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How to construct totally disconnected Markov sets?

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Abstract We deduce a polynomial estimate on a compact planar set from a polynomial estimate on its circular projection, which enables us to prove Markov and Bernstein-Walsh type inequalities for certain sets. We construct

- totally disconnected Markov sets that are scattered around zero in different directions;
- a Markov set $E \subset \mathbb{R}$ such that neither $E \cap [0, +\infty)$ nor $E \cap (-\infty, 0]$ admit Markov's inequality;
- a Markov set that is not uniformly perfect.

Finally, we propose a construction based on a generalization of iterated function systems: a way of obtaining a big family of uniformly perfect sets.

Keywords Markov inequality · Exceptional sets · Leja-Siciak extremal function · Green function · Iterated function systems · Attractors

Mathematics Subject Classification (2000) 41A17 · 30A10 · 28A80 · 34B27

1 Introduction

Our way of answering the question stated in the title is mostly based on the following fact.

Theorem 1.1 *Let $K \subset \mathbb{C}$ be a compact set. Fix $z_0 \in K$, $n \in \{1, 2, \dots\}$ and $j \in \{1, \dots, n\}$. Put $T = T(z_0) := \{ t \geq 0 : \exists z \in K : |z - z_0| = t \}$. If every polynomial $Q \in \mathcal{P}_n$ satisfies*

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the inequality

$$|Q^{(j)}(0)| \leq C_{j,n}\,||Q||_T \tag{1.1}$$

with some constant $C_{j,n}$ *independent of* Q*, then for every* $P \in \mathcal{P}_n$

$$|P^{(j)}(z_0)| \leq 2n\,C_{j,n}\,||P||_K. \tag{1.2}$$

Here and henceforth $\mathcal{P}_n$ denotes the family of all polynomials of degree at most n. We put also $||f||_K := \sup_{w\in K} |f(w)|$ for a bounded function f and a compact set K.

The theorem says that one can deduce some polynomial estimates for the set K from some estimates for a special circular projection of the set. The implication was inspired by a problem solved in [5]. The second result is similar.

Theorem 1.2 *Let* $K \subset \mathbb{C}$ *be a compact set. Fix* $w_0 \in \mathbb{C}\backslash K, n \in \{1, 2, \dots\}$. *Set* $T = T(w_0) := \{\, t \geq 0 \,:\, \exists\, z \in K \,:\, |z - w_0| = t \,\}$.

If every polynomial $Q \in \mathcal{P}_n$ *satisfies the inequality*

$$|Q(0)| \leq C_n\,||Q||_T$$

with some constant C_n *independent of* Q*, then for every* $P \in \mathcal{P}_n$

$$|P(w_0)| \leq (n+1)\,C_n\,||P||_K.$$

Before showing proofs and applications of these two results let us recall definitions and facts concerned with Markov sets.

In 1889, A.A.Markov proved that for every polynomial P

$$\max_{t\in[-1,1]} |P'(t)| \leq (\deg P)^2 \max_{t\in[-1,1]} |P(t)|.$$

This inequality and its counterparts for other sets became soon an interesting subject of extensive research owing to their various generalizations and numerous applications in different domains.

Definition 1.3 Let $m \geq 1$. A compact set $E \subset \mathbb{K}^N$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is said to have the *Markov property* (*with exponent* m) if there exist positive constants M and m such that for every polynomial P of N variables

$$\|\,\nabla P\,\|_E \leq M(\deg P)^m\;\|P\|_E, \tag{GMI}$$

where $\nabla P := (\frac{\partial P}{\partial z_1}, \dots, \frac{\partial P}{\partial z_N})$ and $|\nabla P| = (\sum_{j=1}^N |\frac{\partial P}{\partial z_j}|^2)^{1/2}$. If E satisfies the aforementioned property, we say that it is a *Markov set* (*with exponent* m) and we write $E \in MP(m)$.

Markov sets play a key role in Bernstein type characterization of $\mathcal{C}^\infty$ functions (see e.g. [18,21]). Moreover, Pawłucki and Pleśniak (see e.g. [20]) proved that the Markov property is equivalent to the existence of a continuous linear extension operator from $\mathcal{C}^\infty(E)$ with Jackson's topology to $\mathcal{C}^\infty(\mathbb{R}^N)$ with the natural topology. Markov sets have been a subject of investigations conducted by Totik (see e.g. [24]), Goetgheluck (e.g. [11]), Bos and Milman (e.g. [7]), Goncharov (e.g. [12]) and others.

It is well known that every non-singular connected compact subset of the complex plane is Markov (see e.g. [22]). However, if E has infinitely many components, it is difficult to tell whether E has the Markov property. Stated in the beginning Theorems 1.1 and 1.2 allow us to prove Markov or Bernstein-Walsh type inequalities on some such sets.

Another application is related to a local Markov inequality defined as follows.

Definition 1.4 Let $m \geq 1$. A compact set $E \subset \mathbb{C}$ admits the *local Markov inequality at the point* $z_0 \in E$ *with exponent* m (in short $E \in LMI(z_0, m)$) if there exist $M, s > 0$ such that for every $r \in (0, 1]$, $n \in \{1, 2, \dots\}$ and $P \in \mathcal{P}_n$

$$|P'(z_0)| \leq \frac{M\, n^s}{r^m} ||P||_{E \cap D(z_0, r)}, \tag{LMI}$$

where $D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| \leq r\}$. If inequality (LMI) is verified for every $z_0 \in E$ and the constants M, m, s are independent of z_0, then we write $E \in LMI(m)$.

The above inequality turned out to be very useful in theory of function spaces (see [14]). For instance, we can identify the duals of Hardy spaces as Lipschitz spaces under the hypothesis that $LMI(1)$ holds on E [14, Chap.IV]. Sets satisfying (LMI) have been studied also by Bos and Milman (e.g. [7]), Wallin and Wingren (e.g. [25]), Lithner ([17]) and others.

One may ask about the relationship between inequalities (GMI) and (LMI). Of course, LMI implies the Markov property. Bos and Milman [7] proved that on every Markov set in $\mathbb{R}$ the local Markov inequality holds with some exponent $m \geq 1$ depending on the set. On the other hand, Goncharov and Uzun [12, Example 2] proposed an example of a Markov set $F \subset \mathbb{R}$ that does not admit $LMI(x, 1)$ at certain point $x \in F$. We give a refinement of this example. Namely, we construct a Cantor-type set with Markov's property which does not admit the local Markov inequality (with the exponent 1) at any point. This set and the example in [12] satisfy also a Hölder continuity condition of the Leja-Siciak extremal function (or equivalently, of the Green function) that is a more general property than MP.

Let E be a compact subset of $\mathbb{C}$, $z \in \mathbb{C}$ and

$$\Phi_E(z) := \sup\left\{\left(\frac{|P(z)|}{||P||_E}\right)^{1/n} : P \in \mathcal{P}_n, n \in \{1, 2, \dots\}, P_{/E} \not\equiv 0\right\}$$
$$= \lim_{n\to\infty} \sup\left\{\left(\frac{|P(z)|}{||P||_E}\right)^{1/n} : P \in \mathcal{P}_n, P_{/E} \not\equiv 0\right\}.$$

We call Φ_E the *Leja-Siciak extremal function* of the set E. This function is strictly related to the potential theory: for every non-polar compact set $E \subset \mathbb{C}$, the Green function of the unbounded component of the complement of E is equal to $\log \Phi_E$ (see e.g. [15, Th. 5.1.7]).

Definition 1.5 Let $\alpha \in (0, 1]$. A compact set $E \subset \mathbb{C}$ is said to have the *Hölder continuity property with exponent* α ($HCP(\alpha)$ in short) if there exists a $C > 0$ such that

$$\Phi_E(z) \leq 1 + C\,[\mathrm{dist}\,(z, E)]^\alpha \quad \text{if } \mathrm{dist}\,(z, E) \leq 1, \tag{HCP}$$

where $\mathrm{dist}\,(z, E) := \inf\{|z - w| : w \in E\}$.

One of the most interesting open problems is about the relation between the Hölder continuity property and the Markov property. An easy computation shows that $HCP(\alpha) \Longrightarrow MP(\frac{1}{\alpha})$ (it is sufficient to use Cauchy's integral formula) and $HCP(1) \Longleftrightarrow MP(1)$ (by Taylor's formula). However, the question about this equivalence in the general case remains an open problem.

We give one more example of applications of Theorem 1.1. Namely, for a fixed $m \geq 1$, we construct a set $E = E(m) \subset \mathbb{R}$ such that

- $E \in LMI(1) \subset LMI(m)$, but $E \cap [0, +\infty) \notin LMI(m)$ and $E \cap (-\infty, 0] \notin LMI(m)$ (Example 3.2),
- $E \in MP(m)$ but $E \cap [0, +\infty) \notin MP(m)$ and $E \cap (-\infty, 0] \notin MP(m)$ (Remark 3.3).

This example gains in interest in comparison with [10, Corollary 1.12] concerning the Hölder continuity property of subsets of $[-1, 1]$.

Let us give the last definition of this section. We will denote by $\mathrm{diam}(K)$ the diameter of a bounded set K.

Definition 1.6 A compact planar set K is said to be *uniformly perfect* if there exists a constant $B > 0$ such that

$$K \cap \{z : Br \leq |z - w| \leq r\} \neq \emptyset, \qquad w \in K, 0 < r \leq \mathrm{diam}(K).$$

The uniform perfectness of planar compact sets implies the Hölder continuity property for them ([9], the proof can be found e.g. in [17]). On the other hand, $LMI(1)$ implies the uniform perfectness. Let us also mention that uniform perfect sets in $\mathbb{R}^N$ have positive Hausdorff dimension ([25]).

2 Proofs of Theorems 1.1 and 1.2

We start with a technical lemma.

Lemma 2.1 *Let $z_0, z_1, \ldots, z_n \in \mathbb{C}$ be arbitrary pairwise distinct points. If P is a polynomial of degree n, then*

$$\frac{1}{j} P^{(j)}(z_0) = \sum_{k=1}^{n} \left[\frac{P(z_k) - P(z_0)}{z_k - z_0} \left(\frac{d^{j-1}}{dz^{j-1}} \prod_{l=1, l\neq k}^{n} \frac{z - z_l}{z_k - z_l} \right)_{/z=z_0} \right]$$

for $j \in \{1, \ldots, n\}$. In particular,

$$P'(z_0) = \sum_{k=1}^{n} \left[\frac{P(z_k) - P(z_0)}{z_k - z_0} \prod_{l=1, l\neq k}^{n} \frac{z_0 - z_l}{z_k - z_l} \right].$$

Proof Put

$$R(z) := \frac{P(z) - P(z_0)}{z - z_0}.$$

It is evident that R is a polynomial of degree $n - 1$. By Taylor's formula applied to P

$$R(z) = \sum_{j=1}^{n} \frac{1}{j!} P^{(j)}(z_0)(z - z_0)^{j-1}.$$

Thus $R^{(j-1)}(z_0) = \frac{1}{j} P^{(j)}(z_0)$, $j = 1, \ldots, n$. By the Lagrange interpolation formula,

$$R(z) = \sum_{k=1}^{n} \left[R(z_k) \prod_{l=1, l\neq k}^{n} \frac{z - z_l}{z_k - z_l} \right], \quad z \in \mathbb{C}.$$

□

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 Let $\{t_0, t_1, \dots, t_n\}$ be a Fekete n-tuple of extremal points for T, i.e.

$$|V(t_0, t_1, \dots, t_n)| = \max\{|V(x_0, x_1, \dots, x_n)| \ : \ x_0, x_1, \dots, x_n \in T\},$$

where

$$V(x_0, x_1, \dots, x_n) := \prod_{0 \le k < l \le n} (x_l - x_k)$$

(see e.g. Chap.5 in [23] for more details). In view of (1.1) the set T contains at least $n+1$ points and, consequently, $t_k \neq t_l$ for $k \neq l$. We may assume that $t_0 < t_1 < \dots < t_n$. By the definition of the Fekete n-tuple of extremal points $t_0 = 0$. Find $z_1, \dots, z_n \in K$ such that $|z_l - z_0| = t_l$ for $l = 1, \dots, n$. Since $t_k \neq t_l$ for $k \neq l$, we have $z_k \neq z_l$. Fix a polynomial $P \in \mathcal{P}_n$. Lemma 2.1 leads to

$$|P^{(j)}(z_0)| \le 2j||P||_K \sum_{k=1}^{n} \left[\frac{1}{\prod_{l=0, l\neq k}^{n} |z_k - z_l|} \left| \left(\frac{d^{j-1}}{dz^{j-1}} \prod_{l=1, l\neq k}^{n} (z - z_l) \right)_{/z=z_0} \right| \right].$$

An easy computation shows that

$$\left| \left(\frac{d^{j-1}}{dz^{j-1}} \prod_{l=1, l\neq k}^{n} (z - z_l) \right)_{/z=z_0} \right| \le S\left(\frac{(n-1)!}{(n-j)!}, n-j, |z_0 - z_l| \right),$$

where $S\left(\frac{(n-1)!}{(n-j)!}, n-j, |z_0 - z_l|\right)$ is a sum of $\frac{(n-1)!}{(n-j)!}$ components that are products of $n-j$ factors of the form $|z_0 - z_l|$. Since $|z_k - z_l| \ge |t_k - t_l|$ for each k and l, we have

$$|P^{(j)}(z_0)| \le 2||P||_K \sum_{k=1}^{n} \left[\frac{j}{\prod_{l=0, l\neq k}^{n} |t_k - t_l|} \, S\left(\frac{(n-1)!}{(n-j)!}, n-j, t_l \right) \right].$$

Put

$$L_k(t) := \frac{V(t_0, \dots, t_{k-1}, t, t_{k+1}, \dots, t_n)}{V(t_0, \dots, t_n)} = \prod_{l=0,\, l\neq k}^{n} \frac{t - t_l}{t_k - t_l}$$

for $k = 0, 1, \dots, n$. It is clear that $||L_k||_T = 1$ and therefore $|L_k^{(j)}(0)| \le C_{j,n}$ by the assumptions. It suffices to prove that

$$|L_k^{(j)}(0)| = \frac{j}{\prod_{l=0,\, l\neq k}^{n} |t_k - t_l|} S\left(\frac{(n-1)!}{(n-j)!}, n-j, t_l \right) \tag{2.1}$$

for $k = 1, \dots, n$. But it is an immediate consequence of Lemma 2.1 for the points $t_0 = 0, t_1, \dots, t_n$. □

We proceed now to the proof of Theorem 1.2.

Proof of Theorem 1.2 Let $\{t_0, t_1, \dots, t_n\}$ be a Fekete n-tuple of extremal points for T. Find $z_0, z_1, \dots, z_n \in K$ such that $|z_l - w_0| = t_l$ for $l = 0, 1, \dots, n$. Since $t_k \neq t_l$ for $k \neq l$, we have $z_k \neq z_l$. Fix a polynomial $P \in \mathcal{P}_n$. By the Lagrange interpolation formula

$$|P(w_0)| \le ||P||_K \sum_{k=0}^{n} \prod_{l=0, l\neq k}^{n} \frac{|w_0 - z_l|}{|z_k - z_l|} \le ||P||_K \sum_{k=0}^{n} \prod_{l=0, l\neq k}^{n} \frac{t_l}{|t_k - t_l|},$$

since $|z_k - z_l| \geq |t_k - t_l|$, $k, l \in \{0, 1, \ldots, n\}$. As in the proof of Theorem 1.1

$$|P(w_0)| \leq ||P||_K \sum_{k=0}^{n} |L_k(0)| \leq (n+1)C_n ||P||_K.$$

□

From Theorem 1.2 one can deduce the well-known result.

Corollary 2.2 *Every non-singular connected compact subset of the complex plane has the Hölder continuity property with the exponent* $1/2$*.*

Proof Let K be a non-singular connected compact set in the complex plane and $T(z)$, $z \in K$, be defined as in Theorem 1.1.

Find $r > 0$ such that $\operatorname{diam}(T(z)) \geq r$ for every $z \in K$. There exist $r_0 > 0$ and an open neighborhood U of K such that $\operatorname{diam}(T(z)) \geq r_0$ for every $z \in U$. Fix $z \in U \setminus K$. Put $d := \operatorname{dist}(z, K) = \operatorname{dist}(0, T(z))$. Since every interval has the Hölder continuity property with exponent $1/2$ (this is an easy consequence of the classical Chebyshev inequality and some properties of Chebyshev polynomials), it follows for d small enough that

$$|Q(0)| \leq (1 + C\sqrt{d}\,)^n \, ||Q||_{[d, d+r_0]}$$

where C is a positive constant independent of d and of the polynomial $Q \in \mathcal{P}_n$. The set $T(z)$ is a segment containing $[d, d + r_0]$, therefore

$$|Q(0)| \leq (1 + C\sqrt{d}\,)^n \, ||Q||_{T(z)}.$$

Theorem 1.2 yields now

$$|P(z)| \leq (n + 1)\,(1 + C\sqrt{d}\,)^n \, ||P||_K$$

for any polynomial $P \in \mathcal{P}_n$. Combining the above inequality with the definition of the Leja-Siciak extremal function we obtain HCP(1/2). □

3 Examples

Example 3.1 Let F denote the classical Cantor ternary set in $[0, 1]$ and $(\alpha_k)_{k \in \{0,1,\ldots\}}$ be a sequence of real numbers. The set

$$E := \{0\} \cup \bigcup_{k=0}^{\infty} \left\{ te^{i\,\alpha_k} \;:\; t \in \left[\frac{2}{3^{k+1}}, \frac{1}{3^k}\right] \cap F \right\} \subset \mathbb{C}$$

satisfies the Markov property.

Proof The set $F \subset \mathbb{R}$ has the Markov property (see [2]). Thus there exist $M, m > 0$ such that for every $z \in F$, $n \in \{1, 2, \ldots\}$, $j \in \{1, \ldots, n\}$ and $Q \in \mathcal{P}_n$

$$|Q^{(j)}(z)| \leq M^j n^{mj} ||Q||_F. \tag{3.1}$$

Fix $n \in \{1, 2, \ldots\}$. Observe that $T(0) = F$. Since inequality (3.1) holds for $z = 0$ in particular, it follows by Theorem 1.1 that for all $j \in \{1, \ldots, n\}$ and $P \in \mathcal{P}_n$

$$|P^{(j)}(0)| \leq 2\, M^j\, n^{mj+1} ||P||_E.$$

Fix $z \in E \setminus \{0\}$ and $P \in \mathcal{P}_n$. By Taylor's formula,

$$|P'(z)| \leq \sum_{j=0}^{n-1} \frac{1}{j!} |P^{(j+1)}(0)| |z|^j \leq 2Mn^{m+1} \sum_{j=0}^{n-1} \frac{1}{j!} M^j n^{mj} |z|^j ||P||_E.$$

For $|z| \leq n^{-m}$ we have

$$|P'(z)| \leq 2 M e^M n^{m+1} ||P||_E.$$

It remains to deal with the case $|z| > n^{-m}$. Choose $k_0 \in \{1, 2, \dots\}$ such that $3^{-k_0} \leq n^{-m} < 3^{-k_0+1}$. Of course, the point z belongs to an interval $[2 \cdot 3^{-k-1}, 3^{-k}]$ of length at least 3^{-k_0}. Since the Cantor ternary set is self-similar, inequality (3.1) yields

$$|P'(z)| \leq M\, 3^{k_0} n^m ||P||_E.$$

Consequently,

$$|P'(z)| \leq 3M n^{2m} ||P||_E$$

and E is a Markov set. □

The classical ternary Cantor set is not the only one which can be used in the construction proposed in Example 3.1. One can exchange it with some sets given e.g. in [3,12,20,24] thus obtaining different Markov sets.

Observe also that if $F \subset [0, 1]$ is a Markov set such that $0 \in F$ and $E := \{0\} \cup \{te^{if(t)} : t \in F\}$, where $f : F \to \mathbb{R}$ is an arbitrary function, then inequality LMI holds with $z_0 = 0$, $r = \mathrm{diam}(E)$.

Example 3.2 Fix $m \geq 1$ and put

$$E_1 = E_1(m) := \{0\} \cup \bigcup_{k=1}^{\infty} [-b_k, -a_k],$$

$$E_2 = E_2(m) := \{0\} \cup \bigcup_{k=1}^{\infty} [b_{k+1}, a_k],$$

where $A := m^2 + \frac{1}{4}$, $B := m$, $M := m^2 + 1$, $a_k := 2^{-A \cdot M^k}$, $b_k := 2^{-B \cdot M^k}$ for $k = 1, 2, \dots$. Then

$$E_1, E_2 \notin LMI(m) \quad \text{and} \quad E := E_1 \cup E_2 \in LMI(1) \subset LMI(m).$$

Proof Note that $b_{k+1} < a_k < b_k$ for $k = 1, 2, \dots$. Hence the set E_1 (resp. E_2) is the union of one singleton ($\{0\}$) and separated intervals $[-b_k, -a_k]$ (resp. $[b_{k+1}, a_k]$) $k = 1, 2, \dots$.

We show first that $E_1 \notin LMI(m)$. It is sufficient to prove that $E_1 \notin LMI(0, m)$. For a fixed $k \in \{1, 2, \dots\}$ put $r_k := 2^{-C \cdot M^k}$ with $C := m^2 + \frac{1}{2}$. Then $b_{k+1} < r_k < a_k$. Take $P(z) = z$. If $E_1 \in LMI(0, m)$, we would have for some $M_0 > 0$

$$1 = |P'(0)| \leq \frac{M_0}{r_k^m} ||P||_{E_1 \cap D(0, r_k)} = \frac{M_0}{r_k^m} b_{k+1} = M_0 2^{(Cm - BM)M^k}.$$

Since $Cm - BM = -\frac{1}{2}m < 0$, it is a contradiction for sufficiently large k.

To prove that $E_2 \notin LMI(m)$ put $r_k := 2^{-D \cdot M^k}$, where $D := m + \frac{1}{8m}$. We have $a_k < r_k < b_k$ and we can proceed as in the case above.

It remains to show that $E \in LMI(1)$. Fix an $r \in (0, \varrho] \subset (0, 1]$ for a sufficiently small $\varrho > 0$. We begin by proving that $E \in LMI(0, 1)$. Apply Theorem 1.1 to $K = E \cap D(0, r)$. Since $T = [0, r]$, for every $P \in \mathcal{P}_n$, $j \in \{1, \ldots, n\}$

$$|P^{(j)}(0)| \le 2n \left(\frac{2n^2}{r}\right)^j \ ||P||_{E\cap D(0,r)}, \tag{3.2}$$

because by the classical Markov inequality

$$|Q^{(j)}(0)| \le \left(\frac{2n^2}{r}\right)^j \ ||Q||_{[0,r]}$$

for every $Q \in \mathcal{P}_n$, $j \in \{1, \ldots, n\}$.

Fix $z \in E\backslash\{0\}$ and $P \in \mathcal{P}_n$. By Taylor's formula and by inequality (3.2) with $\frac{r}{2}$ instead of r, we obtain

$$|P'(z)| \le \sum_{j=0}^{n-1} \frac{1}{j!}|P^{(j+1)}(0)|\,|z|^j \le \sum_{j=0}^{n-1} \frac{2n}{j!}\left(\frac{4n^2}{r}\right)^{j+1} ||P||_{E\cap D(0,\frac{r}{2})}|z|^j.$$

If $|z| \le \frac{r}{4n^2}$,

$$|P'(z)| \le \frac{8\,e\,n^3}{r}||P||_{E\cap D(z,r)}. \tag{3.3}$$

We now turn to the more complicated case $|z| > \frac{r}{4n^2}$. If $z \in E_1$, we can find the positive integer k such that $z \in [-b_k, -a_k]$. Note that $b_k - a_k = b_k(1 - 2^{-(m-\frac{1}{2})^2M^k}) \ge b_k\, c$ with $c := 1 - 2^{-(m-\frac{1}{2})^2M}$. By the classical Markov inequality

$$|P'(z)| \le \frac{2n^2}{b_k - a_k}\ ||P||_{[-b_k,-a_k]} \le \frac{2n^2}{c\, b_k}\ ||P||_{[-b_k,-a_k]}.$$

Since $\frac{r}{4n^2} < |z| \le b_k$,

$$|P'(z)| \le \frac{8n^4}{c\,r}\ ||P||_{[-b_k,-a_k]}$$

and it follows for $r \ge b_k - a_k$ that

$$|P'(z)| \le \frac{8n^4}{c\,r}\ ||P||_{E\cap D(z,r)}. \tag{3.4}$$

If $r < b_k - a_k$, then

$$|P'(z)| \le \frac{4n^2}{r}\ ||P||_K \le \frac{4n^2}{r}\ ||P||_{E\cap D(z,r)}, \tag{3.5}$$

where K denotes one of the intervals $[z, z + \frac{r}{2}]$, $[z - \frac{r}{2}, z]$ which is contained in $[-b_k, -a_k]$.

If $z \in E_2$ (and $|z| > \frac{r}{4n^2}$), we take $k \in \{1, 2, \ldots\}$ such that $z \in [b_{k+1}, a_k]$. One can easily check that

$$a_k - b_{k+1} = a_k(1 - 2^{-[(m-1)(m^2+1)+3/4]M^k}) \ge a_k d$$

where $d := 1 - 2^{-[(m-1)(m^2+1)+3/4]M}$. We can now proceed analogously to the proof of the previous case and obtain

$$|P'(z)| \le \frac{8n^4}{d\,r}\ ||P||_{E\cap D(z,r)}. \tag{3.6}$$

From (3.2), (3.3), (3.4), (3.5) and (3.6) we conclude that $E \in LMI(1)$. □

The same proof remains valid for the sets $E_1{}' := \{ z \in \mathbb{C} : z = e^{i\alpha} x, x \in E_1 \}$ and $E_2{}' := \{ z \in \mathbb{C} : z = e^{i\beta} x, x \in E_2 \}$ with fixed real numbers α, β.

Remark 3.3 Every compact set $K \subset \mathbb{C}$ admitting the local Markov inequality at each point $z_0 \in K$ is a Markov set. Bos and Milman [7] proved that all Markov sets in $\mathbb{R}$ admit LMI. An inspection of the proof given in [8] shows that if $F \subset \mathbb{R}$ has the Markov property with exponent $m \geq 1$ then $F \in LMI(\mu)$ for any

$$\mu > m + 5. \tag{3.7}$$

From the above it follows that for fixed $m \geq 4$ and $\mu > m+5$, the sets $E_1 = E_1(\mu)$, $E_2 = E_2(\mu)$ constructed in Example 3.2 do not admit $LMI(\mu)$, and so $E_1, E_2 \notin MP(m)$. On the other hand, $E_1 \cup E_2 \in LMI(1)$ and the constant s in inequality (LMI) equals 4 as is shown in the proof of Example 3.2. Consequently, $E_1 \cup E_2 \in MP(4) \subset MP(m)$.

Finally, it is worth noting that estimate (3.7) can be improved, namely $\mu > m + 1$ (see [6]).

Let $(l_k)_{k=0,1,2,\ldots}$ be a sequence of positive numbers such that for every k

$$l_k < \frac{1}{2} l_{k-1} \quad \text{and } l_0 = 1.$$

Set $I_{0,1}$ =[0,1]. Let $\{F_k\}_{k=0,1,2,\ldots}$ be a family of subsets of [0,1] such that every F_k is the union of 2^k disjoint intervals $I_{k,1}, \ldots, I_{k,2^k}$ of length l_k each and F_{k+1} is obtained by deleting the open concentric subinterval of length $l_k - 2l_{k+1}$ from each interval $I_{k,n}$, $n = 1, \ldots, 2^k$. The set

$$E = \bigcap_{k=0}^{\infty} \bigcup_{n=1}^{2^k} I_{k,n}$$

is called a *Cantor-type set* associated with $(l_k)_{k=0,1,2,\ldots}$.

Example 3.4 Let E be a Cantor-type set associated with $(l_k)_{k=0,1,2,\ldots}$ defined by

$$l_0 := 1, \quad l_k := \begin{cases} \frac{1}{2+\sqrt{k}}\, l_{k-1} : & \sqrt{k} \in \mathbb{Z}, \\ \frac{1}{3}\, l_{k-1} \quad : & \sqrt{k} \notin \mathbb{Z} \end{cases} \quad \text{for} \quad k = 1, 2, \ldots .$$

Then E has the Hölder continuity property and so E is a Markov set but $E \notin LMI(x, 1)$ for any $x \in E$. In particular, E is not uniformly perfect.

Proof We first prove that E satisfies (HCP). In view of [24], it suffices to show that

$$\limsup_{k \to \infty} \frac{\log l_k^{-1}}{k} < \infty. \tag{3.8}$$

To this end fix a $k \in \{1, 2, \ldots\}$ and take a positive integer n such that $n^2 \leq k < (n+1)^2$. Then

$$l_k = \frac{2}{3^{k-n}\,(n+2)!}.$$

Consequently,

$$l_k \geq \frac{2}{3^k} \cdot \frac{3^n}{e(\frac{n+2}{2})^{n+2}} \geq \frac{2}{3^k} \cdot \frac{3^{\sqrt{k}-1}}{e(\frac{\sqrt{k}+2}{2})^{\sqrt{k}+2}}.$$

From this we get (3.8).

Our next goal is to show that $E \notin LMI(x, 1)$ for any point $x \in E$. Fix an $x \in E$ and a $k \geq 2$ such that $\sqrt{k} \in \mathbb{Z}$. Take an $n \in \{1, \ldots, 2^k\}$ with $x \in I_{k,n}$. Clearly, $l_k < \frac{1}{3} l_{k-1}$. Put $r_k := l_{k-1} - 2l_k - \varepsilon$ with an arbitrary $\varepsilon \in (0, l_{k-1} - 3l_k)$. Since $r_k > l_k$, it follows that $I_{k,n} \subset D(x, r_k)$. Moreover,

$$\operatorname{dist}(x, E \setminus I_{k,n}) \geq l_{k-1} - 2l_k > r_k.$$

Take $P(z) = z - x$. If E admitted $LMI(x, 1)$, we would have

$$1 = |P'(x)| \leq \frac{M}{r_k} ||P||_{E \cap D(x, r_k)} \leq \frac{M}{r_k} l_k = M \frac{l_k}{l_{k-1} - 2l_k - \varepsilon}.$$

Hence, if $\varepsilon \to 0$, we obtain

$$1 \leq M \frac{l_k}{l_{k-1} - 2l_k} < M \frac{l_k}{l_{k-1} - \frac{2}{3} l_{k-1}} = 3M \frac{1}{2 + \sqrt{k}},$$

where the right hand side tends to zero as $k \to \infty$, which is impossible. □

4 Attractors of matrices of affine mappings

We will now propose a construction which generalizes the one of Cantor-type sets mentioned earlier.

Fix a function $\varrho : \mathbb{N} \to \mathbb{N}$ and define $\mathbb{N}_\varrho := \{(n, j) \in \mathbb{N}^2 : j \leq \varrho(n)\}$. Let $\Gamma = [\Gamma_{n,j}]_{(n,j) \in \mathbb{N}_\varrho}$ be a matrix of affine contractions, namely

$$\Gamma_{n,j} : \mathbb{C} \ni z \mapsto a_{n,j} z + b_{n,j} \in \mathbb{C},$$

where $|a_{n,j}| < 1$, $(n, j) \in \mathbb{N}_\varrho$. Assume that

$$a := \sup_{(n,j) \in \mathbb{N}_\varrho} |a_{n,j}| < 1 \quad \text{and} \quad b := \sup_{(n,j) \in \mathbb{N}_\varrho} |b_{n,j}| < \infty. \tag{4.1}$$

If K is a non-empty planar compact set, define

$$\Psi_n(K) := \bigcup_{j \leq \varrho(n)} \Gamma_{n,j}(K), \quad n \in \mathbb{N}.$$

It was shown in [16] that the sequence $((\Psi_1 \circ \cdots \circ \Psi_n)(K))_{n \in \mathbb{N}}$ of compact sets converges (in the Hausdorff metric) to a compact set E independent of the choice of the set K. This limit is called the *attractor* of the matrix Γ. By [16, Lemma 4.3] and the remark below it, we have

$$E = \bigcap_{n \in \mathbb{N}} (\Psi_1 \circ \cdots \circ \Psi_n)(D(0, R)), \quad \text{where} \quad R := \frac{b}{1 - a}. \tag{4.2}$$

One can also use a greater radius in the formula (4.2). It was shown in the last section of [16] that Cantor-type sets defined in the end of the previous section are attractors of such matrices as mentioned earlier. On the other hand, such a construction is a generalization of iterated function systems (see e.g. [1,13]).

The aim of our paper is to construct planar Markov sets, which are in particular not polar (see [4]). However, the attractor constructed above can be polar, moreover, it can be even finite. Namely, if for a matrix Γ there exist $k \in \mathbb{N}$ and $c \in \mathbb{C}$ such that

$$\forall n > k \ \forall j \leq \varrho(n) : \ \Gamma_{n,j}(c) = c, \tag{4.3}$$

then its attractor $E = \{(\Gamma_{1,j_1} \circ \cdots \circ \Gamma_{k,j_k})(c) : j_1 \leq \varrho(1), \ldots, j_k \leq \varrho(k)\}$ is a finite set (see [16]). One can also obtain a finite attractor if the matrix Γ has for example all entries in one row constant. Attractors which are Cantor-type sets provide some examples of polar attractors too, for instance if $l_n = 2^{-2^n}$ (see [19]). To avoid such situation, we will assume that

$$\delta := \inf\{|a_{n,j}| : (n, j) \in \mathbb{N}_\varrho\} > 0. \tag{4.4}$$

This assumption implies in particular invertibility of all entries of the matrix Γ. The examples of the Cantor-type sets (see [16]) show, however, that this condition is not necessary for non-polarity. Namely, if for instance $l_n = 1/(n+2)!$, the Cantor-type set is regular, hence not polar (see [19]). Note that in this case $\delta = \inf_{n\in\mathbb{N}}(n+2)^{-1} = 0$. However, the obtained set is not Markov, and we will show that some matrices satisfying i.a. the condition $\delta > 0$ have Markov sets as attractors.

We will need another assumption. Let

$$d_n := \min\left(1; \min_{j\neq k} |b_{n,j} - b_{n,k}| - 2aR\right) > 0, \qquad n \in \mathbb{N}. \tag{4.5}$$

This condition guarantees that no two mappings in one row of the matrix Γ have the same fixed point. Namely, if we have two affine invertible contractions $h(z) = \alpha z + \beta$ and $g(z) = \eta z + \varepsilon$ satisfying $|\beta - \varepsilon| > 2\max(|\alpha|, |\eta|)R$, where $R = \frac{\max(|\beta|,|\varepsilon|)}{1-\max(|\alpha|,|\eta|)}$, then $|\beta - \varepsilon| > |\alpha\varepsilon| + |\beta\eta| \geq |\beta\eta - \alpha\varepsilon|$. While if h and g had the same fixed point, then we would have equality $\beta - \varepsilon = \beta\eta - \alpha\varepsilon$. Hence a matrix satisfying (4.5) cannot fulfill the condition (4.3).

For our convenience, we will denote

$$\delta_n := \min\{|a_{n,j}| : j \leq \varrho(n)\}, \quad n \in \mathbb{N}. \tag{4.6}$$

Note that these numbers are positive in view of (4.4). We have also $\delta = \inf \delta_n$.

Finally, we define

$$U(\Gamma) := \prod_{n\geq 1} d_n^{\frac{\varrho(n)-1}{\varrho(1)\ldots\varrho(n)}} \qquad \text{and} \qquad V(\Gamma) := \prod_{n\geq 1} \delta_n^{\frac{1}{\varrho(1)\ldots\varrho(n)}}. \tag{4.7}$$

Note that if

$$d := \inf\{d_n : n \in \mathbb{N}\} > 0, \tag{4.8}$$

then

$$U(\Gamma) \geq d > 0, \tag{4.9}$$

since $\sum_{j=1}^{n} \frac{\varrho(n)-1}{\varrho(1)\ldots\varrho(j-1)} = 1$. On the other hand, if $s := \sum_{n\geq 1} \frac{1}{\varrho(1)\ldots\varrho(n)}$ is finite, then

$$V(\Gamma) \geq \delta^s > 0. \tag{4.10}$$

To this end, we can assume for example that $\varrho \geq 2$. Namely, in that case

$$s = \sum_{n\geq 1} \frac{1}{\varrho(1)\ldots\varrho(n)} \leq \sum_{n\geq 1} \frac{1}{2^n} = 1.$$

We could also assume less, but we have to be careful not to make the set $\varrho^{-1}(\{1\})$ too large. In particular, if the sequence $(\varrho(1)\ldots\varrho(n))_{n\in\mathbb{N}}$ does not tend to infinity, but is bounded by an integer L, then it is easy to check that the obtained attractor has at most L points.

If $\sigma \in \Sigma_\varrho := \{\sigma \in \mathbb{N}^{\mathbb{N}} : \sigma \le \varrho\}$ and $n \in \mathbb{N}$, the set

$$(\Gamma_{1,\sigma(1)} \circ \cdots \circ \Gamma_{n,\sigma(n)})(D(0, R))$$

is the disc with the center

$$a_{1,\sigma(1)}a_{2,\sigma(2)} \ldots a_{n-1,\sigma(n-1)}b_{n,\sigma(n)} + \cdots + a_{1,\sigma(1)}b_{2,\sigma(2)} + b_{1,\sigma(1)}$$

and the radius $|a_{1,\sigma(1)}a_{2,\sigma(2)} \ldots a_{n-1,\sigma(n-1)}a_{n,\sigma(n)}|R$.

The set $\Psi_1(D(0, R))$ is the union of $\varrho(1)$ disjoint discs contained in $D(0, R)$ and the distance between two of such discs, say $\Gamma_{1,j}(D(0, R))$ and $\Gamma_{1,l}(D(0, R))$ with $j \ne l$, equals

$$|b_{1,j} - b_{1,l}| - |a_{1,j}|R - |a_{1,l}|R \ge d_1 > 0.$$

Note that in step two, we obtain $\varrho(2)$ disks in each of the ones from $\Psi_1(D(0, R))$, and so on.

We need to be able to skip finite number of first rows from the matrix Γ. For each $N \in \mathbb{N}$, we define $\varrho_N : \mathbb{N} \ni n \mapsto \varrho(n + N) \in \mathbb{N}$ and $\Gamma^{+N} = [\Gamma_{n+N,j}]_{n\in\mathbb{N}, j\le\varrho_N(n)}$. Note that the new matrix Γ^{+N} satisfies the same assumptions as Γ, for any $N \in \mathbb{N}$. Moreover,

$$U(\Gamma^{+N}) = \left(U(\Gamma)\prod_{n\le N} d_n^{-\frac{\varrho(n)-1}{\varrho(1)\ldots\varrho(n)}}\right)^{\varrho(1)\ldots\varrho(N)} \ge U(\Gamma) \tag{4.11}$$

$$V(\Gamma^{+N}) = \left(V(\Gamma)\prod_{n\le N} \delta_n^{-\frac{1}{\varrho(1)\ldots\varrho(n)}}\right)^{\varrho(1)\ldots\varrho(N)} \ge V(\Gamma). \tag{4.12}$$

For the convenience assume now that $|a_{n,1}| = \delta_n, n \in \mathbb{N}$ (if it is not so, renumber the entries of each row). Fix for the time being $n \in \mathbb{N}$ and denote by $B(1\ldots n) = (\Psi_1 \circ \cdots \circ \Psi_n)(D(0, R))$. The set $B(1\ldots n)$ is the union of $\varrho(1)$ sets

$$B_j = B(1\ldots n) \cap \Gamma_{1,j}(D(0, R)),\ j \le \varrho(1),$$

and every set B_j is the image of B_1 (the smallest of them) under the affine mapping

$$z \mapsto \frac{a_{1,j}}{a_{1,1}}z + b_{1,j} - \frac{a_{1,j}}{a_{1,1}}b_{1,1}.$$

On the other hand, the distance between B_j and B_k is not smaller than d_1 if $j \ne k$.

The set B_1 is actually the image of the set

$$B(2\ldots n) = (\Psi_2 \circ \cdots \circ \Psi_n)(D(0, R))$$

under the mapping $\Gamma_{1,1}$. Now we deal with the set $B(2\ldots n)$ in the same way as with the set $B(1\ldots n)$ and so on.

5 Special cases and some subcases

First, it is easy to check that a big family of Cantor-type sets (contained in an interval) fulfill our assumptions.

Second, fix an $r > 0$ and $\varrho : \mathbb{N} \to \mathbb{N}$. Let $\Gamma = [\Gamma_{n,j}]_{(n,j)\in\mathbb{N}_\varrho}$ with entries $\Gamma_{n,j}(z) = a_{n,j}z + b_{n,j}$, $(n, j) \in \mathbb{N}_\varrho$, satisfy assumptions about a, b and δ and be such that $b \le r(1 - a)$. Then Γ satisfies our assumptions and $r \ge R$.

All conditions stated here are fulfilled by a matrix Γ with all rows being the same, consisting of at least two affine invertible contractions of $\mathbb{C}$ with values at zero, each two of which are distanced as required. For instance we can take family $\mathcal{F}_1 = \{g_1, h_1\}$ with $g_1(z) = 0.25z + 0.75$, $h_1(z) = -0.25z + 0.75i$ (with $R_1 = 1$), but family $\mathcal{F}_2 = \{g_2, h_2\}$ with $g_2(z) = 0.5z + 0.5$, $h_2(z) = -0.25z + 1$ is not allowed (here $R_2 = 2$ and (4.5) is not satisfied). The attractors of such matrices with constant rows are actually the attractors of iterated function systems and their potential properties were thoroughly studied in [1].

We will present now another family of examples associated with so called family of Cantor-type sets studied by Lithner in [17].

Let $q \in (0, 1/3]$ be a fixed constant. Consider the family $\mathcal{T}$ of all matrices $\Gamma = [\Gamma_{n,j}]_{n\in\mathbb{N}, j\in\{1,2\}}$ with $\Gamma_{n,j}(z) = a_{n,j}z + b_{n,j}$ such that

$$\forall (n, j) \in \mathbb{N} \times \{1, 2\} : \quad |a_{n,j}| = q, \quad |b_{n,j}| \le \frac{1}{2} - q, \quad |b_{n,1} - b_{n,2}| \ge 2q.$$

If $\Gamma \in \mathcal{T}$, then $a = q$ and $b \le \frac{1}{2} - q$, hence $a + b \le 1/2$ and $R \le 1/2$.

If $\Gamma \in \mathcal{T}$ and $D = D(0, 1/2)$,

$$(\Psi_1 \circ \cdots \circ \Psi_n)(D) = \bigcup_{k=1}^{2^n} B_{n,k} \subset D,$$

where $B_{n,k} = (\Gamma_{1,j_1} \circ \cdots \circ \Gamma_{n,j_n})(D)$ is a disk with the radius $q^n/2$ and the center

$$a_{1,j_1} \ldots a_{n-1,j_{n-1}} b_{n,j_n} + \cdots + a_{1,j_1} b_{2,j_2} + b_{1,j_1}.$$

Furthermore,

$$\begin{aligned} &\operatorname{dist}\big((\Gamma_{1,j_1} \circ \cdots \circ \Gamma_{n-1,j_{n-1}} \circ \Gamma_{n,1})(D); (\Gamma_{1,j_1} \circ \cdots \circ \Gamma_{n-1,j_{n-1}} \circ \Gamma_{n,2})(D)\big) \\ &\quad = |a_{1,j_1} \ldots a_{n-1,j_{n-1}} (b_{n,1} - b_{n,2})| - q^n \ge q^n. \end{aligned}$$

Therefore by [16, Lemma 4.3], the family of all attractors of matrices from $\mathcal{T}$ coincides with $\mathcal{E}$, the special family of Cantor-type sets defined in [17].

In the end of this section, let us observe that our setting leads to examples, which are not included in any of the special subcases: e.g. we can change the number of mappings at each step.

6 Uniform perfectness of the attractors

We will now deal with the logarythmic capacity of (parts of) attractors from our setting. If E is a compact set, by $c(E)$ we denote its logarythmic capacity. For the definition and properties, we refer the reader to [23].

Theorem 6.1 *Let $\varrho : \mathbb{N} \to \mathbb{N}$ be a function such that $\varrho(1) \ldots \varrho(n) \longrightarrow \infty$. Let $\Gamma = [\Gamma_{n,j}]_{(n,j)\in\mathbb{N}_\varrho}$ with*

$$\Gamma_{n,j} : \mathbb{C} \ni z \mapsto a_{n,j} z + b_{n,j} \in \mathbb{C}, \quad (n, j) \in \mathbb{N}_\varrho,$$

satisfy (4.1), (4.4) and (4.5). *Let E be the attractor of* Γ. *Take R as in* (4.2).

Then $c(E) \ge U(\Gamma)V(\Gamma)$, where $U(\Gamma)$ and $V(\Gamma)$ are given by (4.7). *Hence, if $U(\Gamma)V(\Gamma) > 0$, the attractor is not polar.*

Proof Recall that if $h : \mathbb{C} \ni z \mapsto \alpha z + \beta \in \mathbb{C}$ and $K \subset \mathbb{C}$, then $c(h(K)) = |\alpha| c(K)$ (see e.g. [23, 5.2.5]).

We will use the setting from Sect. 4.

Fix an $n \in \mathbb{N}$. Consider $B(1\dots n)$ and its $\varrho(1)$ disjoint subsets B_j as discussed before. We have $c(B_j) \geq c(B_1)$, $j \leq \varrho(n)$, since B_1 was the smallest of the sets. Of course $c(B(1\dots n)) \geq c(B_1)$. Note that $c(B_1)$ is positive, since there is at least one closed disc contained in B_1. If $c(B(1\dots n)) \geq d_1$, then it is obvious that

$$\log c(B(1\dots n)) \geq \left(1 - \frac{1}{\varrho(1)}\right)\log d_1 + \frac{1}{\varrho(1)}\log c(B_1). \tag{6.1}$$

If $c(B(1\dots n)) < d_1$, we use [23, Theorem 5.1.4 (b)] to show that

$$\frac{1}{\log^+ \frac{d_1}{c(B(1\dots n))}} \geq \sum_{j=1}^{\varrho(1)} \frac{1}{\log^+ \frac{d_1}{c(B_j)}} \geq \frac{\varrho(1)}{\log^+ \frac{d_1}{c(B_1)}}$$

and consequently we obtain once again inequality (6.1).

Since $c(B_1) = \delta_1 c(B(2\dots n))$, it follows from (6.1) that

$$\log c(B(1\dots n)) \geq \left(1 - \frac{1}{\varrho(1)}\right)\log d_1 + \frac{1}{\varrho(1)}\log \delta_1 + \frac{1}{\varrho(1)}\log c(B(2\dots n)).$$

Now we deal with the set $B(2\dots n)$ in the same way as with the set $B(1\dots n)$ and obtain

$$\log c(B(2\dots n)) \geq \left(1 - \frac{1}{\varrho(2)}\right)\log d_2 + \frac{1}{\varrho(2)}\log \delta_2 + \frac{1}{\varrho(2)}\log c(B(3\dots n)).$$

Consequently repeating the argument gives

$$\log c((\Psi_{\Gamma_1} \circ \dots \circ \Psi_{\Gamma_n})(D(0,R)) \geq \sum_{j=1}^{n} \frac{\varrho(j)-1}{\varrho(1)\dots\varrho(j)}\log d_j +$$

$$+ \sum_{j=1}^{n} \frac{1}{\varrho(1)\dots\varrho(j)}\log \delta_n + \frac{1}{\varrho(1)\dots\varrho(n)}\log c(D(0,R)).$$

When $n \to \infty$, it yields

$$\log c(E) \geq \sum_{j=1}^{\infty} \frac{\varrho(j)-1}{\varrho(1)\dots\varrho(j)}\log d_j + \sum_{j=1}^{\infty} \frac{1}{\varrho(1)\dots\varrho(j)}\log \delta_n,$$

as required. □

Using [23, Theorem 5.1.4 (a)] we can obtain in a similar way also an upper bound for the capacity, but we will not go into the details.

Corollary 6.2 *Let $\varrho : \mathbb{N} \to \mathbb{N}$ be a function. Let $\Gamma = [\Gamma_{n,j}]_{(n,j)\in\mathbb{N}_\varrho}$ with*

$$\Gamma_{n,j} : \mathbb{C} \ni z \mapsto a_{n,j}z + b_{n,j} \in \mathbb{C}, \quad (n,j) \in \mathbb{N}_\varrho,$$

satisfy (4.1). *Take R as in* (4.2)*, assume that conditions* (4.4) *and* (4.8) *hold. Let E be the attractor of Γ. Assume that $s := \sum_{n\geq 1} \frac{1}{\varrho(1)\dots\varrho(n)} < \infty$.*

Then $c(E) \geq d\delta^s > 0$.

Proof It follows from Theorem 6.1 combined with (4.9) and (4.10). □

Corollary 6.3 *Let $\varrho : \mathbb{N} \to \mathbb{N}$ be a function. Let $\Gamma = [\Gamma_{n,j}]_{(n,j)\in\mathbb{N}_\varrho}$ with*

$$\Gamma_{n,j} : \mathbb{C} \ni z \mapsto a_{n,j} z + b_{n,j} \in \mathbb{C}, \quad (n, j) \in \mathbb{N}_\varrho,$$

satisfy (4.1)*. Take R as in* (4.2)*, assume that conditions* (4.4) *and* (4.5) *hold and that the number $U(\Gamma)$ defined by* (4.7) *is positive. Let E be the attractor of Γ. Finally assume that $s := \sum_{n\geq 1} \frac{1}{\varrho(1)\ldots\varrho(n)}$ if finite.*

Then there exists a positive constant C depending only on Γ such that

$$\forall w \in E\, \forall r \in (0, \operatorname{diam}(E)] : \ c(E \cap D(w, r)) > Cr.$$

In particular the set E is uniformly perfect.

Proof Pick $w \in E$. By the choice of R

$$w \in \bigcap_{n\in\mathbb{N}} (\Psi_1 \circ \cdots \circ \Psi_n)\,(D(0, R)).$$

Thus, there exists a $\sigma \in \Sigma_\varrho$ such that $w \in (\Gamma_{1,\sigma(1)} \circ \cdots \circ \Gamma_{N,\sigma(N)})(D(0, R))$ for any $N \in \mathbb{N}$. The set $(\Gamma_{1,\sigma(1)} \circ \cdots \circ \Gamma_{N,\sigma(N)})(D(0, R))$ is a ball of radius $|a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}|R$. Because of the assumption on distance between the values of zeroes

$$E \cap (\Gamma_{1,\sigma(1)} \circ \cdots \circ \Gamma_{N,\sigma(N)})(D(0, R)) = (\Gamma_{1,\sigma(1)} \circ \cdots \circ \Gamma_{N,\sigma(N)})(E_N),$$

where E_N denotes the attractor of the matrix Γ^{+N}. Furthermore,

$$(\Gamma_{1,\sigma(1)} \circ \cdots \circ \Gamma_{N,\sigma(N)})(D(0, R)) \subset D(w, 2|a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}|R).$$

Hence combining the equality above with Theorem 6.1 and estimates (4.10), (4.11) and (4.12) gives

$$c(E \cap D(w, 2|a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}|R) \geq |a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}|U(\Gamma)\delta^s.$$

Now take a positive number $r \leq \operatorname{diam}(E)$. Then there exists $N \in \mathbb{N}$ with

$$2|a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}|R < r \leq 2|a_{1,\sigma(1)} \ldots a_{N-1,\sigma(N-1)}|R.$$

We conclude that

$$c(E \cap D(w, r)) \geq |a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}|U(\Gamma)\delta^s \geq \frac{r}{2R}|a_{N,\sigma(N)}|U(\Gamma)\delta^s \geq Cr,$$

where the constant $C = U(\Gamma)\delta^{s+1}(2R)^{-1}$ depends neither on N nor on r.

Finally, we conclude that the set E is uniformly perfect, since

$$E \cap \left\{z \in \mathbb{C} : \frac{C}{2} r \leq |z - w| \leq r\right\} \neq \emptyset.$$

□

Remark 6.4 If we take now a matrix Γ satisfying the assumptions from the last corollary and for which the attractor is contained in a line, we can construct, as in Example 3.1, another examples of totally disconnected Markov set that are not uniformly perfect.

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