-continuous (defined below), $\alpha > 0$, α is the quotient of odd positive integers.

When $\mathbb{T} = \mathbb{R}$, the dynamic Eq. (1.1) is the second-order Emden–Fowler differential equation

$$x''(t) + p(t)x^{\alpha}(t) = 0. \tag{1.2}$$

The Emden–Fowler Eq. (1.2) has several interesting physical applications in astrophysics (cf. Bellman [8] and Fowler [13]). Moore and Nehari [14] established the following: If $p(t)$ is positive and continuous and $\alpha \geq 1$, then (1.1) has solutions for which

$$\lim_{t\to\infty} \frac{x(t)}{t} = A > 0$$

if and only if

$$\int^{\infty} t^{\alpha} p(t) \mathrm{d}t < \infty. \tag{1.3}$$

This is related to results of Atkinson [1] who showed that if $\alpha > 1$, $p(t) \geq 0$ and is non-increasing, then (1.3) implies that all solutions of (1.1) are nonoscillatory. We refer to [3,6] and [7] for additional results for the oscillation of (1.1). Wong [15, Theorem 2] established the sufficiency part of the above Moore–Nehari theorem without an assumption as to the sign of $p(t)$.

In this paper, by using a generalized Gronwall's inequality on time scales and an idea used by Wong [15], we prove that if

$$\int^{\infty} t^{\alpha} |p(t)| \Delta t < \infty,$$

(and when $\alpha = 1$ we assume $\lim_{t\to\infty} tp(t)\mu(t) = 0$), then Eq. (1.1) has a solution for which

$$\lim_{t\to\infty} \frac{x(t)}{t} = A > 0.$$

Since we do not make an assumption concerning the sign of the coefficient p, a fixed point approach is not of use.

For completeness (see [9] and [10] for elementary results for the time scale calculus), we recall some basic results for dynamic equations and the calculus on time scales. Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup \mathbb{T} = \infty$. The forward jump operator is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\sup \varnothing = \inf \mathbb{T}$, where $\varnothing$ denotes the empty set. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $\sigma(t) = t$ we say t is right-dense, while if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$ we say t is left-dense. Given a time scale interval $[c, d]_{\mathbb{T}} := \{t \in \mathbb{T} : c \leq t \leq d\}$ in $\mathbb{T}$ the notation $[c, d]_{\mathbb{T}}^{\kappa}$ denotes the interval $[c, d]_{\mathbb{T}}$ in case $\rho(d) = d$ and denotes the interval $[c, d)_{\mathbb{T}}$ in case $\rho(d) < d$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. We say $p : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ is rd-continuous and write $p \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ provided p is continuous at each right-dense point in $[t_0, \infty)_{\mathbb{T}}$ and at each left-dense point in $(t_0, \infty)_{\mathbb{T}}$ the left hand limit of p exists (finite). We say that $x : \mathbb{T} \to \mathbb{R}$ is (delta) differentiable at $t \in \mathbb{T}$ provided

$$x^{\Delta}(t) := \lim_{s \to t} \frac{x(t) - x(s)}{t - s},$$

exists when $\sigma(t) = t$ (here by $s \to t$ it is understood that s approaches t in the time scale) and when x is continuous at t and $\sigma(t) > t$

$$x^{\Delta}(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}.$$

Note that if $\mathbb{T} = \mathbb{R}$, then the delta derivative is just the standard derivative, and when $\mathbb{T} = \mathbb{Z}$ the delta derivative is just the forward difference operator. Hence, our results contain the discrete and continuous cases as special cases and generalize these results to many other time scales. Section 2 is devoted to a few preliminary results, the main result is in Sect. 3 and we include several examples in Sect. 4. We remark also that many results dealing with Sturmian Theory for the dynamic Eq. (1.1) consider the case when the term $p(t)x^{\alpha}(t)$ is replaced by $p(t)x^{\alpha}(\sigma(t))$. In this regard, a comprehensive analysis of the linear and half-linear equation may be found in the book of Došlý and Řehák [12], where there are also many references to the literature.

2 Preliminary lemmas

Let $\hat{\mathbb{T}} := \{t \in \mathbb{T} : \mu(t) > 0\}$ and let χ denote the characteristic function of $\hat{\mathbb{T}}$. The following condition, which will be needed later, imposes a lower bound on the graininess function $\mu(t)$, for $t \in \hat{\mathbb{T}}$. More precisely, we introduce the following (see [11] and [5]).

Condition (C) We say that $\mathbb{T}$ satisfies condition (C) if there is an $M > 0$ such that

$$\chi(t) \le M\mu(t), \quad t \in \mathbb{T}.$$

We note that if $\mathbb{T}$ satisfies condition (C), then the set

$$\check{\mathbb{T}} = \{t \in \mathbb{T} |\ t > 0 \text{ is isolated or right-scattered or left-scattered}\}$$

is necessarily countable.

We let $\mathcal{R}$ denote the set of rd-continuous functions p on $\mathbb{T}$ satisfying the regressivity condition $1 + \mu(t)p(t) \neq 0$ on $\mathbb{T}$. Recall the definition of the generalized exponential function $e_p(t, t_0)$ ([9], Page 57)

$$e_p(t, t_0) = \exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right), \tag{2.1}$$

where

$$\xi_h(p(\tau)) = \begin{cases} \frac{1}{h} Log(1 + hp(\tau)), & h > 0 \\ p(\tau), & h = 0, \end{cases} \tag{2.2}$$

and Log is the principal logarithm function. We will use this representation in the proof of Lemma 2.1 below.

The following lemma is [9, Corollary 6.7].

Lemma 2.1 *Suppose that $y(t)$ and $p(t)$ are rd-continuous and $p \geq 0$. Then,*

$$y(t) \leq y_0 + \int_{t_0}^{t} p(s)y(s)\Delta s \quad \text{for all} \quad t \geq t_0, \tag{2.3}$$

implies

$$y(t) \leq y_0 e_p(t, t_0) \quad \text{for all} \quad t \geq t_0.$$

Lemma 2.2 *Suppose that $p \in \mathcal{R}$ and $\lim_{t\to\infty} p(t)\mu(t) = 0$. Then, there is a $T \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$|e_p(t, T)| \leq \exp\left[\int_{T}^{t} 2|p(\tau)|\Delta\tau\right]$$

for $t \in [T, \infty)_{\mathbb{T}}$. If, in addition, $\int_{t_0}^{\infty} |p(s)|\Delta s < \infty$, then $e_p(t, t_0)$ is bounded on $[t_0, \infty)_{\mathbb{T}}$.

Proof Since $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$, there is a $\delta > 0$ such that $|\log(1+x)| \leq 2|x|$, for $|x| < \delta$. Using the hypothesis $\lim_{t\to\infty} p(t)\mu(t) = 0$, there is a $T \in [t_0, \infty)_{\mathbb{T}}$ such that

$$|p(t)\mu(t)| < \delta, \quad t \in [T, \infty)_{\mathbb{T}}.$$

Assume $\tau \in [T, \infty)_{\mathbb{T}}$. If, in addition, $\mu(\tau) > 0$, we have that

$$\begin{aligned} \left|\xi_{\mu(\tau)}(p(\tau))\right| &= \left|\frac{Log[1 + \mu(\tau)p(\tau)]}{\mu(\tau)}\right| \\ &= \frac{|\log(1 + \mu(\tau)p(\tau))|}{\mu(\tau)} \leq 2|p(\tau)|. \end{aligned}$$

On the other hand, if $\mu(\tau) = 0$, we have

$$|\xi_{\mu(\tau)}(p(\tau))| = |p(\tau)| \leq 2|p(\tau)|.$$

Hence, for all $\tau \in [T, \infty)_{\mathbb{T}}$, we have

$$|\xi_{\mu(\tau)}(p(\tau))| \leq 2|p(\tau)|.$$

Therefore, by (2.1) and (2.2), we get that for $t \in [T, \infty)_{\mathbb{T}}$

$$\begin{aligned} |e_p(t, T)| &= |e^{\int_T^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau}| \\ &\leq \left|\exp\left[\int_{T}^{t} 2|p(\tau)|\Delta\tau\right]\right| \\ &\leq \exp\left[\int_{t_0}^{t} 2|p(\tau)|\Delta\tau\right]. \end{aligned}$$

The last statement in this lemma follows from this last inequality and the semi group property [9, Theorem 2.36] $e_p(t, t_0) = e_p(t, T)e_p(T, t_0)$. □

Lemma 2.3 *Suppose that $[t_0, \infty)_{\mathbb{T}}$ satisfies condition* (C)*, $x^{\Delta}(t)$ is rd-continuous, and $f : (0, \infty) \to (0, \infty)$ is continuous and nonincreasing with $F'(x) = f(x), x \in \mathbb{R}$. Then,we* *have that*

$$\int_{t_0}^{t} f(x(s))x^{\Delta}(s)\Delta s \geq F(x(t)) - F(x(t_0)), \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

Proof Since $[t_0, \infty)_{\mathbb{T}}$ satisfies property (C), $[t_0, \infty)_{\mathbb{T}} = \cup_{i=0}^{\infty}[t_i, t_{i+1}]_{\mathbb{T}}$, where for each $i \geq 0$ either $\sigma(t_i) = t_{i+1} > t_i$ or $[t_i, t_{i+1}]_{\mathbb{T}} = [t_i, t_{i+1}]_{\mathbb{R}}$. From the additivity of the integral, it suffices to show that

$$\int_{t_i}^{t_{i+1}} f(x(s))x^{\Delta}(s)\Delta s \geq F(x(t_{i+1})) - F(x(t_i)) \tag{2.4}$$

for each $i \geq 0$. First consider the case $\sigma(t_i) = t_{i+1} > t_i$ and consider the subcase $x(t_i) \leq x(t_{i+1})$. Using the fact that f is nonincreasing, we have

$$\begin{aligned}
\int_{t_i}^{t_{i+1}} f(x(s))x^{\Delta}(s)\Delta s &= \int_{t_i}^{\sigma(t_i)} f(x(s))x^{\Delta}(s)\Delta s \\
&= f(x(t_i))x^{\Delta}(t_i)\mu(t_i) \\
&= f(x(t_i))[x(t_{i+1}) - x(t_i)] \\
&\geq \int_{x(t_i)}^{x(t_{i+1})} f(x)\mathrm{d}x \\
&= F(x(t_{i+1})) - F(x(t_i))
\end{aligned}$$

and so (2.4) holds in this case. Next consider the subcase $x(t_i) > x(t_{i+1})$. In this case,

$$\begin{aligned}
\int_{t_i}^{t_{i+1}} f(x(s))x^{\Delta}(s)\Delta s &= \int_{t_i}^{\sigma(t_i)} f(x(s))x^{\Delta}(s)\Delta s \\
&= f(x(t_i))x^{\Delta}(t_i)\mu(t_i) \\
&= -f(x(t_i))[x(t_i) - x(t_{i+1})] \\
&\geq -\int_{x(t_{i+1})}^{x(t_i)} f(x)\mathrm{d}x \\
&= F(x(t_{i+1})) - F(x(t_i))
\end{aligned}$$

and so also in the subcase $x(t_i) > x(t_{i+1})$, it follows that (2.4) holds.

Finally, if $[t_i, t_{i+1}]_{\mathbb{T}} = [t_i, t_{i+1}]_{\mathbb{R}}$ then

$$\int_{t_i}^{t_{i+1}} f(x(t))x^{\Delta}(t)\Delta t = \int_{t_i}^{t_{i+1}} f(x(t))x'(t)dt = F(x(t_{i+1})) - F(x(t_i))$$

and so (2.4) holds in this case as well. □

Our next lemma is a sublinear analogue of the Gronwall inequality.

Lemma 2.4 *Assume* $[t_0,\infty)_{\mathbb{T}}$ *satisfies condition* (C), $p(t)\ge 0$, $q(t)\ge 0$ *are* rd*-continuous,* $y_0>0$, *and* $0<\alpha<1$. *If* $y(t)\ge 0$, *satisfies*

$$y(t)\le y_0+\int_{t_0}^{t} p(s)y(s)\Delta s+\int_{t_0}^{t} q(s)y^{\alpha}(s)\Delta s$$

for $t\in[t_0,\infty)_{\mathbb{T}}$, *then*

$$y(t)\le e_p(t,t_0)\left\{y_0^{1-\alpha}+(1-\alpha)\int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s,t_0)\frac{q(s)}{1+\mu(s)p(s)}\Delta s\right\}^{\frac{1}{1-\alpha}}$$

for $t\in[t_0,\infty)_{\mathbb{T}}$.

Proof Let

$$z(t):=y_0+\int_{t_0}^{t} p(s)y(s)\Delta s+\int_{t_0}^{t} q(s)y^{\alpha}(s)\Delta s>0$$

for $t\in[t_0,\infty)_{\mathbb{T}}$. Then, by hypothesis $y(t)\le z(t)$ on $[t_0,\infty)_{\mathbb{T}}$ and

$$\begin{aligned} z^{\Delta}(t)&=p(t)y(t)+q(t)y^{\alpha}(t)\\ &\le p(t)z(t)+q(t)z^{\alpha}(t)\\ &=p(t)[z^{\sigma}(t)-\mu(t)z^{\Delta}(t)]+q(t)z^{\alpha}(t). \end{aligned}$$

It follows that

$$z^{\Delta}(t)-\frac{p(t)}{1+\mu(t)p(t)}z^{\sigma}(t)\le\frac{q(t)}{1+\mu(t)p(t)}z^{\alpha}(t).$$

Hence,

$$z^{\Delta}(t)+(\ominus p)(t)z^{\sigma}(t)\le\frac{q(t)}{1+\mu(t)p(t)}z^{\alpha}(t).$$

Multiplying by the integrating factor $e_{\ominus p}(t,t_0)$, we get

$$\begin{aligned} \left[e_{\ominus p}(t,t_0)z(t)\right]^{\Delta}&\le e_{\ominus p}(t,t_0)\frac{q(t)}{1+\mu(t)p(t)}z^{\alpha}(t)\\ &=e_{\ominus p}^{1-\alpha}(t,t_0)\frac{q(t)}{1+\mu(t)p(t)}[e_{\ominus p}(t,t_0)z(t)]^{\alpha}. \end{aligned}$$

Letting

$$v(t):=e_{\ominus p}(t,t_0)z(t)>0,$$

we have

$$\frac{v^{\Delta}(t)}{v^{\alpha}(t)}\le e_{\ominus p}^{1-\alpha}(t,t_0)\frac{q(t)}{1+\mu(t)p(t)}.$$

Integrating from t_0 to t, we obtain

$$\int_{t_0}^{t}\frac{v^{\Delta}(s)}{v^{\alpha}(s)}\Delta s\le\int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s,t_0)\frac{q(s)}{1+\mu(s)p(s)}\Delta s.$$

Applying Lemma 2.3 (with $f(x) = \frac{1}{x^\alpha}$) to the left-hand side of this last inequality gives

$$\int_{t_0}^{t} \frac{v^\Delta(s)}{v^\alpha(s)} \Delta s \geq \frac{v^{1-\alpha}(t)}{1-\alpha} - \frac{v^{1-\alpha}(t_0)}{1-\alpha}.$$

It then follows that

$$\frac{v^{1-\alpha}(t)}{1-\alpha} - \frac{v^{1-\alpha}(t_0)}{1-\alpha} \leq \int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s, t_0) \frac{q(s)}{1+\mu(s)p(s)} \Delta s \tag{2.5}$$

and consequently

$$v^{1-\alpha}(t) \leq y_0^{1-\alpha} + (1-\alpha) \int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s, t_0) \frac{q(s)}{1+\mu(s)p(s)} \Delta s.$$

Then,

$$v(t) = e_{\ominus p}(t, t_0) z(t) \leq \left\{ y_0^{1-\alpha} + (1-\alpha) \int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s, t_0) \frac{q(s)}{1+\mu(s)p(s)} \Delta s \right\}^{\frac{1}{1-\alpha}}$$

which gives us the desired result

$$y(t) \leq z(t) \leq e_p(t, t_0) \left\{ y_0^{1-\alpha} + (1-\alpha) \int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s, t_0) \frac{q(s)}{1+\mu(s)p(s)} \Delta s \right\}^{\frac{1}{1-\alpha}}.$$

□

Remark 2.5 When $p(t) \equiv 0$ in Lemma 2.4 we have that

$$y(t) \leq y_0 + \int_{t_0}^{t} q(s) y^\alpha(s) \Delta s \quad \text{for all} \quad t \in [t_0, \infty)_{\mathbb{T}}$$

implies

$$y(t) \leq \left\{ y_0^{1-\alpha} + (1-\alpha) \int_{t_0}^{t} q(s) \Delta s \right\}^{\frac{1}{1-\alpha}},$$

for $t \in [t_0, \infty)_{\mathbb{T}}$.

The superlinear analogue of Lemma 2.4 is the following result:

Lemma 2.6 *Assume in Lemma 2.4 we replace* $0 < \alpha < 1$ *by* $\alpha > 1$ *and that we can pick* $y_0 > 0$ *such that*

$$y_0^{1-\alpha} > (\alpha - 1) \int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s, t_0) \frac{q(s)}{1 + \mu(s)p(s)} \Delta s$$

for $t \in [t_0, \infty)_{\mathbb{T}}$. *Suppose* $y(t) \geq 0$, *satisfies*

$$y(t) \leq y_0 + \int_{t_0}^{t} p(s)y(s)\Delta s + \int_{t_0}^{t} q(s)y^{\alpha}(s)\Delta s$$

for $t \in [t_0, \infty)_{\mathbb{T}}$. *Then we have*

$$y(t) \leq \frac{e_p(t, t_0)}{\left\{y_0^{1-\alpha} - (\alpha - 1) \int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s, t_0) \frac{q(s)}{1+\mu(s)p(s)} \Delta s\right\}^{\frac{1}{\alpha - 1}}}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$.

Proof The proof starts out the same as in the proof of Lemma 2.4 until we get (2.5). Solving this equation for $v(t)$, we get (here we use the assumption $y_0^{1-\alpha} > (\alpha - 1) \int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s, t_0) \frac{q(s)}{1+\mu(s)p(s)} \Delta s$)

$$v(t) \leq \frac{1}{\left\{y_0^{1-\alpha} - (\alpha - 1) \int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s, t_0) \frac{q(s)}{1+\mu(s)p(s)} \Delta s\right\}^{\frac{1}{\alpha - 1}}}.$$

Since $v(t) = e_{\ominus p}(t, t_0)z(t)$ and $e_{\ominus p}(t, t_0) = \frac{1}{e_p(t, t_0)}$ we have that

$$y(t) \leq z(t) \leq \frac{e_p(t, t_0)}{\left\{y_0^{1-\alpha} - (\alpha - 1) \int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s, t_0) \frac{q(s)}{1+\mu(s)p(s)} \Delta s\right\}^{\frac{1}{\alpha - 1}}}.$$

□

3 Asymptotic behavior of solutions

We now prove our main result

Theorem 3.1 *Assume* $[t_0, \infty)_{\mathbb{T}}$ *satisfies condition* (*C*), $\alpha > 0$ *is the quotient of odd positive integers, and*

$$\int_{t_0}^{\infty} t^{\alpha}|p(t)|\Delta t < \infty,$$

(*and if* $\alpha = 1$ *we assume* $\lim_{t\to\infty} tp(t)\mu(t) = 0$). *Then*

$$x^{\Delta\Delta} + p(t)x^{\alpha}(t) = 0, \tag{3.1}$$

has a solution satisfying $\lim_{t\to\infty} \frac{x(t)}{t} = A \neq 0$.

Proof Without loss of generality we can assume $t_0 \geq 1$. Assume $x(t)$ is a solution of (3.1) with $x(t_0) \neq 0$ and let

$$k(t_0) := |x(t_0)| + |x^{\Delta}(t_0)| > 0.$$

Now Eq. (3.1) is equivalent to the integral equation

$$x(t) = x(t_0) + x^{\Delta}(t_0)(t - t_0) - \int_{t_0}^{t} (t - \sigma(s))p(s)x^{\alpha}(s)\Delta s. \tag{3.2}$$

Thus for $t \in [t_0, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} |x(t)| &\leq |x(t_0)| + |x^{\Delta}(t_0)|t + t\int_{t_0}^{t} |p(s)x^{\alpha}(s)|\Delta s \\ &\leq k(t_0)t + t\int_{t_0}^{t} |p(s)x^{\alpha}(s)|\Delta s, \quad (\text{using} \quad t \geq 1) \\ &= k(t_0)t + t\int_{t_0}^{t} s^{\alpha}|p(s)| \left(\frac{|x(s)|}{s}\right)^{\alpha} \Delta s. \end{aligned}$$

Letting $y(t) := \frac{|x(t)|}{t}$, we obtain

$$y(t) \leq k(t_0) + \int_{t_0}^{t} s^{\alpha}|p(s)|y^{\alpha}(s)\Delta s, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{3.3}$$

Consider first the case $0 < \alpha < 1$. By Remark 2.5, we get

$$\begin{aligned} y(t) &\leq \left\{k^{1-\alpha}(t_0) + (1-\alpha)\int_{t_0}^{t} s^{\alpha}|p(s)|\Delta s\right\}^{\frac{1}{1-\alpha}} \\ &\leq \left\{k^{1-\alpha}(t_0) + (1-\alpha)\int_{t_0}^{\infty} s^{\alpha}|p(s)|\Delta s\right\}^{\frac{1}{1-\alpha}} =: B. \end{aligned} \tag{3.4}$$

So we have $y(t) \leq B$, that is $|x(t)| \leq Bt$. Since

$$x^{\Delta}(t) = x^{\Delta}(t_0) - \int_{t_0}^{t} p(s)x^{\alpha}(s)\Delta s \tag{3.5}$$

and

$$\int_{t_0}^{t} \left|p(s)x^{\alpha}(s)\right| \Delta s \leq B^{\alpha} \int_{t_0}^{\infty} |p(s)|s^{\alpha}\Delta s < \infty, \tag{3.6}$$

we have that $\lim_{t\to\infty} x^{\Delta}(t) = A$ exists. Therefore if $\epsilon > 0$ is given, then there exists a $T \in [t_0, \infty)_{\mathbb{T}}$, such that $A - \epsilon < x^{\Delta}(t) < A + \epsilon$, for $t \in [T, \infty)_{\mathbb{T}}$. By the time scales

Mean Value Theorem [10, Theorem 1.14], we get that if $t \in [T, \infty)_{\mathbb{T}}$ with $t > T$ there exist $\tau_t, \xi_t \in [T, t)_{\mathbb{T}}$ such that

$$A - \epsilon < x^{\Delta}(\tau_t) \le \frac{x(t) - x(T)}{t - T} \le x^{\Delta}(\xi_t) < A + \epsilon.$$

This implies for all sufficiently large $t \in [T, \infty)_{\mathbb{T}}$ that

$$(A - \epsilon)\left(1 - \frac{T}{t}\right) + \frac{x(T)}{t} < \frac{x(t)}{t} < (A + \epsilon)\left(1 - \frac{T}{t}\right) + \frac{x(T)}{t}.$$

Therefore we get that

$$\lim_{t \to \infty} \frac{x(t)}{t} = A. \tag{3.7}$$

From (3.5) and (3.6), we have

$$|x^{\Delta}(t)| \ge |x^{\Delta}(t_0)| - B^{\alpha} \int_{t_0}^{\infty} s^{\alpha} |p(s)| \Delta s. \tag{3.8}$$

We want to show that we can find a solution $x(t)$ so that the constant A in (3.7) is nonzero. To see this, we still assume $t_0 \ge 1$ let $t_1 \in [t_0, \infty)_{\mathbb{T}}$ be fixed but arbitrary and let $x_{t_1}(t)$ be a family of solutions of (1.1) whose initial conditions satisfy

$$|x_{t_1}(t_1)| = C > 0, \quad |x_{t_1}^{\Delta}(t_1)| = D > 0, \quad t_1 \in [t_0, \infty)_{\mathbb{T}},$$

where C and D are constants (do not depend on t_1). Then by the proof of (3.8) we obtain

$$\begin{aligned} |x_{t_1}^{\Delta}(t)| &\ge |x_{t_1}^{\Delta}(t_1)| - B^{\alpha}(t_1) \int_{t_1}^{\infty} s^{\alpha} |p(s)| \Delta s \\ &= D - B^{\alpha}(t_1) \int_{t_1}^{\infty} s^{\alpha} |p(s)| \Delta s \end{aligned} \tag{3.9}$$

where

$$B(t_1) := \left\{ (C + D)^{1-\alpha} + (1 - \alpha) \int_{t_1}^{\infty} s^{\alpha} |p(s)| \Delta s \right\}^{\frac{1}{1-\alpha}}.$$

Since

$$\lim_{t_1 \to \infty} B^{\alpha}(t_1) \int_{t_1}^{\infty} s^{\alpha} |p(s)| \Delta s = 0$$

we can pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ sufficiently large so that (using (3.9)) we have that

$$|x_{t_1}^{\Delta}(t)| \ge \frac{1}{2} D > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

It is then easy to see that for such a t_1 we have that

$$\lim_{t \to \infty} \frac{x_{t_1}(t)}{t} = A \ne 0.$$

This completes the proof when $0 < \alpha < 1$.

Now consider the case when $\alpha = 1$. The proof is the same as the above proof up to (3.3). That is we get (3.3) with $\alpha = 1$, namely

$$y(t) \leq k(t_0) + \int_{t_0}^{t} s|p(s)|y(s)\Delta s, \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

Then we use Lemma 2.1 to obtain

$$y(t) \leq k(t_0)e_r(t, t_0), \quad \text{where} \quad r = r(t) := t|p(t)|. \tag{3.10}$$

Since $\int_{t_0}^{\infty} r(t)\Delta t < \infty$ and $\lim_{t\to\infty} r(t)\mu(t) = 0$ we get from Lemma 2.2 that $e_r(t, t_0)$ is bounded and hence there is a constant $B > 0$ such that $y(t) \leq B$ for $t \in [t_0, \infty)_{\mathbb{T}}$. The rest of the proof is the same as in the case $0 < \alpha < 1$ after (3.4).

Next assume $\alpha > 1$. At the outset of this proof, assume that $x(t)$ is a solution of (1.1), with $x^{\Delta}(t_0) = 0$ and $x_0 := x(t_0)$ is chosen so that $k(t_0) = |x(t_0)| > 0$ satisfies

$$|x_0|^{1-\alpha} = k^{1-\alpha}(t_0) > (\alpha - 1) \int_{t_0}^{\infty} s^{\alpha}|p(s)|\Delta s$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$ (we can do this because we are assuming $\int_{t_0}^{\infty} s^{\alpha}|p(s)|\Delta s < \infty$). It follows that

$$|x_0|^{1-\alpha} = k^{1-\alpha}(t_0) > (\alpha - 1) \int_{t_0}^{t} s^{\alpha}|p(s)|\Delta s$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$. Using the fact that (3.4) holds we get from Lemma 2.6 that

$$\begin{aligned} y(t) &\leq \frac{1}{\left\{|x_0|^{1-\alpha} - (\alpha - 1)\int_{t_0}^{t} s^{\alpha}|p(s)|\Delta s\right\}^{\frac{1}{\alpha-1}}} \\ &\leq \frac{1}{\left\{|x_0|^{1-\alpha} - (\alpha - 1)\int_{t_0}^{\infty} s^{\alpha}|p(s)|\Delta s\right\}^{\frac{1}{\alpha-1}}} =: B \end{aligned}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$. It follows that $|x(t)| \leq Bt$ for $t \in [t_0, \infty)_{\mathbb{T}}$ and the proof now proceeds as in the case $0 < \alpha < 1$ following (3.4). □

4 Examples

We consider in this section several examples to illustrate the results obtained (we deal first with the more challenging case when $p(t)$ is not of one sign.)

Example 4.1 Let $0 < \alpha \leq 1$, $\mathbb{T} = \mathbb{N}$ and

$$p(n) := a(-1)^n n^{-b}.$$

Consider the equation

$$\Delta^2 x(n) + p(n)x^{\alpha}(n) = 0, \quad 0 < \alpha < 1. \tag{4.1}$$

From Theorem 3.1, we conclude that if $b > \alpha + 1$, then

$$\sum_{n_0}^{\infty} n^{\alpha}|p(n)| < \infty$$

and therefore it follows that (4.1) has a solution satisfying

$$\lim_{n\to\infty} \frac{x(n)}{n} = A \neq 0. \tag{4.2}$$

Note also that (4.2) holds in the case $\alpha = 1$ since $b > 2$ implies that $\lim_{n\to\infty} np(n) = 0$.

Example 4.2 Let $\alpha > 1$ and let

$$p(n) := \frac{a}{(n+1)n^b} + \frac{c(-1)^n}{n^b}, \quad n \in \mathbb{N}.$$

We note that (4.1) has a nonoscillatory solution such that (4.2) holds if $b > \alpha + 1$, since

$$\sum_{n=1}^{\infty} n^{\alpha}|p(n)| \le a\sum_{n=1}^{\infty} n^{\alpha-b-1} + c\sum_{n=1}^{\infty} n^{\alpha-b} < \infty.$$

We also observe that for the case when the term $p(n)x^{\alpha}(n)$ in (4.1) is replaced by $p(n)x^{\alpha}(n+1)$, additional oscillation and nonoscillation results can be found in [3] and [4].

Example 4.3 As an example of a different sort, suppose that a population is modeled by a growth law of the form

$$x''(t) + k_n x^{\alpha}(t) = 0, \quad 0 < \alpha, t \in [a_n, b_n] \tag{4.3}$$

where $a_n < b_n < a_{n+1}$, with $\lim_{n\to\infty} a_n = \infty$. If $a_{n+1} - b_n \ge \delta > 0$ for some positive δ, this represents a population growth model with non-overlapping generations and with a positive length of time between generations. We can write this model as a dynamic equation

$$x^{\Delta\Delta}(t) + p(t)x^{\alpha}(t) = 0, \quad t \in \mathbb{T} := \bigcup_{n=1}^{\infty}[a_n, b_n]. \tag{4.4}$$

With a_n, b_n restricted as above and with $p(t) = k_n$, $t \in [a_n, b_n]$, condition (C) clearly holds and we have for $a_n \le t \le b_n$

$$\int_{a_1}^{t} s^{\alpha}|p(s)|\Delta s \le \sum_{j=1}^{n} k_j \frac{(b_j^{\alpha+1} - a_j^{\alpha+1})}{\alpha+1} + \sum_{j=1}^{n-1} k_j b_j^{\alpha}(a_{j+1} - b_j).$$

Also, if $b_n < t < a_{n+1}$ then we have

$$\int_{a_1}^{t} s^{\alpha}|p(s)|\Delta s \le \sum_{j=1}^{n} k_j \frac{(b_j^{\alpha+1} - a_j^{\alpha+1})}{\alpha+1} + \sum_{j=1}^{n} k_j b_j^{\alpha}(a_{j+1} - b_j).$$

Consequently, if

$$\sum_{j=1}^{\infty} k_j \frac{(b_j^{\alpha+1} - a_j^{\alpha+1})}{\alpha+1} + \sum_{j=1}^{\infty} k_j b_j^{\alpha}(a_{j+1} - b_j) < +\infty,$$

then (4.4) has a solution with

$$\lim_{t\to\infty} \frac{x(t)}{t} = A > 0.$$

Remark 4.4 It is easy to give additional examples for the q-difference equation case using related ideas for oscillation and nonoscillation in the references [2–5] and [6]. We note also that in the case $\mathbb{T} = \mathbb{R}$, the differential equation

$$x'' + p(t)x^\alpha = 0 \tag{4.5}$$

has a solution $x(t)$ satisfying

$$\lim_{t\to\infty} \frac{x(t)}{t} = A \neq 0 \tag{4.6}$$

if $0 < \alpha \leq 1$, where

$$p(t) = \frac{\sin t}{t^b}, \quad t \in [1, \infty)$$

and where $b > \alpha + 1$.

Also, in the superlinear case $\alpha > 1$ and with

$$p(t) = \frac{a}{t^{b+1}} + \frac{c \sin t}{t^b},$$

then (4.5) has a solution satisfying (4.6) if $a > 0$, $c \neq 0$ and $b > \alpha + 1$.

References

1. Atkinson, F.V.: On second order nonlinear oscillations. Pac. J. Math. **5**, 643–647 (1955)
2. Baoguo, J., Erbe, L., Peterson, A.: Oscillation of a family of q-difference equations. Appl. Math. Lett. **22**, 871–875 (2009)
3. Baoguo, J., Erbe, L., Peterson, A.: Nonoscillation for second order sublinear dynamic equations on time scales. J. Comput. Appl. Math. **232**, 594–599 (2009)
4. Baoguo, J., Erbe, L., Peterson, A.: Oscillation of sublinear Emden–Fowler dynamic equations on time scales. J. Differ. Equs. Appl. **16**, 217–226 (2010)
5. Baoguo, J., Erbe, L., Peterson, A.: A Wong-type oscillation theorem for second order linear dynamic equations on time scales. J. Differ. Equs. Appl. **16**, 15–36 (2010)
6. Baoguo, J., Erbe, L., Peterson, A.: Kiguradze-type oscillation theorems for second order superlinear dynamic equations on time scales. Canad. Math. Bull. (to appear)
7. Baoguo, J., Erbe, L., Peterson, A.: Belohorec-type oscillation theorems for second order sublinear dynamic equations on time scales. Mathematische Nachrichten (to appear)
8. Bellman, R.: Stability Theory of Differential Equations, Dover Books on Intermediate and Advanced Mathematics. Dover Publications Inc, New York (1953)
9. Bohner, M., Peterson, A.: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston (2001)
10. Bohner, M., Peterson, A. (eds.): Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston (2003)
11. Erbe, L.: Oscillation criteria for second order linear equations on a time scale. Canad. Appl. Math. Quart. **9**, 346–375 (2001)
12. Došlý, O., Řehák, P.: Half-Linear Differential Equations. Elsevier, Amsterdam (2005)
13. Fowler, R.H.: Further studies of Emden's and similar differential equations. Quart. J. Math. **2**, 259–288 (1931)
14. Moore, R.A., Nehari, Z.: Nonoscillation theorems for a class of nonlinear differential equations. Trans. Amer. Math. Soc. **93**, 30–52 (1959)
15. Wong James, S.W.: On two theorems of Waltman. SIAM J. Appl. Math. **14**, 724–728 (1966)