Inequivalent measures of noncompactness

John Mallet-Paret · Roger D. Nussbaum

Abstract   Two homogeneous measures of noncompactness β and γ on an infinite dimensional Banach space X are called "equivalent" if there exist positive constants b and c such that $b\beta(S) \leq \gamma(S) \leq c\beta(S)$ for all bounded sets $S \subset X$. If such constants do not exist, the measures of noncompactness are "inequivalent." We ask a foundational question which apparently has not previously been considered: For what infinite dimensional Banach spaces do there exist inequivalent measures of noncompactness on X? We provide here the first examples of inequivalent measures of noncompactness. We prove that such inequivalent measures exist if X is a Hilbert space; or if (Ω, Σ, μ) is a general measure space, $1 \leq p \leq \infty$, and $X = L^p(\Omega, \Sigma, \mu)$; or if K is a compact Hausdorff space and $X = C(K)$; or if K is a compact metric space, $0 < \lambda \leq 1$, and $X = C^{0,\lambda}(K)$, the Banach space of Hölder continuous functions with Hölder exponent λ. We also prove the existence of such inequivalent measures of noncompactness if Ω is an open subset of $\mathbb{R}^n$ and X is the Sobolev space $W^{m,p}(\Omega)$. Our motivation comes from questions about existence of eigenvectors of homogeneous, continuous, order-preserving cone maps $f : C \to C$ and from the closely related issue of giving the proper definition of the "cone essential spectral radius" of such maps. These questions are considered in the companion paper [28]; see, also, [27].

Keywords   Kuratowski measure of noncompactness · Inequivalent measures of noncompactness · Classical Banach spaces

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1 Introduction

If (X, d) is a metric space and S is a bounded subset of X, then K. Kuratowski [22] has defined $\alpha(S)$, the **Kuratowski measure of noncompactness** (or **MNC**) of S, by

$$\alpha(S) := \inf\{\delta > 0 \mid S = \bigcup_{i=1}^{n} S_i \text{ for some } S_i \text{ with } \operatorname{diam}(S_i) \le \delta, \text{ for } 1 \le i \le n < \infty\}.$$

As usual, the diameter of a bounded set $T \subset X$ is defined by

$$\operatorname{diam}(T) := \sup\{d(x, y) \mid x, y \in T\}.$$

If (X, d) is a complete metric space, then one easily verifies the following fundamental fact:

(A1) $\alpha(S) = 0$ if and only if $\overline{S}$ is compact, for all bounded sets $S \subset X$.

Property (A1) explains the terminology "measure of noncompactness." It is also straightforward to verify the following properties, which hold whether or not (X, d) is complete:

(A2) $\alpha(S) \le \alpha(T)$ for all bounded sets $S \subset T \subset X$;
(A3) $\alpha(S \cup \{x_0\}) = \alpha(S)$ for all bounded sets $S \subset X$ and all $x_0 \in X$; and
(A4) $\alpha(\overline{S}) = \alpha(S)$ for all bounded sets $S \subset X$.

If $(X, \| \cdot \|)$ is a normed linear space and S and T are bounded subsets of X, we shall denote by $\operatorname{co}(S)$, the convex hull of S, namely the smallest convex set containing S, and we shall write

$$S + T := \{s + t \mid s \in S \text{ and } t \in T\}, \qquad \lambda S := \{\lambda s \mid s \in S\},$$

where λ is any scalar. If $d(x, y) := \|x - y\|$ for $x, y \in X$ and α denotes the Kuratowski MNC on X, then G. Darbo [12] observed the following properties also hold:

(A5) $\alpha(\operatorname{co}(S)) = \alpha(S)$ for all bounded sets $S \subset X$;
(A6) $\alpha(S + T) \le \alpha(S) + \alpha(T)$ for all bounded sets $S, T \subset X$; and
(A7) $\alpha(\lambda S) = |\lambda|\alpha(S)$ for all bounded sets $S \subset X$ and all scalars λ.

Property (A7) is sometimes referred to as the "homogeneity of α." Note that properties (A4) and (A5), along with (A1), already imply a classical theorem of Mazur (see [29], and [11] page 180): If X is a Banach space and $S \subset X$ is compact, then $\overline{\operatorname{co}(S)}$, the closure of $\operatorname{co}(S)$, is compact.

One further property satisfied by the Kuratowski MNC is the so-called "set-additivity property," namely

(A8) $\alpha(S \cup T) = \max\{\alpha(S), \alpha(T)\}$ for all bounded sets $S, T \subset X$.

Although the Kuratowski MNC is a useful tool in analysis, it is not nearly as widely known as it should be. The utility of the Kuratowski MNC is particularly apparent in fixed point theory. To illustrate we recall Darbo's fixed point theorem [12].

Theorem 1.1 *(Darbo [12]). Let G be a closed, bounded, convex set in a Banach space X and $f : G \to G$ a continuous map. Assume that there is a constant $c < 1$ such that $\alpha(f(S)) \le c\alpha(S)$ for all $S \subset G$, where α denotes the Kuratowski MNC. Then f has a fixed point in G and the set of fixed points of f in G is compact.*

Corollary 1.2 *(Darbo [12]). Let G be a closed, bounded convex set in a Banach space X. Assume that $U : G \rightarrow X$ satisfies $\|U(x) - U(y)\| \leq c\|x - y\|$ for all $x, y \in G$, where $c < 1$. Assume that $C : G \rightarrow X$ is a compact, continuous map. Define $f(x) := U(x) + C(x)$ and assume that $f(x) \in G$ for all $x \in G$. Then f has a fixed point in G and the set of fixed points of f in G is compact.*

The above corollary generalizes both the Schauder fixed point theorem and, in a weak sense, the contraction mapping principle. It is striking that there is no known proof of the corollary (in the stated generality) which does not use the Kuratowski MNC.

If $(X, \|\cdot\|)$ is a Banach space, we shall denote by $\mathcal{B}(X)$ the set of all bounded subsets of X. We say a map $\beta : \mathcal{B}(X) \rightarrow [0, \infty)$ is a **homogeneous measure of noncompactness on X** or **homogeneous MNC** if β satisfies properties (A1)–(A7) with β replacing α in these formulas. Many authors (see [2,4,5]) place a more restrictive condition in their definition and additionally require the set-additivity property (A8). Clearly, if β satisfies properties (A1) and (A8), then it satisfies properties (A2) and (A3), but the converse is false. Most authors (see [2,4,5]) say that a map $\beta : \mathcal{B}(X) \rightarrow [0, \infty)$ is a "measure of noncompactness" (homogeneity is understood) if β satisfies properties (A1) and (A4)–(A8). However, we shall not demand that our homogeneous MNC's satisfy property (A8). We are aware of few applications in analysis which require the use of property (A8) as opposed to properties (A2) and (A3). Furthermore, as we shall discuss in Sect. 2, there are important examples of maps $\beta : \mathcal{B}(X) \rightarrow [0, \infty)$ which satisfy properties (A1)–(A7) but not necessarily property (A8). A more flexible axiomatic treatment of measures of noncompactness, closer in spirit to our approach here, is given by Banaś and Goebel in [7]. See, also, [40] for a related discussion.

For the purposes of this paper, the issue of set-additivity will be unimportant. We shall prove in Sect. 2 that there is a canonical procedure which assigns to each homogeneous MNC β (in our sense) a homogeneous MNC ξ which satisfies property (A8). Furthermore, $\beta = \xi$ if β already satisfies property (A8). Moreover, in all of our constructions, whenever we obtain inequivalent homogeneous MNC's β_1 and β_2, our canonical construction will give inequivalent homogeneous MNC's ξ_1 and ξ_2 which satisfy the set-additivity property of property (A8).

There are many examples of homogeneous MNC's; see [2,4,5,7] and references there. For the reader's convenience, we recall two examples. If (X, d) is a metric space and $r > 0$, we shall always write

$$B_r(x) := \{y \in X \mid d(y, x) < r\}. \tag{1.1}$$

If $(X, \|\cdot\|)$ is a Banach space and $S \in \mathcal{B}(X)$, we define $\widetilde{\alpha}(S)$, the so-called "ball MNC" or "Hausdorff MNC" of S, by

$$\widetilde{\alpha}(S) := \inf\{r > 0 \mid S \subset \bigcup_{i=1}^{n} B_r(x_i) \text{ for some } x_i \in X, \text{ for } 1 \leq i \leq n < \infty\}.$$

One can check that $\widetilde{\alpha}$ satisfies properties (A1)–(A8) and that for all $S \in \mathcal{B}(X)$,

$$\widetilde{\alpha}(S) \leq \alpha(S) \leq 2\widetilde{\alpha}(S),$$

where α is the Kuratowski MNC on X. However, one should note that, for applications in fixed point theory, there are important differences between α and $\widetilde{\alpha}$; see Section 3 of [34] and [31].

For our second example, let (K, d) be a compact metric space and let $C(K)$ denote the usual Banach space of continuous maps $f : K \rightarrow \mathbb{R}$ with norm $\|f\| := \sup_{x \in K} |f(x)|$. If S is a bounded subset of $C(K)$ and $\delta > 0$, we define

$$\omega_\delta(S) := \sup\{|f(x) - f(y)| \mid f \in S \text{ and } x, y \in K \text{ satisfy } d(x, y) \leq \delta\},$$

$$\omega(S) := \lim_{\delta \to 0^+} \omega_\delta(S).$$

One can prove that ω is a homogeneous MNC on $C(K)$ and ω satisfies property (A8). Furthermore, it is a special case of results in [32] (see, also, [3] for related theorems) that for every bounded $S \subset C(K)$ we have

$$\frac{\omega(S)}{2} \leq \alpha(S) \leq \omega(S),$$

where α denotes the Kuratowski MNC. Note that property (A1) for ω implies the Ascoli-Arzelà theorem. The homogeneous MNC ω plays an important role in [24,36,37].

The restriction of ω to any closed linear subspace Y of $C(K)$ also gives a homogeneous MNC on Y. An old result of Banach [6] implies that for any separable Banach space Z, there is a linear isometry $L : Z \to Y \subset C([0, 1])$ which maps Z onto a closed linear subspace Y of $C([0, 1])$. Using L, one can define a homogeneous MNC $\widetilde{\omega}$ on Z by

$$\widetilde{\omega}(S) := \omega(LS).$$

If, abusing notation, α denotes the Kuratowski MNC on Z, one has, for all bounded sets $S \subset Z$,

$$\frac{\widetilde{\omega}(S)}{2} \leq \alpha(S) \leq \widetilde{\omega}(S).$$

In general, if β and γ are homogeneous MNC's on a Banach space $(X, \| \cdot \|)$, we shall say that β **dominates** γ if there exists $c > 0$ such that

$$\gamma(S) \leq c\beta(S)$$

for all $S \in \mathcal{B}(X)$. We shall say that β and γ are **equivalent** if β dominates γ and γ dominates β, that is, if there exist positive constants b and c such that, for all $S \in \mathcal{B}(X)$,

$$b\beta(S) \leq \gamma(S) \leq c\beta(S). \tag{1.2}$$

If β and γ are not equivalent, we shall say they are **inequivalent**. A great deal of effort has been expended in proving that various measures of noncompactness are equivalent and in finding the optimal constants b and c in Eq. (1.2). Here, we raise the following basic problem:

Fundamental Question *For what infinite dimensional Banach spaces $(X, \| \cdot \|)$, do there exist inequivalent homogeneous measures of noncompactness β_1 and β_2 on X?*

Despite its basic nature, it seems that the Fundamental Question has not been raised before and poses significant difficulties.

We believe that the Fundamental Question is of considerable intrinsic interest, and we shall study the question in its own right here. However, our original motivation comes from problems in nonlinear analysis, in particular, from the problem of generalizing the classical linear Krein-Rutman theorem (see [10,21,24,35–38,41,42] and references there) to the case of continuous, homogeneous, order-preserving maps f which take a closed cone C into itself. Such maps arise in numerous applications, for example, in the study of max-plus operators, which in turn arise in problems in delay-differential equations; see [24–26]. Moreover, such maps typically are not compact, and this gives rise to significant difficulties in their analysis. Often one wants to find an eigenvector of f with eigenvalue equal to $r_C(f) = r$, the so-called "cone spectral radius of f." This problem (see [24,35,36]) is closely related to measures of

noncompactness and to correctly defining the "(cone) essential spectral radius of f," denoted $\rho_C(f)$. In [28], we explore these issues further, and in particular, our explicit construction of inequivalent MNC's in the present paper plays a significant role in the analysis in [28]. See also [35] and [36] for definitions, references to the literature and related theorems.

If X is an infinite dimensional complex Banach space and $L : X \to X$ is a bounded linear map, there are several definitions of ess(L), the essential spectrum of L; and these definitions are, in general, inequivalent. See [19] for one definition. However, it is known that the essential spectral radius of L, namely the quantity $\rho(L)$ defined by

$$\rho(L) := \sup\{|\lambda| \mid \lambda \in \text{ess}(L)\},$$

is independent of the particular definition of ess(L). If β is any homogeneous MNC on X (for example, if $\beta = \alpha$) one can define

$$\beta(L) := \inf\{c \geq 0 \mid \beta(LS) \leq c\beta(S) \text{ for all bounded } S \subset X\}.$$

More generally if C is a closed cone in a Banach space X and $f : C \to C$ is continuous, homogeneous and order-preserving, one can define

$$\beta_C(f) := \inf\{c \geq 0 \mid \beta(f(S)) \leq c\beta(S) \text{ for all bounded } S \subset C\}.$$

It is proved in [30] that for the Kuratowski MNC α,

$$\rho(L) = \lim_{n \to \infty} \alpha(L^n)^{1/n} = \inf_{n \geq 1} \alpha(L^n)^{1/n},$$

and it follows easily that if β is a homogeneous MNC equivalent to α, then

$$\rho(L) = \lim_{n \to \infty} \beta(L^n)^{1/n}.$$

See, also, [15] for related results. For a general continuous, homogeneous, and order-preserving map $f : C \to C$, where C is a cone in a Banach space X, and for a homogeneous MNC β on X, one can follow the approach in [24,35,36] and try to define $\rho_C(f)$, the cone essential spectral radius of f, by

$$\rho_C(f) = \lim_{n \to \infty} \beta_C(f^n)^{1/n}. \tag{1.3}$$

One would hope that for such a definition one would have that (a) $\rho_C(f)$ is independent of β; (b) $\rho_C(f) \leq r_C(f)$; and (c) $\rho_C(f)$ is always defined and finite. However, with the aid of results in this paper, it is shown in [28] (see also [27]) that, in general, the definition of Eq. (1.3) has serious failings and that properties (a), (b) and (c) may fail even for continuous maps f which are linear on C where C is a total cone. Indeed, we prove in [28] that for a general homogeneous MNC β, even the inequality (b) above may fail. Revised definitions of the cone essential spectral radius and new theorems about existence of eigenvectors are given in [28].

A brief outline of this paper may be helpful. In Sect. 2, we give some general theorems which provide methods for finding inequivalent homogeneous MNC's which satisfy the set-additivity property (A8). In Sects. 3–6, we apply the results of Sect. 2 to various special cases. Our generic theorem is the following: For a "suitable" Banach space X, we prove that there exists a so-called "graded family" of homogeneous, set-additive MNC's, namely, an uncountable set $\{\beta_t\}_{t>0}$ of homogeneous, set-additive MNC's which are pairwise inequivalent, and with β_s dominating β_t if $0 < s < t$. The formal definition is given in Sect. 2. In Sect. 3, we assume that (Ω, Σ, μ) is a general measure space and $1 \leq p < \infty$, and we prove this theorem for $X = L^p(\Omega, \Sigma, \mu)$ provided that X is infinite dimensional. In Sect. 4, we

assume that K is a compact Hausdorff space with infinitely many points, and we prove the theorem for $X = C(K)$. As a corollary, we obtain the theorem for $X = L^\infty(\Omega, \Sigma, \mu)$. In Sect. 5, we assume that (K, d) is a compact, metric space with infinitely many points and that $0 < \lambda \le 1$, and we prove the theorem for $X = C^{0,\lambda}(K)$, the Banach space of Hölder continuous functions with Hölder exponent λ. In Sect. 6, we assume that Ω is an open subset of $\mathbb{R}^n$, that m is a positive integer, and that $1 \le p \le \infty$, and we prove the theorem for the Sobolev space $X = W^{m,p}(\Omega)$.

In fact, using these theorems and results of Sect. 2, one can easily obtain a wide variety of more general results. Thus, if Y is a Banach space, K is a compact Hausdorff space with infinitely many points and $X = C(K; Y)$ denotes the Banach space of continuous functions $f : K \to Y$, our generic theorem holds for X. This follows from Theorem 2.12 below and the fact that $C(K; \mathbb{R})$ is linearly isomorphic to a closed, complemented linear subspace of $C(K; Y)$.

2 Generating inequivalent measures of noncompactness

In view of our interest in cones and cone mappings [24,35–38], we shall initially define (weakly) homogeneous MNC's on "wedges," although our main interest here will be in MNC's on Banach spaces.

Let $(X, \|\cdot\|)$ be a normed linear space. A set $C \subset X$ will be called a **wedge** if C is a convex set and $\lambda C \subset C$ for all $\lambda \ge 0$. A wedge will be called a **cone** if additionally $C \cap (-C) = \{0\}$. A wedge C will be called a **complete wedge** in $(X, \|\cdot\|)$ if C is a complete metric space in the metric derived from the norm on X. If C is a wedge in a normed linear space $(X, \|\cdot\|)$, then $\mathcal{B}(C)$ will denote the collection of bounded subsets of C and as before, $\mathcal{B}(X)$ will denote the collection of bounded subsets of X.

If C is a complete wedge in a normed linear space $(X, \|\cdot\|)$, a map $\beta : \mathcal{B}(C) \to [0, \infty)$ may satisfy certain properties:

(B1) $\beta(S) = 0$ if and only if $\overline{S}$ is compact, for every $S \in \mathcal{B}(C)$;
(B2) $\beta(S) \le \beta(T)$ for every $S, T \in \mathcal{B}(C)$ with $S \subset T$;
(B3) $\beta(S \cup \{x_0\}) = \beta(S)$ for every $S \in \mathcal{B}(C)$ and $x_0 \in C$;
(B4) $\beta(\overline{S}) = \beta(S)$ for every $S \in \mathcal{B}(C)$;
(B5) $\beta(\text{co}(S)) = \beta(S)$ for every $S \in \mathcal{B}(C)$;
(B6) $\beta(S + T) \le \beta(S) + \beta(T)$ for every $S, T \in \mathcal{B}(C)$; and
(B7w) $\beta(\lambda S) = \lambda \beta(S)$ for every $S \in \mathcal{B}(C)$ and every $\lambda \ge 0$.

If $C = X$ and $(X, \|\cdot\|)$ is a Banach space over $\mathbb{R}$ or $\mathbb{C}$, it may also be true that

(B7) $\beta(\lambda S) = |\lambda| \beta(S)$ for every $S \in \mathcal{B}(X)$ and every scalar λ.

Definition If C is a complete wedge in a normed linear space $(X, \|\cdot\|)$, a map $\beta : \mathcal{B}(C) \to [0, \infty)$ which satisfies properties (B1)–(B7w) will be called a **weakly homogeneous measure of noncompactness on C**. If $C = X$ and X is a Banach space and if β satisfies properties (B1)–(B7), then β will be called a **homogeneous measure of noncompactness on X**.

If $0 < r \le 1$ and $\beta : \mathcal{B}(C) \to [0, \infty)$ satisfies properties (B1)–(B6), and if $\gamma(S) := \beta(S)^r$, then γ also satisfies properties (B1)–(B6). Thus, properties (B7w) and (B7) are useful normalizing conditions.

We shall say that a map $\beta : \mathcal{B}(C) \to [0, \infty)$ satisfies the **set-additivity property** if

(B8) $\beta(S \cup T) = \max\{\beta(S), \beta(T)\}$ for every $S, T \in \mathcal{B}(C)$.

Of course property (B8), with (B1) for $S = \{x_0\}$, implies properties (B2) and (B3) and implies, for $n < \infty$, that $\beta(\bigcup_{i=1}^{n} S_i) = \max_{1 \le i \le n} \beta(S_i)$. For brevity, we shall henceforth refer to a (weakly) homogeneous MNC which satisfies the set-additivity property (B8) as a **(weakly) homogeneous, set-additive MNC**.

The main results of this paper concern the existence of graded families of homogeneous MNC's on various spaces, defined as follows.

Definition Let X be a Banach space. By a **graded family of homogeneous MNC's**, we mean an uncountable collection $\{\beta_t\}_{t>0}$ of homogeneous MNC's on X, indexed by $t > 0$, such that if $0 < s < t$ then β_s dominates β_t but β_s is inequivalent to β_t. Further, if all the MNC's β_t are set-additive, we say that $\{\beta_t\}_{t>0}$ is a **graded family of homogeneous, set-additive MNC's**.

Recall that for most authors (see [2,4,5]) an MNC is a map $\beta : \mathcal{B}(X) \to [0, \infty)$ which satisfies properties (B1) and (B4)–(B8). For us, this is a homogeneous, set-additive MNC. Unlike most authors [2,4,5], we do not assume that our MNC's necessarily satisfy property (B8) because we are aware of few applications in which property (B8), as opposed to properties (B2) and (B3), plays a crucial role. Furthermore, property (B8) is not preserved by a variety of simple algorithms which generate homogeneous MNC's.

For example, suppose that $(X, \|\cdot\|)$ is a normed linear space and $C \subset X$ is a complete wedge. Suppose that β is a weakly homogeneous MNC on C and that $L : X \to X$ is a linear map such that $LC \subset C$ and $L|C$ is continuous. For $n \ge 1$ a fixed integer, define $\gamma : \mathcal{B}(C) \to [0, \infty)$ by

$$\gamma(S) := \sum_{j=0}^{n} \beta(L^j S). \tag{2.1}$$

The construction in Eq. (2.1) plays an important role in [35] and [36], and the reader may easily verify that γ is a weakly homogeneous MNC on C, and also that γ is homogeneous if $C = X$ and β is homogeneous. However, γ need not satisfy property (B8) even if β is the Kuratowski MNC. (Note that it is incorrectly claimed in [35] and [36] that γ satisfies property (B8), but property (B8) is not needed in the proofs there.)

If $(X, \|\cdot\|)$ is a Banach space and β is a homogeneous MNC on X, properties (B1)–(B7) are not independent, as the following three results show.

Proposition 2.1 *Let C be a complete wedge in a normed linear space $(X, \|\cdot\|)$ and assume that $\beta : \mathcal{B}(C) \to [0, \infty)$ satisfies properties (B1), (B2) and (B6). Then, for all $S, T \in \mathcal{B}(C)$ with $\beta(T) = 0$, we have*

$$\beta((S \cup \{0\}) \cup T) = \beta(S \cup \{0\}). \tag{2.2}$$

In addition, we have that

$$\beta(S \cup T) = \beta(S) \tag{2.3}$$

if C is a linear subspace of X.

Proof The reader can check that

$$(S \cup \{0\}) \cup T \subset (S \cup \{0\}) + (T \cup \{0\}).$$

Property (B1) implies that $\overline{T}$ is compact so $T \cup \{0\}$ has compact closure and (using property (B1) again), $\beta(T \cup \{0\}) = 0$. Properties (B2) and (B6) now give

$$\beta((S \cup \{0\}) \cup T) \leq \beta(S \cup \{0\}) + \beta(T \cup \{0\}) = \beta(S \cup \{0\}).$$

Property (B2) implies that $\beta(S \cup \{0\}) \leq \beta((S \cup \{0\}) \cup T)$, so we obtain Eq. (2.2).

Of course, if we assume property (B3), Eq. (2.3) follows immediately from Eq. (2.2). The point is that, if C is a linear subspace, Eq. (2.3) and, of course, property (B3), follow from our given assumptions. If C is a linear subspace of X, select $x_0 \in S$ and note that $-x_0 \in C$. The reader can verify that

$$S \cup T \subset S + (T \cup \{x_0\}) + \{-x_0\}$$

It follows from properties (B1), (B2) and (B6) that

$$\beta(S \cup T) \leq \beta(S) + \beta((T \cup \{x_0\}) + \{-x_0\})$$

$$\leq \beta(S) + \beta(T \cup \{x_0\}) + \beta(\{-x_0\}) = \beta(S).$$

Property (B2) implies that $\beta(S) \leq \beta(S \cup T)$, so $\beta(S \cup T) = \beta(S)$. $\square$

Proposition 2.2 *Let $(X, \|\cdot\|)$ be a Banach space and let $\beta : \mathcal{B}(X) \to [0, \infty)$ be a map which satisfies properties (B1), (B2), (B6) and (B7w). Then for all $S \in \mathcal{B}(X)$ and for all $T \in \mathcal{B}(X)$ with $\beta(T) = 0$ we have*

$$\beta(\overline{S}) = \beta(S), \qquad \beta(S + T) = \beta(S).$$

Furthermore, if α denotes the Kuratowski measure of noncompactness on X, there exists $c \geq 0$ such that

$$\beta(S) \leq c\alpha(S)$$

for all $S \in \mathcal{B}(X)$, and so α dominates β.

Proof First, property (B2) implies that $\beta(S) \leq \beta(\overline{S})$. If $\varepsilon > 0$ and $B_\varepsilon(0)$ is as in Eq. (1.1), we have that $\overline{S} \subset S + B_\varepsilon(0)$, so property (B6) implies that $\beta(\overline{S}) \leq \beta(S) + \beta(B_\varepsilon(0))$. Because $B_\varepsilon(0) = \varepsilon B_1(0)$, property (B7w) implies that

$$\beta(\overline{S}) \leq \beta(S) + \varepsilon\beta(B_1(0)).$$

Since ε is arbitrary, we see that $\beta(\overline{S}) \leq \beta(S)$, hence $\beta(\overline{S}) = \beta(S)$.

If $\beta(T) = 0$, property (B1) implies that $\beta(-T) = 0$. It follows from property (B6) that $\beta(S + T) \leq \beta(S) + \beta(T) = \beta(S)$. Since $S \subset (S + T) + (-T)$, we see that

$$\beta(S) \leq \beta((S + T) + (-T)) \leq \beta(S + T) + \beta(-T) = \beta(S + T),$$

so $\beta(S + T) = \beta(S)$.

Define $c := \beta(B_1(0))$. Let $S \in \mathcal{B}(X)$, denote $d := \alpha(S)$, and let $\varepsilon > 0$. Then there exist sets S_i for $1 \leq i \leq n < \infty$, with $S = \bigcup_{i=1}^{n} S_i$ and $\mathrm{diam}(S_i) < d + \varepsilon$. Select $x_i \in S_i$ for each $1 \leq i \leq n$, define $T := \{x_i \mid 1 \leq i \leq n\}$ and note that $S \subset T + B_{d+\varepsilon}(0)$. It follows that

$$\beta(S) \leq \beta(T) + \beta(B_{d+\varepsilon}(0)) = (d + \varepsilon)\beta(B_1(0)) = (\alpha(S) + \varepsilon)c.$$

Since $\varepsilon > 0$ is arbitrary, $\beta(S) \leq c\alpha(S)$. $\square$

Corollary 2.3 *If $(X, \|\cdot\|)$ is a Banach space, then $\beta : \mathcal{B}(X) \to [0, \infty)$ is a homogeneous MNC if and only if β satisfies properties (B1), (B2), (B5), (B6) and (B7). Furthermore, if this is the case then β is dominated by the Kuratowski MNC α.*

Proof This follows directly from Propositions 2.1 and 2.2. □

In contrast to the case of a Banach space, if C is a complete cone in a normed linear space and β is a weakly homogeneous MNC on C, then the interdependence of properties (B1)–(B7w) is unclear, and one can prove [28] that β is not necessarily dominated by the Kuratowski MNC.

We now suppose that $(Z, \| \cdot \|)$ and $(Z_1, \| \cdot \|_1)$ are Banach spaces with $Z \subset Z_1$. We assume that the inclusion map $i : Z \to Z_1$ is continuous, so there exists a constant M with

(C1) $\|x\|_1 \leq M\|x\|$ for all $x \in Z$.

We also assume that there exist continuous linear maps $P_n : Z \to Z$, indexed by integers $n \geq 1$, with the following properties:

(C2) there exists a constant C with $\|P_n x\| \leq C\|x\|$ for all $x \in Z$ and all $n \geq 1$;
(C3) there exists a constant C_1 with $\|P_n x\|_1 \leq C_1\|x\|_1$ for all $x \in Z$ and all $n \geq 1$; and
(C4) for every $n \geq 1$ there exists a constant c_n with $\|P_n x\| \leq c_n\|P_n x\|_1$ for all $x \in Z$.

The following theorem provides the basic construction which will be used to obtain inequivalent homogeneous MNC's. While the hypotheses of this result may seem unnatural, we shall see that it can be applied to many important examples of Banach spaces Z.

Theorem 2.4 *Let $(Z, \| \cdot \|)$ and $(Z_1, \| \cdot \|_1)$ be Banach spaces with $Z \subset Z_1$ such that property (C1) holds for some M. Also assume there exist linear maps $P_n : Z \to Z$, for $n \geq 1$, for which properties (C2)–(C4) are satisfied. Let α (respectively, α_1) denote the Kuratowski MNC on Z (respectively, on Z_1). With $\mathcal{B}(Z)$ denoting the collection of bounded subsets of Z, define $\mathcal{A}(Z) \subset \mathcal{B}(Z)$ by*

$$\mathcal{A}(Z) := \{S \in \mathcal{B}(Z) \mid \lim_{n \to \infty} \alpha((I - P_n)S) = 0\}. \tag{2.4}$$

For M as in property (C1) and $S \in \mathcal{B}(Z)$, define

$$\beta(S) := \inf\{\alpha_1(A) + M\alpha(B) \mid S \subset A + B, \text{ for some } A \in \mathcal{A}(Z) \text{ and } B \in \mathcal{B}(Z)\}. \tag{2.5}$$

Then β is a homogeneous MNC on Z, with $\beta(S) = \alpha_1(S)$ for all $S \in \mathcal{A}(Z)$ and $\beta(S) \leq M\alpha(S)$ for all $S \in \mathcal{B}(Z)$. If additionally there exists a sequence of sets $S_n \in \mathcal{A}(Z)$ for $n \geq 1$ such that $\alpha(S_n) > 0$ for all $n \geq 1$ and

$$\lim_{n \to \infty} \left(\frac{\alpha_1(S_n)}{\alpha(S_n)}\right) = 0, \tag{2.6}$$

then β is inequivalent to α.

Proof For convenience, we shall denote $\mathcal{A} := \mathcal{A}(Z)$ and $\mathcal{B} := \mathcal{B}(Z)$. The reader can verify, using the linearity of the maps P_n, that (a) if $\alpha(S) = 0$, then $S \in \mathcal{A}$; (b) if $A_1 \in \mathcal{A}$ and $A_2 \in \mathcal{A}$, then $A_1 + A_2 \in \mathcal{A}$; (c) if $A \in \mathcal{A}$, then $\mathrm{co}(A) \in \mathcal{A}$; and (d) if $A \in \mathcal{A}$ and λ is a scalar, then $\lambda A \in \mathcal{A}$. In particular, $\{0\} \in \mathcal{A}$, and for $S \in \mathcal{B}$ we obtain from Eq. (2.5) that

$$\beta(S) \leq \alpha_1(\{0\}) + M\alpha(S) = M\alpha(S). \tag{2.7}$$

We next show that $\beta(S) = \alpha_1(S)$ for all $S \in \mathcal{A}$. If we take $A = S$ and $B = \{0\}$ in Eq. (2.5), we see that $\beta(S) \leq \alpha_1(S)$. Conversely, suppose that $S \subset A + B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Using property (C1), it follows that

$$\alpha_1(S) \leq \alpha_1(A) + \alpha_1(B) \leq \alpha_1(A) + M\alpha(B),$$

and this, with Eq. (2.5), implies that $\alpha_1(S) \leq \beta(S)$. Thus, $\alpha_1(S) = \beta(S)$.

To prove that β is a homogeneous MNC on Z, by Corollary 2.3 it is enough to prove that β satisfies properties (B1), (B2), (B5), (B6) and (B7). We begin with property (B1). If $\overline{S}$ is compact, then (2.7) implies that $\beta(S) = 0$. Conversely, suppose that $\beta(S) = 0$. We have to prove that $\alpha(S) = 0$. Given $\varepsilon > 0$, Eq. (2.5) implies that there exist $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $S \subset A + B$ and $\alpha_1(A) + M\alpha(B) < \varepsilon$. Because $A \in \mathcal{A}$, there exists N such that $\alpha((I - P_n)A) < \varepsilon$ for all $n \geq N$. Because we have

$$(I - P_N)S \subset (I - P_N)A + (I - P_N)B,$$

it follows that

$$\alpha((I - P_N)S) \leq \alpha((I - P_N)A) + \alpha((I - P_N)B) < \varepsilon + \alpha((I - P_N)B).$$

Using property (C2), we deduce that $\alpha((I - P_N)B) \leq \alpha(B) + \alpha(P_N B) \leq (1 + C)\alpha(B)$ and so

$$\alpha((I - P_N)S) < \varepsilon + (1 + C)\alpha(B) < \varepsilon + \frac{(1 + C)\varepsilon}{M}.$$

Let N be as above and let c_N be as in property (C4), where we can assume that $c_N \geq 1$. Since $\beta(S) = 0$, select $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$ with $S \subset A' + B'$ and

$$\alpha_1(A') + M\alpha(B') < \frac{\varepsilon}{c_N}.$$

Using property (C3) we see that

$$\alpha_1(P_N A') \leq C_1\alpha_1(A') < \frac{C_1\varepsilon}{c_N},$$

and using (C4), we derive from this that

$$\alpha(P_N A') \leq c_N\alpha_1(P_N A') < C_1\varepsilon.$$

Property (C2) yields that

$$\alpha(P_N B') \leq C\alpha(B') < \frac{C\varepsilon}{Mc_N} \leq \frac{C\varepsilon}{M}.$$

Since $P_N S \subset P_N A' + P_N B'$, we see that

$$\alpha(P_N S) \leq \alpha(P_N A') + \alpha(P_N B') < \left(C_1 + \frac{C}{M}\right)\varepsilon.$$

Finally, because $S \subset P_N S + (I - P_N)S$ we conclude that

$$\alpha(S) \leq \alpha(P_N S) + \alpha((I - P_N)S) < \left(C_1 + \frac{C}{M}\right)\varepsilon + \left(1 + \frac{1 + C}{M}\right)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain that $\alpha(S) = 0$ and β satisfies property (B1).

The fact that β satisfies property (B2) is straightforward and is left to the reader.

To verify property (B5), note that property (B2) implies that $\beta(S) \leq \beta(\mathrm{co}(S))$. On the other hand, given $\varepsilon > 0$ and $S \in \mathcal{B}$, select $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $S \subset A + B$ and

$$\alpha_1(A) + M\alpha(B) < \beta(S) + \varepsilon.$$

Recall that $\mathrm{co}(A) \in \mathcal{A}$. It is well known that the sum of convex sets is convex so $\mathrm{co}(A) + \mathrm{co}(B)$ is convex and thus $\mathrm{co}(S) \subset \mathrm{co}(A) + \mathrm{co}(B)$. It follows that

$$\beta(\text{co}(S)) \leq \alpha_1(\text{co}(A)) + M\alpha(\text{co}(B)) = \alpha_1(A) + M\alpha(B) < \beta(S) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\beta(\text{co}(S)) \leq \beta(S)$ and hence $\beta(\text{co}(S)) = \beta(S)$.

To prove property (B6), let S, $T \in \mathcal{B}$ and $\varepsilon > 0$. Then there exist A, $A' \in \mathcal{A}$ and B, $B' \in \mathcal{B}$ with $S \subset A + B$ and $T \subset A' + B'$, and both

$$\alpha_1(A) + M\alpha(B) < \beta(S) + \varepsilon, \qquad \alpha_1(A') + M\alpha(B') < \beta(T) + \varepsilon.$$

Recall that $A + A' \in \mathcal{A}$ and note that $S + T \subset (A + A') + (B + B')$. It follows that

$$\beta(S + T) \leq \alpha_1(A + A') + M\alpha(B + B')$$
$$\leq (\alpha_1(A) + M\alpha(B)) + (\alpha_1(A') + M\alpha(B')) < \beta(S) + \beta(T) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we see that $\beta(S + T) \leq \beta(S) + \beta(T)$.

The proof that β is homogeneous (property (B7)) is straightforward and is left to the reader.

Finally, if there exists a sequence of sets $S_n \in \mathcal{A}$ for $n \geq 1$, as in the statement of the theorem, we have

$$\lim_{n \to \infty} \left(\frac{\beta(S_n)}{\alpha(S_n)} \right) = \lim_{n \to \infty} \left(\frac{\alpha_1(S_n)}{\alpha(S_n)} \right) = 0,$$

so β and α are inequivalent. □

We shall show below that, in general, the homogeneous MNC β constructed in Theorem 2.4 need not satisfy the set-additivity property. However, because α and α_1 in Theorem 2.4 do satisfy the set-additivity property, β as constructed in this theorem "almost satisfies" this property. More precisely, we have the following result.

Proposition 2.5 *With the notation and assumptions of Theorem 2.4, let $S_i \in \mathcal{B}(Z)$ for $1 \leq i \leq n$ and $S := \bigcup_{i=1}^n S_i$. Then*

$$\frac{\beta(S)}{2} \leq \max_{1 \leq i \leq n} \beta(S_i) \leq \beta(S) \tag{2.8}$$

where β is the homogeneous MNC given by (2.5).

Proof Because $S_i \subset S$ for $1 \leq i \leq n$ and β is a homogeneous MNC, it holds that $\beta(S_i) \leq \beta(S)$ for $1 \leq i \leq n$. This gives the second inequality in (2.8).

To prove the first inequality in (2.8), fix $\varepsilon > 0$ and for each i with $1 \leq i \leq n$ select $A_i \in \mathcal{A}(Z)$ and $B_i \in \mathcal{B}(Z)$ such that $S_i \subset A_i + B_i$ and

$$\alpha_1(A_i) + M\alpha(B_i) < \beta(S_i) + \varepsilon.$$

Such A_i and B_i exist from the definition (2.5) of β. Let $A := \bigcup_{i=1}^n A_i$ and $B := \bigcup_{i=1}^n B_i$. Then the reader can easily verify that $A \in \mathcal{A}(Z)$ and $B \in \mathcal{B}(Z)$, and that $S \subset A + B$, and so

$$\beta(S) \leq \alpha_1(A) + M\alpha(B), \tag{2.9}$$

again using (2.5). Now $\alpha_1(A) = \max_{1 \leq i \leq n} \alpha_1(A_i)$ and $\alpha(B) = \max_{1 \leq i \leq n} \alpha(B_i)$, and so there exist j and k, with $1 \leq j, k \leq n$, such that $\alpha_1(A) = \alpha_1(A_j)$ and $\alpha(B) = \alpha(B_k)$. Therefore,

$$\alpha_1(A) + M\alpha(B) \leq \begin{cases} 2(\alpha_1(A_k) + M\alpha(B)) < 2(\beta(S_k) + \varepsilon), & \text{if } \alpha_1(A) \leq M\alpha(B), \\ 2(\alpha_1(A) + M\alpha(B_j)) < 2(\beta(S_j) + \varepsilon), & \text{if } \alpha_1(A) \geq M\alpha(B). \end{cases} \tag{2.10}$$

Combining (2.9) and (2.10), we have that

$$\beta(S) < 2 \left(\max_{1 \leq i \leq n} \beta(S_i) + \varepsilon \right),$$

and as ε is arbitrary, the proposition is proved. □

To prove that, in general, the homogeneous MNC β constructed in Theorem 2.4 is not set-additive, we shall need to construct Banach spaces which are infinite direct sums of other Banach spaces. This general construction will also be used later, in Theorem 2.13, to obtain Banach spaces which possess graded families of homogeneous MNC's.

Let $(Y_n, \| \cdot \|_n)$, for $n \geq 1$, be a sequence of infinite dimensional Banach spaces over the same scalar field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let Z denote the space of all infinite sequences $y = (y_1, y_2, y_3, \ldots)$, where $y_n \in Y_n$ for each $n \geq 1$, such that $\lim_{n \to \infty} \| y_n \|_n = 0$, and let

$$\|y\| := \sup_{n \geq 1} \|y_n\|_n \tag{2.11}$$

denote the norm on Z. One easily checks that $(Z, \| \cdot \|)$ is a Banach space, and we sometimes denote $Z = (\oplus_{n=1}^{\infty} Y_n)_{c_0}$ for this space.

Alternatively, an ℓ^p direct sum can also be defined as follows. With $(Y_n, \| \cdot \|_n)$ as above, let p be fixed satisfying $1 \leq p \leq \infty$. Let Z denote the space of all infinite sequences $y = (y_1, y_2, y_3, \ldots)$, where $y_n \in Y_n$ for each $n \geq 1$, such that $\|y\| < \infty$, where

$$\|y\| := \begin{cases} \left(\sum_{n=1}^{\infty} \|y_n\|_n^p \right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \sup_{n \geq 1} \|y_n\|_n, & \text{if } p = \infty, \end{cases} \tag{2.12}$$

denotes the norm on Z. Again one easily checks that $(Z, \| \cdot \|)$ is a Banach space, and we sometimes denote $Z = (\oplus_{n=1}^{\infty} Y_n)_{\ell^p}$ for this space. As will be noted in Sect. 3, the space $\ell^p(\mathbb{N})$ is an example of such a space Z.

We also consider the following variation on the above constructions. With $(Y_n, \| \cdot \|_n)$ as before, fix a nonincreasing sequence a_n of positive reals satisfying $a_n \leq 1$ for $n \geq 1$. (Later we shall additionally assume that $\lim_{n \to \infty} a_n = 0$, although we do not need this condition at present.) If $Z = (\oplus_{n=1}^{\infty} Y_n)_{c_0}$ then let Z_1 denote the space of infinite sequences $y = (y_1, y_2, y_3, \ldots)$, where $y_n \in Y_n$ for each $n \geq 1$, such that $\lim_{n \to \infty} a_n \| y_n \|_n = 0$, and let

$$\|y\|_1 := \sup_{n \geq 1} a_n \|y_n\|_n \tag{2.13}$$

denote the norm on Z_1. If on the other hand $Z = (\oplus_{n=1}^{\infty} Y_n)_{\ell^p}$, then let Z_1 denote the space of infinite sequences $y = (y_1, y_2, y_3, \ldots)$, where $y_n \in Y_n$ for each $n \geq 1$, such that $\|y\|_1 < \infty$, where

$$\|y\|_1 := \begin{cases} \left(\sum_{n=1}^{\infty} a_n^p \|y_n\|_n^p \right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \sup_{n \geq 1} a_n \|y_n\|_n, & \text{if } p = \infty, \end{cases} \tag{2.14}$$

denotes the norm on Z_1. In any case, it is easily verified that $(Z_1, \| \cdot \|_1)$ is a Banach space. Clearly $\|y\|_1 \leq \|y\|$ for all $y \in Z$, and so we have the continuous inclusion $Z \subset Z_1$.

With Z and Z_1 as above, define continuous linear projections $P_n : Z \to Z$ and $Q_n : Z \to Z$ by

$$(P_n y)_j := \begin{cases} y_j, & \text{for } 1 \le j \le n, \\ 0, & \text{for } j > n, \end{cases} \qquad (Q_n y)_j := \begin{cases} y_j, & \text{for } j = n, \\ 0, & \text{for } j \ne n. \end{cases} \quad (2.15)$$

Note that

$$\|P_n y\| \le \|y\|, \qquad \|P_n y\|_1 \le \|y\|_1, \qquad \|P_n y\| \le a_n^{-1} \|P_n y\|_1,$$

hold for every $y \in Z$ and $n \ge 1$. Thus, the hypotheses (C1)–(C4) of Theorem 2.4 hold with $M = 1$ as in the statement of that result, and so it follows upon defining

$$\beta(S) := \inf\{\alpha_1(A) + \alpha(B) \mid S \subset A + B, \text{ for some } A \in \mathcal{A}(Z) \text{ and } B \in \mathcal{B}(Z)\}, \quad (2.16)$$

that β is a homogeneous MNC on Z.

Lemma 2.6 *With $(Z, \|\cdot\|)$ and $(Z_1, \|\cdot\|_1)$ the sequence spaces as above, and with the norms* (2.11) *and* (2.13)*, or else* (2.12) *and* (2.14)*, let $m \ge 1$ be any integer and let c and d be real numbers satisfying $0 < d < c$. Define the set*

$$S(c, d; m) := \{y \in Q_m Z \mid \|y\| \le c\} \cup \{y \in Z \mid \|y\| \le d\},$$

with Q_m as in (2.15)*. Then*

$$\beta(S(c, d; m)) = 2(a_m(c - d) + d)$$

for the homogeneous MNC β given by (2.16)*.*

Proof Let us make several observations before proceeding with the proof. First note that

$$\alpha((I - P_n)T) \le \alpha(T), \qquad \alpha(Q_n T) \le \alpha(T), \qquad \alpha_1(Q_n T) \le \alpha_1(T), \quad (2.17)$$

hold for every $T \in \mathcal{B}(Z)$ and $n \ge 1$, for the projections P_n and Q_n in (2.15). These are a consequence of the inequalities $\|(I - P_n)y\| \le \|y\|$, $\|Q_n y\| \le \|y\|$, and $\|Q_n y\|_1 \le \|y\|_1$, which hold for every $y \in Z$. Also note that

$$\alpha_1(T) = a_n \alpha(T), \qquad \text{if } T \subset Q_n Z \text{ and } T \in \mathcal{B}(Z), \quad (2.18)$$

which holds because $\|y\|_1 = a_n \|y\|$ for every $y \in Q_n Z$.

For simplicity, let us write $S := S(c, d; m)$. We first prove that

$$\beta(S) \le 2(a_m(c - d) + d). \quad (2.19)$$

Let

$$A := \{y \in Q_m Z \mid \|y\| \le c - d\}, \qquad B := \{y \in Z \mid \|y\| \le d\}.$$

Then the reader can verify that $A \in \mathcal{A}(Z)$ and $B \in \mathcal{B}(Z)$, and further that $S \subset A + B$. Also, $\alpha_1(A) = a_m \alpha(A)$ from (2.18), as $A \subset Q_m Z$. Thus

$$\beta(S) \le \alpha_1(A) + \alpha(B) = a_m \alpha(A) + \alpha(B). \quad (2.20)$$

Recall (see Proposition 5, Section A, in [33], or [14]) that if T is a ball of radius r in an infinite dimensional Banach space $(W, \|\cdot\|)$ and α_W denotes the Kuratowski MNC on W, then $\alpha_W(T) = 2r$. Since $Q_m Z$ and Z are both infinite dimensional Banach spaces, we have that $\alpha(A) = 2(c - d)$ and $\alpha(B) = 2d$, and with (2.20) this gives (2.19).

We now prove that

$$\beta(S) \geq 2(a_m(c - d) + d). \tag{2.21}$$

Fix any $\varepsilon > 0$. Then there exist $A' \in \mathcal{A}(Z)$ and $B' \in \mathcal{B}(Z)$ such that $S \subset A' + B'$ and

$$\alpha_1(A') + \alpha(B') < \beta(S) + \varepsilon. \tag{2.22}$$

Define sets A'' and B'' by

$$A'' := (Q_m A') \cup \{0\}, \qquad B'' := (Q_m B') \cup \{y \in Z \mid \|y\| \leq d\},$$

noting that $A'' \in \mathcal{A}(Z)$ and $B'' \in \mathcal{B}(Z)$, with $S \subset A'' + B''$. We claim that the chain of inequalities

$$a_m \alpha(A'') + \alpha(B'') \leq a_m \alpha(A'') + \alpha(B') = \alpha_1(A'') + \alpha(B') \leq \alpha_1(A') + \alpha(B') \tag{2.23}$$

holds. First note that the equality in (2.23) follows from (2.18) applied to the set $A'' \subset Q_m Z$. The final inequality in (2.23) follows from the final inequality in (2.17) applied to the set A'. The first inequality in (2.23) will hold once we have proved that

$$\alpha(B'') \leq \alpha(B'). \tag{2.24}$$

To establish (2.24), we first note that

$$(I - P_n)S \subset (I - P_n)A' + (I - P_n)B' \tag{2.25}$$

for every $n \geq 1$. As $(I - P_n)Z$ is an infinite dimensional Banach space, and as

$$(I - P_n)S = \{y \in (I - P_n)Z \mid \|y\| \leq d\}$$

for $n \geq m$, it follows that $\alpha((I - P_n)S) = 2d$ for $n \geq m$. Also, $\lim_{n \to \infty} \alpha((I - P_n)A') = 0$ because $A' \in \mathcal{A}(Z)$. Additionally, $\alpha((I - P_n)B') \leq \alpha(B')$ for all n from the first inequality in (2.17). Thus, it follows from (2.25) with these observations that

$$2d \leq \alpha(B'). \tag{2.26}$$

From the definition of B'', we now have that

$$\alpha(B'') = \max\{\alpha(Q_m B'), 2d\} \leq \max\{\alpha(B'), 2d\} = \alpha(B'),$$

where we have used the second inequality in (2.17) and the inequality (2.26). This proves (2.24) and establishes (2.23). Upon combining (2.22) with (2.23), we obtain

$$a_m \alpha(A'') + \alpha(B'') < \beta(S) + \varepsilon. \tag{2.27}$$

Let us now establish the two inequalities

$$2c \leq \alpha(A'') + \alpha(B''), \qquad 2d \leq \alpha(B''). \tag{2.28}$$

The first inequality in (2.28) holds because $Q_m S \subset Q_m A'' + Q_m B''$ and $\alpha(Q_m S) = 2c$. The second inequality in (2.28) holds because $\{y \in Z \mid \|y\| \leq d\} \subset B''$. With this, it follows immediately from (2.27) and (2.28), that

$$\inf\{a_m u + v \mid u \geq 0, v \geq 2d, \text{ and } u + v \geq 2c\} < \beta(S) + \varepsilon.$$

However, it is a simple exercise to prove that

$$\inf\{a_m u + v \mid u \geq 0, v \geq 2d, \text{ and } u + v \geq 2c\} = 2(a_m(c - d) + d),$$

with the infimum achieved at $u = 2(c - d)$ and $v = 2d$. This proves that

$$2(a_m(c - d) + d) < \beta(S) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (2.21) holds, which completes the proof of the lemma. □

The next result shows that the homogeneous MNC β constructed in Theorem 2.4 need not satisfy the set-additivity property. Also, this result shows that the first inequality in (2.8) is sharp in the sense that the denominator 2 in the first term cannot, in general, be decreased.

Proposition 2.7 *Let $(Z, \|\cdot\|)$ and $(Z_1, \|\cdot\|_1)$ be as in Lemma 2.6, with the notation as in the statement of that result.*

(1) Let $m \geq 1$ be any integer for which $a_m < 1$ and let c_i and d_i be real numbers satisfying $0 < d_i < c_i$, for $i = 1, 2$, and also $c_2 < c_1$ and $d_1 < d_2$. Also let $S_i = S(c_i, d_i; m)$ for $i = 1, 2$, and $S = S_1 \cup S_2$. Then

$$\beta(S) > \max\{\beta(S_1), \beta(S_2)\}. \tag{2.29}$$

(2) Let $m \geq 1$ be any integer for which $a_m < 1$ and fix a real number θ satisfying $a_m < \theta < 1$. Then there exist c_i and d_i, and sets S_i and S as in (1) above, such that

$$\frac{\beta(S)}{2 - \theta} = \beta(S_1) = \beta(S_2) \tag{2.30}$$

holds.

Proof (1) It is easy to see that $S = S(c_1, d_2; m)$. It follows from Lemma 2.6 that

$$\beta(S) = 2(a_m(c_1 - d_2) + d_2), \quad \beta(S_1) = 2(a_m(c_1 - d_1) + d_1), \quad \beta(S_2) = 2(a_m(c_2 - d_2) + d_2).$$

From this, using the properties of c_i and d_i, one easily verifies (2.29).

(2) Now with θ as in the statement of the proposition, fix any $c_1 > 0$. With $\varepsilon > 0$ a small parameter, let

$$d_1 := \varepsilon, \quad c_2 := c_1\theta - \left(\frac{(1 - \theta)(1 - a_m)}{a_m}\right)\varepsilon, \quad d_2 := \frac{(1 - \theta)a_m c_1}{1 - a_m} + (2 - \theta)\varepsilon.$$

By choosing ε sufficiently small, one easily verifies that all the required conditions are satisfied and that (2.30) holds. □

Although the homogeneous MNC β constructed in Theorem 2.4 need not satisfy the set-additivity property, our next theorem shows that there is a canonical construction which assigns to a general homogeneous MNC β on a Banach space X a homogeneous, set-additive MNC γ on X, and for which β dominates γ. In the particular case that β arises from the construction in Theorem 2.4, we have in fact that β and γ are equivalent.

Theorem 2.8 *Let C be a complete wedge in a normed linear space $(X, \|\cdot\|)$ and let β be a weakly homogeneous MNC on C. If $S \in \mathcal{B}(C)$, define $\gamma(S)$ by*

$$\gamma(S) := \inf\left\{\max_{1 \leq i \leq n} \beta(S_i) \mid S = \bigcup_{i=1}^{n} S_i \text{ for some } S_i \text{ with } 1 \leq i \leq n < \infty\right\}. \tag{2.31}$$

Then γ is a weakly homogeneous, set-additive MNC on C, and $\gamma(S) \leq \beta(S)$ for all $S \in \mathcal{B}(C)$. If $C = X$ and β is homogeneous, then γ is homogeneous.

Suppose additionally that we are in the setting of Theorem 2.4, with β given by Eq. (2.5) in the statement of that result, and with $X = Z$ a Banach space. Then it is the case that

$$\frac{\beta(S)}{2} \leq \gamma(S) \leq \beta(S) \tag{2.32}$$

for every $S \in \mathcal{B}(Z)$, and so β and γ are equivalent MNC's. Also,

$$\gamma(S) = \beta(S) = \alpha_1(S) \tag{2.33}$$

for every $S \in \mathcal{A}(Z)$.

Remark In the Fundamental Question posed in the Introduction, we asked whether there exist inequivalent homogeneous MNC's β_1 and β_2 on a given infinite dimensional Banach space X. If such MNC's exist, then we may take one of them to be the Kuratowski MNC on X, say $\beta_2 = \alpha$. Then, α dominates β_1 by Proposition 2.2, and Theorem 2.8 implies that there is a homogeneous, set-additive MNC γ_1 on X with β_1 dominating γ_1. It follows that γ_1 and α are inequivalent homogeneous, set-additive MNC's on X. Thus, whenever there exist inequivalent homogeneous MNC's on a Banach space X, there exist inequivalent homogeneous, set-additive MNC's on X.

Proof of Theorem 2.8 If $S = \bigcup_{i=1}^{n} S_i$, it is immediate from property (B2) that $\beta(S_i) \leq \beta(S)$ for $1 \leq i \leq n$, so $\gamma(S) \leq \beta(S)$ and β dominates γ.

We now verify that γ satisfies the properties (B1)–(B6), along with (B7w) or (B7), and (B8), as claimed. We begin with (B1). First, if $S \subset C$ and $\overline{S}$ is compact, then $\beta(S) = 0$, and thus $\gamma(S) = 0$ since β dominates γ. Now assume that $\gamma(S) = 0$. We must prove that $\overline{S}$ is compact. To this end, let $x_k \in S$ for $k \geq 1$ be any sequence in S. Then, it suffices to prove that this sequence has a convergent subsequence. If the set $A := \{x_k \mid k \geq 1\}$ has only finitely many distinct elements, then certainly a convergent subsequence exists, so assume that A has infinitely many distinct elements. Since $\gamma(S) = 0$, for each $n \geq 1$, there exist a finite collection of sets $S_{i,n}$, for $1 \leq i \leq N(n)$, such that $S = \bigcup_{i=1}^{N(n)} S_{i,n}$ and $\beta(S_{i,n}) < \frac{1}{n}$ for each i. For $n = 1$, it follows that there exists an integer i_1 with $1 \leq i_1 \leq N(1)$ such that $T_1 := S_{i_1,1}$ contains infinitely many elements of A. Let us define sets T_j for $j \geq 1$, inductively as follows. Suppose, for some $m > 1$, that we have found sets $T_1 \supset T_2 \supset \cdots \supset T_{m-1}$ such that for $1 \leq j \leq m - 1$ it is the case that $T_j \subset S_{i_j,j}$ for some i_j with $1 \leq i_j \leq N(j)$, and that T_j contains infinitely many elements of A. If we note that

$$T_{m-1} = \bigcup_{i=1}^{N(m)} (S_{i,m} \cap T_{m-1}),$$

it follows that there exists an integer i_m satisfying $1 \leq i_m \leq N(m)$, such that $S_{i_m,m} \cap T_{m-1}$ contains infinitely many elements of A. If we define $T_m := S_{i_m,m} \cap T_{m-1}$, then $T_m \subset T_{m-1}$ and $T_m \subset S_{i_m,m}$, and T_m contains infinitely many elements of A. We have thus defined T_j inductively for all $j \geq 1$.

Since each T_j contains infinitely many elements of A, we can choose a strictly increasing sequence of integers k_j for $j \geq 1$, such that $y_j := x_{k_j} \in T_j$. We define the set $B := \{y_j \mid j \geq 1\}$. Repeated application of property (B3) for β shows that if we define $B_n := \{y_j \mid j \geq n\}$ for positive integers n, then $\beta(B) = \beta(B_n)$. Since $B_n \subset T_n$ and $\beta(T_n) < \frac{1}{n}$, we see that $\beta(B) < \frac{1}{n}$ for all $n \geq 1$. Thus, $\beta(B) = 0$ and $\overline{B}$ is compact. It follows that the sequence y_j, for $j \geq 1$, has a convergent subsequence, which is thus a convergent subsequence of x_k, as desired. Thus $\overline{S}$ is compact.

To prove property (B2) for γ, suppose that $S, T \in \mathcal{B}(C)$ and $S \subset T$. Given $\varepsilon > 0$, there exist sets T_j, for $1 \leq j \leq m < \infty$, with

$$T = \bigcup_{j=1}^{m} T_j, \qquad \max_{1 \leq j \leq m} \beta(T_j) < \gamma(T) + \varepsilon. \tag{2.34}$$

Property (B2) for β implies that $\beta(T_j \cap S) \leq \beta(T_j)$, and since $S = \bigcup_{j=1}^{m}(T_j \cap S)$ we obtain that

$$\gamma(S) \leq \max_{1 \leq j \leq m} \beta(T_j \cap S) \leq \max_{1 \leq j \leq m} \beta(T_j) < \gamma(T) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\gamma(S) \leq \gamma(T)$.

Next, we prove that γ satisfies the set-additivity property (B8). If $S, T \in \mathcal{B}(C)$, then γ satisfies property (B2), so

$$\max\{\gamma(S), \gamma(T)\} \leq \gamma(S \cup T).$$

Given $\varepsilon > 0$, there exist sets S_i, for $1 \leq i \leq n < \infty$, with

$$S = \bigcup_{i=1}^{n} S_i, \qquad \max_{1 \leq i \leq n} \beta(S_i) < \gamma(S) + \varepsilon. \tag{2.35}$$

Similarly, there exist sets T_j, for $1 \leq j \leq m < \infty$ satisfying (2.34). Thus $S \cup T = \left(\bigcup_{i=1}^{n} S_i\right) \cup \left(\bigcup_{j=1}^{m} T_j\right)$ and

$$\gamma(S \cup T) \leq \max \left\{ \max_{1 \leq i \leq n} \beta(S_i), \max_{1 \leq j \leq m} \beta(T_j) \right\} < \max\{\gamma(S) + \varepsilon, \gamma(T) + \varepsilon\}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\gamma(S \cup T) \leq \max\{\gamma(S), \gamma(T)\},$$

which proves property (B8) for γ.

Property (B3) holds for γ, as it is a special case of property (B8), using (B1).

To prove property (B4) for γ, we have to show that $\gamma(S) = \gamma(\overline{S})$ for all $S \in \mathcal{B}(C)$. Property (B2) implies that $\gamma(S) \leq \gamma(\overline{S})$. Given $\varepsilon > 0$, there exist sets S_i, for $1 \leq i \leq n < \infty$ satisfying (2.35). Since n is finite, $\overline{S} = \bigcup_{i=1}^{n} \overline{S}_i$ and

$$\gamma(\overline{S}) \leq \max_{1 \leq i \leq n} \beta(\overline{S}_i) = \max_{1 \leq i \leq n} \beta(S_i) < \gamma(S) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain that $\gamma(\overline{S}) \leq \gamma(S)$, so $\gamma(S) = \gamma(\overline{S})$.

We next show that γ satisfies property (B5). We have to prove that $\gamma(\text{co}(S)) = \gamma(S)$, and since $S \subset \text{co}(S)$ and $\gamma(S) \leq \gamma(\text{co}(S))$, it suffices to prove that $\gamma(\text{co}(S)) \leq \gamma(S)$. Given $\varepsilon > 0$, there exist sets S_i, for $1 \leq i \leq n < \infty$ satisfying (2.35). Define $\Delta \subset \mathbb{R}^n$ to be the simplex given by

$$\Delta := \left\{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \mid \lambda_i \geq 0 \text{ for } 1 \leq i \leq n \text{ and } \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

and define $T_i := \text{co}(S_i)$. If we define a set T by

$$T := \left\{ \sum_{i=1}^{n} \lambda_i x_i \mid (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Delta \text{ and } x_i \in T_i \text{ for } 1 \leq i \leq n \right\},$$

we leave to the reader the exercise of proving that $S \subset T \subset \mathrm{co}(S)$ and that T is convex, and so $T = \mathrm{co}(S)$. Let $\| \cdot \|_\infty$ denote the sup norm on $\mathbb{R}^n$, so $\|\lambda\|_\infty := \max_{1 \leq i \leq n} |\lambda_i|$, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Select κ with $0 < \kappa < \frac{\varepsilon}{n}$ and for $\lambda \in \Delta$ define

$$B_\kappa(\lambda) := \{\mu \in \Delta \mid \|\mu - \lambda\|_\infty < \kappa\}.$$

Let $B_\kappa(\lambda^j)$, for $1 \leq j \leq p$, be a finite covering of the compact set Δ, where $\lambda^j := (\lambda_1^j, \lambda_2^j, \ldots, \lambda_n^j) \in \Delta$ for each j. Define $\Gamma_i := \mathrm{co}(T_i \cup \{0\})$ and note that

$$(\lambda_i^j + \kappa)\Gamma_i \supset \{sy \mid 0 \leq s \leq \lambda_i^j + \kappa \text{ and } y \in T_i\}.$$

It follows that

$$\mathrm{co}(S) = T \subset \bigcup_{j=1}^{p} \left(\sum_{i=1}^{n} (\lambda_i^j + \kappa)\Gamma_i \right),$$

so

$$\mathrm{co}(S) = \bigcup_{j=1}^{p} \left(\left(\sum_{i=1}^{n} (\lambda_i^j + \kappa)\Gamma_i \right) \cap \mathrm{co}(S) \right). \tag{2.36}$$

Using the properties of β, we see that

$$\beta(\Gamma_i) = \beta(T_i \cup \{0\}) = \beta(T_i) = \beta(S_i),$$

so

$$\beta\left(\left(\sum_{i=1}^{n} (\lambda_i^j + \kappa)\Gamma_i \right) \cap \mathrm{co}(S) \right) \leq \beta\left(\sum_{i=1}^{n} (\lambda_i^j + \kappa)\Gamma_i \right)$$

$$\leq \sum_{i=1}^{n} (\lambda_i^j + \kappa)\beta(\Gamma_i) = \sum_{i=1}^{n} (\lambda_i^j + \kappa)\beta(S_i) \leq \sum_{i=1}^{n} (\lambda_i^j + \kappa)(\gamma(S) + \varepsilon)$$

$$= (1 + n\kappa)(\gamma(S) + \varepsilon) < (1 + \varepsilon)(\gamma(S) + \varepsilon).$$

The above estimate and Eq. (2.36) imply that $\gamma(\mathrm{co}(S)) < (1+\varepsilon)(\gamma(S)+\varepsilon)$, and since $\varepsilon > 0$ is arbitrary, we obtain that $\gamma(\mathrm{co}(S)) \leq \gamma(S)$, as desired.

We now prove that γ satisfies property (B6). If $S, T \in \mathcal{B}(C)$, then given $\varepsilon > 0$, there exist sets T_j, for $1 \leq j \leq m < \infty$, and sets S_i, for $1 \leq i \leq n < \infty$, satisfying (2.34) and (2.35), respectively. We then have $S + T = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (S_i + T_j)$. Furthermore, since β satisfies property (B6), we obtain

$$\beta(S_i + T_j) \leq \beta(S_i) + \beta(T_j) < \gamma(S) + \gamma(T) + 2\varepsilon$$

for $1 \leq i \leq n$ and $1 \leq j \leq m$. It follows that $\gamma(S + T) < \gamma(S) + \gamma(T) + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\gamma(S + T) \leq \gamma(S) + \gamma(T)$.

The proof of weak homogeneity (property (B7w)) of γ follows easily from Eq. (2.31) using the weak homogeneity of β. Similarly, if $C = X$ and β is homogeneous (property (B7)), one easily sees that γ is also homogeneous. We leave the details to the reader.

Lastly, we prove the claims in the final paragraph of the statement of the theorem, where β is given by Theorem 2.4. For any $S \in \mathcal{B}(Z)$, we have (2.32) by Proposition 2.5. Now, let

$S \in \mathcal{A}(Z)$, and suppose that $S = \bigcup_i^n S_i$ for some S_i, for $1 \leq i \leq n < \infty$. Then $S_i \in \mathcal{A}(Z)$ for each i, and so $\beta(S) = \alpha_1(S)$ and $\beta(S_i) = \alpha_1(S_i)$ for every i, by Theorem 2.4. Thus

$$\beta(S) = \alpha_1(S) = \max_{1 \leq i \leq n} \alpha_1(S_i) = \max_{1 \leq i \leq n} \beta(S_i).$$

From this and from (2.31), one easily sees that (2.33) holds, as claimed. $\qquad\square$

With the aid of Theorem 2.8, we can give a refinement of Theorem 2.4.

Theorem 2.9 *Let the assumptions and notation of the statement of Theorem 2.4 hold, including the existence of sets $S_n \in \mathcal{A}(Z)$ for $n \geq 1$ with $\alpha(S_n) > 0$ such that the limit (2.6) holds. Then there exists a homogeneous, set-additive MNC γ on Z such that γ is inequivalent to α, and for which $\gamma(S) = \alpha_1(S)$ for all $S \in \mathcal{A}(Z)$. Moreover, $\gamma(S) \leq M\alpha(S)$ for all $S \in \mathcal{B}(Z)$ where M is as in condition (C1).*

Proof Theorem 2.4 implies that the homogeneous MNC β given by Eq. (2.5) is inequivalent to α, and that $\beta(S) = \alpha_1(S)$ for all $S \in \mathcal{A}(Z)$. Theorem 2.8 implies that if γ is defined by Eq. (2.31), then γ is a homogeneous, set-additive MNC on Z which is equivalent to β, and is thus inequivalent to α. Also, $\gamma(S) = \beta(S) = \alpha_1(S)$ for all $S \in \mathcal{A}(Z)$, again by Theorem 2.8. Finally, $\gamma(S) \leq \beta(S) \leq M\alpha(S)$ for all $S \in \mathcal{B}(Z)$ by Theorems 2.4 and 2.8. $\qquad\square$

In our applications, we shall actually have more information. For the reader's convenience, in Theorems 2.10 and 2.12 we explicitly describe two situations we shall encounter.

Theorem 2.10 *Let Z be a Banach space with norm $\| \cdot \|$ and let α denote the Kuratowski MNC on Z. Suppose for each $t > 0$ that $(Z_t, \| \cdot \|_t)$ is a Banach space with Kuratowski MNC α_t, and that $Z \subset Z_t$ with the inclusion map continuous. Assume further that for each integer $n \geq 1$, there is a continuous linear map $P_n : Z \to Z$, and that for each $t \geq 0$ there is a constant $C(t)$, such that*

$$\|P_n x\|_t \leq C(t)\|x\|_t$$

for every $x \in Z$, where for $t = 0$ we denote $(Z_0, \| \cdot \|_0) := (Z, \| \cdot \|)$. Also assume that for each $n \geq 1$ and $t > 0$, there is a constant $c_n(t)$ with

$$\|P_n x\| \leq c_n(t)\|P_n x\|_t$$

for every $x \in Z$. Define $\mathcal{A}(Z)$ as in Eq. (2.4). For each ordered pair (s,t) with $0 < s < t$, assume that there exists a sequence $S_n \in \mathcal{A}(Z)$, for $n \geq 1$, with $\alpha_s(S_n) > 0$ for every $n \geq 1$ and

$$\lim_{n \to \infty} \left(\frac{\alpha_t(S_n)}{\alpha_s(S_n)} \right) = 0. \tag{2.37}$$

Then for each $t > 0$, there exists a homogeneous, set-additive MNC γ_t on Z such that γ_s and γ_t are inequivalent whenever $0 < s < t$. Furthermore, if $Z_s \subset Z_t$ whenever $0 < s < t$ and if the inclusion map is continuous, then $\{\gamma_t\}_{t>0}$ is a graded family of homogeneous, set-additive MNC's on Z.

Proof If, for fixed $t > 0$, we let Z_t take the role of Z_1 in Theorem 2.4, Eq. (2.5) gives a homogeneous MNC β_t on Z. If we then apply Theorem 2.8 to β_t, we obtain from Eq. (2.31) a homogeneous, set-additive MNC γ_t on Z such that $\gamma_t(S) = \beta_t(S) = \alpha_t(S)$ for all $S \in \mathcal{A}(Z)$.

If, whenever $0 < s < t$, there exists a sequence of sets $S_n \in \mathcal{A}(Z)$ for $n \geq 1$ as in the statement of the theorem, it follows that

$$\lim_{n\to\infty}\left(\frac{\gamma_t(S_n)}{\gamma_s(S_n)}\right) = \lim_{n\to\infty}\left(\frac{\alpha_t(S_n)}{\alpha_s(S_n)}\right) = 0,$$

so γ_t and γ_s are inequivalent. Finally, if $Z_s \subset Z_t$ with a continuous inclusion map whenever $0 < s < t$, it follows easily from the explicit formula (2.5) that β_s dominates β_t. Further, as γ_s is equivalent to β_s and γ_t is equivalent to β_t by Theorem 2.8, it follows that γ_s dominates γ_t. Thus, $\{\gamma_t\}_{t>0}$ is a graded family of homogeneous, set-additive MNC's on Z, as desired. □

Our next proposition is straightforward, but it will prove useful in our subsequent work.

Proposition 2.11 *Let $(X, \|\cdot\|)$ be a Banach space and let $P : X \to X$ be a continuous linear projection. Define $X_1 = PX$ and $X_2 = (I - P)X$, so X_1 and X_2 are closed linear subspaces of X, and let γ_j and β_j, for $j = 1, 2$, be homogeneous MNC's on X_j. If $\beta(S) := \beta_1(PS) + \beta_2((I - P)S)$ and $\gamma(S) := \gamma_1(PS) + \gamma_2((I - P)S)$ for $S \in \mathcal{B}(X)$, then γ and β are homogeneous MNC's on X. Further, if β_1 and γ_1 are inequivalent MNC's on X_1, then β and γ are inequivalent MNC's on X.*

Proof The fact that β and γ satisfy properties (B2), (B5), (B6) and (B7) follows easily from the linearity and continuity of P and the fact that β_j and γ_j are homogeneous MNC's. If $S \in \mathcal{B}(X)$ and $\overline{S}$ is compact, then, by the continuity of P, the sets $P\overline{S}$ and $(I - P)\overline{S}$ are compact, so $\beta_1(P\overline{S}) = 0 = \beta_2((I - P)\overline{S})$. It follows that $\beta_1(PS) = 0 = \beta_2((I - P)S)$ and $\beta(S) = 0$. Conversely, if $\beta(S) = 0$, then $\beta_1(PS) = 0$ and $\beta_2((I - P)S) = 0$, so $\overline{PS}$ and $\overline{(I - P)S}$ are compact. It follows that $T := \overline{PS} + \overline{(I - P)S}$ is compact; and since $S \subset T$, the set S has compact closure. Using Corollary 2.3, we conclude that β and γ are homogeneous MNC's.

If β_1 and γ_1 are inequivalent, then, because $\beta(S) = \beta_1(S)$ and $\gamma(S) = \gamma_1(S)$ for every $S \subset X_1$, it follows that β and γ must be inequivalent. (This is the only part of the proof which uses that P is a projection.)   □

If X_0 is a closed linear subspace of a Banach space $(X, \|\cdot\|)$, recall that X_0 is called **complemented** if there exists a continuous linear projection P of X onto X_0. It sometimes happens that for each $t > 0$ we have a homogeneous, set-additive MNC ξ_t on X_0 such that ξ_s is inequivalent to ξ_t whenever $0 < s < t$. Our next theorem shows that in this situation we obtain homogeneous, set-additive MNC's γ_t on X for $t > 0$ such that γ_s is inequivalent to γ_t whenever $0 < s < t$.

Theorem 2.12 *Let $(X, \|\cdot\|)$ be a Banach space and X_0 a closed, complemented linear subspace of X. Assume that for each $t > 0$, there exists a homogeneous, set-additive MNC ξ_t on X_0 such that ξ_s is inequivalent to ξ_t whenever $0 < s < t$. Then, for each $t > 0$, there exists a homogeneous, set-additive MNC γ_t on X such that γ_s is inequivalent to γ_t whenever $0 < s < t$.*

Furthermore, if ξ_s dominates ξ_t for some s and t, then γ_s dominates γ_t. Thus, if $\{\xi_t\}_{t>0}$ is a graded family of homogeneous, set-additive MNC's on X_0, then $\{\gamma_t\}_{t>0}$ is a graded family of homogeneous, set-additive MNC's on X.

Proof Let $P : X \to X_0$ be a continuous linear projection of X onto X_0 and let α denote the Kuratowski MNC on X. If $S \in \mathcal{B}(X)$ and $t > 0$, define

$$\eta_t(S) := \xi_t(PS) + \alpha((I - P)S).$$

Proposition 2.11 implies that η_t is a homogeneous MNC on X, but η_t does not necessarily satisfy the set-additivity property. Thus, we use Theorem 2.8 and define $\gamma_t(S)$ by

$$\gamma_t(S) := \inf \left\{ \max_{1 \le i \le n} \eta_t(S_i) \mid S = \bigcup_{i=1}^{n} S_i \text{ for some } S_i \text{ with } 1 \le i \le n < \infty \right\}.$$

Theorem 2.8 implies that γ_t is a homogeneous, set-additive MNC on X. If $S \subset X_0 = PX$, note that $\eta_t(S) = \xi_t(S)$. Since ξ_t satisfies the set-additivity property, it follows that for such S that

$$\gamma_t(S) = \eta_t(S) = \xi_t(S). \tag{2.38}$$

Since ξ_s is inequivalent to ξ_t whenever $0 < s < t$, it follows from Eq. (2.38) that γ_s is inequivalent to γ_t. The proof that γ_s dominates γ_t if ξ_s dominates ξ_t follows from our explicit formulas for η_t and γ_t. Details are left to the reader. □

With the aid of the theorems of this section, we can now describe some large classes of Banach spaces which possess many inequivalent homogeneous measures of noncompactness.

Theorem 2.13 *Let $(Y_n, \| \cdot \|_n)$, for $n \ge 1$, be a sequence of infinite dimensional Banach spaces over the same scalar field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Either let $Z := (\oplus_{n=1}^{\infty} Y_n)_{c_0}$, namely the Banach space of all infinite sequences $y = (y_1, y_2, y_3, \ldots)$, where $y_n \in Y_n$ for each $n \ge 1$ and $\lim_{n \to \infty} \|y_n\|_n = 0$, and with the norm $\| \cdot \|$ given by (2.11); or let $Z := (\oplus_{n=1}^{\infty} Y_n)_{\ell^p}$ for some p with $1 \le p \le \infty$, namely the Banach space of all infinite sequences $y = (y_1, y_2, y_3, \ldots)$, where $y_n \in Y_n$ for each $n \ge 1$ and $\|y\| < \infty$, where $\| \cdot \|$ is the norm given by (2.12). Then there exists a graded family of homogeneous, set-additive MNC's $\{\gamma_t\}_{t>0}$ on Z.*

Proof Let a_n, for $n \ge 1$, be a nonincreasing sequence of positive reals with $a_n \le 1$ for $n \ge 1$ and $\lim_{n \to \infty} a_n = 0$. For $t > 0$, let $(Z_t, \| \cdot \|_t)$ denote the space of infinite sequences $y = (y_1, y_2, y_3, \ldots)$ with $y_n \in Y_n$ for each $n \ge 1$, and which satisfies the following properties. If $Z = (\oplus_{n=1}^{\infty} Y_n)_{c_0}$ then $\lim_{n \to \infty} a_n^t \|y_n\|_n = 0$ and

$$\|y\|_t := \sup_{n \ge 1} a_n^t \|y_n\|_n.$$

If $Z = (\oplus_{n=1}^{\infty} Y_n)_{\ell^p}$ then $\|y\|_t < \infty$, where

$$\|y\|_t := \begin{cases} \left(\displaystyle\sum_{n=1}^{\infty} a_n^{tp} \|y_n\|_n^p \right)^{1/p}, & \text{if } 1 \le p < \infty, \\[2ex] \displaystyle\sup_{n \ge 1} a_n^t \|y_n\|_n, & \text{if } p = \infty. \end{cases}$$

In any case, with $\| \cdot \|_t$ the norm on Z_t, the reader can easily verify that $(Z_t, \| \cdot \|_t)$ is a Banach space. Also, $Z \subset Z_t$ for all $t > 0$, with $Z_s \subset Z_t$ whenever $0 < s < t$, and all the inclusions are continuous, having norm less than or equal to 1.

Let us also define linear maps $P_n : Z \to Z$ and $Q_n : Z \to Z$ by the formulas (2.15). Then

$$\|P_n y\| \le \|y\|, \qquad \|P_n y\|_t \le \|y\|_t, \qquad \|P_n y\| \le a_n^{-t} \|P_n y\|_t,$$

for all $n \geq 1$ and $y \in Z$, and every $t > 0$. Define $S_n \subset Z$ by

$$S_n := \{y \in Q_n Z \mid \|y\| \leq 1\}$$

for $n \geq 1$, and note that $S_n \in \mathcal{A}(Z)$, where $\mathcal{A}(Z)$ is defined by Eq. (2.4). The set S_n can be considered as a subset of Z_t, and in the norm in this space, S_n is isometric to the closed ball of radius a_n^t, centered at 0, in the infinite dimensional Banach space Y_n with norm $a_n^t \| \cdot \|_n$. Thus, $\alpha_t(S_n) = 2a_n^t$ where α_t denotes the Kuratowski MNC on Z_t. It follows that if $0 < s < t$, then the limit (2.37) holds. We have thus verified the hypotheses of Theorem 2.10, and Theorem 2.13 follows directly.   □

3 Inequivalent measures of noncompactness on $\mathbf{L^p}$ spaces

Throughout this section, (Ω, Σ, μ) will denote a general measure space. Thus, Ω is a set, Σ is a σ-algebra of subsets of Ω with $\Omega \in \Sigma$ and μ is a measure on Ω, with measurable sets being the elements of Σ. We shall denote by $L^p(\Omega, \Sigma, \mu)$, where $1 \leq p \leq \infty$, the usual Banach space whose elements are equivalence classes of measurable functions. For the most part we consider the case of $p < \infty$. Discussion of the case $p = \infty$ will be deferred to the next section; see Corollary 4.7.

The main result of this section is the following theorem.

Theorem 3.1 *Let (Ω, Σ, μ) be a measure space and assume that $1 \leq p < \infty$. If the space $L^p := L^p(\Omega, \Sigma, \mu)$ is infinite dimensional, then there exists a graded family of homogeneous, set-additive MNC's $\{\gamma_t\}_{t>0}$ on L^p.*

A necessary and sufficient condition for the space $L^p(\Omega, \Sigma, \mu)$ to be infinite dimensional is given below in Lemma 3.3.

Remark Theorem 3.1 can be stated in a more elegant way. Suppose that $1 \leq p < \infty$, and recall that a Banach lattice X for which $\|x + y\|^p = \|x\|^p + \|y\|^p$ whenever $x \wedge y = 0$, is called an **abstract L^p space**. We refer to pages 1–15 of [23] for further details and definitions. A classical result of S. Kakutani (see [17] or Theorem 1.b.2 on page 15 of [23]) implies that an abstract L^p space X, where $1 \leq p < \infty$, is linearly isometric to $L^p(\Omega, \Sigma, \mu)$ for some measure space (Ω, Σ, μ). Furthermore, the linear isometry $\Lambda : X \to L^p(\Omega, \Sigma, \mu)$ can be chosen so that Λ and Λ^{-1} respect the partial orderings on X and $L^p(\Omega, \Sigma, \mu)$. It follows from Theorem 3.1 that on an infinite dimensional abstract L^p space X, where $1 \leq p < \infty$, there exists a graded family of homogeneous, set-additive MNC's $\{\gamma_t\}_{t>0}$.

Let us begin by treating the special case of the space $\ell^p(\mathbb{N})$, namely, the case where $\Omega = \mathbb{N}$ is the set of positive integers and μ is the counting measure on $\mathbb{N}$. Thus for $1 \leq p < \infty$, the set $\ell^p(\mathbb{N})$ is the collection of maps $x : \mathbb{N} \to \mathbb{R}$ such that $\|x\|_p := (\sum_{i=1}^{\infty} |x(i)|^p)^{1/p}$ is finite, where here $\| \cdot \|_p$ denotes the norm. We shall also consider the Banach space $\ell^p(\mathbb{N} \times \mathbb{N})$ of maps $y : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ in the norm $\|y\|_p := (\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |y(i, j)|^p)^{1/p}$. The analogous spaces for $p = \infty$ are also considered.

Proposition 3.2 *Let $1 \leq p \leq \infty$. Then there exists a graded family of homogeneous, set-additive MNC's $\{\gamma_t\}_{t>0}$ on $\ell^p(\mathbb{N})$.*

Proof There is a one-one map σ of $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$. Fixing such a map σ, define a linear map $L_\sigma : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N} \times \mathbb{N})$ by $L_\sigma x = y$, where $y(i, j) = x(\sigma(i, j))$. One can check that L_σ

is an isometry of $\ell^p(\mathbb{N})$ onto $\ell^p(\mathbb{N} \times \mathbb{N})$. Next note that $\ell^p(\mathbb{N} \times \mathbb{N})$ is linearly isometric to $(\oplus_{n=1}^{\infty} Y_n)_{\ell^p}$ where $Y_n := \ell^p(\mathbb{N})$ for all $n \geq 1$. Thus, we are in the situation of Theorem 2.13, from which Proposition 3.2 follows. □

Let us now recall a few measure theoretic generalities. For a measure space (Ω, Σ, μ), a set $E \in \Sigma$ is an **atom** if both $\mu(E) > 0$ and there do not exist disjoint measurable sets E' and E'' with $E = E' \cup E''$ and both $\mu(E') > 0$ and $\mu(E'') > 0$. If E_1 and E_2 are atoms, it follows that either $\mu(E_1 \cap E_2) = 0$ or else both $\mu(E_1 \backslash (E_1 \cap E_2)) = 0$ and $\mu(E_2 \backslash (E_1 \cap E_2)) = 0$ hold. We shall say that E_1 and E_2 are **equivalent atoms** if $\mu(E_1 \cap E_2) > 0$. It is easy to see that this is an equivalence relation on the set $\mathcal{E}$ of all atoms.

If $f : \Omega \rightarrow \mathbb{R}$ is measurable, one defines (see [39], page 73) the **essential range** of f, denoted $R(f)$, as follows: $r \in \mathbb{R}$ is an element of $R(f)$ if and only if, for every $\varepsilon > 0$, the set $f^{-1}((r - \varepsilon, r + \varepsilon))$ has positive measure. If $[c, d]$ is a compact interval such that $\mu(f^{-1}([c, d])) > 0$, one can prove by repeated bisections of $[c, d]$ that $[c, d] \cap R(f) \neq \emptyset$. It follows, by taking $[c, d] = [-n, n]$ with n sufficiently large, that $R(f) \neq \emptyset$ must always hold. (We avoid the trivial case of $\mu(\Omega) = 0$.) We further claim that if $E \subset \Omega$ is an atom then there exists some $r \in \mathbb{R}$ such that $f(x) = r$ for almost every $x \in E$. Indeed, take any $r \in R(f|E)$, where $R(f|E)$ denotes the essential range of the restriction of f to E. Then for every $\varepsilon > 0$, the set $f^{-1}((r - \varepsilon, r + \varepsilon)) \cap E$ has positive measure, and thus the set $E \backslash f^{-1}((r - \varepsilon, r + \varepsilon))$ has measure zero. It follows that $|f(x) - r| < \varepsilon$ for almost every $x \in E$, and as ε is arbitrary, that $f(x) = r$ almost everywhere on E.

Our next lemma is obvious for most measure spaces, but we have not found a reference for the general case. We are indebted to Shelley Goldstein for a suggestion which simplified our original argument.

Lemma 3.3 *Let (Ω, Σ, μ) be a measure space and assume that $1 \leq p < \infty$. Then $L^p(\Omega, \Sigma, \mu)$ is infinite dimensional if and only if there exist infinitely many pairwise disjoint measurable sets Ω_n, for $n \geq 1$, with $0 < \mu(\Omega_n) < \infty$ for all n.*

Proof Denote $L^p := L^p(\Omega, \Sigma, \mu)$ for simplicity of notation. If sets Ω_n, for $n \geq 1$, as in the statement of the lemma exist and if χ_{Ω_n} is the characteristic function of Ω_n, then any finite subcollection of $\{\chi_{\Omega_n} \mid n \geq 1\}$ is linearly independent and $\chi_{\Omega_n} \in L^p$ for $1 \leq p \leq \infty$. Thus, L^p is infinite dimensional.

Conversely, suppose that $1 \leq p < \infty$ and L^p is infinite dimensional. With $\mathcal{E}$ denoting the set of all atoms in Σ, recall the equivalence relation on $\mathcal{E}$ described above. Note that two atoms E_1 and E_2 are equivalent if and only if their characteristic functions χ_{E_1} and χ_{E_2} are equal almost everywhere. Now let $\mathcal{E}_0 \subset \mathcal{E}$ denote the set of all atoms E for which $\mu(E) < \infty$ and consider this equivalence relation restricted to $\mathcal{E}_0$. If the number of equivalence classes in $\mathcal{E}_0$ is infinite, there exist atoms $E_k \in \mathcal{E}_0$ for $k \geq 1$ with $\mu(E_j \cap E_k) = 0$ whenever $j \neq k$. If we define $\Omega_1 = E_1$, and $\Omega_n = E_n \backslash (\bigcup_{k=1}^{n-1} E_k)$ for $n \geq 2$, the sets Ω_n satisfy the conditions of the lemma and we are done.

Thus, we can assume that the number of equivalence classes in $\mathcal{E}_0$ is finite. If there are N equivalence classes, select an atom E_k from each equivalence class, and define $F = \Omega \backslash \bigcup_{k=1}^{N} E_k$. Now suppose there exists a measurable set $G_1 \subset F$ such that $0 < \mu(G_1) < \infty$. Then G_1 is not an atom, being disjoint from each E_k, and so there exists a measurable set $G_2 \subset G_1$ such that $0 < \mu(G_2) < \mu(G_1)$. Continuing in this fashion, we obtain a sequence of measurable sets $G_1 \supset G_2 \supset G_3 \supset \cdots$ such that $0 < \mu(G_{n+1}) < \mu(G_n) < \infty$ for every $n \geq 1$. Upon setting $\Omega_n = G_n \backslash G_{n+1}$, we again see that we are done.

Thus we may assume that whenever $G \subset F$ is a measurable set then either $\mu(G) = 0$ or $\mu(G) = \infty$. It follows that if $f \in L^p$ and $Q_\varepsilon := \{r \in \mathbb{R} \mid |r| > \varepsilon\}$ for $\varepsilon > 0$, then

$\mu(f^{-1}(Q_\varepsilon) \cap F) = 0$; otherwise we would contradict $\int_\Omega |f|^p \, d\mu < \infty$. Thus $|f(x)| \leq \varepsilon$ almost everywhere on F, and since $\varepsilon > 0$ is arbitrary it follows that $f(x) = 0$ almost everywhere on F. Further, since E_k is an atom for each k, with $1 \leq k \leq N$, there exists $r_k \in \mathbb{R}$, with r_k dependent on f, such that $f(x) = r_k$ almost everywhere on E_k. We have thus shown that for every $f \in L^p$, there exist real numbers r_k, for $1 \leq k \leq N$, such that $f = \sum_{k=1}^N r_k \chi_{E_k}$ in L^p, where χ_{E_k} denotes the characteristic function of E_k. This contradicts the assumption that L^p is infinite dimensional, and completes the proof.   □

Lemma 3.4 *Assume that $1 \leq p \leq \infty$ and that (Ω, Σ, μ) is a measure space, and denote $L^p := L^p(\Omega, \Sigma, \mu)$. Assume that for each integer $n \geq 1$, there is a measurable set Ω_n such that $0 < \mu(\Omega_n) < \infty$ for all $n \geq 1$ and $\Omega_m \cap \Omega_n$ is empty whenever $m \neq n$. Define a linear map $P : L^p \to L^p$ by*

$$(Pf)(x) := \begin{cases} \dfrac{1}{\mu(\Omega_n)} \displaystyle\int\limits_{\Omega_n} f \, d\mu, & \text{for } x \in \Omega_n, \\[4mm] 0, & \text{for } x \in \Omega \backslash \displaystyle\bigcup_{n=1}^\infty \Omega_n. \end{cases}$$

Then P is a continuous linear projection, with $\|P\| = 1$, and whose range $PL^p \subset L^p$ is linearly isometric to $\ell^p(\mathbb{N})$.

Proof We consider only the case that $1 \leq p < \infty$, the case $p = \infty$ being treated similarly but with slight modifications. The linearity of P is obvious, as is the fact that $P^2 = P$. Also, Hölder's inequality gives

$$|(Pf)(x)| \leq \left(\frac{1}{\mu(\Omega_n)} \int\limits_{\Omega_n} |f|^p \, d\mu \right)^{1/p}$$

for every $x \in \Omega_n$. It follows that

$$\int\limits_\Omega |Pf|^p \, d\mu = \sum_{n=1}^\infty \int\limits_{\Omega_n} |Pf|^p \, d\mu \leq \sum_{n=1}^\infty \int\limits_{\Omega_n} |f|^p \, d\mu = \int\limits_\Omega |f|^p \, d\mu,$$

so $\|P\| \leq 1$. If f is the characteristic function of Ω_n for some $n \geq 1$, then $Pf = f$, so $\|P\| = 1$.

If $g \in PL^p$, we know that there exist quantities a_n for $n \geq 1$ such that $g(x) = a_n$ for all $x \in \Omega_n$ and $g(x) = 0$ for $x \in \Omega \backslash \bigcup_{n=1}^\infty \Omega_n$. Upon defining $Vg \in \ell^p(\mathbb{N})$ by $(Vg)_n := a_n \mu(\Omega_n)^{1/p}$, one can easily check that $\|Vg\| = \|g\|$ and that $V : PL^p \to \ell^p(\mathbb{N})$ is onto. □

We now prove the main theorem of this section.

Proof of Theorem 3.1 By Lemma 3.3, there exist pairwise disjoint, measurable sets Ω_n for $n \geq 1$ with $0 < \mu(\Omega_n) < \infty$ for all $n \geq 1$. Lemma 3.4 implies that there is a continuous linear projection $P : L^p \to L^p$ such that PL^p is linearly isometric to $\ell^p(\mathbb{N})$, and by Proposition 3.2, there exists a graded family of homogeneous, set-additive MNC's $\{\xi_t\}_{t>0}$ on PL^p. Thus by Theorem 2.12, there exists a graded family of homogeneous, set-additive MNC's $\{\gamma_t\}_{t>0}$ on L^p.   □

Theorem 3.1 implies a corresponding result for any infinite dimensional Hilbert space.

Corollary 3.5 *Let H be an infinite dimensional Hilbert space. Then there exists a graded family of homogeneous, set-additive MNC's $\{\gamma_t\}_{t>0}$ on H.*

Proof If H is separable, H is linearly isometric to $L^2([0, 1])$ with Lebesgue measure, so Corollary 3.5 follows from Theorem 3.1 in this case.

Thus, we assume that H is not separable and use Theorem 2.12 to reduce to the separable case. There exists a countable, infinite family of orthonormal vectors $\{e_n \mid n \geq 1\}$ in H. If H_0 denotes the closed linear span of $\{e_n \mid n \geq 1\}$, then H_0 is separable and there exists an orthogonal linear projection P of H onto H_0. Since H_0 is separable, there exists a graded family of homogeneous, set-additive MNC's $\{\xi_t\}_{t>0}$ on H_0. Corollary 3.5 now follows from Theorem 2.12. □

4 Inequivalent measures of noncompactness on C(K)

If K is a compact Hausdorff space, $C(K)$ will denote the Banach space of continuous maps $f : K \to \mathbb{R}$ with the norm $\|f\| := \sup_{x \in K} |f(x)|$. Our main goal in this section is to prove the following result.

Theorem 4.1 *Let K be a compact Hausdorff space with infinitely many elements. Then there exists a graded family of homogeneous, set-additive MNC's $\{\gamma_t\}_{t>0}$ on $C(K)$.*

We shall prove Theorem 4.1 by considering two separate cases, namely, the case where the set K_* of accumulation points in K is infinite, and the case where K_* is finite. Here, $K_* \subset K$ is defined by

$$K_* := \{x \in K \mid \{x\} \text{ is not an open set}\}. \tag{4.1}$$

With K a compact Hausdorff space, let us observe that K_* is closed and thus compact. Also, $K_* \neq \emptyset$ if and only if K is an infinite set; in particular, if K is infinite but $K_* = \emptyset$ then $\{\{x\} \mid x \in K\}$ would be an open cover of K without a finite subcovering, contradicting the compactness of K. Let us note further that if $x \in K_*$, then every neighborhood of x contains infinitely many points of K.

With the next two lemmas, we treat the case that K_* is an infinite set.

Lemma 4.2 *Let K be a compact Hausdorff space. Assume that there exists a decreasing sequence of nonempty open sets $U_n \subset K$, for $n \geq 1$, with $U_1 = K$ and $\overline{U}_{n+1} \subset U_n$ for all $n \geq 1$, and where $U_n \backslash \overline{U}_{n+1}$ is an infinite set for all $n \geq 1$. Then there exists a graded family of homogeneous, set-additive MNC's $\{\gamma_t\}_{t>0}$ on $C(K)$.*

Proof We define $A := \bigcap_{n=1}^{\infty} \overline{U}_n = \bigcap_{n=1}^{\infty} U_n$. Since $\overline{U}_n$ is a decreasing sequence of compact, nonempty sets, A is compact and nonempty. We define $Z := C(K)$ with the usual norm $\|\cdot\|$, and for convenience we set $B_n := U_n \backslash U_{n+1}$.

Our strategy now is to use Theorem 2.10 to prove our lemma. Let a_n, for $n \geq 1$, be a nonincreasing sequence of positive reals with $a_1 \leq 1$ and $\lim_{n\to\infty} a_n = 0$. For $t > 0$, let Z_t denote the set of functions $f : K \to \mathbb{R}$ such that $f|A$ is continuous, $f|(K \backslash A)$ is continuous, and $\|f\|_t < \infty$, where

$$\|f\|_t := \max \left\{ \sup_{x \in A} |f(x)|, \sup_{n \geq 1} \left(a_n^t \sup_{x \in B_n} |f(x)| \right) \right\}$$

denotes the norm on Z_t. Again, one can easily verify that $(Z_t, \|\cdot\|_t)$ is a real Banach space; in proving this fact, it is useful to note that for every open neighborhood V of A, there is an

integer $N = N(V)$ such that $\overline{U}_n \subset V$ for all $n \geq N(V)$. One also has that $Z \subset Z_t$ for $t > 0$, with $Z_s \subset Z_t$ whenever $0 < s < t$, with all the inclusion maps having norm less than or equal to 1.

Since K is a normal space, for each $n \geq 1$ there exists a continuous map $\psi_n : K \to [0, 1]$ with $\psi_n(x) = 0$ for all $x \in \overline{U}_{n+1}$ and $\psi_n(x) = 1$ for all $x \in K \setminus U_n$. Define linear maps $P_n : Z \to Z$ for $n \geq 1$ by

$$(P_n f)(x) := \psi_n(x) f(x),$$

and note that

$$\|P_n f\| \leq \|f\|, \qquad \|P_n f\|_t \leq \|f\|_t, \qquad \|P_n f\| \leq a_n^{-t} \|P_n f\|_t,$$

hold for all $n \geq 1$ and $f \in Z$, and every $t > 0$. (We remark that, in contrast to our earlier application of Theorem 2.10, the maps P_n here are not projections.)

For each $n \geq 1$, define

$$Y_n := \{f \in Z \mid f(x) = 0 \text{ for all } x \in \overline{U}_{n+1} \cup (K \setminus U_n)\}.$$

Note that Y_n is a closed linear subspace of $(Z, \|\cdot\|)$, and also of $(Z_t, \|\cdot\|_t)$ for all $t > 0$, and thus is a Banach space in each of these norms. Further, $\|f\|_t = a_n^t \|f\|$ for all $f \in Y_n$, and because $U_n \setminus \overline{U}_{n+1}$ is an infinite set, Y_n is infinite dimensional.

Again with $\mathcal{A}(Z)$ defined by Eq. (2.4), where α is the Kuratowski MNC on Z, define $S_n := \{f \in Y_n \mid \|f\| \leq 1\} \subset Z$ for $n \geq 1$. As before, $S_n \in \mathcal{A}(Z)$. Then S_n, considered as a subset of $(Y_n, \|\cdot\|_t)$, is the closed ball of radius a_n^t, so $\alpha_t(S_n) = 2a_n^t$ where α_t denotes the Kuratowski MNC on Z_t. Thus, if $0 < s < t$, the limit (2.37) holds. We have verified the hypotheses of Theorem 2.10, so Lemma 4.2 follows directly. $\qquad\square$

The hypotheses of Lemma 4.2 will be satisfied if K has infinitely many accumulation points.

Lemma 4.3 *Let K be a compact Hausdorff space and let K_*, the set of accumulation points, be given by* (4.1). *Assume that K_* has infinitely many elements. Then there exists a graded family of homogeneous, set-additive MNC's $\{\gamma_t\}_{t>0}$ on $C(K)$.*

Proof Under the assumption that K_* is an infinite set, there exists $x_* \in K_*$ such that every open neighborhood U of x_* contains infinitely many elements of K_*. If not, then for every $x \in K_*$ there exists an open neighborhood V_x of x such that x is the only element of K_* in V_x. But then $\{V_x \mid x \in K_*\}$ is an open covering of the compact space K_*, and this open covering has no finite refinement, a contradiction.

We may thus fix $x_* \in K_*$ so that every open neighborhood U of x_* contains infinitely many elements of K_*. Recall that for any $x \in K_*$, every open neighborhood of x contains infinitely many elements of K. Let $U_1 := K$ and fix any $x_1 \in K_*$ with $x_1 \neq x_*$. Next select an open neighborhood U_2 of x_* with $x_1 \notin \overline{U}_2$ and select $x_2 \in K_* \cap U_2$ with $x_2 \neq x_*$. In general, we proceed by induction. Suppose we have found open neighborhoods U_j of x_* for $1 \leq j \leq n$ such that $\overline{U}_{j+1} \subset U_j$ for $1 \leq j < n$ and such that there exists $x_j \in (K_* \cap U_j) \setminus \overline{U}_{j+1}$ for $1 \leq j < n$. Then select $x_n \in K_* \cap U_n$ with $x_n \neq x_*$ and select U_{n+1} to be an open neighborhood of x_* such that $\overline{U}_{n+1} \subset U_n$ and $x_n \notin \overline{U}_{n+1}$. Since $U_n \setminus \overline{U}_{n+1}$ is an open neighborhood of $x_n \in K_*$, the set $U_n \setminus \overline{U}_{n+1}$ contains infinitely many elements of K. Thus, Lemma 4.3 follows from Lemma 4.2. $\qquad\square$

To complete the proof of Theorem 4.1, it suffices, by virtue of Lemma 4.3, to assume that K_* is a finite set. Here it is useful to consider the Banach spaces $c(\Theta)$ and $c_0(\Theta)$, where Θ is

an infinite set. (The set Θ here is not endowed with a topology, but is simply an index set.) We recall some definitions. Let $b(\Theta)$ denote the Banach space of all bounded maps $f : \Theta \to \mathbb{R}$ in the norm $\|f\| := \sup_{\theta \in \Theta} |f(\theta)|$. We say that $f \in b(\Theta)$ is an element of $c(\Theta)$ if and only if for every $\varepsilon > 0$, there exists a finite set $S = S(\varepsilon, f)$ such that

$$\sup\{|f(\theta_1) - f(\theta_2)| \mid \theta_1, \theta_2 \in \Theta \backslash S\} < \varepsilon.$$

One can easily check that $c(\Theta)$ is a closed linear subspace of $b(\Theta)$ and hence a Banach space. Given $f \in c(\Theta)$ and any finite subset $S \subset \Theta$, define $A(f; S)$ to be the closure of $\{f(\theta) \mid \theta \in \Theta \backslash S\}$. If $\mathcal{F}(\Theta)$ denotes the collection of all finite subsets $S \subset \Theta$ and if $f \in c(\Theta)$, it is easy to show that there is a unique real number r such that

$$\bigcap_{S \in \mathcal{F}(\Theta)} A(f; S) = \{r\}. \tag{4.2}$$

For $f \in c(\Theta)$ and r as in Eq. (4.2) define Lf by

$$Lf := r. \tag{4.3}$$

One can check that $L : c(\Theta) \to \mathbb{R}$ is a continuous linear functional and $\|L\| = 1$. We define $c_0(\Theta) \subset c(\Theta)$ by

$$c_0(\Theta) := \{f \in c(\Theta) \mid Lf = 0\}.$$

Then $c_0(\Theta) \subset c(\Theta)$ is a closed complemented subspace, in fact with a one-dimensional complement spanned by $e : \Theta \to \mathbb{R}$, the function identically equal to 1.

Proposition 4.4 *Let Θ be an infinite set. Then there exists a graded family of homogeneous, set-additive MNC's $\{\gamma_t\}_{t>0}$ on $c(\Theta)$, and similarly on $c_0(\Theta)$.*

Proof By Theorem 2.12, it is enough to prove the result for $c_0(\Theta)$. As is well known, Θ and $\Theta \times \mathbb{N}$ have the same cardinality (see [20], page 280), so there is a one-one map σ of $\Theta \times \mathbb{N}$ onto Θ. Fixing such a map σ, define a linear map $U_\sigma : c_0(\Theta) \to c_0(\Theta \times \mathbb{N})$ by $U_\sigma f = g$, where $g(\theta, n) = f(\sigma(\theta, n))$. One can check that U_σ is an isometry of $c_0(\Theta)$ onto $c_0(\Theta \times \mathbb{N})$. Thus, it suffices to prove the result for $c_0(\Theta \times \mathbb{N})$.

One now easily checks that for a function $g : \Theta \times \mathbb{N} \to \mathbb{R}$, one has that $g \in c_0(\Theta \times \mathbb{N})$ if and only if both $g(\cdot, n) \in c_0(\Theta)$ for every $n \in \mathbb{N}$ and $\lim_{n \to \infty} \|g(\cdot, n)\|_{c_0(\Theta)} = 0$. From this, one sees directly that $c_0(\Theta \times \mathbb{N})$ is linearly isometric to $(\oplus_{n=1}^\infty Y_n)_{c_0}$ where $Y_n := c_0(\Theta)$ for all $n \geq 1$. The result now follows from Theorem 2.13. □

We now prove the main result of this section.

Proof of Theorem 4.1 It is enough to consider the case where K_* is a nonempty finite set. We consider first the special case that K_* is a singleton, so $K_* = \{x_*\}$ for some $x_* \in K$. We claim that $C(K)$ is linearly isometric to $c(\Theta)$ where $\Theta := K \backslash \{x_*\}$. By virtue of Proposition 4.4, the proof of Theorem 4.1 will be complete in this case. We begin by proving that if $f \in C(K)$, then $f|\Theta \in c(\Theta)$. First observe that $K \backslash U$ is a finite set for any neighborhood U of x_* as every point in $K \backslash U$ is both open and closed. From this observation, and from the continuity of f at x_*, one directly shows that $f|\Theta \in c(\Theta)$. Thus define $H : C(K) \to c(\Theta)$ by setting $Hf := f|\Theta$ for $f \in C(K)$. It is clear that H is linear and that $\|Hf\| = \|f\|$ for all $f \in C(K)$. To see that H is onto, given $f \in c(\Theta)$, we define a map $g : K \to \mathbb{R}$ by $g(x) := f(x)$ for $x \in \Theta$ and $g(x_*) := Lf$, where L is defined as in Eqs. (4.2), (4.3). We leave to the reader the verification that g is continuous and $Hg = f$.

Now suppose more generally that K_* has n elements, say $K_* = \{x_1, x_2, \ldots, x_n\}$, where $1 \le n < \infty$. Let U_1 be an open neighborhood of x_1 such that $x_j \notin \overline{U}_1$ for $2 \le j \le n$. Then $\overline{U}_1 \backslash U_1$, being disjoint from K_*, is an open set. Thus, $\overline{U}_1$ is an open set, and of course, a closed set. We define $\psi(x) = 1$ for $x \in \overline{U}_1$ and $\psi(x) = 0$ for $x \in K \backslash \overline{U}_1$. Since $\overline{U}_1$ is both open and closed, ψ is a continuous function. If we define $P : C(K) \rightarrow C(K)$ by

$$(Pf)(x) := \psi(x)f(x),$$

then P is a continuous linear projection whose range $P(C(K))$ is clearly linearly isometric to $C(S)$, where $S := \overline{U}_1$. Noting that S is a compact Hausdorff space and an infinite set, and that for every $x \in S$ with $x \ne x_1$ the set $\{x\}$ is both open and closed, we conclude from the first part of this proof (where $n = 1$) that the space $C(S)$ and hence $P(C(K))$ each possess a graded family of homogeneous, set-additive MNC's. Thus, by Theorem 2.12, the space $C(K)$ possesses a graded family of homogeneous, set-additive MNC's. With this, the theorem is proved. $\square$

If $[a, b]$ is a finite interval of reals and $1 \le m < \infty$, then $C^m([a, b])$ denotes, as usual, the Banach space of m times continuously differential functions $f : [a, b] \rightarrow \mathbb{R}$ in the norm

$$\|f\| := \sup_{a \le t \le b} |f^{(m)}(t)| + \sum_{j=0}^{m-1} |f^{(j)}(a)|.$$

We have the following result for this space.

Corollary 4.5 *There exists a graded family of homogeneous, set-additive MNC's $\{\gamma_t\}_{t>0}$ on $C^m([a, b])$.*

Proof It is well known that $C^m([a, b])$ is linearly isomorphic to the Banach space $C([a, b]) \times \mathbb{R}^m$ by the map

$$Jf := (f^{(m)}, f(a), f^{(1)}(a), f^{(2)}(a), \ldots, f^{(m-1)}(a)).$$

There is a continuous linear projection of $C([a, b]) \times \mathbb{R}^m$ onto $C([a, b])$, so the result follows by Theorems 2.12 and 4.1. $\square$

We recall that $C(K)$ is a **Banach lattice**, and we refer the reader to [23] or [42] for basic definitions and theorems about Banach lattices. In general, if X and Y are Banach lattices, one says that X and Y are **linearly order isometric** (see [23]) if there is a linear isometry Λ of X onto Y such that Λ and Λ^{-1} preserve the natural partial ordering on X and Y associated with the lattice structures. A Banach lattice X is called an **abstract M-space** if $\|x + y\| = \max(\|x\|, \|y\|)$ whenever $x \wedge y = 0$. Of course $C(K)$ is an abstract M-space for K a compact Hausdorff space. An element e of a Banach lattice X is called a **strong unit** of X provided that, for every $x \in X$, it is the case that $\|x\| \le 1$ if and only if $|x| \le e$. If e denotes the function identically equal to 1 on K, then e is a strong unit in $C(K)$; however, not every abstract M-space has a strong unit. If X is an abstract M-space, a classical result of S. Kakutani asserts (see [24], Theorem 1.b.6, page 16 and [18]), that X is linearly order isometric to a Banach sublattice of $C(K)$ for some compact Hausdorff space K. If, in addition, X has a strong unit, then X is linearly order isometric to $C(K)$ for some compact Hausdorff space K.

Kakutani (see [13,18,23]) has also explicitly described the closed sublattices of $C(K)$ for K a compact Hausdorff space. If X is a closed linear subspace of $C(K)$, then X is a sublattice

if and only if there is a collection $\mathcal{F}$ of ordered triples (k_1, k_2, λ) with $k_1, k_2 \in K$ and $\lambda \geq 0$ such that

$$X = \{f \in C(K) \mid f(k_1) = \lambda f(k_2) \text{ for every } (k_1, k_2, \lambda) \in \mathcal{F}\}. \tag{4.4}$$

See Theorem 1.b.5 in [23]. More generally, if K is a compact Hausdorff space, and $\mathcal{F}$ is a collection of ordered triples (k_1, k_2, λ) with $k_1, k_2 \in K$ and $\lambda \in \mathbb{R}$, and if X is defined by (4.4), then X is called a **G-space**. Thus, any Banach sublattice of $C(K)$ is a G-space. A. Grothendieck [16] introduced G-spaces, and later Y. Benjamini [8] proved that every separable G-space X is linearly isomorphic to $C(S)$ for some compact Hausdorff space S. (Of course, all linear isomorphisms are understood to be continuous).

In view of these remarks, we easily obtain the following result.

Theorem 4.6 *Let X be an infinite dimensional Banach space and assume that X satisfies at least one of the following conditions: (a) X is an abstract M-space with a strong unit; (b) X is a separable abstract M-space; (c) X is a separable G-space; or (d) $X \subset C(K)$ is a G-space, where K is a compact metric space. Then there exists a graded family of homogeneous, set-additive MNC's $\{\gamma_t\}$ on X.*

Proof If (a) is satisfied, Kakutani's theorem implies that X is linearly order isometric to $C(K)$ for some compact Hausdorff space K. Since X is infinite dimensional, K must have infinitely many points. The conclusions of Theorem 4.6 now follow from Theorem 4.1. If X satisfies condition (c), Benyamini's theorem [8] implies that X is linearly isomorphic to $C(S)$, for some compact Hausdorff space S, so we again obtain Theorem 4.6 from Theorem 4.1. If X satisfies condition (b), Kakutani's theorem implies that X is linearly order isometric to a separable Banach sublattice of $C(K)$ for some compact Hausdorff space K, so X is linearly isomorphic to a separable G-space, and case (b) reduces to case (c). If K is a compact metric space, it is well known that $C(K)$ is separable, so if $X \subset C(K)$ is a G-space, then X is separable, and case (d) reduces to case (c). $\square$

The following result extends Theorem 3.1 to the case of $p = \infty$.

Corollary 4.7 *Let (Ω, Σ, μ) be a measure space. If the space $L^\infty := L^\infty(\Omega, \Sigma, \mu)$ is infinite dimensional, then there exists a graded family of homogeneous, set-additive MNC's $\{\gamma_t\}_{t>0}$ on L^∞.*

Proof We see that L^∞ is an infinite dimensional abstract M-space with a strong unit. Thus the result follows from part (a) of Theorem 4.6. $\square$

Remark It is natural to conjecture that Theorem 4.6 remains true if X is any infinite dimensional abstract M-space, but this question remains open. Y. Benyamini [9] has given an example of a nonseparable abstract M-space Z such that Z is not linearly isomorphic to a closed, complemented linear subspace Y of $C(K)$ for any compact Hausdorff space K. Thus, Benyamini's space Z falls outside the scope of Theorem 4.6. However, Z is a special case of the class of Banach spaces considered in Theorem 2.13, so the conclusions of Theorem 4.6 also hold for Benyamini's M-space Z.

5 Inequivalent measures of noncompactness on Hölder spaces

Let (K, d) be a compact metric space with metric d. For a given real number λ with $0 < \lambda \leq 1$, we are interested here in $C^{0,\lambda}(K)$, the Banach space of Hölder continuous functions

$f : K \to \mathbb{R}$ with Hölder exponent λ. Recall that a continuous function $f : K \to \mathbb{R}$ is Hölder continuous with Hölder exponent λ if and only if $\|f\|_\lambda < \infty$ where

$$\|f\|_\lambda := \sup_{x \in K} |f(x)| + \sup_{\substack{x,y \in K \\ x \neq y}} \left(\frac{|f(x) - f(y)|}{d(x,y)^\lambda} \right) \tag{5.1}$$

is the norm on $C^{0,\lambda}(K)$. Given $\delta > 0$, it is sometimes convenient to use the equivalent norm

$$\|f\|_{\lambda,\delta} := \sup_{x \in K} |f(x)| + \sup_{\substack{x,y \in K \\ 0 < d(x,y) \leq \delta}} \left(\frac{|f(x) - f(y)|}{d(x,y)^\lambda} \right).$$

It is known (see Lemma 5.2 of [37]) that the Kuratowski MNC obtained from $\|\cdot\|_\lambda$ equals the Kuratowski MNC obtained from $\|\cdot\|_{\lambda,\delta}$. We remark that part of our interest here stems from questions about the essential spectral radius and the cone essential spectral radius of so-called linear "Perron-Frobenius operators" on $C^{0,\lambda}(K)$; see Sections 5 and 6 of [37].

Our goal in this section is to prove the following result.

Theorem 5.1 *Let (K, d) be a compact metric space with infinitely many points and let $0 < \lambda \leq 1$. Then there exists a graded family of homogeneous, set-additive MNC's $\{\gamma_t\}_{t>0}$ on $C^{0,\lambda}(K)$.*

For the remainder of this section, (K, d) will denote a compact metric space with infinitely many points and λ will denote a fixed real number with $0 < \lambda \leq 1$. Our assumptions imply that (K, d) has an accumulation point. Let $x_* \in K$ denote any accumulation point, which will remain fixed throughout this section. For each $n \geq 1$, we select $x_n \in K$ with $d(x_n, x_*) > 0$ and

$$d(x_{n+1}, x_*) \leq \frac{d(x_n, x_*)}{10},$$

and for $n \geq 1$ we denote

$$\varepsilon_n := \frac{d(x_n, x_*)}{10}.$$

For the remainder of this section, the above points x_n will remain fixed.

We shall denote by X the Banach space

$$X := \{f \in C^{0,\lambda}(K) \mid f(x_*) = 0\},$$

which we note is a closed subspace of $C^{0,\lambda}(K)$ of codimension one, and is thus a complemented subspace. Thus by Theorem 2.12, it suffices to prove Theorem 5.1 with X replacing $C^{0,\lambda}(K)$. Let us also observe that

$$\|f\| := \sup_{\substack{x,y \in K \\ x \neq y}} \left(\frac{|f(x) - f(y)|}{d(x,y)^\lambda} \right) \tag{5.2}$$

is a norm on the space X, and in fact the norms given by Eqs. (5.1) and (5.2) are equivalent on X. Below we shall always use the norm (5.2) on X.

Our strategy in proving Theorem 5.1 will be to show that $\ell^\infty(\mathbb{N})$ is linearly isomorphic to a closed, complemented linear subspace of X, and thus Theorem 5.1 will follow from

Theorem 2.12 and Corollary 4.7. To this end, we define linear maps $R : X \to \ell^\infty(\mathbb{N})$ and $E : \ell^\infty(\mathbb{N}) \to X$ by

$$(Rf)(n) := \frac{f(x_n)}{\varepsilon_n^\lambda}$$

for every $n \geq 1$, where $f \in X$, and

$$(Eg)(x) := \begin{cases} \left(\varepsilon_n^\lambda - d(x, x_n)^\lambda\right) g(n), & \text{for } x \in B_n, \\ \\ 0, & \text{for } x \in K \setminus \bigcup_{n=1}^\infty B_n, \end{cases} \quad (5.3)$$

where $g \in \ell^\infty(\mathbb{N})$, and where

$$B_n := B_{\varepsilon_n}(x_n) \subset K$$

in the notation (1.1). Note that $B_m \cap B_n = \emptyset$ for $m \neq n$, so Eg is well defined as a function on K. Also note the estimate

$$|(Eg)(x)| \leq \varepsilon_n^\lambda |g(n)| \leq \varepsilon_n^\lambda \|g\| \quad (5.4)$$

for every $x \in B_n$, for all $n \geq 1$. We think of R as the operation of restriction of f to the set $S := \{x_n \mid n \geq 1\} \cup \{x_*\}$, and E as the operation of extending a map g, defined on S, to all of K, with a weight factor of $\varepsilon_n^{-\lambda}$ or ε_n^λ. It still must be shown that the above operators indeed map into the indicated spaces, and that they are continuous operators.

Lemma 5.2 *The operator R is a continuous linear map of X into $\ell^\infty(\mathbb{N})$.*

Proof Linearity of R is clear. If $f \in X$, we have

$$|(Rf)(n)| = \frac{|f(x_n)|}{\varepsilon_n^\lambda} = \frac{10^\lambda |f(x_n) - f(x_*)|}{d(x_n, x_*)^\lambda} \leq 10^\lambda \|f\|$$

for all $n \geq 1$, and so $\|Rf\| \leq 10^\lambda \|f\|$, so R is a continuous linear map. □

Lemma 5.3 *The operator E is a continuous linear map of $\ell^\infty(\mathbb{N})$ into X. Also, we have that $RE = I$, the identity operator on $\ell^\infty(\mathbb{N})$.*

Proof The linearity of E is clear. We also see from the formulas for R and E that $(REg)(n) = \varepsilon_n^{-\lambda}(Eg)(x_n) = g(n)$ for any $n \geq 1$, and thus $RE = I$. We have to prove that $Eg \in X$ for every $g \in \ell^\infty(\mathbb{N})$, and that $E : \ell^\infty(\mathbb{N}) \to X$ is continuous. We shall in fact show that for every such g, and denoting $f := Eg$, that we have

$$\frac{|f(x) - f(y)|}{d(x, y)^\lambda} \leq (1 + 10^{-\lambda})\|g\| \quad (5.5)$$

for every $x, y \in K$ with $x \neq y$. The estimate (5.5) implies that $f \in C^{0,\lambda}(K)$, and noting that $x_* \notin B_n$ for all $n \geq 1$, we have that $f(x_*) = 0$ and thus $f \in X$. Therefore, (5.5) implies that $Eg \in X$ and $\|Eg\| \leq (1 + 10^{-\lambda})\|g\|$, as desired.

Thus fix g and f as above, and take any $x, y \in K$ with $x \neq y$. We consider four cases, based on whether x and/or y belong to the balls B_n.

Case 1 There exists $n \geq 1$ such that $x, y \in B_n$. In this case, our formula (5.3) for f gives

$$\frac{|f(x) - f(y)|}{d(x, y)^\lambda} = \left(\frac{|d(x, x_n)^\lambda - d(y, x_n)^\lambda|}{d(x, y)^\lambda}\right) |g(n)| \leq \left(\frac{|d(x, x_n)^\lambda - d(y, x_n)^\lambda|}{d(x, y)^\lambda}\right) \|g\|.$$

We have $|d(x, x_n) - d(y, x_n)| \leq d(x, y)$ by the triangle inequality, so $|d(x, x_n) - d(y, x_n)|^\lambda \leq d(x, y)^\lambda$. A simple calculus argument shows that $|u^\lambda - v^\lambda| \leq |u - v|^\lambda$ whenever u and v are nonnegative reals and $0 < \lambda \leq 1$, so it follows that $|d(x, x_n)^\lambda - d(y, x_n)^\lambda| \leq d(x, y)^\lambda$. Using this estimate, we see that

$$\frac{|f(x) - f(y)|}{d(x, y)^\lambda} \leq \|g\|,$$

which verifies the desired estimate (5.5) for Case 1.

Case 2 There exist $m \geq 1$ and $n \geq 1$ with $m \neq n$ such that $x \in B_m$ and $y \in B_n$. By symmetry in the roles of x and y, we can assume that $m < n$. Using (5.4), we have that

$$\frac{|f(x) - f(y)|}{d(x, y)^\lambda} \leq \frac{|f(x)| + |f(y)|}{d(x, y)^\lambda} \leq \left(\frac{\varepsilon_m^\lambda + \varepsilon_n^\lambda}{d(x, y)^\lambda}\right) \|g\| \leq \left(\frac{(1 + 10^{-\lambda})\varepsilon_m^\lambda}{d(x, y)^\lambda}\right) \|g\|, \quad (5.6)$$

where we have used the fact that $\varepsilon_n \leq \frac{1}{10}\varepsilon_m$ in the final inequality. By the triangle inequality, we have

$$10\varepsilon_m = d(x_*, x_m) \leq d(x_*, x_n) + d(x_n, y) + d(y, x) + d(x, x_m) < 10\varepsilon_n + \varepsilon_n + d(y, x) + \varepsilon_m$$

and thus

$$d(x, y) > 9\varepsilon_m - 11\varepsilon_n \geq 9\varepsilon_m - \frac{11\varepsilon_m}{10} > \varepsilon_m,$$

and so it follows from this and from (5.6) that

$$\frac{|f(x) - f(y)|}{d(x, y)^\lambda} \leq (1 + 10^{-\lambda})\|g\|.$$

This verifies the estimate (5.5) for Case 2.

Case 3 One of the points x and y lies in some ball B_m, while the other does not lie in any ball B_n. For definiteness, assume that $x \in B_m$ for some $m \geq 1$, while $y \notin B_n$ for all $n \geq 1$. Our formula for f gives $f(y) = 0$ and so with (5.3) we have

$$\frac{|f(x) - f(y)|}{d(x, y)^\lambda} = \frac{|f(x)|}{d(x, y)^\lambda} = \left(\frac{\varepsilon_m^\lambda - d(x, x_m)^\lambda}{d(x, y)^\lambda}\right) |g(m)| \leq \left(\frac{\varepsilon_m^\lambda - d(x, x_m)^\lambda}{d(x, y)^\lambda}\right) \|g\|.$$
$$(5.7)$$

By the triangle inequality, we have $d(x, y) \geq d(y, x_m) - d(x, x_m) \geq \varepsilon_m - d(x, x_m) > 0$, and so

$$d(x, y)^\lambda \geq (\varepsilon_m - d(x, x_m))^\lambda \geq \varepsilon_m^\lambda - d(x, x_m)^\lambda.$$

It follows from this and from (5.7) that

$$\frac{|f(x) - f(y)|}{d(x, y)^\lambda} \leq \|g\|,$$

to give (5.5) in Case 3.

Case 4 We have both $x \notin B_n$ for every $n \geq 1$, and $y \notin B_n$ for every $n \geq 1$. In this case $f(x) = f(y) = 0$ and the estimate (5.5) trivially holds. With this final case, (5.5) is established, and the proof of the lemma is complete. □

With these preliminaries, we are almost ready to prove Theorem 5.1, but it is convenient to state one more lemma first.

Lemma 5.4 *The operator $P := ER$ is a continuous linear projection of X onto a closed linear subspace $X_0 := PX \subset X$. The restriction of R to X_0 is a one-one, continuous linear map of X_0 onto $\ell^\infty(\mathbb{N})$.*

Proof By Lemma 5.3, we have $RE = I$, and so $P^2 = (ER)^2 = E(RE)R = ER = P$. Thus, P is a projection on X.

Now let $h \in X_0$, say $h = ERf$ for some $f \in X$. If $Rh = 0$ then $RERf = Rf = 0$, so $h = 0$. Thus, we see that $R|X_0$ is one-one. Next let $g \in \ell^\infty(\mathbb{N})$ and set $f = Eg$. Then $RERf = (RE)^2 g = g$. Since $ERf \in X_0$, this shows that $R|X_0$ maps onto $\ell^\infty(\mathbb{N})$. □

Proof of Theorem 5.1 As noted earlier in this section, there exists a graded family of homogeneous, set-additive MNC's $\{\eta_t\}_{t>0}$ on $\ell^\infty(\mathbb{N})$ In the notation of Lemma 5.4, the restriction $R|X_0$ is a linear homeomorphism of X_0 onto $\ell^\infty(\mathbb{N})$, so there exists a graded family of homogeneous, set-additive MNC's $\{\xi_t\}_{t>0}$ on X_0. Lemma 5.4 implies that P is a continuous linear projection of X onto X_0, so Theorem 2.12 implies that there exists a graded family of homogeneous, set-additive MNC's $\{\gamma_t\}_{t>0}$ on X. As previously noted, this implies the corresponding result for $C^{0,\lambda}(K)$. □

6 Inequivalent measures of noncompactness on Sobolev spaces

In this section, Ω will always denote a fixed, open subset of $\mathbb{R}^n$, while m will denote a fixed positive integer and p is either a real number with $1 \leq p < \infty$ or $p = \infty$. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers and $u \in L^p(\Omega)$, then $D^\alpha u$ will denote the distributional partial derivative of u. We shall write $|\alpha| = \sum_{i=1}^{n} \alpha_i$. As usual (see [1], pages 44–45) the Sobolev space $W^{m,p}(\Omega)$ is given by

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\}.$$

The following theorem is the main result of this section.

Theorem 6.1 *There exists a graded family of homogeneous, set-additive MNC's $\{\gamma_t\}_{t>0}$ on $W^{m,p}(\Omega)$.*

Fix any point $x_0 \in \Omega$ and quantities $0 < a < b$ such that $\overline{A} \subset \Omega$, where

$$A := \{x \in \Omega \mid a < \|x - x_0\| < b\}.$$

We keep the point x_0 and the quantities a and b fixed for the remainder of this section. By using so-called extension theorems for Sobolev spaces (see [1], pages 83–94), we see

that there exists a continuous linear map $E : W^{m,p}(A) \to W^{m,p}(\Omega)$ with the property that $(Eu)(x) = u(x)$ for almost every $x \in A$. If $v \in W^{m,p}(\Omega)$, we define the restriction map $R : W^{m,p}(\Omega) \to W^{m,p}(A)$ by $(Rv)(x) = v(x)$ for $x \in A$. The map R is continuous and linear. Clearly, $RE = I$, the identity map on $W^{m,p}(A)$.

With these preliminaries, the proof of the next lemma is essentially identical to the proof of Lemma 5.4 and is left to the reader.

Lemma 6.2 *The operator $P := ER$ is a continuous linear projection of $W^{m,p}(\Omega)$ onto a closed linear subspace $X := P(W^{m,p}(\Omega)) \subset W^{m,p}(\Omega)$. The restriction of R to X is a one-one, continuous linear map of X onto $W^{m,p}(A)$.*

With this, we now prove our main theorem for Sobolev spaces.

Proof of Theorem 6.1 By using Theorem 2.12 and Lemma 6.2, we see that the problem of finding a graded family of homogeneous, set-additive MNC's on $W^{m,p}(\Omega)$ reduces to the same problem on $W^{m,p}(A)$.

We make a further reduction by projecting onto the space of radial functions in $W^{m,p}(A)$, which forms a closed, complemented subspace of $W^{m,p}(A)$. Recall that if $p < \infty$ then $Z := C^m(A) \cap W^{m,p}(A)$ is dense in $W^{m,p}(A)$ in the norm on $W^{m,p}(A)$. Define $Q : Z \to Z$ by

$$(Qu)(x) := \frac{1}{c_{n-1}} \int\limits_{|\omega|=1} u(|x|\omega) \, d\omega \tag{6.1}$$

for $u \in Z$ and $x \in A$, where $d\omega$ denotes surface area on the unit sphere and c_{n-1} denotes the surface area of the unit sphere in $\mathbb{R}^n$. The map Q can be shown to extend to a continuous linear projection of $W^{m,p}(A)$ into $W^{m,p}(A)$ whose range $Y := Q(W^{m,p}(A)) \subset W^{m,p}(A)$ consists of the radially symmetric functions in $W^{m,p}(A)$. If $p = \infty$ then $W^{m,\infty}(A) \subset C(\overline{A})$, and again Q as in (6.1) defines such a projection onto a subspace Y. Thus in any case, it suffices to find a graded family of homogeneous, set-additive MNC's on Y.

The space Y is, in turn, linearly isomorphic to the Sobolev space $W^{m,p}(a, b)$. If $u : (a, b) \to \mathbb{R}$, it is known that $u \in W^{m,p}(a, b)$ if and only if u has $m - 1$ continuous, bounded derivatives on (a, b) with the function $u^{(m-1)}$ absolutely continuous (and consequently differentiable almost everywhere), and with $u^{(m)} \in L^p(a, b)$. Let c be a fixed element of (a, b) and define a linear map $J : W^{m,p}(a, b) \to L^p(a, b) \times \mathbb{R}^m$ by

$$Ju := (u^{(m)}, u(c), u^{(1)}(c), u^{(2)}(c), \ldots, u^{(m-1)}(c)).$$

It is relatively easy to show that J is a linear isomorphism of $W^{m,p}(a, b)$ onto $L^p(a, b) \times \mathbb{R}^m$, so we conclude that Y is linearly isomorphic to $L^p(a, b) \times \mathbb{R}^m$.

Thus, it is enough to find a graded family of homogeneous, set-additive MNC's on $L^p(a, b) \times \mathbb{R}^m$. By Theorem 2.12, it is enough to find a graded family of homogeneous, set-additive MNC's on the subspace $L^p(a, b)$. However, such exists by Theorem 3.1 if $p < \infty$, and by Corollary 4.7 if $p = \infty$.   □

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