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Asymptotic stability for quasi-linear systems whose linear approximation is not assumed to be uniformly attractive

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Abstract Sufficient conditions are obtained for uniform stability and asymptotic stability of the zero solution of two-dimensional quasi-linear systems under the assumption that the zero solution of linear approximation is not always uniformly attractive. A class of quasi-linear systems considered in this paper includes a planar system equivalent to the damped pendulum $x'' + h(t)x' + \sin x = 0$, where $h(t)$ is permitted to change sign. Some suitable examples are included to illustrate the main results.

Keywords Asymptotic stability · Uniform stability · Quasi-linear systems · Weakly integrally positive · Discontinuous coefficients

Mathematics Subject Classification (2000) 34D05 · 34D10 · 34D20 · 37C75

1 Introduction and motive

The purpose of this paper is to give some sufficient conditions for the zero solution of a class of two-dimensional quasi-linear systems to be uniformly stable and asymptotically stable (for the definition, see Sect. 2).

Let $\mathbf{x}_0(t)$ be any solution on $0 \le t < \infty$ of the non-linear system

$$\mathbf{x}' = \mathbf{F}(t, \mathbf{x}), \qquad (N)$$

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where the prime denotes d/dt, and $\mathbf{F} = (F_1, F_2)$, $\mathbf{F}$ is continuous in $(t, \mathbf{x}) \in [0, \infty) \times \mathbb{R}^2$ and has continuous first-order partial derivatives with respect to the components of $\mathbf{x}$. Putting $\mathbf{y} = \mathbf{x} - \mathbf{x}_0(t)$, we can transform system (N) into the quasi-linear system

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t, \mathbf{y}), \tag{QL}$$

where $A(t) = \mathbf{F}_\mathbf{x}(t, \mathbf{x}_0(t))$, $\mathbf{F}_\mathbf{x}(t, \mathbf{x}_0(t))$ is the continuous matrix whose element in the i th row and j th column is $\partial F_i/\partial x_j(t, \mathbf{x}_0(t))$ $(i = 1, 2;\ j = 1, 2)$. Under the assumption that $\mathbf{f}$ is a continuous function in $(t, \mathbf{y})$ with the property

$$\lim_{\|\mathbf{y}\|\to 0} \frac{\|\mathbf{f}(t, \mathbf{y})\|}{\|\mathbf{y}\|} = 0 \tag{1.1}$$

uniformly in t, so that $\mathbf{f}(t, \mathbf{0}) \equiv \mathbf{0}$, if the zero solution of (QL) is uniformly stable or asymptotically stable, then the same will be true of the solution $\mathbf{x}_0(t)$ of (N) (refer to [2, pp. 160–161]). For this reason, the stability of the zero solution of quasi-linear systems, such as (QL), has been studied by a considerable number of authors from old times. For example, the reader is referred to the classical books [2–5, 8, 9, 16, 24].

If $\mathbf{f}(t, \mathbf{y})$ has the property (1.1), then it may be expected the zero solution of (QL) shows the same stability behavior as the zero solution of the linear approximation

$$\mathbf{y}' = A(t)\mathbf{y}. \tag{L}$$

In the case that the zero solution is uniformly asymptotically stable for system (L), the expectation is actually fulfilled; in other words, we may neglect the perturbed term $\mathbf{f}(t, \mathbf{y})$ of (QL) in the study of stability. The most fundamental theorem of this type is as follows (for example, see Theorem 1.7 in [8, pp. 59–61] or Theorem 2.4 in [9, pp. 86–87]):

Theorem A *If the zero solution of* (L) *is uniformly asymptotically stable, then the zero solution of* (QL) *is uniformly asymptotically stable.*

The concept of uniform asymptotic stability is a combination of the concepts of uniform stability and uniform attractivity (we give the definitions in the first paragraph of Sect. 2). The two "uniformity" for system (L) are essential roles in proving Theorem A. We cannot drop the term "uniformly" in the statement of Theorem A. Indeed, by a rather intricate analysis, Perron [21] has proved that even the asymptotic stability of the zero solution of (L) does not imply that the zero solution of (QL) is asymptotically stable in the case that

$$A(t) = \begin{pmatrix} -a & 0 \\ 0 & \sin\log(1+t) + \cos\log(1+t) - 2a \end{pmatrix}, \tag{1.2}$$

where $1 < 2a < 1 + e^{-\pi}$, and

$$\mathbf{f}(t, \mathbf{y}) = \begin{pmatrix} 0 \\ x^2 \end{pmatrix}.$$

For detailed arguments, see the classical books [1, pp. 42–43], [2, pp. 169–170], [5, p. 71], [27, pp. 92–93], [29, pp. 315–317], etc.

To be precise, in Perron's example, the zero solution is asymptotically stable for system (L), but it is neither uniformly stable nor uniformly attractive. This means that the matrix $A(t)$ given by (1.2) is not suitable at all for the zero solution of (QL) to be uniformly stable or asymptotically stable, namely the quality of $A(t)$ is too poor for uniformly stability or asymptotic stability of the zero solution of (QL).

The following question then arises: what happens if $A(t)$ has a better quality? What kind of condition on $A(t)$ will guarantee uniform stability and asymptotic stability of the zero solution of (QL) under the assumption that the zero solution of (L) is uniformly stable and asymptotically stable?

To answer our question mentioned above, we develop an idea of Sugie and Onitsuka [25]. They have considered a system of the form

$$\begin{aligned} x' &= -\ e(t)x + f(t)\phi_{p^*}(y), \\ y' &= -\ g(t)\phi_p(x) - h(t)y, \end{aligned} \tag{HL}$$

where $p > 1$, $1/p + 1/p^* = 1$, $\phi_q(z) \overset{\text{def}}{=} |z|^{q-2}z$ ($q = p$ or $q = q^*$); $e(t)$, $f(t)$, $g(t)$ and $h(t)$ are continuous for $t \geq 0$ and defined the crucial weakly integrally positive function $\psi(t)$ (playing the same role of the function $\psi(t)$ in the present paper). System (HL) is referred to as a half-linear system. In the special case that $p = 2$, system (HL) becomes the linear system (L). However, system (HL) does not correspond to the quasi-linear system (QL) in any case. Hence, we cannot apply results in [25] to system (QL). In [25], the function $\psi(t)$ is assumed to be non-negative for $t \geq 0$. We would like to be able to deal with a possibly sign-changing function $\psi(t)$. For this purpose, we improve the assumption of $\psi(t)$ to assumptions that are stated in the term of the positive part $\psi_+(t)$ and the negative part $\psi_-(t)$ of $\psi(t)$.

In Sect. 2, we prove the main theorem with some lemmas. To explain an advantage of the main result, we consider a damped pendulum equation and apply our main result to the equation in Sect. 3. In addition, we compare a result obtained from our main theorem with results of Hatvani [10]. For the illustration of our main result, we take some concrete examples and draw positive orbits of examples in Sect. 4.

2 Uniform stability and asymptotic stability

To begin with, we give some definitions. The zero solution of (QL) is said to be *stable* if, for any $\varepsilon > 0$ and any $t_0 \geq 0$, there exists a $\delta(\varepsilon, t_0) > 0$ such that $\|\mathbf{y}_0\| < \delta$ implies $\|\mathbf{y}(t; t_0, \mathbf{y}_0)\| < \varepsilon$ for all $t \geq t_0$. The zero solution is *uniformly stable* if it is stable and δ can be chosen independent of t_0. The zero solution is said to be *asymptotically stable* if it is stable and there exists a $\delta_0(t_0) > 0$ such that $\|\mathbf{y}_0\| < \delta_0$ implies $\|\mathbf{y}(t; t_0, \mathbf{y}_0)\| \to 0$ as $t \to \infty$. The zero solution is *uniformly attractive* if δ_0 in the definition of asymptotic stability can be chosen independent of t_0, and for every $\eta > 0$ there is a $T(\eta) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{y}_0\| < \delta_0$ imply $\|\mathbf{y}(t; t_0, \mathbf{y}_0)\| < \eta$ if $t \geq t_0 + T(\eta)$. The zero solution is *uniformly asymptotically stable* if it is uniformly stable and is uniformly attractive.

Consider a system of differential equations of the form

$$\begin{aligned} x' &= f(t)y, \\ y' &= -g(t)\left(x - \gamma(x)\right) - h(t)y, \end{aligned} \tag{E}$$

where $h(t)$ is piecewise continuous on $[0, \infty)$, and $f(t)$ and $g(t)$ are non-diminishing piecewise continuous on $[0, \infty)$; $\gamma(x)$ is continuous in the neighborhood of $x = 0$. We say that $\phi(t)$ is *piecewise continuous* on $[0, \infty)$ if there exists a sequence $\{t_n\}$ such that $\phi(t)$ is continuous on each of the open subintervals (t_n, t_{n+1}), and we say that $\phi(t)$ is *non-diminishing piecewise continuous* on $[0, \infty)$ if it is piecewise continuous and each length of the open subintervals (t_n, t_{n+1}) is not less than some $d > 0$. We assume that

$$f(t)g(t) > 0 \quad \text{for} \quad t \geq 0 \qquad \text{and} \qquad \liminf_{t\to\infty} f(t)g(t) > 0 \tag{2.1}$$

and

$$\lim_{|x|\to 0} \frac{|\gamma(x)|}{|x|} = 0. \tag{2.2}$$

Note that the coefficients $f(t)$, $g(t)$ and $h(t)$ are allowed to change sign and the non-linear function $\gamma(x)$ is not assumed to be the signum condition $x\gamma(x) > 0$ if $x \neq 0$. A typical case of (E) is the system

$$\begin{aligned} x' &= y, \\ y' &= -\sin x - h(t)y. \end{aligned}$$

In this case, $f(t) = g(t) = 1$ and $\gamma(x) = x - \sin x$.

In addition, we assume that $g(t)/f(t)$ is differentiable for $t \geq 0$. Let

$$\psi(t) = 2h(t) + \frac{f(t)}{g(t)}\left(\frac{g(t)}{f(t)}\right)'$$

and define

$$\psi_+(t) = \max\{0, \psi(t)\} \quad \text{and} \quad \psi_-(t) = \max\{0, -\psi(t)\}.$$

We here introduce concepts that play major roles in this paper. A non-negative function ϕ is said to be *integrally positive* if

$$\int_I \phi(t)\mathrm{d}t = \infty$$

for every set $I = \bigcup_{n=1}^{\infty}[\tau_n, \sigma_n]$ such that $\tau_n + \omega \leq \sigma_n < \tau_{n+1}$ for some $\omega > 0$ (refer to [6,10–14,19,28]). The integral positivity is rather stringent restriction than

$$\lim_{t\to\infty} \int^t \phi(s)\mathrm{d}s = \infty.$$

If, in addition, the set I satisfies $\tau_{n+1} \leq \sigma_n + \Omega$ for some $\Omega > 0$, the function ϕ is said to be *weakly integrally positive*. For example, $1/(1+t)$ and $\sin^2 t/(1+t)$ are weakly integrally positive, but they are not integrally positive (refer to [12,25,26]).

Remark 2.1 The notions given in Hatvani [12] are more general than the above-mentioned integral positivity and weak integral positivity. Roughly speaking, for two non-negative functions $a(t)$ and $b(t)$, if certain assumptions are satisfied, then a is called integrally positive with respect to b or weakly integrally positive with respect to b. A function ϕ is weakly integrally positive (in the sense described above) if and only if ϕ is weakly integrally positive with respect to $b \equiv 1$ (in the sense of [12]).

Our main result is stated as follows:

Theorem 2.1 *Suppose that conditions* (2.1) *and* (2.2) *hold. Suppose also that* $f(t)$, $g(t)$ *and* $h(t)$ *are bounded for* $t \geq 0$. *If* $\psi(t)$ *has the properties*:

$$\psi_+(t) \text{ is weakly integrally positive}; \tag{2.3}$$

$$\int_0^\infty \psi_-(t)dt < \infty, \tag{2.4}$$

then the zero solution of (E) *is uniformly stable and asymptotically stable.*

Before giving the proof of Theorem 2.1, we prepare some lemmas.

Lemma 2.2 *Suppose that condition (2.4) holds. Let $v(t)$ be non-negative and piecewise continuously differentiable on $[t_0, \infty)$ for some $t_0 \geq 0$. If*

$$v'(t) \leq \psi_-(t)v(t) \quad \textit{for} \quad t \geq t_0,$$

then $v'(t)$ is absolutely integrable.

Proof Integrating the both sides of $v'(t) \leq \psi_-(t)v(t)$ from t_0 to ∞, we have

$$v(t) \leq v(t_0) \exp\left(\int_{t_0}^{\infty} \psi_-(t)\mathrm{d}t\right),$$

and therefore,

$$v'(t) \leq v(t_0) \exp\left(\int_{t_0}^{\infty} \psi_-(t)\mathrm{d}t\right)\psi_-(t).$$

Since the right-hand side of this inequality is non-negative for $t \geq t_0$, we get

$$v'_+(t) \leq v(t_0) \exp\left(\int_{t_0}^{\infty} \psi_-(t)\mathrm{d}t\right)\psi_-(t).$$

Integrating both sides from t_0 to ∞, we obtain

$$\int_{t_0}^{\infty} v'_+(t)\mathrm{d}t \leq v(t_0) \exp\left(\int_{t_0}^{\infty} \psi_-(t)\mathrm{d}t\right)\int_{t_0}^{\infty} \psi_-(t)\mathrm{d}t.$$

Hence, by (2.4), we have

$$\int_{t_0}^{\infty} v'_+(t)\mathrm{d}t < \infty.$$

On the other hand, since $v(t) \geq 0$ for $t \geq t_0$, we get

$$\int_{t_0}^{\infty} v'_-(t)\mathrm{d}t \leq v(t_0) + \int_{t_0}^{\infty} v'_+(t)\mathrm{d}t < \infty.$$

We therefore conclude that

$$\int_{t_0}^{\infty} |v'(t)|\mathrm{d}t = \int_{t_0}^{\infty} (v'_+(t) + v'_-(t))\mathrm{d}t < \infty,$$

as required. This completes the proof of Lemma 2.2. □

Define

$$\Gamma(x) = \int_0^x \gamma(\zeta)d\zeta.$$

Then, we have the following lemma.

Lemma 2.3 *If $\gamma(x)$ satisfies condition (2.2), then*

$$|x-\gamma(x)| \geq \frac{1}{2}x^2 - \Gamma(x) \geq 0$$

for $|x|$ sufficiently small (the equalities hold if and only if $x=0$).

Proof From (2.2), it follows that for any $\varepsilon > 0$, there exists a $\rho(\varepsilon) > 0$ such that

$$-\varepsilon|x| \leq \gamma(x) \leq \varepsilon|x| \quad \text{for } |x| < \rho. \tag{2.5}$$

Hence, we obtain

$$-\frac{\varepsilon}{2}x^2 \leq \Gamma(x) \leq \frac{\varepsilon}{2}x^2 \quad \text{for } |x| < \rho. \tag{2.6}$$

We may assume without loss of generality that $\varepsilon < 1$ and $\rho < 2(1-\varepsilon)/(1+\varepsilon)$. Then, we have

$$\frac{1}{2}x^2 - \Gamma(x) \geq 0 \quad \text{for } |x| < \rho.$$

If $0 \leq x < \rho$, then by (2.5) and (2.6), we get

$$x - \gamma(x) - \frac{1}{2}x^2 + \Gamma(x) \geq x - \varepsilon x - \frac{1}{2}x^2 - \frac{\varepsilon}{2}x^2 \geq \frac{1}{2}x\left\{2(1-\varepsilon) - (1+\varepsilon)\rho\right\} \geq 0.$$

Similarly, if $-\rho < x \leq 0$, then

$$x - \gamma(x) + \frac{1}{2}x^2 - \Gamma(x) \leq x - \varepsilon x + \frac{1}{2}x^2 + \frac{\varepsilon}{2}x^2 \leq \frac{1}{2}x\left\{2(1-\varepsilon) - (1+\varepsilon)\rho\right\} \leq 0.$$

We therefore conclude that

$$|x-\gamma(x)| \geq \frac{1}{2}x^2 - \Gamma(x) \quad \text{for } |x| < \rho.$$

Thus, the proof of Lemma 2.3 is complete. □

We are now ready to prove the main result.

Proof of Theorem 2.1 We will divide the proof into two parts: (a) uniform stability of the zero solution; and (b) asymptotic stability of the zero solution. The proof of part (a) is carried out by means of a classical Lyapunov's direct method (as to the direct method of Lyapunov, for example, see [8,9,15,20,23,30]). The significant thing is to demonstrate part (b) rather than part (a).

Part (a): Define

$$V(t,x,y) = \frac{1}{2}x^2 - \Gamma(x) + \frac{f(t)}{2g(t)}y^2$$

and

$$U(t,x,y) = V(t,x,y)\exp\left(-\int_0^t \psi_-(s)\mathrm{d}s\right)$$

on $D \stackrel{\text{def}}{=} [0,\infty)\times\mathbb{R}^2$. From (2.1) and the boundedness of $f(t)$ and $g(t)$, we can choose numbers $k > 0$ and $K > 0$ such that

$$k \leq \frac{f(t)}{g(t)} \leq K \quad \text{for } t \geq 0. \tag{2.7}$$

Let $L = \int_0^\infty \psi_-(t)\mathrm{d}t$ (because of (2.4), such an L exists). Then, we have

$$\left(\frac{1}{2}x^2 - \Gamma(x) + \frac{k}{2}y^2\right)e^{-L} \le V(t,x,y)e^{-L} \le U(t,x,y)$$
$$\le V(t,x,y) \le \frac{1}{2}x^2 - \Gamma(x) + \frac{K}{2}y^2.$$

Differentiate $V(t,x,y)$ along any solution of (E) to obtain

$$\dot{V}_{(E)}(t,x,y) = -\frac{1}{2}\psi(t)\frac{f(t)}{g(t)}y^2 \le \frac{1}{2}\psi_-(t)\frac{f(t)}{g(t)}y^2.$$

By Lemma 2.3, we get

$$\dot{V}_{(E)}(t,x,y) \le \psi_-(t)V(t,x,y)$$

for $(t,x,y) \in D$, $|x|$ sufficiently small. Hence, we have

$$\dot{U}_{(E)}(t,x,y) = \left\{\dot{V}_{(E)}(t,x,y) - \psi_-(t)V(t,x,y)\right\}\exp\left(-\int_0^t \psi_-(s)\mathrm{d}s\right) \le 0$$

for $(t,x,y) \in D$, $|x|$ sufficiently small. Thus, $U(t,x,y)$ is positive definite and decrescent, and $\dot{U}_{(E)}(t,x,y)$ is non-positive. We therefore conclude that the zero solution of (E) is uniformly stable by using a Lyapunov-type theorem due to Persidski [22] (refer also to Theorem 1.7 in [23, p. 14] or to Theorem 8.2 in [30, p. 32]).

Part (b): Recall that (2.2) yields (2.5). It follows from part (a) that for any $\varepsilon > 0$, there exists a $\delta(\rho(\varepsilon)) > 0$ such that $t_0 \ge 0$ and $\|\mathbf{y}_0\| < \delta$ imply

$$\|\mathbf{y}(t;t_0,\mathbf{y}_0)\| < \rho \quad \text{for} \quad t \ge t_0, \tag{2.8}$$

where $\rho(\varepsilon)$ is the number given in (2.5). For brevity's sake, we write $(x(t),y(t)) = \mathbf{y}(t;t_0,\mathbf{y}_0)$ and define

$$u(t) = \frac{f(t)}{2g(t)}y^2(t) \quad \text{and} \quad v(t) = V(t,x(t),y(t)).$$

Then, we have

$$v(t) = \frac{1}{2}x^2(t) - \Gamma(x(t)) + u(t) \ge \frac{1}{2}x^2(t) - \Gamma(x(t)) + \frac{k}{2}y^2(t) \tag{2.9}$$

and

$$v'(t) = -\psi(t)u(t) \le \psi_-(t)v(t) \tag{2.10}$$

for $t \ge t_0$. As proved in part (a), $U'(t,x(t),y(t)) \le 0$ for $t \ge t_0$. Hence, $U(t,x(t),y(t))$ is non-increasing for $t \ge t_0$ and, therefore, it has a non-negative limiting value u_0. From (2.4), we see that $v(t)$ has a limiting value $v_0 = u_0e^L \ge 0$. If $v_0 = 0$, then by (2.9) and Lemma 2.3, the solution $(x(t),y(t))$ tends to $(0,0)$ as $t \to \infty$. This completes the proof of part (b) (we may choose δ as the number δ_0 in the definition of asymptotic stability). Hereafter, we consider only the case in which $v_0 > 0$.

From (2.7) and (2.8), we see that $u(t)$ is bounded for $t \ge t_0$. Hence, $u(t)$ has the inferior limit and the superior limit. First, we will show that the inferior limit of $u(t)$ is zero, and we will then show that the superior limit of $u(t)$ is also zero.

Suppose that $\liminf_{t\to\infty} u(t) > 0$. Then, there exist an $\varepsilon_1 > 0$ and a $T_1 \ge t_0$ such that $u(t) > \varepsilon_1$ for $t \ge T_1$. From (2.10) and Lemma 2.2, it follows that

$$\infty > \int_{t_0}^{\infty} |v'(t)|\mathrm{d}t = \int_{t_0}^{\infty} |\psi(t)|u(t)\mathrm{d}t \ge \int_{t_0}^{\infty} \psi_+(t)u(t)\mathrm{d}t > \varepsilon_1 \int_{T_1}^{\infty} \psi_+(t)\mathrm{d}t.$$

This contradicts (2.3). Thus, we see that $\liminf_{t\to\infty} u(t) = 0$.

Suppose that $\limsup_{t\to\infty} u(t) > 0$. From (2.1) and the boundedness of $g(t)$ and $h(t)$, we can choose numbers $\underline{g} > 0$ and $\overline{h} > 0$ such that

$$|g(t)| \ge \underline{g} \quad \text{and} \quad |h(t)| \le \overline{h} \tag{2.11}$$

for $t \ge 0$. Since the limiting value v_0 of $v(t)$ is positive, there exists a $T_2 \ge t_0$ such that

$$0 < \frac{v_0}{2} < v(t) < \frac{3v_0}{2} \quad \text{for} \quad t \ge T_2. \tag{2.12}$$

Let $\lambda = \limsup_{t\to\infty} u(t)$, and let $\varepsilon_2 > 0$ be so small that $\varepsilon_2 < \lambda/2$ and

$$\overline{h}\sqrt{\frac{2\varepsilon_2}{k}} < \underline{g}\Big(\frac{v_0}{2} - \varepsilon_2\Big). \tag{2.13}$$

Noticing that $\liminf_{t\to\infty} u(t) = 0 < \limsup_{t\to\infty} u(t)$, we can find two divergent sequences $\{\tau_n\}$ and $\{\sigma_n\}$ with $T_2 < \tau_n < \sigma_n < \tau_{n+1}$ such that $u(\tau_n) = u(\sigma_n) = \varepsilon_2$,

$$u(t) \ge \varepsilon_2 \quad \text{for} \quad \tau_n < t < \sigma_n, \tag{2.14}$$

$$0 \le u(t) \le \varepsilon_2 \quad \text{for} \quad \sigma_n < t < \tau_{n+1}. \tag{2.15}$$

From (2.11) and the second equation in system (E), we obtain

$$|y'(t)| \ge |g(t)\,(x(t) - \gamma(x(t)))| - |h(t)y(t)| \ge \underline{g}\,|x(t) - \gamma(x(t))| - \overline{h}\,|y(t)|$$

for $t \ge t_0$. It follows from (2.8) that $|x(t)| < \rho$ for $t \ge t_0$. Hence, by Lemma 2.3, we have

$$|x(t) - \gamma(x(t))| \ge \frac{1}{2}x^2(t) - \Gamma(x(t))$$

for $t \ge t_0$. By (2.7), (2.9), (2.12) and (2.15), we get

$$|y(t)| \le \sqrt{\frac{2}{k}u(t)} \le \sqrt{\frac{2\varepsilon_2}{k}}, \tag{2.16}$$

$$\frac{1}{2}x^2(t) - \Gamma(x(t)) = v(t) - u(t) \ge \frac{v_0}{2} - \varepsilon_2$$

for $\sigma_n \le t \le \tau_{n+1}$. We therefore conclude that

$$|x(t) - \gamma(x(t))| \ge \frac{v_0}{2} - \varepsilon_2 \quad \text{and} \quad |y'(t)| \ge \underline{g}\Big(\frac{v_0}{2} - \varepsilon_2\Big) - \overline{h}\sqrt{\frac{2\varepsilon_2}{k}} \stackrel{\text{def}}{=} M$$

for $\sigma_n \le t \le \tau_{n+1}$. From (2.13), we see that M is a positive number. Note that M is independent of n. Since $x(t)$ is continuous, we may assume without loss of generality that

$$x(t) - \gamma(x(t)) \ge \frac{v_0}{2} - \varepsilon_2 \quad \text{for} \quad \sigma_n \le t \le \tau_{n+1}. \tag{2.17}$$

Recall that $g(t)$ is assumed to be non-diminishing piecewise continuous on $[0,\infty)$. Hence, there are two cases to consider: (i) $g(t)$ is continuous on $[\sigma_n, \tau_{n+1}]$; and (ii) $g(t)$ is discontinuous on $[\sigma_n, \tau_{n+1}]$. Let N be the number of discontinuous points of $g(t)$ on (σ_n, τ_{n+1}) and let

$\mu_2, \mu_3, \ldots, \mu_{N+1}$ be the discontinuous points with $\sigma_n < \mu_2 < \mu_3 < \cdots < \mu_{N+1} < \tau_{n+1}$. Write $\mu_1 = \sigma_n$ and $\mu_{N+2} = \tau_{n+1}$. Of course, in case (i), $N = 0$ and $(\mu_1, \mu_2) = (\sigma_n, \tau_{n+1})$. In case (ii), $g(t)$ is continuous for $t \in (\mu_i, \mu_{i+1})$ with $i = 1, 2, \ldots, N+1$. In any case, because of (2.11), we see that $g(t) \geq \underline{g}$ or $g(t) \leq -\underline{g}$ on each subinterval (μ_i, μ_{i+1}) with $i = 1, 2, \ldots, N+1$. If there exists a $j \in \{1, 2, \ldots, N+1\}$ such that $g(t) \geq \underline{g}$ on (μ_j, μ_{j+1}), then by (2.16) and (2.17), we have

$$\begin{aligned} y'(t) &= -g(t)\,(x(t) - \gamma(x(t))) - h(t)y(t) \\ &\leq -\underline{g}\left(\frac{v_0}{2} - \varepsilon_2\right) + |h(t)||y(t)| \\ &\leq -\underline{g}\left(\frac{v_0}{2} - \varepsilon_2\right) + \overline{h}\sqrt{\frac{2\varepsilon_2}{k}} = -M \end{aligned}$$

for $\mu_j < t < \mu_{j+1}$. Integrate the above inequality from μ_j to μ_{j+1} to obtain

$$|y(\mu_{j+1})| + |y(\mu_j)| \geq \left| \int_{\mu_j}^{\mu_{j+1}} y'(t)\mathrm{d}t \right| \geq M(\mu_{j+1} - \mu_j).$$

Similarly, if there exists a $j \in \{1, 2, \ldots, N+1\}$ such that $g(t) \leq -\underline{g}$ on (μ_j, μ_{j+1}), then by (2.16) and (2.17), we get

$$y'(t) \geq M \quad \text{for} \quad \mu_j < t < \mu_{j+1},$$

and therefore, $|y(\mu_{j+1})| + |y(\mu_j)| \geq M(\mu_{j+1} - \mu_j)$. We therefore conclude that

$$|y(\mu_{i+1})| + |y(\mu_i)| \geq M(\mu_{i+1} - \mu_i)$$

for all $i \in \{1, 2, \ldots, N+1\}$. Adding up these inequalities and using (2.8), we obtain

$$\begin{aligned} 2(N+1)\rho &> |y(\sigma_n)| + 2\sum_{i=2}^{N+1} |y(\mu_i)| + |y(\tau_{n+1})| \\ &\geq M \sum_{i=1}^{N+1} (\mu_{i+1} - \mu_i) = M(\tau_{n+1} - \sigma_n), \end{aligned}$$

namely,

$$\tau_{n+1} \leq \sigma_n + \frac{2(N+1)\rho}{M}.$$

By means of Lemma 2.2 with (2.10) and (2.14), we have

$$\infty > \int_{t_0}^{\infty} |v'(t)|\mathrm{d}t \geq \int_{t_0}^{\infty} \psi_+(t)u(t)\mathrm{d}t > \varepsilon_2 \int_I \psi_+(t)\mathrm{d}t,$$

where $I = \bigcup_{n=1}^{\infty} [\tau_n, \sigma_n]$. Consequently,

$$\int_I \psi_+(t)\mathrm{d}t < \infty. \tag{2.18}$$

Suppose that there exists an $\omega > 0$ such that $\sigma_n - \tau_n > \omega$ for each $n \in \mathbb{N}$. As proved above, $\tau_{n+1} - \sigma_n \leq 2(N+1)\rho/M \overset{\text{def}}{=} \Omega$ for any $n \in \mathbb{N}$. Hence, from (2.3), we see that

$$\int_I \psi_+(t)\mathrm{d}t = \infty.$$

This contradicts (2.18). Hence, there is no such $\omega > 0$, and therefore,

$$\liminf_{n\to\infty}(\sigma_n - \tau_n) = 0. \tag{2.19}$$

Since $\liminf_{t\to\infty} u(t) = 0$ and $\limsup_{t\to\infty} u(t) = \lambda > 0$, we can select two divergent sequences $\{t_n\}$ and $\{s_n\}$ with $T_2 < t_n < s_n < t_{n+1}$ such that $u(t_n) = \lambda/2$, $u(s_n) = 3\lambda/4$ and

$$\frac{\lambda}{2} < u(t) < \frac{3\lambda}{4} \quad \text{for } t_n < t < s_n.$$

Since $\varepsilon_2 < \lambda/2$, we may consider that $[t_n, s_n] \subset [\tau_n, \sigma_n]$ for $n \in \mathbb{N}$ (if necessary, we can change $\{\tau_n\}$ and $\{\sigma_n\}$ into suitable subsequences of $\{\tau_n\}$ and $\{\sigma_n\}$). Hence, by (2.19), we have

$$\liminf_{n\to\infty}(s_n - t_n) = 0. \tag{2.20}$$

From the boundedness of $f(t)$, we can choose a number $\overline{f} > 0$ such that $|f(t)| \leq \overline{f}$ for $t \geq 0$. Hence, together with (2.5), (2.8) and (2.9), we get

$$\begin{aligned} u'(t) &= v'(t) - f(t)y(t)\left(x(t) - \gamma(x(t))\right) \\ &\leq |v'(t)| + \overline{f}|y(t)|\left(|x(t)| + |\gamma(x(t))|\right) \\ &< |v'(t)| + \overline{f}\rho^2(1+\varepsilon) \end{aligned}$$

for $t \geq t_0$. Integrating this inequality from t_n to s_n, we obtain

$$\frac{\lambda}{4} = u(s_n) - u(t_n) \leq \int_{t_n}^{s_n} |v'(t)|\mathrm{d}t + \overline{f}\rho^2(1+\varepsilon)(s_n - t_n)$$

for each $n \in \mathbb{N}$. This contradicts (2.20). Thus, we conclude that $\lambda = 0$, and therefore, $\limsup_{t\to\infty} u(t) = 0$.

Since $\liminf_{t\to\infty} u(t) = \limsup_{t\to\infty} u(t) = 0$, it follows that $\lim_{t\to\infty} u(t) = 0$. Hence, there exists a $T_3 \geq t_0$ such that

$$u(t) < \varepsilon_2 \quad \text{for} \quad t \geq T_3.$$

As in the same argument of the preceding paragraph, we conclude that

$$|x(t) - \gamma(x(t))| \geq \frac{v_0}{2} - \varepsilon_2 \quad \text{for} \quad t \geq T_3.$$

We may assume without loss of generality that

$$x(t) - \gamma(x(t)) \geq \frac{v_0}{2} - \varepsilon_2 \quad \text{for} \quad t \geq T_3. \tag{2.21}$$

Since $g(t)$ is non-diminishing piecewise continuous on $[0, \infty)$, there exists a greatest lower bound of lengths of subintervals on which $g(t)$ is continuous. Let d be the greatest lower bound. We may consider that $\overline{h} > 2/d$, where $\overline{h}$ is the number given in (2.11). Choose $\varepsilon_3 > 0$ with

$$\varepsilon_3 < \frac{\underline{g}}{2\overline{h}}\left(\frac{v_0}{2} - \varepsilon_2\right). \tag{2.22}$$

From (2.7) and the fact that $\lim_{t\to\infty} u(t) = 0$, it turns out that $y(t)$ also tends to zero as $t \to \infty$. Hence, there exists a $T_4 \geq T_3$ such that $|y(t)| < \varepsilon_3$ for $t \geq T_4$. By (2.11) again, we can find an interval $[a, b] \subset [T_4, \infty)$ such that $g(t) \geq \underline{g}$ or $g(t) \leq -\underline{g}$ for $a \leq t \leq b$. In the former, by (2.11) and (2.21), we have

$$\begin{aligned} y'(t) &\leq -g(t)\,(x(t) - \gamma(x(t))) + |h(t)||y(t)| \\ &\leq -\underline{g}\Big(\frac{v_0}{2} - \varepsilon_2\Big) + \overline{h}\varepsilon_3 < -\frac{\underline{g}}{2}\Big(\frac{v_0}{2} - \varepsilon_2\Big) \end{aligned}$$

for $a \leq t \leq b$. Hence, we obtain

$$\begin{aligned} 2\varepsilon_3 > |y(b) - y(a)| &= \left|\int_a^b y'(t)\mathrm{d}t\right| \\ &= \int_a^b |y'(t)|\mathrm{d}t > \frac{\underline{g}}{2}\Big(\frac{v_0}{2} - \varepsilon_2\Big)(b-a) \geq \frac{\underline{g}}{2}\Big(\frac{v_0}{2} - \varepsilon_2\Big)d. \end{aligned}$$

Since $\overline{h} > 2/d$, it follows that $\varepsilon_3 > \underline{g}(v_0/2 - \varepsilon_2)/(2\overline{h})$. This contradicts (2.22). We can carry out the latter in the same manner as the former, and we then reach a contradiction. Thus, the case of $v_0 > 0$ does not happen. We therefore conclude that the zero solution of (E) is asymptotically stable.

The proof of Theorem 2.1 is now complete. □

When we apply a Lyapunov-type theorem on asymptotic stability to a concrete problem, we have to find a Lyapunov function whose total derivative is negative definite. As such a theorem, we can quote the following result given by Haddock [7] (his original result can be applied to n-dimensional non-linear systems).

Theorem B *Suppose that there exist positive numbers A and M such that $\|\mathbf{F}(t, \mathbf{x})\| \leq M$ for all $(t, \mathbf{x}) \in [0, \infty) \times S(A)$, $S(A) = \{\mathbf{x} : \|\mathbf{x}\| < A\}$. Suppose also that there exists a differentiable function $V : [0, \infty) \times G \to [0, \infty)$, G is an open subset of $\mathbb{R}^2$, which satisfies the following conditions*:

(i) $V(t, \mathbf{0}) \equiv 0$;
(ii) $V(t, \mathbf{x}) \geq 0$ *for all* $(t, \mathbf{x}) \in [0, \infty) \times S(A)$;
(iii) *the total derivative* $\dot{V}_{(N)}(t, \mathbf{x})$ *is negative definite.*

Then the zero solution of (N) is asymptotically stable.

Haddock [7] also showed that, under the assumptions of Theorem B, $V(t, \mathbf{x})$ is positive definite. Hence, Theorem B is essentially the same as Theorem 4 of Maratschkow [17] though it is a little easier to use (see also [18]).

In general, it is very difficult to compose a Lyapunov function satisfying the assumption (ii) of Theorem B. This is a weak point of Theorem B. Since the Lyapunov functions $V(t, x, y)$ and $U(t, x, y)$ given in the proof of Theorem 2.1 are energy functions for system (E), it is safe to say that they are appropriate. However, the total derivatives $\dot{V}_{(N)}(t, \mathbf{x})$ and $\dot{U}_{(N)}(t, \mathbf{x})$ are not negative definite. Hence, we cannot prove Theorem 2.1 by means of Theorem B.

The linear approximation of (E) is system (L) with

$$A(t) = \begin{pmatrix} 0 & f(t) \\ -g(t) & -h(t) \end{pmatrix}. \tag{2.23}$$

Needless to say, Theorem 2.1 can be applied to this linear system. If $f(t)$, $g(t)$ and $h(t)$ are bounded for $t \geq 0$ and they satisfy conditions (2.1), (2.2) and (2.3), then the zero solution of (L) with (2.23) is uniformly stable and asymptotically stable, in other words, the matrix $A(t)$ given by (2.23) has a better quality. Theorem 2.1 shows that if $A(t)$ has such a good quality, then the zero solution of the quasi-linear system

$$\mathbf{y}' = \begin{pmatrix} 0 & f(t) \\ -g(t) & -h(t) \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 \\ g(t)\gamma(x) \end{pmatrix},$$

namely system (E), is uniformly stable and asymptotically stable for arbitrary non-linear function $\gamma(x)$ satisfying condition (2.1). This is an answer to our question raised in Sect. 1.

Although we assume the boundedness of $f(t)$, $g(t)$ and $h(t)$ in Theorem 2.1, we can remove the assumption by changing variables

$$\tau = \mathcal{F}(t) \overset{\text{def}}{=} \int_0^t \sqrt{f(s)g(s)}\,\mathrm{d}s \quad \text{and} \quad z = \sqrt{\frac{f(t)}{g(t)}}\,y.$$

We assume that

$$f(t)g(t) > 0 \quad \text{for } t \geq 0 \quad \text{and} \quad \lim_{t\to\infty} \mathcal{F}(t) = \infty, \tag{2.1$'$}$$

instead of condition (2.1). Then, $\mathrm{d}\tau/\mathrm{d}t = \sqrt{f(t)g(t)}$, and there exists the inverse function $\mathcal{F}^{-1}(\tau)$ satisfying $\lim_{\tau\to\infty} \mathcal{F}^{-1}(\tau) = \infty$. By a straightforwards calculation, we can transform system (E) into the system

$$\begin{aligned} \dot{x} &= z, \\ \dot{z} &= -\,(x - \gamma(x)) - \tilde{h}(\tau)z, \end{aligned} \tag{$\widetilde{E}$}$$

where $\dot{} = \mathrm{d}/\mathrm{d}\tau$ and

$$\tilde{h}(\tau) = \frac{\psi(\mathcal{F}^{-1}(\tau))}{2\sqrt{f(\mathcal{F}^{-1}(\tau))g(\mathcal{F}^{-1}(\tau))}}.$$

For the sake of convenience, let

$$\Psi(t) = \frac{\psi(t)}{\sqrt{f(t)g(t)}}.$$

Then, $\tilde{h}(\tau) = \Psi(t)/2$. If $\gamma(x)$ satisfies condition (2.2), and $\Psi(t)$ is bounded for $t \geq 0$ and has the properties

$$\Psi_+(t) \text{ is weakly integrally positive;} \tag{2.3$'$}$$

$$\int_0^\infty \Psi_-(t)\mathrm{d}t < \infty, \tag{2.4$'$}$$

then, we can apply Theorem 2.1 to system ($\widetilde{E}$). Hence, the zero solution of ($\widetilde{E}$) is uniformly stable and asymptotically stable. If, in addition, $g(t)/f(t)$ is bounded for $t \geq 0$, then the zero solution of (E) is uniformly stable and asymptotically stable.

In summary, we have the following result.

Theorem 2.4 *Suppose that conditions (2.1)′, (2.2), (2.3)′ and (2.4)′. Suppose also that $g(t)/f(t)$ and $\Psi(t)$ are bounded for $t \geq 0$. Then, the zero solution of (E) is uniformly stable and asymptotically stable.*

Remark 2.2 If at least one of $f(t)$, $g(t)$ and $h(t)$ is unbounded, then Theorem 2.1 and Theorem B are inapplicable for system (E). On the other hand, Theorem 2.4 can be applied even to that case.

3 Damped pendulum

Let us consider a pendulum with time-varying friction described by the equation

$$x'' + h(t)x' + \sin x = 0, \tag{P}$$

where $h(t)$ is continuous for $t \geq 0$. Then, we can rewrite equation (P) as the system

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -1 & -h(t) \end{pmatrix}\mathbf{y} + \begin{pmatrix} 0 \\ x - \sin x \end{pmatrix} \tag{3.1}$$

with $\mathbf{y} = {}^t(x, y)$. System (3.1) is of type (QL). In the special case that $h(t) = 2/(1+t)$, we can find a fundamental matrix for the linear approximation

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -1 & -h(t) \end{pmatrix}\mathbf{y}. \tag{3.2}$$

A fundamental matrix for system (3.2) with $h(t) = 2/(1+t)$ is given by

$$\Phi(t) = \begin{pmatrix} \dfrac{\sin t}{1+t} & \dfrac{\cos t}{1+t} \\ \dfrac{\cos t}{1+t} - \dfrac{\sin t}{(1+t)^2} & -\dfrac{\sin t}{1+t} - \dfrac{\cos t}{(1+t)^2} \end{pmatrix}.$$

From considerations of the fundamental matrix, we see that the zero solution of (3.2) with $h(t) = 2/(1+t)$ is uniformly stable and asymptotically stable, but it is not uniformly attractive; accordingly, it is not uniformly asymptotically stable. Hence, in this case, Theorem A is inapplicable to system (3.2), and therefore, we cannot decide whether the equilibrium $x = x' = 0$ of (P) is uniformly stable or asymptotically stable.

However, it is true that the equilibrium of (P) with $h(t) = 2/(1+t)$ is asymptotically stable. In fact, by means of Hatvani's result [10, Corollary 3.1], we can verify that if $h(t)$ is non-negative and weakly integrally positive, then the equilibrium of (P) is asymptotically stable (see also [11]).

Theorem 2.1 is also useful for verifying the fact above. Comparing system (3.1) with system (E), we see that $f(t) = g(t) = 1$ and $\gamma(x) = x - \sin x$. Hence, conditions (2.1) and (2.2) are satisfied. If $h(t) = 2/(1+t)$, then it is easy to show that $\psi_+(t) = 4/(1+t)$ and $\psi_-(t) = 0$; accordingly, $\psi(t)$ has the properties (2.3) and (2.4). Thus, it follows from Theorem 2.1 that the zero solution of (3.1) is uniformly stable and asymptotically stable. Of course, the equilibrium of (P) is uniformly stable and asymptotically stable.

In the above results [10, 11], the friction $h(t)$ is assumed to be non-negative for $t \geq 0$. Is the assumption essential to show the asymptotic stability of the equilibrium of (P)? The answer is in the negative. There are cases in which the equilibrium of (P) is uniformly stable and asymptotically stable even if there exists a sequence $\{t_n\}$ such that $h(t_n) < 0$. For example, consider system (3.1) with

$$h(t) = \frac{\sin^2 t}{1+t} - \frac{\cos^2 t}{(1+t)^2}.$$

Then, it is easy to verify that $\psi(t) = 2h(t)$ and $\psi(t)$ have the properties (2.3) and (2.4). Hence, by Theorem 2.1, the zero solution is uniformly stable and asymptotically stable, and therefore, the equilibrium of (P) is uniformly stable and asymptotically stable. In this example, $h(t)$ is negative at $t = n\pi$ for $n \in \mathbb{N}$. Hence, $\psi_-(t) = 2h_-(t) \not\equiv 0$.

As mentioned above, Theorem 2.1 has a big advantage of being applicable to the case that $h(t)$ is not necessarily positive for $t \geq 0$.

4 Example and simulation

To illustrate Theorems 2.1 and 2.4, we give two examples: the coefficients $f(t)$, $g(t)$ and $h(t)$ of (E) are piecewise continuous in the first example; and $f(t)$ and $g(t)$ are piecewise continuous but $h(t)$ is continuous in the second example.

The result of Sugie and Onitsuka [25] quoted in Sect. 1 cannot be applied to the quasi-linear system (E) unless $\gamma(x) \not\equiv 0$ and $f(t)$, $g(t)$ and $h(t)$ are continuous for $t \geq 0$. Hence, in Examples 4.1 and 4.2 below, we cannot judge whether the zero solution is uniformly stable and asymptotically stable by using the result in [25].

Let $n \in \mathbb{N}$ and let $\eta(t)$ be an on–off function defined by

$$\eta(t) = \begin{cases} 0 & \text{if } 2(n-1)\pi \leq t < 2n\pi - \frac{1}{n}, \\ 1 & \text{if } 2n\pi - \frac{1}{n} \leq t < 2n\pi. \end{cases}$$

Then, we have the following example.

Example 4.1 Consider system (E) with

$$f(t) = \begin{cases} 1 + \frac{1}{2}\sin t & \text{if } 2(n-1)\pi \leq t < (2n-1)\pi, \\ -1 - \frac{1}{2}\sin t & \text{if } (2n-1)\pi \leq t < 2n\pi, \end{cases}$$

$$g(t) = \begin{cases} 2 - \sin t & \text{if } 2(n-1)\pi \leq t < (2n-1)\pi, \\ -2 + \sin t & \text{if } (2n-1)\pi \leq t < 2n\pi, \end{cases}$$

$$h(t) = \frac{1 - 2\eta(t)}{1+t} + \frac{2\cos \pi t}{3 + \cos^2 \pi t} \quad \text{and} \quad \gamma(x) = x - \sin x.$$

Then, the zero solution is uniformly stable and asymptotically stable.

Note that $h(t)$ is piecewise continuous and $f(t)$ and $g(t)$ are non-diminishing piecewise continuous with $d = 1$. It is clear that conditions (2.1) and (2.2) are satisfied and that $f(t)$, $g(t)$ and $h(t)$ are bounded for $t \geq 0$. Since

$$\psi(t) = 2h(t) + \frac{f(t)}{g(t)}\left(\frac{g(t)}{f(t)}\right)' = \frac{2(1 - 2\eta(t))}{1+t},$$

we see that

$$\psi_+(t) = \frac{2(1 - \eta(t))}{1+t} \quad \text{and} \quad \psi_-(t) = \frac{2\eta(t)}{1+t}.$$

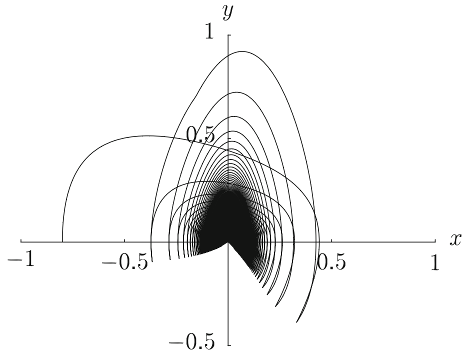

Fig. 1 A positive orbit of Example 4.1

Hence, $\psi_+(t)$ is weakly integrally positive and

$$\int_0^\infty \psi_-(t)\mathrm{d}t < \sum_{n=1}^{\infty} \frac{2}{2\pi n^2 + n - 1} < \sum_{n=1}^{\infty} \frac{1}{\pi n^2} < \infty.$$

Thus, by means of Theorem 2.1, we conclude that the zero solution is uniformly stable and asymptotically stable.

In Fig. 1, we draw a positive orbit of Example 4.1. The starting point $\mathbf{y}_0$ is $(-0.8, 0)$, and the initial time t_0 is 0. The positive orbit seems to consist of two groups of circular arcs whose central angles are obtuse, arcs that are longer horizontally than vertically and arcs that are longer vertically than horizontally. The positive orbit alternately generates the two arc groups and finally approaches the origin $\mathbf{0}$.

In the above example, $\psi(t)$ is piecewise continuous for $t \geq 0$. As shown in the following example, however, it not always necessary for $\psi(t)$ to be discontinuous.

Example 4.2 Consider system (E) with

$$f(t) = \begin{cases} 2+t & \text{if } 2(n-1) \leq t < 2n-1, \\ -2-t & \text{if } 2n-1 \leq t < 2n, \end{cases}$$

$$g(t) = \begin{cases} 1+t & \text{if } 2(n-1) \leq t < 2n-1, \\ -1-t & \text{if } 2n-1 \leq t < 2n, \end{cases}$$

$$h(t) = \sqrt{\frac{2+t}{1+t}} \quad \text{and} \quad \gamma(x) = x - \sin x.$$

Then, the zero solution is uniformly stable and asymptotically stable.

Since $f(t)$ and $g(t)$ are unbounded, Theorem 2.1 cannot be applied to Example 4.2. We use Theorem 2.4 in substitution for Theorem 2.1. From

$$f(t)g(t) = \sqrt{(1+t)(2+t)} \geq \sqrt{2}$$

for $t \geq 0$, it follows that

$$\tau = \mathcal{F}(t) = \frac{2t+3}{4}\sqrt{(1+t)(2+t)} - \frac{1}{8}\log\Big(2t+3+2\sqrt{(1+t)(2+t)}\Big),$$

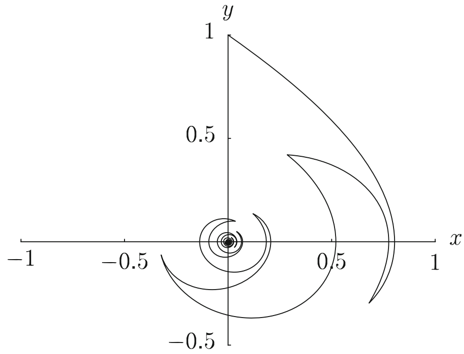

Fig. 2 A positive orbit of Example 4.2

which tends to ∞ as $t \to \infty$. Hence, condition $(2.1)'$ holds. It is easy to verify that condition (2.2) is satisfied and $g(t)/f(t)$ is bounded for $t \geq 0$. Since

$$\psi(t) = 2\sqrt{\frac{2+t}{1+t}} + \frac{1}{(1+t)(2+t)}$$

and

$$\Psi(t) = \frac{\psi(t)}{\sqrt{f(t)g(t)}} = \frac{2}{1+t} + \frac{1}{\{(1+t)(2+t)\}^{3/2}}$$

in this example. Hence, $\Psi_+(t) = \Psi(t)$ and $\Psi_-(t) = 0$; accordingly, $\Psi_+(t)$ is weakly integrally positive and

$$\int_0^\infty \Psi_-(t)\mathrm{d}t = 0.$$

We therefore conclude that the zero solution is uniformly stable and asymptotically stable.

Figure 2 indicates a positive orbit of Example 4.2. The starting point $\mathbf{y}_0$ is (0, 1), and the initial time t_0 is 0. The positive orbit moves round the origin $\mathbf{0}$ in a clockwise and a counter-clockwise direction by turns, because $f(t)$ and $h(t)$ change their sign. The positive orbit approaches the origin $\mathbf{0}$ as it goes up and down.

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