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An example of chaotic behaviour in presence of a sliding homoclinic orbit

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Abstract We present an example on the chaotic behaviour of a 3-dimensional quasiperiodically perturbed discontinuous differential equation whose unperturbed part has a homoclinic orbit that is a solution homoclinic to a hyperbolic fixed point with a part belonging to a discontinuity plane. Melnikov type analysis is applied.

Keywords Bernoulli shift · Chaotic behaviour · Discontinuous systems

Mathematics Subject Classification (2000) 34C23 · 34C37 · 37G20

1 Introduction

The purpose of this paper is twofold. First, we construct an example of a 3-dimensional discontinuous differential equation (3DDE) with a sliding homoclinic solution, that is a homoclinic orbit a part of which lies on the discontinuity plane. The sliding homoclinic orbit tends asymptotically to a hyperbolic equilibrium. Then, we take quasiperiodic perturbations of this 3DDE and using analytical methods based on a Melnikov type analysis from [7], we show chaos for the perturbed 3DDE.

Bifurcation from homoclinic orbits in perturbed smooth differential equations is well developed [8,23,24,42]. Recently, attempts have been made to extend the theory of chaos

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to differential equations with discontinuous right-hand sides: planar DDE are investigated in [1,26–28,30], piecewise linear 3DDE are studied in [5,35] and weakly discontinuous systems are considered in [3,17,18]. Melnikov type analysis is also presented for DDE in [11,14,25,33,44]. An overview of some aspects of chaotic dynamics in hybrid systems is given in [12]. A survey of controlling chaotic differential equations is presented in [19]. The switchability of flows of general DDE is discussed in [36]. Note DDE appear in many important situations such as in mechanical systems with dry frictions or with impacts or in control theory, electronics, economics, medicine and biology (see [10,13,16,21,28,30–32] for more references).

We started our analysis of homoclinic bifurcation for DDE in [5], where we assume the homoclinic orbit crosses transversally discontinuity levels. Then in [6], we proved that, in such DDEs, a Melnikov type condition implies in fact a Smale horseshoe-type chaos after a time perturbation with some recurrence properties. After this, it was natural to study the homoclinic sliding bifurcation (see [7]). Typical examples of sliding motions are in a relay controller and a stick-slip friction system. The stick motion corresponds to sliding. Many nonsmooth models can be found in [4,9,20,27–29,34,38–40]. Sliding homoclinic solutions to pseudo-saddles (saddles lying on discontinuity curves/lines) of planar DDE are studied in [22,30] both numerically and analytically. A theoretical discussion on sliding homoclinic solutions to saddles of planar DDE is presented in [30]. Also in [1], there is a sliding periodic motion for an oscillator with dry friction, while a planar sliding homoclinic is studied in [2]. So it is natural to expect homoclinic sliding in certain mechanical oscillators. Although, in principle, examples of sliding homoclinic orbits that satisfy the conditions of the main results of [7] can be constructed, we have not found in literature any which is analytically solved, except in [2] where an example is given with two discontinuity lines. Maybe the problem is that in the open set the hyperbolic equilibrium belongs to, the DDE must be nonlinear. So any example should be highly nonlinear, and this fact makes computations harder. We think that this paper is the first with an example of a system of DDE with homoclinic sliding structure with a rigorous analytical proof of chaotic behaviour.

The plan of our paper is as follows. In Sect. 2, we recall the problem, and basic assumptions along with main results of [7] for application. Then in Sect. 3, we present our main results. We start from a two-dimensional system studied in [15], and we modify it in such a way that it has a sliding homoclinic solution. We decided to start with this system since it is quite easy, in spite of its nonlinear character, to describe stable and unstable manifolds of the fixed point. Then, we add a third equation and a discontinuity level in such a way that the (unperturbed) system satisfies the condition of the main result in [7]. Finally, we add a quasiperiodic perturbation, consisting of two periodic functions the ratio of whose periods is irrational, and apply Theorem 1 to obtain chaotic behaviour. In principle, our result gives chaos only for values of the parameters close to zero; however, a careful analysis of the Melnikov function shows that chaotic behaviour exists also when the frequency of one of the two periodic functions becomes larger, at least in a simple case.

2 Preliminary results

In this section, we recall basic results of [7] that we apply to a concrete example discussed in next Sect. 3. Let $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ with corresponding projections $P_z : \mathbb{R}^n \to \mathbb{R}$ and $P_y : \mathbb{R}^n \to \mathbb{R}^{n-1}$. For $x \in \mathbb{R}^n$, we write $x = (z, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$.

Consider a discontinuous system in $\mathbb{R}^n$ with a small parameter such as:

$$\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon)\,, \tag{1}$$

where

$$f(x) = \begin{cases} f_+(z, y) & \text{for } z > 0, \\ f_-(z, y) & \text{for } z < 0, \end{cases}$$

with $f_\pm : \Omega \to \mathbb{R}^n$, $f_\pm \in C_b^r(\Omega)$ and $g : \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}^n$, $g \in C_b^r(\mathbb{R} \times \Omega \times \mathbb{R})$, Ω being a bounded open subset of $\mathbb{R}^n$ that has nonempty intersection with the hyperplane $z = 0$. Note we allow the possibility that $f_+(0, y) \neq f_-(0, y)$. We also assume that the r-th order derivatives of $f_\pm(x)$ and $g(t, x, \varepsilon)$ are uniformly continuous. We set

$$\Omega_\pm = \{x = (z, y) \in \Omega \mid \pm z > 0\}, \quad \Omega_0 = \{x = (z, y) \in \Omega \mid z = 0\}.$$

By putting

$$f_\pm = (h_\pm(z, y), k_\pm(z, y)),$$

where $h_\pm : \Omega \to \mathbb{R}$ and $k_\pm : \Omega \to \mathbb{R}^{n-1}$, we assume that

(H1) For any $(0, y) \in \Omega_0$ it results:

$$h_-(0, y) - h_+(0, y) > 0.$$

Then, we set (see [32, Eq. (2.12)]))

$$H(y) := V(y)\frac{k_+(0, y) - k_-(0, y)}{2} + \frac{k_+(0, y) + k_-(0, y)}{2}$$

where

$$V(y) = \frac{h_+(0, y) + h_-(0, y)}{h_-(0, y) - h_+(0, y)}$$

and, for $(0, y) \in \Omega_0$, we consider the equation

$$\dot{y} = H(y). \tag{2}$$

We suppose that

(H2) The unperturbed equation

$$\dot{x} = f_-(x)$$

has a hyperbolic fixed point $x_0 \in \Omega_-$ and two solutions $\gamma_\pm(t)$, defined respectively, for $t \geq \bar{T}$ and $t \leq -\bar{T}$, such that $\lim_{t\to\pm\infty} \gamma_\pm(t) = x_0$ and $\gamma_\pm(\pm\bar{T}) \in \Omega_0$.

(H3) Equation (2) has a solution $y_0(t)$ such that $\gamma_0(t) := (0, y_0(t)) \in \Omega_0$ for $-\bar{T} \leq t \leq \bar{T}$ and

$$\gamma_-(-\bar{T}) = \gamma_0(-\bar{T}), \quad \gamma_+(\bar{T}) = \gamma_0(\bar{T}).$$

Moreover the following hold:

$h_+(\gamma_0(t)) < 0$ for any $t \in [-\bar{T}, \bar{T}]$;
$h_-(\gamma_0(t)) > 0$ for any $t \in [-\bar{T}, \bar{T})$;
$h_-(\gamma_0(\bar{T})) = 0$ and $k_-(\gamma_0(\bar{T}))$ is not orthogonal to $\nabla_y h_-(\gamma_0(\bar{T})) \neq 0$. Here $\nabla_y h_-(\gamma_0(\bar{T}))$ is the gradient of $h_-(0, y)$ at the point $\gamma_0(\bar{T}) \in \Omega_0$.

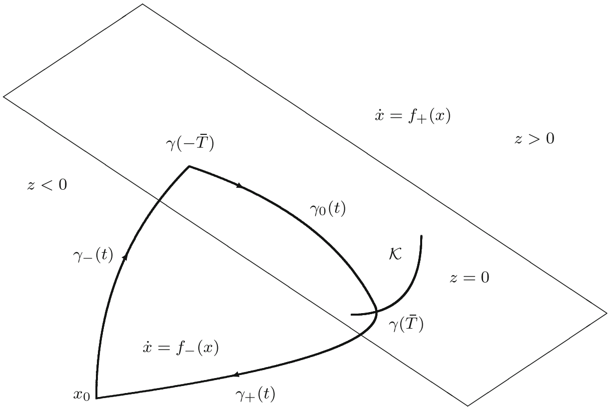

Fig. 1 A homoclinic sliding orbit $\gamma(t)$ of (1) with $\varepsilon = 0$ to the hyperbolic equilibrium $x = x_0$

Remark 1 From (H3), it follows that $h_-^{-1}(0)$ is a submanifold $\mathcal{K}$ of Ω_0 of codimension 1 near the point $\gamma_0(\bar{T})$ (here we consider the restriction $h_- : \Omega_0 \to \mathbb{R}$). Moreover, since $V(y_0(\bar{T})) = -1$ we get

$$H(y_0(\bar{T})) = k_-(\gamma_0(\bar{T})),$$

so $\dot{\gamma}_0(\bar{T}) = (0, H(y_0(\bar{T}))) = (0, k_-(\gamma_0(\bar{T}))) = f_-(\gamma_0(\bar{T}))$. Thus, condition (H3) means that $\dot{\gamma}_0(\bar{T})$ is transverse to $\mathcal{K}$ in Ω_0. Next, from (H3), it follows immediately that

$$\nabla_y h_-(0, y_0(\bar{T}))\dot{y}_0(\bar{T}) < 0.$$

Note $\nabla_y h_-(0, y_0(t))\dot{y}_0(t) = h'_-(\gamma_0(t))\dot{\gamma}_0(t)$ for $t \in [-\bar{T}, \bar{T}]$. Finally, for the validity of the results of this paper, it is enough that condition (H1) holds for y in a neigbourhood of $y_0(t)$, $-\bar{T} \le t \le \bar{T}$ see ([32, eq (2.12)]).

We set:

$$\gamma(t) = \begin{cases} \gamma_-(t) & \text{if } t \le -\bar{T} \\ \gamma_0(t) & \text{if } -\bar{T} \le t \le \bar{T} \\ \gamma_+(t) & \text{if } t \ge \bar{T} \end{cases}$$

and will refer to $\gamma(t)$ as the *sliding homoclinic* solution of (1) when $\varepsilon = 0$ (see Fig. 1).

We note that $\gamma(t)$ is C^1-smooth also at $t = \bar{T}$ (see [7]). Recalling $x = (z, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we set

$$f_\pm(x) + \varepsilon g(t, x, \varepsilon) = (h_\pm(t, z, y, \varepsilon), k_\pm(t, z, y, \varepsilon)).$$

We are interested in the chaotic dynamics of (1) near $\gamma(t)$ for $\varepsilon \neq 0$ small.

Definition 1 By a *sliding* solution $x(t)$ of (1), we mean a function $x : \mathbb{R} \to \mathbb{R}^n$ for which the following hold:

There exists an increasing sequence $\{\widetilde{T}_m\}$ (possibly finite or defined either for $m \le m_0 \in \mathbb{Z}$, or for $m \ge m_0 \in \mathbb{Z}$, with $m_0 \in \mathbb{Z}$, or for $m \in \mathbb{Z}$) such that $x(t)$ is C^1-smooth for any $t \in \mathbb{R}\backslash\{\widetilde{T}_{2m}\}$ and possesses right and left derivatives at $t = \widetilde{T}_{2m}$;

if $t \in (\widetilde{T}_{2m-1}, \widetilde{T}_{2m})$ then $x(t) \in \Omega_-$ and satisfies the equation $\dot{x} = f_-(x) + \varepsilon g(t, x, \varepsilon)$;
if $t \in (\widetilde{T}_{2m}, \widetilde{T}_{2m+1})$ then $x(t) = (0, y(t)) \in \Omega_0$ and $y(t)$ satisfies the equation $\dot{y} = H(t, y, \varepsilon)$;
at $t = \widetilde{T}_{2m+1}$, the equation $h_-(\widetilde{T}_{2m+1}, 0, y(\widetilde{T}_{2m+1}), \varepsilon) = 0$ is satisfied.

Since x_0 is a hyperbolic fixed point of $\dot{x} = f_-(x)$, the linear system

$$\dot{x} = f'_-(\gamma_+(t))x$$

has an exponential dichotomy on $[\bar{T}, \infty)$ with projection P_+ and denote by $X_+(t)$ its fundamental matrix with $X_+(\bar{T}) = \mathbb{I}$ [8]. Similarly, the equation

$$\dot{x} = f'_-(\gamma_-(t))x$$

has an exponential dichotomy on $(-\infty, -\bar{T}]$ with projection P_- and denote by $X_-(t)$ its fundamental matrix with $X_-(-\bar{T}) = \mathbb{I}$. Let:

$$\mathcal{S}' = \mathcal{N}P_- \cap P_y(\mathbb{R}^n) \subset \mathbb{R}^{n-1} \tag{3}$$

that is

$$\mathcal{S}' = \{y \in \mathbb{R}^{n-1} \mid (0, y) \in \mathcal{N}P_-\}.$$

Since $\dot{\gamma}_-(-\bar{T}) \in \mathcal{N}P_-$, it follows that $\dim \mathcal{S}' = \dim \mathcal{N}P_- - 1$. Next, we define projections Q and R as follows:

$Q : \mathbb{R}^n \to \mathbb{R}^n$ is the projection on $\mathbb{R}^n$ with $\mathcal{R}Q = \{0\} \times \mathbb{R}^{n-1}$ and $\mathcal{N}Q = \operatorname{span}\{\dot{\gamma}_-(-\bar{T})\}$,
$R : \mathcal{R}P_y \to \mathcal{R}P_y$ is the projection on $\mathcal{R}P_y$ such that $\mathcal{R}R = \mathcal{N}\nabla_y h_-(0, y_0(\bar{T}))$ and $\mathcal{N}R = \operatorname{span}\{\dot{y}_0(\bar{T})\}$.

Let $Y_0(t)$ be the fundamental solution of $\dot{y} = H'(y_0(t))y$, with $Y_0(-\bar{T}) = \mathbb{I}$. Note [7]

$$\dim \left[\begin{pmatrix} 0 \\ RY_0(\bar{T})\mathcal{S}' \end{pmatrix} + \mathcal{R}P_+ \right] \le n - 1.$$

Our next assumption is as follows:

(H4) $\begin{pmatrix} 0 \\ RY_0(\bar{T})\mathcal{S}' \end{pmatrix} + \mathcal{R}P_+$ has codimension 1 in $\mathbb{R}^n$.

It follows from (H4) that a unitary vector $\psi \in \mathbb{R}^n$ exists such that

$$\{\psi\}^\perp = \begin{pmatrix} 0 \\ RY_0(\bar{T})(\mathcal{S}') \end{pmatrix} + \mathcal{R}P_+.$$

Using this vector, we define the function

$$\psi(t) = \begin{cases} X_-^{-1}(t)^* P_-^* Q^* P_y^* Y_0(\bar{T})^* R^* P_y \psi & \text{for } t \le -\bar{T} \\ P_y^* Y_0^{-1}(t)^* Y_0(\bar{T})^* P_y \psi \\ \quad - \frac{k_+(0,y_0(t)) + k_-(0,y_0(t))}{h_+(0,y_0(t)) - h_-(0,y_0(t))} P_z^* Y_0^{-1}(t)^* Y_0(\bar{T})^* P_y \psi & \text{for } -\bar{T} \le t \le \bar{T} \\ X_+^{-1}(t)^* (\mathbb{I} - P_+^*)\psi & \text{for } t \ge \bar{T}. \end{cases}$$

Set

$$\mathcal{M}(\alpha) := \int_{-\infty}^{\infty} \psi^*(t) g(t + \alpha, \gamma(t), 0) dt.$$

Remark 2 It has been proved in [7] that if $\dim \mathcal{N}P_- = n - 1$ and $\dot{y}_0(-\bar{T}) \notin \mathcal{S}'$, then (H4) holds and $\psi = e_1 = (1, 0, \ldots, 0)$, and so $P_y\psi = 0$. Hence,

$$\psi(t) = \begin{cases} 0 & \text{for } t \le \bar{T} \\ X_+^{-1}(t)^* (\mathbb{I} - P_+^*)\psi & \text{for } t \ge \bar{T} \end{cases} \tag{4}$$

and

$$\mathcal{M}(\alpha) = \int_{\bar{T}}^{\infty} \psi^*(t) g(t+\alpha, \gamma(t), 0) dt. \tag{5}$$

As a consequence, in this case, the Melnikov function contains only the $\gamma_+(t)$ part of $\gamma(t)$.

We recall that $g(t, x, \varepsilon)$ is quasiperiodic in t, if the following holds:

(H5) $g(t, x, \varepsilon) = q(\omega_1 t, \ldots, \omega_m t, x, \varepsilon)$ for $\omega_1, \ldots, \omega_m \in \mathbb{R}$ with $q \in C_b^r(\mathbb{R}^{m+n+1}, \mathbb{R}^n)$ and $q(\theta_1, \ldots, \theta_m, x, \varepsilon)$ is 1-periodic in each θ_i, $i = 1, 2, \ldots, m$. Moreover, ω_i, $i = 1, 2, \ldots, m$ are linearly independent over $\mathbb{Z}$, i.e. if $\sum_{i=1}^m l_i \omega_i = 0$, $l_i \in \mathbb{Z}$, $i = 1, 2, \ldots, m$, then $l_i = 0$, $i = 1, 2, \ldots, m$.

Finally, let $\mathcal{E} := \{e : \mathbb{Z} \to \{0, 1\}\}$ be the set of doubly infinite sequences of 0 and 1. It is well known that $\mathcal{E}$ becomes a totally disconnected compact metric space if we take on $\mathcal{E}$ the distance:

$$d(e', e'') = \sum_{m \in \mathbb{Z}} \frac{|e'_m - e''_m|}{2^{m+1}}.$$

The Bernoulli shift $\sigma : \mathcal{E} \to \mathcal{E}$ is defined as $\sigma(e) := \{e_{m+1}\}_{m \in \mathbb{Z}}$ [24,42].

The following result has been proved in [7]:

Theorem 1 *Assume (H1)–(H5) hold. If $\mathcal{M}$ has a simple zero α_0, i.e. $\mathcal{M}(\alpha_0) = 0$ and $\mathcal{M}'(\alpha_0) \neq 0$, then for any $\varepsilon \neq 0$ sufficiently small, there are sequences $\{T_k^\varepsilon\}_{k \in \mathbb{Z}} \subset \mathbb{R}$, $\{\mathcal{S}_k^\varepsilon\}_{k \in \mathbb{Z}}$, $\{\Phi_k^\varepsilon\}_{k \in \mathbb{Z}}$ such that*

a) $\inf_{m \in \mathbb{Z}} (T_{m+1}^\varepsilon - T_m^\varepsilon) \to \infty$ *as* $\varepsilon \to 0$,
b) $\mathcal{S}_k^\varepsilon \subset \mathbb{R}^n$ *are compact,*
c) $\Phi_k^\varepsilon : \mathcal{E} \mapsto \mathcal{S}_k^\varepsilon$ *are homeomorphisms,*
d) *Let $F_k^\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ be defined so that $F_k^\varepsilon(\xi)$ is the value at time $T_{2(k+1)}^\varepsilon$ of the solution of Eq. (1) such that $z(T_{2k}^\varepsilon) = \xi$. Then, the following diagrams commute*:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\sigma"] \arrow[d, "\Phi_k^\varepsilon"'] & \mathcal{E} \arrow[d, "\Phi_{k+1}^\varepsilon"] \\ \mathcal{S}_k^\varepsilon \arrow[r, "F_k^\varepsilon"'] & \mathcal{S}_{k+1}^\varepsilon \end{tikzcd}$$

for all $k \in \mathbb{Z}$.

If in addition $g(t, z, \varepsilon)$ is p-periodic in t, then for any $\varepsilon > 0$, there exists $r_\varepsilon \in \mathbb{N}$ such that for any $k \in \mathbb{Z}$ we have $F_k^\varepsilon = F^\varepsilon = \varphi_\varepsilon^{r_\varepsilon} = \varphi_\varepsilon \circ \ldots \circ \varphi_\varepsilon = (r_\varepsilon$ times), the r_εth iterate of the p-period map φ_ε of (1), $\mathcal{S}_k^\varepsilon = \mathcal{S}^\varepsilon$ and $\Phi_k^\varepsilon = \Phi^\varepsilon$, that is in the periodic case the above diagram is independent of $k \in \mathbb{Z}$. Finally, all these chaotic solutions are sliding ones orbitally located close to $\gamma(t)$.

Theorem 1 generalizes results of [24,37,41,42] to the DDE (1).

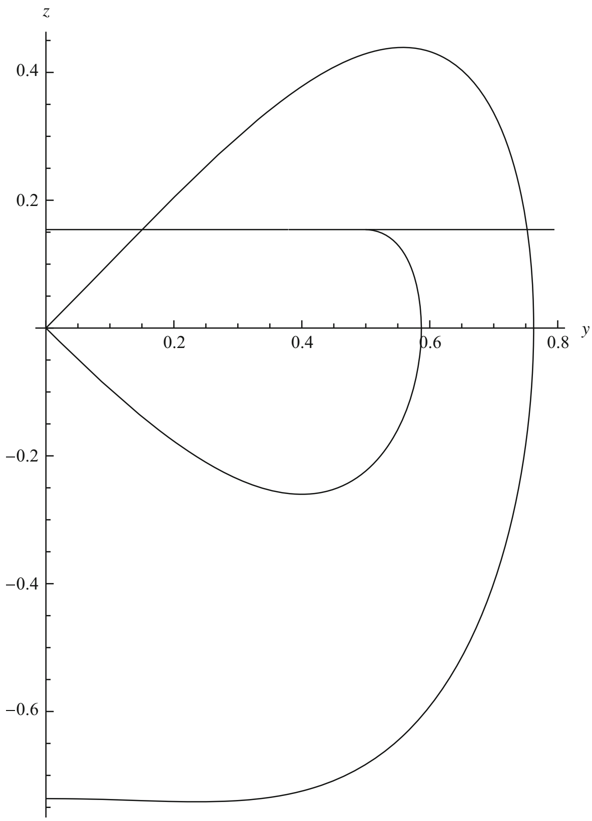

Fig. 2 The stable and unstable manifolds of system (6)

3 Example of 3DDE with homoclinic sliding chaos

This section is devoted to a construction of a concrete example (cf. (14), (15), (18)) of (1) where the above theory is applied. Then, we proceed with a more particular perturbation (cf. Theorems 3, 4).

In order to construct our example, we start from (see [15]):

$$\begin{aligned} \dot{z} &= y - \beta y^3 + yz, \\ \dot{y} &= z, \end{aligned} \tag{6}$$

for $\beta > 1/8$. Then $(0, 0)$ is hyperbolic and $(1/\sqrt{\beta}, 0)$ is an unstable focus. Since $(0, 0)$ is hyperbolic, it has one-dimensional stable and unstable manifolds. In the following, we first show that these two manifolds have the structure depicted in Fig. 2 where the stable manifold is tangent (and the unstable manifold is transverse) to the horizontal straight line.

Performing the transformation $u = 1 - \beta y^2$, $y > 0$, $v = z$, we get

$$\begin{aligned} \dot{u} &= -2\beta v \frac{\sqrt{1-u}}{\sqrt{\beta}}, \quad z < 1 \\ \dot{v} &= (u + v)\frac{\sqrt{1-u}}{\sqrt{\beta}}. \end{aligned} \tag{7}$$

Note that $(0, 0)$ corresponds to $(1, 0)$ and $(1/\sqrt{\beta}, 0)$ to $(0, 0)$. Let $' = \frac{d}{d\theta}$ and consider the linear system

$$\begin{aligned} u' &= -2\beta v \\ v' &= u + v. \\ u(0) = 1&, \quad v(0) = 0 \end{aligned} \tag{8}$$

whose solution has the form

$$\begin{aligned} u_\tau(\theta) &= \mathrm{e}^{\theta/2}\cos(\tau\theta) - \frac{1}{2\tau}\,\mathrm{e}^{\theta/2}\sin(\tau\theta) \\ v_\tau(\theta) &= \frac{1}{\tau}\,\mathrm{e}^{\theta/2}\sin(\tau\theta) \end{aligned} \tag{9}$$

with

$$\tau = \frac{\sqrt{8\beta - 1}}{2}, \tag{10}$$

and so $\beta = \frac{4\tau^2+1}{8}$. Note that $u_\tau'(\theta) = -2\beta v_\tau(\theta) = -2\beta\frac{1}{\tau}\,\mathrm{e}^{\theta/2}\sin(\tau\theta)$ has the opposite sign to $\sin(\tau\theta)$ thus $u_\tau(\theta) \le u_\tau(0) = 1$ for any $\theta \in \left(-\frac{\pi}{\tau}, \frac{\pi}{\tau}\right)$. On the other hand, if $\tau\theta \le -\pi$, we have

$$u_\tau(\theta) \le \mathrm{e}^{-\frac{\pi}{2\tau}}\frac{\sqrt{4\tau^2+1}}{2\tau} = \mathrm{e}^{-\frac{\pi}{2\tau}}\sqrt{1 + \left(\frac{1}{2\tau}\right)^2} < 1$$

since $\mathrm{e}^{\pi s} > \sqrt{1+s^2}$ for any $s > 0$. As a consequence, $(u_\tau(\theta), v_\tau(\theta))$ is tangent to the line $u = 1$ from the left at $\theta = 0$ for $\theta \in (-\infty, \theta_\tau^+)$. Here $\theta_\tau^+ > 0$ is the least positive value such that $u_\tau(\theta) = 1$. Next, let θ_τ^- be the greatest negative value for which $v_\tau'(\theta) = 0$ and $v_\tau(\theta) > 0$. Then θ_τ^- solves the following system:

$$\begin{aligned} \cos(\tau\theta_\tau^-) + \frac{1}{2\tau}\sin(\tau\theta_\tau^-) &= 0 \\ \sin(\tau\theta_\tau^-) &> 0 \end{aligned}$$

so

$$\tau\theta_\tau^- = -\arctan 2\tau - \pi.$$

Given $\bar{T} > 0$ (we will fix it later), we consider the solution $\theta^-(t)$ of the equation:

$$\dot{\theta} = \sqrt{\frac{1 - u_\tau(\theta)}{\beta}}, \quad \theta(\bar{T}) = \theta_\tau^-. \tag{11}$$

Separating variables we see that

$$\int_{\theta_\tau^-}^{\theta^-(t)} \frac{d\theta}{\sqrt{1 - u_\tau(\theta)}} = \frac{t - \bar{T}}{\sqrt{\beta}}$$

or

$$\theta^{-}(t) = \Theta_{-}^{-1}\left(\frac{t-\bar{T}}{\sqrt{\beta}}\right)$$

where

$$\Theta_{-}(\theta) = \int_{\theta_\tau^-}^{\theta} \frac{d\theta}{\sqrt{1-u_\tau(\theta)}}.$$

From (9), we easily see that

$$1 - u_\tau(\theta) = \beta\theta^2 + o(\theta^2)$$

as $\theta \to 0$. As a consequence, $\Theta_{-}(\theta)$ is an increasing function that tends to $+\infty$ as $\theta \to 0$. Thus, $\theta^{-}(t)$ is increasing and tends to 0 as $t \to \infty$. Moreover, since $u_\tau(\theta) < 1$ for $\theta < 0$, we also see that $\theta(t) \to -\infty$ as $t \to -\infty$. Summarizing $\theta^{-}(t)$ is an increasing function defined on $(-\infty, \infty)$, taking values on $(-\infty, 0)$ and such that $\theta^{-}(\bar{T}) = \theta_\tau^-$. Setting

$$y_+(t) = \sqrt{\frac{1-u_\tau(\theta^-(t))}{\beta}}$$
$$z_+(t) = v_\tau(\theta^-(t))$$

we see that $(z_+(t), y_+(t))$ is a solution of Eq. (6) such that

$$\lim_{t\to\infty}(z_+(t), y_+(t)) = \lim_{\theta\to 0}\left(v_\tau(\theta), \sqrt{\frac{1-u_\tau(\theta)}{\beta}}\right) = (0,0)$$

and

$$\lim_{t\to-\infty}(z_+(t), y_+(t)) = \lim_{\theta\to-\infty}\left(v_\tau(\theta), \sqrt{\frac{1-u_\tau(\theta)}{\beta}}\right) = \left(0, \sqrt{\frac{1}{\beta}}\right)$$

that is $(z_+(t), y_+(t))$ is a heteroclinic connection from $\left(0, \sqrt{\frac{1}{\beta}}\right)$ to $(0,0)$. Next, we know that θ_τ^- is the greatest negative value such that $v'(\theta) = 0$ and $v(\theta) > 0$. This means that at $t = \bar{T}$, we have

$$z_+(\bar{T}) = v_\tau(\theta_\tau^-) := \Omega_\tau > 0, \quad \dot{z}_+(\bar{T}) = 0$$

and these two conditions are not satisfied when $t > \bar{T}$. Note that:

$$\Omega_\tau = \frac{1}{\tau}\,\mathrm{e}^{\theta_\tau^-/2}\sin(\tau\theta_\tau^-) = 2\,\mathrm{e}^{\theta_\tau^-/2}\sqrt{\frac{1}{1+4\tau^2}} = \mathrm{e}^{\theta_\tau^-/2}\sqrt{\frac{1}{2\beta}}, \tag{12}$$

moreover

$$y_+(\bar{T}) = \sqrt{\frac{1-u_\tau(\theta_\tau^-)}{\beta}} = \sqrt{\frac{1+v_\tau(\theta_\tau^-)}{\beta}} = \sqrt{\frac{1+\Omega_\tau}{\beta}}. \tag{13}$$

Now we consider the solution $(z_-(t), y_-(t))$ of Eq. (6) that belongs to the unstable manifold of the saddle $(0,0)$. Since $(z_-(t), y_-(t)) \to (0,0)$ as $t \to -\infty$, it follows that we have to look for a solution $(u(t), v(t))$ of (7) such that $(u(t), v(t)) \to (1, 0)$ as $t \to -\infty$. Thus, we consider again Eq. (8) but with $\theta \in (0, \theta_\tau^+)$. Thus, $\theta = \theta^+(t)$ is again a solution of

$$\dot{\theta} = \sqrt{\frac{1 - u_\tau(\theta)}{\beta}}$$

but with the initial condition $\theta(0) = \theta_\tau^+$. So we obtain:

$$\int_{\theta_\tau^+}^{\theta^+(t)} \frac{d\theta}{\sqrt{1 - u_\tau(\theta)}} = \frac{t}{\sqrt{\beta}}$$

that is

$$\theta^+(t) = \Theta_+^{-1}\left(\frac{t}{\sqrt{\beta}}\right),$$

where

$$\Theta_+(\theta) = \int_{\theta_\tau^+}^{\theta} \frac{d\theta}{\sqrt{1 - u_\tau(\theta)}}.$$

Obviously, $\Theta_+(\theta)$ is an increasing function and since $\theta \in (0, \theta_\tau^+)$, $\Theta_+(\theta) < 0$ for $0 \le \theta < \theta_\tau^+$. Arguing as before, we see that

$$\lim_{\theta \to 0} \Theta_+(\theta) = -\infty$$

and hence

$$\lim_{t \to -\infty} \theta^+(t) = 0.$$

For $t \in (-\infty, 0]$ (and hence $\theta^+(t) \in (0, \theta_\tau^+]$), we set:

$$y_-(t) = \sqrt{\frac{1 - u_\tau(\theta^+(t))}{\beta}}$$
$$z_-(t) = v_\tau(\theta^+(t))$$

and note that the following hold:

$$y_-(0) = 0, \quad z_-(0) = v_\tau(\theta_\tau^+)$$
$$\lim_{t \to -\infty} (z_-(t), y_-(t)) = \lim_{\theta \to 0} \left(v_\tau(\theta), \sqrt{\frac{1 - u_\tau(\theta)}{\beta}} \right) = (0, 0).$$

Now, since $u_\tau(\theta) < 1$ for $\theta \in (0, \theta_\tau^+)$, we see that $u'_\tau(\theta_\tau^+) \ge 0$ and then $z_-(0) = v_\tau(\theta_\tau^+) \le 0$. But it must be $z_-(0) = v_\tau(\theta_\tau^+) < 0$ otherwise $(z_-(0), y_-(0)) = (0, 0)$ because of uniqueness. Next, $(z_-(t), y_-(t))$ belongs to the unstable manifold of the equilibrium (0,0) and $y_-(t) > 0$ for any $t \in (-\infty, 0)$, thus

$$\frac{(\dot{z}_-(t), \dot{y}_-(t))}{\sqrt{\dot{z}_-(t)^2 + \dot{y}_-(t)^2}} \to v_-$$

as $t \to -\infty$, v_- being the eigenvector of the positive eigenvalue of the linearization of Eq. (6) at (0,0), i.e.:

$$\dot{z} = y$$
$$\dot{y} = z$$

having a positive second component. Hence, $z_-(t) = \dot{y}_-(t)$ is eventually positive for $t \to -\infty$. Thus, the curve $(z_-(t), y_-(t))$ has to pass from the first quadrant to the fourth, and this can be realized only by passing above the line $z = z_+(\bar{T})$ because otherwise it would intersect the curve $(z_+(t), y_+(t))$. As a consequence, $t_0 < 0$ must exist so that $z_-(t_0) = z_+(\bar{T}) = \Omega_\tau$ and $z_-(t) < \Omega_\tau$ for any $t < t_0$. We set

$$\bar{y}_\tau := \sqrt{\frac{1 - u_\tau(\theta_\tau^+(t_0))}{\beta}}.$$

Shifting time, we can suppose without loss of generality that $t_0 = -\bar{T}$. Thus, we have found solutions $\tilde{\gamma}_\pm(t) = (z_\pm(t), y_\pm(t))$ of (6) such that

$$\begin{aligned} \tilde{\gamma}_-(t) &\to (0,0) \quad \text{as } t \to -\infty \\ \tilde{\gamma}_+(t) &\to (0,0) \quad \text{as } t \to +\infty \\ z_-(-\bar{T}) &= z_+(\bar{T}) = \Omega_\tau. \end{aligned}$$

The graphs of the above-mentioned invariant manifolds of (6) and the line $z = \Omega_\tau$ in the right half plane for $\beta = 37/8$, i.e. $\tau = 3$, are as in Fig. 2.

We are now able to construct our example. We take

$$\begin{aligned} \dot{z} &= y_1 - \beta y_1^3 + z y_1 + y_2^2 \\ \dot{y}_1 &= z \\ \dot{y}_2 &= y_2(1+z) \end{aligned} \tag{14}$$

for $z < \Omega_\tau$ and

$$\begin{aligned} \dot{z} &= -z \\ \dot{y}_1 &= 0 \\ \dot{y}_2 &= 0 \end{aligned} \tag{15}$$

when $z > \Omega_\tau$, that is we take:

$$\begin{aligned} f_+(z, y_1, y_2) &= \begin{pmatrix} -z \\ 0 \\ 0 \end{pmatrix} \quad \text{for } z > \Omega_\tau \\ f_-(z, y_1, y_2) &= \begin{pmatrix} y_1 - \beta y_1^3 + z y_1 + y_2^2 \\ z \\ y_2(1+z) \end{pmatrix} \quad \text{for } z < \Omega_\tau. \end{aligned} \tag{16}$$

Then,

$$h_-(y_1, y_2) = y_1 - \beta y_1^3 + \Omega_\tau y_1 + y_2^2, \; h_+(y_1, y_2) = -\Omega_\tau$$

and

$$H(y_1, y_2) = \frac{\Omega_\tau}{y_1 - \beta y_1^3 + \Omega_\tau(y_1+1) + y_2^2} \begin{pmatrix} \Omega_\tau \\ y_2(1+\Omega_\tau) \end{pmatrix}.$$

We note that $h_-(y_1,y_2)-h_+(y_1,y_2)=y_1(1-\beta y_1^2+\Omega_\tau)+y_2^2+\Omega_\tau>0$ if $0\le y_1\le\sqrt{\frac{1+\Omega_\tau}{\beta}}$. Then, we take the solution $\tilde y_1(t)$ of

$$\dot y_1=\frac{\Omega_\tau^2}{y_1-\beta y_1^3+\Omega_\tau(y_1+1)}$$

such that $y_1(0)=\bar y_\tau$ and let $\tilde T$ be such that $y_1(\tilde T)=\sqrt{\frac{1+\Omega_\tau}{\beta}}$. Note that, according to the previous remark, $h_-(\Omega_\tau,y_1,y_2)-h_+(\Omega_\tau,y_1,y_2)>0$ in a neighborhood of $\tilde y_1(t)$, $0\le t\le\tilde T$. Thus, we are in position to apply Remark 1.

Now, we define $\bar T=\frac{\tilde T}{2}$ and set

$$\begin{aligned}\gamma_0(t)&=(\Omega_\tau,\tilde y_1(t+\bar T),0)\\ \gamma_-(t)&=(\tilde\gamma_-(t),0)\\ \gamma_+(t)&=(\tilde\gamma_+(t),0)\end{aligned}$$

and

$$\gamma(t)=\begin{cases}\gamma_-(t) & \text{if } t\le-\bar T\\ \gamma_0(t) & \text{if } -\bar T\le t\le\bar T\\ \gamma_+(t) & \text{if } t\ge\bar T\end{cases}$$

is a sliding homoclinic orbit for the system (14), (15).

For concrete values of $\tau>0$, we take $\beta=\frac{1}{8}+\frac{\tau^2}{2}$, compute Ω_τ and we solve (6) with initial values $z_s(\bar T)=\sqrt{\frac{1+\Omega_\tau}{\beta}}$, $y_s(\bar T)=\Omega_\tau$ to get $\widetilde{\gamma}_+(t)$ and $\gamma_+(t)$.

We now verify that system (14), (15) and $\gamma(t)$ satisfy conditions (H1)–(H4) of this paper. We have already seen that (H1) is satisfied (see also Remark 1). Condition (H2) is also satisfied with $x_0=\left(z^0,y_1^0,y_2^0\right)=(0,0,0)$. Note that in this example the discontinuity level is at $z=\Omega_\tau$ and not at $z=0$, but of course this fact does not make any difference. Now we verify (H3). It is trivial to verify that $h_+(\gamma(t))<0$ for $-\bar T\le t\le\bar T$, $h_-(\gamma(t))>0$ for $-\bar T\le t<\bar T$ and $h_-(\gamma(\bar T))=y_+(\bar T)(1-\beta y_+(\bar T)^2+\Omega_\tau)=0$. So we check the last condition in (H3). We have

$$\nabla_y h_-(\gamma_0(\bar T))=-2\begin{pmatrix}1+\Omega_\tau\\0\end{pmatrix}\quad\text{and}\quad k_-\left(\gamma_0(\bar T)\right)=\begin{pmatrix}\Omega_\tau\\0\end{pmatrix}$$

from which we obtain $\nabla_y h_-(\gamma_0(\bar T)k_-(\gamma_0(\bar T))^*=-2\Omega_\tau(1+\Omega_\tau)\ne0$. Finally, we check (H4). By Remark 2, it is enough to prove that

$$\left(\dot y_1\left(-\bar T\right),0\right)=\left(\frac{\Omega_\tau^2}{\bar y_\tau u_\tau(\theta_\tau^+(-\bar T))+\Omega_\tau},0\right)\notin\mathcal S'$$

(see (3) for the definition of $\mathcal S'$) or, equivalently, that $(1,0)\notin\mathcal S'$. Now, the variational system of (14) along $\gamma_-(t)$ is given by:

$$\begin{aligned}\dot z&=y_-(t)z+\left(1-3\beta y_-(t)^2+z_-(t)\right)y_1\\ \dot y_1&=z\\ \dot y_2&=(1+z_-(t))y_2\end{aligned}\tag{17}$$

Since (17) has the bounded solution at $-\infty$: $\left(0,0,\mathrm{e}^{t+y_-(t)}\right)$ and $\dim\mathcal S'=1$, it follows that $\mathcal S'=\operatorname{span}\{(0,1)\}$ and hence $(1,0)\notin\mathcal S'$. Thus, (H4) holds.

Finally, we add a perturbation

$$\varepsilon g(t) = \varepsilon \begin{pmatrix} q(\omega t) \\ 0 \\ q_1(\omega_1 t) \end{pmatrix} \tag{18}$$

to (14), (15) and compute the Melnikov function. Here, ω, ω_1 are positive constants and q_1, q_2 are C^2-functions with bounded derivatives and their second-order derivatives are uniformly continuous. To this end, we need to compute the solution $\psi(t)$ of the adjoint variational system:

$$\begin{aligned} \dot{z} &= -y_+(t)z - y_1 \\ \dot{y}_1 &= -\left(1 - 3\beta y_+(t)^2 + z_+(t)\right) z \\ \dot{y}_2 &= -(1 + z_+(t))y_2 \end{aligned} \tag{19}$$

with $\psi(0) = (1, 0, 0)$ (see (4)). Since $y_2 = 0$ is invariant for system (19), we get $\psi(t) = (\psi_1(t), \psi_2(t), 0)$ where $(z, y) = (\psi_1(t), \psi_2(t))$ is a bounded (at $+\infty$) solution of

$$\begin{aligned} \dot{z} &= -y_+(t)z - y_1 \\ \dot{y}_1 &= -\left(1 - 3\beta y_+(t)^2 + z_+(t)\right) z \end{aligned}$$

that is

$$\psi(t) = \begin{pmatrix} \dot{y}_+(t) \\ -\dot{z}_+(t) \\ 0 \end{pmatrix} \mathrm{e}^{-\int_{\bar{T}}^t y_+(s)ds}$$

and the Melnikov function is (see (5))

$$\mathcal{M}(\alpha) = \int_{\bar{T}}^{\infty} \dot{y}_+(t)\,\mathrm{e}^{-\int_{\bar{T}}^t y_+(s)ds} q(\omega t + \alpha)dt.$$

Since

$$\begin{aligned} \lim_{\omega\to 0} \mathcal{M}(\alpha) &= q(\alpha) \int_{\bar{T}}^{\infty} \dot{y}_+(t)\,\mathrm{e}^{-\int_{\bar{T}}^t y_+(s)ds} dt \\ \lim_{\omega\to 0} \mathcal{M}'(\alpha) &= q'(\alpha) \int_{\bar{T}}^{\infty} \dot{y}_+(t)\,\mathrm{e}^{-\int_{\bar{T}}^t y_+(s)ds} dt \end{aligned}$$

we see that if $q(\alpha)$ has a simple zero at some $\alpha = \alpha_0$ and

$$\int_{\bar{T}}^{\infty} \dot{y}_+(t)\,\mathrm{e}^{-\int_{\bar{T}}^t y_+(s)ds} dt \neq 0, \tag{20}$$

then $\mathcal{M}(\alpha)$ will have a simple zero at some α near to α_0 for $\omega > 0$ small. To check condition (20), we recall that

$$y_+(t) = \sqrt{\frac{1 - u_\tau(\theta^-(t))}{\beta}} = \dot{\theta}^-(t),$$

so:

$$\int_{\bar{T}}^{t} y_+(s)ds = \theta^-(t) - \theta^-(\bar{T}) = \theta^-(t) - \theta_\tau^-. \tag{21}$$

Now, let $Y(\theta) = \sqrt{\frac{1-u_\tau(\theta)}{\beta}}$. Then,

$$y_+(t) = Y(\theta^-(t))$$

and

$$\dot{y}_+(t) = Y'(\theta^-(t))\dot{\theta}^-(t). \tag{22}$$

Plugging (21), (22) into (20), we obtain

$$\begin{aligned}
\int_{\bar{T}}^{\infty} \dot{y}_+(t)\,\mathrm{e}^{-\int_{\bar{T}}^{t} y_+(s)ds}dt &= \mathrm{e}^{\theta_\tau^-}\int_{\bar{T}}^{\infty} \mathrm{e}^{-\theta^-(t)}Y'(\theta^-(t))\dot{\theta}^-(t)dt \\
&= \mathrm{e}^{\theta_\tau^-}\int_{\theta_\tau^-}^{0} \mathrm{e}^{-\theta}Y'(\theta)d\theta = \mathrm{e}^{\theta_\tau^-}\left[Y(\theta)\,\mathrm{e}^{-\theta}\Big|_{\theta_\tau^-}^{0} + \int_{\theta_\tau^-}^{0} \mathrm{e}^{-\theta}Y(\theta)d\theta\right] \\
&= \mathrm{e}^{\theta_\tau^-}\int_{\theta_\tau^-}^{0} \mathrm{e}^{-\theta}Y(\theta)d\theta - Y(\theta_\tau^-) = \int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- - \theta}Y(\theta)d\theta - \sqrt{\frac{1+\Omega_\tau}{\beta}} \\
&= \frac{1}{\sqrt{\beta}}\left(\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- - \theta}\sqrt{1-u_\tau(\theta)}d\theta - \sqrt{1+\Omega_\tau}\right).
\end{aligned} \tag{23}$$

We prove now that the expression (23) is negative for any $\tau > 0$. Using Cauchy–Schwarz–Bunyakovsky inequality, we get

$$\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- - \theta}\sqrt{1-u_\tau(\theta)}d\theta \le \sqrt{\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- - \theta}d\theta}\sqrt{\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- - \theta}(1-u_\tau(\theta))d\theta}.$$

Next, we integrate

$$\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- - \theta}d\theta = 1 - \mathrm{e}^{\theta_\tau^-}$$

and

$$\begin{aligned}
\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- - \theta}(1-u_\tau(\theta))d\theta &= \int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- - \theta}\left(1 - \mathrm{e}^{\theta/2}\cos(\tau\theta) + \frac{1}{2\tau}\mathrm{e}^{\theta/2}\sin(\tau\theta)\right)d\theta \\
&= 1 - \mathrm{e}^{\theta_\tau^-} + 2\,\mathrm{e}^{\theta_\tau^-/2}\sqrt{\frac{1}{1+4\tau^2}} = 1 - \mathrm{e}^{\theta_\tau^-} + \Omega_\tau.
\end{aligned}$$

Consequently,

$$\int_{\theta_\tau^-}^{0} e^{\theta_\tau^- - \theta}\sqrt{1-u_\tau(\theta)}d\theta \le \sqrt{1-e^{\theta_\tau^-}}\sqrt{1-e^{\theta_\tau^-}+\Omega_\tau} < \sqrt{1+\Omega_\tau}\,, \tag{24}$$

hence the expression (23) is negative for any value of $\tau > 0$. Summarizing, we obtain the following result.

Theorem 2 *Let $\tau > 0$ and let $q(t), q_1(t)$ be almost periodic functions such that $q(t)$ has a simple zero at some $\alpha \in \mathbb{R}$. Then there exist $\omega_0 > 0$ and $\varepsilon_0 > 0$ such that for $0 < |\omega| < \omega_0$ and $0 < |\varepsilon| < \varepsilon_0$, system*

$$\dot{x} = f_\pm(x) + \varepsilon g(t), \quad x \in \Omega_\pm \tag{25}$$

where $x = (z, y_1, y_2) \in \mathbb{R}^3$, $f_\pm(x)$ is as in (16) and $g(t)$ as in (18) is chaotic.

For example, if $q(t) = \cos t$, we get $\mathcal{M}(\alpha) = \int_{\bar{T}}^{\infty} \dot{y}_+(t)\, e^{-\int_{\bar{T}}^{t} y_+(s)ds} \cos(\omega t + \alpha)dt$ and then

$$\mathcal{M}(\alpha) - \imath\mathcal{M}'(\alpha) = e^{\imath\alpha}\Psi_\tau(\omega), \quad \Psi_\tau(\omega) := \int_{\bar{T}}^{\infty} \dot{y}_+(t)\, e^{-\int_{\bar{T}}^{t} y_+(s)ds}\, e^{\imath\omega t}dt.$$

As a consequence, if $\Psi_\tau(\omega) \neq 0$ then $\mathcal{M}(\alpha)$ has a simple zero. Since $\Psi_\tau(0) \neq 0$, $\Psi_\tau(\omega)$ is a nonzero analytical function. From Theorem 2, we know that (25) behaves chaotically for $|\omega| < \omega_0$ (and $|\varepsilon| < \varepsilon_0$) sufficiently small. However, for this particular example ($q(t) = \cos t$), (25) behaves chaotically also when ω is large. As a matter of fact, we have the following:

Theorem 3 *There exist continuous functions $F(\beta), D(\beta) : \left(\frac{1}{8}, \infty\right) \to (0, \infty)$ such that for any given constants $\beta > 1/8, \omega_1 > 0, \omega \in (0, \infty)\backslash[F(\beta), D(\beta)]$ and an almost periodic C^2-function $q_1(t)$ with bounded derivatives and such that its second-order derivative is uniformly continuous, there exists $\varepsilon_0 = \varepsilon_0(\beta, \omega, \omega_1, q_1(\cdot))$ such that for $0 < |\varepsilon| < \varepsilon_0$ and*

$$g(t) = \begin{pmatrix} \cos(\omega t) \\ 0 \\ q_1(\omega_1 t) \end{pmatrix}$$

system (14), (15) is chaotic. Moreover, it holds

$$\lim_{\tau \to 1/8_+} F(\beta) = 0, \quad \lim_{\beta\to\infty} F(\beta) = \frac{2\sqrt{2}}{\pi\left(2\sqrt{2}+1\right)} \doteq 0.235166\,,$$

$$\lim_{\beta\to 1/8_+} D(\beta) = \infty, \quad \lim_{\beta\to\infty} D(\beta) = \frac{3\sqrt{2}\pi}{2} + 4 - \sqrt{2} \doteq 9.25011.$$

Proof Let, for simplicity,

$$\eta(t) = -\tau\theta^-(t) \quad \text{and} \quad \phi_\tau(t) = \Phi_\tau(\eta(t))$$

where

$$\Phi_\tau(\eta) := v_\tau\left(-\frac{\eta}{\tau}\right) e^{\frac{\eta+\tau\theta_\tau^-}{\tau}} = -\frac{1}{\tau}\, e^{\frac{\eta}{2\tau}+\theta_\tau^-} \sin\eta$$

(recall that $\theta^-(t)$ is the solution of (11)). First, we prove few properties of $\eta(t)$ and $\phi_\tau(t)$ that will be used later:

N_1) from (11), we see that $\eta(t)$ satisfies the equation:

$$\dot{\eta} = \varGamma_\tau(\eta) := -\frac{\tau}{\sqrt{\beta}}\sqrt{1-u_\tau\left(-\frac{\eta}{\tau}\right)} = -\frac{\tau}{\sqrt{\beta}}\sqrt{1-\mathrm{e}^{-\frac{\eta}{2\tau}}\left(\cos\eta-\frac{1}{2\tau}\sin\eta\right)}$$
$$\eta(\bar{T}) = \arctan 2\tau + \pi \geq \bar{\eta} > 0; \tag{26}$$

N_2) since $\theta^-(t)$ is increasing and tends to 0 as $t \to \infty$, it follows that $\eta(t)$ is decreasing and tends to 0 as $t \to \infty$;

N_3) since $\lim_{\theta\to 0} u_\tau(\theta) = 1$, it follows from (26) that $\lim_{t\to\infty} \dot{\eta}(t) = 0$;

N_4) since $\eta(t)$ is decreasing, $\eta(\bar{T}) = \pi + \arctan(2\tau) > \pi$ and $\eta(t) \to 0$ as $t \to \infty$, there exists a unique $T_\pi > \bar{T}$ such that $\eta(T_\pi) = \pi$ and $\eta(t) > \pi$ if and only if $t < T_\pi$;

N_5) since $\eta(t)$ is decreasing and $\frac{\eta(\bar{T})}{\tau} = -\theta_\tau^-$ we get $\frac{\eta(t)}{2\tau} + \theta_\tau^- \leq \frac{\theta_\tau^-}{2}$ for any $t \geq \bar{T}$ and the equality holds if and only if $t = \bar{T}$.

Then

F_1) since $y_+(t) = \sqrt{\frac{1-u_\tau(\theta^-(t))}{\beta}} = \dot{\theta}^-(t)$ we get $\int_{\bar{T}}^t y_+(s)ds = \theta^-(t) - \theta_\tau^- = -\frac{\eta(t)+\tau\theta_\tau^-}{\tau}$ and then, using also $\dot{y}_+(t) = z_+(t) = v_\tau(\theta^-(t))$, we obtain

$$\phi_\tau(t) = \dot{y}_+(t)\,\mathrm{e}^{\theta_\tau^- - \theta^-(t)}, \quad \varPsi_\tau(\omega) = \int_{\bar{T}}^{\infty} \phi_\tau(t)\,\mathrm{e}^{\imath\omega t}dt;$$

F_2) $\phi_\tau(\bar{T}) = \varPhi_\tau(-\tau\theta_\tau^-) = \frac{1}{\tau}\,\mathrm{e}^{\frac{\theta_\tau^-}{2}}\sin(\tau\theta_\tau^-) = \varOmega_\tau$

F_3) $\dot{\phi}_\tau(\bar{T}) = \varPhi_\tau'(-\tau\theta_\tau^-)\dot{\eta}(\bar{T}) = -\frac{1}{\tau}\,\mathrm{e}^{\frac{\theta_\tau^-}{2}}\sin(\tau\theta_\tau^-)\sqrt{\frac{1-u_\tau(\theta_\tau^-)}{\beta}} = -\varOmega_\tau\sqrt{\frac{1+\varOmega_\tau}{\beta}}$ since $\cos(\tau\theta_\tau^-) = -\frac{1}{2\tau}\sin(\tau\theta_\tau^-)$.

Now, integrating by parts twice yields (using also F_2), F_3)):

$$\begin{aligned}\varPsi_\tau(\omega) &= \phi_\tau(\bar{T})\frac{\imath\,\mathrm{e}^{\imath\omega\bar{T}}}{\omega} - \dot{\phi}_\tau(\bar{T})\frac{\mathrm{e}^{\imath\omega\bar{T}}}{\omega^2} - \frac{1}{\omega^2}\int_{\bar{T}}^{\infty}\ddot{\phi}_\tau(t)\,\mathrm{e}^{\imath\omega t}dt \\ &= \frac{1}{\omega}\varOmega_\tau\imath\,\mathrm{e}^{\imath\omega\bar{T}} + \frac{1}{\omega^2}\varOmega_\tau\sqrt{\frac{1+\varOmega_\tau}{\beta}}\,\mathrm{e}^{\imath\omega\bar{T}} - \frac{1}{\omega^2}\int_{\bar{T}}^{\infty}\ddot{\phi}_\tau(t)\,\mathrm{e}^{\imath\omega t}dt,\end{aligned}$$

hence

$$|\varPsi_\tau(\omega)| \geq \frac{1}{\omega}\varOmega_\tau - \frac{1}{\omega^2}\varOmega_\tau\sqrt{\frac{1+\varOmega_\tau}{\beta}} - \frac{1}{\omega^2}\int_{\bar{T}}^{\infty}|\ddot{\phi}_\tau(t)|dt.$$

Next, we derive

$$\ddot{\phi}_\tau(t) = \varPhi_\tau''(\eta(t))\varGamma_\tau(\eta(t))\dot{\eta}(t) + \varPhi_\tau'(\eta(t))\varGamma_\tau'(\eta(t))\dot{\eta}(t).$$

First, we have

$$\Gamma_\tau'(\eta) = -\frac{u_\tau'\left(-\frac{\eta}{\tau}\right)}{2\beta\sqrt{\frac{1-u_\tau\left(-\frac{\eta}{\tau}\right)}{\beta}}} = -\frac{\tau v_\tau\left(-\frac{\eta}{\tau}\right)}{\Gamma_\tau(\eta)} = \frac{\mathrm{e}^{-\frac{\eta}{2\tau}}\sin\eta}{\Gamma_\tau(\eta)}.$$

So $\Gamma_\tau'(\eta) < 0$ for $0 < \eta < \pi$, and $\Gamma_\tau'(\eta) > 0$ for $\arctan 2\tau + \pi > \eta > \pi$. Next, from N_5), we obtain

$$|\Phi_\tau'(\eta(t))| = \left|\frac{1}{2\tau^2}\,\mathrm{e}^{\frac{\eta(t)}{2\tau}+\theta_\tau^-}(2\tau\cos\eta(t) + \sin\eta(t))\right| \le \frac{\mathrm{e}^{\theta_\tau^-/2}}{2\tau}(2+\tau^{-1})$$

and

$$\begin{aligned}|\Phi_\tau''(\eta(t))| &= \left|\frac{1}{4\tau^3}\,\mathrm{e}^{\frac{\eta(t)}{2\tau}+\theta_\tau^-}(\sin\eta(t) + 4\tau(\cos\eta(t) - \tau\sin\eta(t)))\right| \\ &\le \frac{\mathrm{e}^{\theta_\tau^-/2}}{4\tau}\left(\tau^{-2} + 4\tau^{-1} + 4\right).\end{aligned}$$

Then, using N_4), $|\Gamma_\tau(\eta)| \le 1$ and the equalities: $\Gamma_\tau(\eta(\bar T)) = -\tau\sqrt{\frac{1-u_\tau(\theta_\tau^-)}{\beta}} = -\tau\sqrt{\frac{1+\Omega_\tau}{\beta}}$ and $u_\tau\left(-\frac{\pi}{\tau}\right) = -\,\mathrm{e}^{-\frac{\pi}{2\tau}}$, we obtain

$$\begin{aligned}\int_{\bar T}^{\infty}|\ddot\phi_\tau(t)|dt &\le -\frac{1}{4\sqrt\beta}\,\mathrm{e}^{\theta_\tau^-/2}(\tau^{-2}+4\tau^{-1}+4)\int_{\bar T}^{\infty}\dot\eta(t)dt \\ &\quad + \frac{1}{2\tau}\,\mathrm{e}^{\theta_\tau^-/2}(2+\tau^{-1})\left(-\int_{\bar T}^{T_\pi}\Gamma_\tau'(\eta(t))\dot\eta(t)dt + \int_{T_\pi}^{\infty}\Gamma_\tau'(\eta(t))\dot\eta(t)dt\right) \\ &\le \frac{1}{4\sqrt\beta}\,\mathrm{e}^{\theta_\tau^-/2}\left(\tau^{-2}+4\tau^{-1}+4\right)(\arctan 2\tau + \pi) \\ &\quad + \frac{1}{2\tau}\,\mathrm{e}^{\theta_\tau^-/2}\left(2+\tau^{-1}\right)\left(\Gamma_\tau(\eta(\bar T)) + \Gamma_\tau(0) - 2\Gamma_\tau(\pi)\right) \\ &\le \frac{\tau}{2\sqrt\beta}\,\mathrm{e}^{\theta_\tau^-/2}\left[\frac{3\pi}{4}\left(\tau^{-3}+4\tau^{-2}+4\tau^{-1}\right) + \left(2\tau^{-1}+\tau^{-2}\right)\right. \\ &\quad \left.\times\left(2\sqrt{1+\,\mathrm{e}^{-\frac{\pi}{2\tau}}} - \sqrt{1+\Omega_\tau}\right)\right] \\ &\le A_\tau := \mathrm{e}^{\theta_\tau^-/2}\left[\frac{3\sqrt2\pi}{8}\left(\tau^{-3}+4\tau^{-2}+4\tau^{-1}\right) + \left(2\tau^{-1}+\tau^{-2}\right)\left(2-\sqrt{1/2}\right)\right],\end{aligned}$$

since $\tau/\sqrt\beta < \sqrt2$. Consequently, we obtain

$$|\Psi_\tau(\omega)| \ge \frac{1}{\omega}\Omega_\tau - \frac{1}{\omega^2}\Omega_\tau\sqrt{\frac{1+\Omega_\tau}{\beta}} - \frac{1}{\omega^2}A_\tau > 0$$

for any

$$\omega > \sqrt{\frac{1+\Omega_\tau}{\beta}} + \frac{A_\tau}{\Omega_\tau}. \tag{27}$$

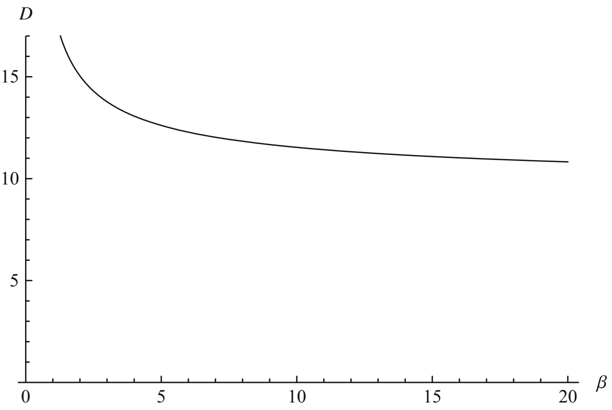

Fig. 3 The graph of $D(\beta)$ over the interval $(1/8, 20]$

Since $\tau^{-1}\,\mathrm{e}^{-\frac{\pi}{2\tau}} > \Omega_\tau$, we have

$$\sqrt{\frac{1+\Omega_\tau}{\beta}} + \frac{A_\tau}{\Omega_\tau} \leq \sqrt{\frac{8(1+\tau^{-1}\,\mathrm{e}^{-\frac{\pi}{2\tau}})}{4\tau^2+1}}$$
$$+\frac{\sqrt{4+\tau^{-2}}}{2}\left(\frac{3\sqrt{2}\pi}{8}(\tau^{-2}+4\tau^{-1}+4)+(2+\tau^{-1})\left(2-\sqrt{1/2}\right)\right) := B(\tau).$$

Consequently, (27) holds if

$$\omega > B(\tau).$$

Note $\lim_{\tau\to 0_+} B(\tau) = \infty$ and $\lim_{\tau\to\infty} B(\tau) = \frac{3\sqrt{2}\pi}{2} + 4 - \sqrt{2} \doteq 9.25011$. Setting

$$D(\beta) := B\left(\sqrt{8\beta-1}/2\right)$$

we see that the statement of the theorem holds for $\omega > D(\beta)$. The graph of $D(\beta)$ over interval $(1/8, 20)$ looks like on Fig. 3.

Next, from Theorem 2, we know that system (25) behaves chaotically also for $\omega > 0$ sufficiently small, but the smallness of ω may depend on β, or τ. In the following, we intend to investigate this dependence. Since

$$\mathrm{e}^{-\imath\omega\bar{T}}\Psi_\tau(\omega) = \int_{\bar{T}}^{\infty}\phi_\tau(t)dt + \imath\omega\int_{\bar{T}}^{\infty}(t-\bar{T})\phi_\tau(t)\int_0^1 \mathrm{e}^{\imath\omega s(t-\bar{T})}ds\,dt,$$

we derive

$$|\Psi_\tau(\omega)| \geq \left|\int_{\bar{T}}^{\infty}\phi_\tau(t)dt\right| - \omega\int_{\bar{T}}^{\infty}(t-\bar{T})|\phi_\tau(t)|dt.$$

Next, from N_2) and F_1) we get $|\phi_\tau(t)| \le |\dot{y}_+(t)|$ for any $t \ge \bar{T}$. Hence

$$\int_{\bar{T}}^{\infty} (t-\bar{T})|\phi_\tau(t)|dt \le \int_{\bar{T}}^{\infty} (t-\bar{T})|\dot{y}_+(t)|dt.$$

As a consequence, using F_1) and (23)–(24):

$$\begin{aligned}|\Psi_\tau(\omega)| &\ge \left|\int_{\bar{T}}^{\infty} \phi_\tau(t)dt\right| - \omega\int_{\bar{T}}^{\infty}(t-\bar{T})|\phi_\tau(t)|dt \\ &\ge \frac{1}{\sqrt{\beta}}\left(\sqrt{1+\Omega_\tau} - \sqrt{1-\mathrm{e}^{\theta_\tau^-}}\sqrt{1-\mathrm{e}^{\theta_\tau^-}+\Omega_\tau}\right) - \omega\int_{\bar{T}}^{\infty}(t-\bar{T})|\dot{y}_+(t)|dt.\end{aligned}$$

Next, since $\dot{y}_+(t) = \frac{1}{\tau}\mathrm{e}^{\theta^-(t)/2}\sin(\tau\theta^-(t))$, we see that $\dot{y}_+(T_\pi) = 0$ and $\dot{y}_+(t) > 0$ for $T_\pi > t \ge \bar{T}$, while $\dot{y}_+(t) < 0$ for $T_\pi < t$. Then, integrating by parts and using $y_+(t) = \dot{\theta}^-(t)$ (see (F_1)) we obtain

$$\begin{aligned}\int_{\bar{T}}^{\infty}(t-\bar{T})|\dot{y}_+(t)|dt &= \int_{\bar{T}}^{T_\pi}(t-\bar{T})\dot{y}_+(t)dt - \int_{T_\pi}^{\infty}(t-\bar{T})\dot{y}_+(t)dt \\ &= 2(T_\pi-\bar{T})y_+(T_\pi) - \int_{\bar{T}}^{T_\pi}\dot{\theta}^-(t)dt + \int_{T_\pi}^{\infty}\dot{\theta}^-(t)dt \\ &= 2(T_\pi-\bar{T})y_+(T_\pi) + \frac{\pi-\arctan 2\tau}{\tau},\end{aligned}$$

since $\theta^-(T_\pi) = -\pi/\tau$. Since $\dot{\theta}^-(t) = y_+(t) \ge y_+(\bar{T})$ for $T_\pi \ge t \ge \bar{T}$, we get

$$\frac{\arctan 2\tau}{\tau} = -\frac{\pi}{\tau} - \theta_\tau^- = \theta^-(T_\pi) - \theta^-(\bar{T}) \ge y_+(\bar{T})(T_\pi-\bar{T}),$$

and so

$$T_\pi - \bar{T} \le \frac{\arctan 2\tau}{\tau y_+(\bar{T})} = \frac{\arctan 2\tau}{\tau}\sqrt{\frac{\beta}{1+\Omega_\tau}}.$$

From $y_+(t) = \dot{\theta}^-(t)$, we obtain immediately

$$y_+(T_\pi) = \sqrt{\frac{1-u_\tau(\theta^-(T_\pi))}{\beta}} = \sqrt{\frac{1-u_\tau(-\pi/\tau)}{\beta}} = \sqrt{\frac{1+\mathrm{e}^{-\frac{\pi}{2\tau}}}{\beta}}$$

and then

$$\begin{aligned}|\Psi_\tau(\omega)| \ge &\frac{1}{\sqrt{\beta}}\left(\sqrt{1+\Omega_\tau} - \sqrt{1-\mathrm{e}^{\theta_\tau^-}}\sqrt{1-\mathrm{e}^{\theta_\tau^-}+\Omega_\tau}\right) \\ &-\omega\left(2\frac{\arctan 2\tau}{\tau}\sqrt{\frac{1+\mathrm{e}^{-\frac{\pi}{2\tau}}}{1+\Omega_\tau}} + \frac{\pi-\arctan 2\tau}{\tau}\right) > 0\end{aligned}$$

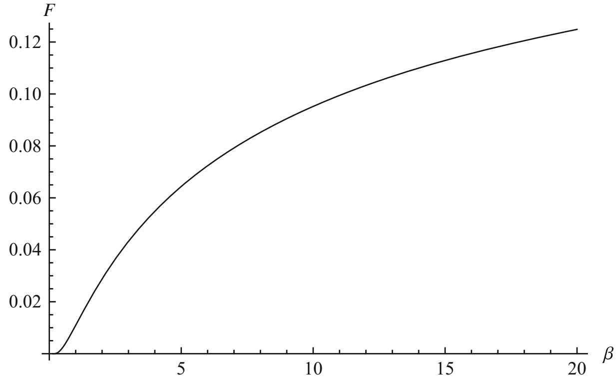

Fig. 4 The graph of $F(\beta) := C\left(\sqrt{8\beta - 1}/2\right)$ over the interval $(1/8, 20]$

for any ω such that

$$0 < \omega < C(\tau) := \frac{2\sqrt{2}\tau}{\sqrt{4\tau^2+1}} \frac{\sqrt{1+\Omega_\tau} - \sqrt{1-e^{\theta_\tau^-}}\sqrt{1-e^{\theta_\tau^-}+\Omega_\tau}}{2\arctan 2\tau\sqrt{\frac{1+e^{-\frac{\pi}{2\tau}}}{1+\Omega_\tau}} + \pi - \arctan 2\tau}.$$

Clearly, $\lim_{\tau\to 0_+} C(\tau) = 0$ and $\lim_{\tau\to\infty} C(\tau) = \frac{2\sqrt{2}}{\pi(2\sqrt{2}+1)} \doteq 0.235166$. The graph of $F(\beta) := C\left(\sqrt{8\beta-1}/2\right)$ over interval $(1/8, 20)$ looks like on Fig. 4.

Thus, system (25) with $f_\pm(x)$ as in (16) and $g(t)$ as in the statement of the Theorem is chaotic also for $0 < \omega < F(\beta)$ provided ε is sufficiently small. The proof is complete.

For instance, a numerical evaluation shows that for $\beta = 25$, and $\omega \in (0,\infty)\backslash[0.13, 10.65]$, system (14), (15) is chaotic for $\varepsilon \neq 0$.

Furthermore, since $\Psi_\tau(\omega)$ is analytical, there is at most a finite number of $\omega_1, \ldots, \omega_{n_\beta} \in [F(\beta), D(\beta)]$ such that for any $\omega > 0$ and $\omega \notin \{\omega_1, \ldots, \omega_{n_\beta}\}$, then there is a chaos like in Theorem 3. An open problem remains to estimate n_β. On the other hand, let $h_\tau(\eta) := 1 - e^{-\frac{\eta}{2\tau}}\cos\eta - \frac{1}{2\tau}e^{-\frac{\eta}{2\tau}}\sin\eta$. Then, using Taylor's formula $h_\tau(\eta) = h_\tau(0) + h_\tau'(0)\eta + \frac{1}{2}h_\tau''(0)\eta^2 + \frac{1}{6}h_\tau'''(\eta_0)\eta^3$ and $h_\tau(0) = h_\tau'(0) = 0$, $h_\tau''(0) = \frac{4\tau^2+1}{4\tau^2}$ along with

$$h_\tau'''(\eta) = \frac{1}{16\tau^4}e^{-\frac{\eta}{2\tau}}\left(\sin\eta - 4\tau(\cos\eta + 4\tau^2\cos\eta + 4\tau^3\sin\tau)\right),$$

for $0 \le \eta \le \pi/2$ and $\tau \ge 1$, we get

$$\begin{aligned} 1 - e^{-\frac{\eta}{2\tau}}\cos\eta - \frac{1}{2\tau}e^{-\frac{\eta}{2\tau}}\sin\eta &\ge \frac{1+4\tau^2}{8\tau^2}\eta^2 - \frac{1+4\tau(1+4\tau^2+4\tau^3)}{16\tau^4}\frac{\eta^3}{6} \\ &\ge \left(\frac{1+4\tau^2}{8\tau^2} - \frac{1+4\tau(1+4\tau^2+4\tau^3)}{16\tau^4}\frac{\pi}{12}\right)\eta^2 \ge \frac{120-37\pi}{192}\eta^2 > 0.01\eta^2 \end{aligned}$$

and for $\pi/2 \le \eta \le 3\pi/2$, $\tau \ge 1$:

$$1 - e^{-\frac{\eta}{2\tau}}\cos\eta - \frac{1}{2\tau}e^{-\frac{\eta}{2\tau}}\sin\eta \ge 1 - \frac{1}{2\tau} \ge \frac{1}{2} \ge \frac{2}{9\pi^2}\eta^2 > 0.01\eta^2.$$

Hence,

$$\Gamma_\tau(\eta) \leq -\frac{\tau}{\sqrt{\beta}}\frac{1}{10}\eta \leq -\frac{2\sqrt{2}}{\sqrt{5}}\frac{1}{10}\eta = -\frac{\sqrt{2}}{5\sqrt{5}}\eta$$

for $\tau \geq 1$ and $0 \leq \eta \leq \arctan 2\tau + \pi$ $\left(\text{note}\,\frac{\tau}{\sqrt{\beta}} = \sqrt{\frac{8\beta-1}{4\beta}} \geq 2\frac{\sqrt{2}}{\sqrt{5}} \text{ for } \beta \geq \frac{5}{8} \iff \tau \geq 1\right)$. Consequently, we obtain

$$\eta(t) \leq (\arctan 2\tau + \pi)\,\mathrm{e}^{-\frac{\sqrt{2}}{5\sqrt{5}}(t-\bar{T})} \leq \frac{3\pi}{2}\,\mathrm{e}^{-\frac{\sqrt{2}}{5\sqrt{5}}(t-\bar{T})} \tag{28}$$

for any $t \geq \bar{T}$ and $\tau \geq 1$. Let $\varphi_\tau(t) := \tau\phi_\tau(t)$ and

$$\eta_\infty(t) := 4\arctan\left[\mathrm{e}^{-t+\bar{T}}\cot\frac{\pi}{8}\right].$$

be the solution of the Cauchy problem:

$$\dot{\eta} = \Gamma_\infty(\eta) := \lim_{\tau\to\infty}\Gamma_\tau(\eta) = -\sqrt{2}\sqrt{1-\cos\eta},$$
$$\eta(\bar{T}) = \lim_{\tau\to\infty}\arctan 2\tau + \pi = 3\pi/2.$$

Then, using the definition of $\phi_\tau(\eta)$, F_5) and (28), we get

$$|\varphi_\tau(t)| \leq |\sin\eta(t)| \leq \frac{3\pi}{2}\,\mathrm{e}^{-\frac{\sqrt{2}}{5\sqrt{5}}(t-\bar{T})} \leq 4\left(1+\sqrt{2}\right)\,\mathrm{e}^{-\frac{\sqrt{2}}{5\sqrt{5}}(t-\bar{T})}, \tag{29}$$

and

$$|\sin\eta_\infty(t)| \leq |\eta_\infty(t)| \leq 4\,\mathrm{e}^{-t+\bar{T}}\cot\frac{\pi}{8} \leq 4\left(1+\sqrt{2}\right)\,\mathrm{e}^{-\frac{\sqrt{2}}{5\sqrt{5}}(t-\bar{T})} \tag{30}$$

for any $t \geq \bar{T}$ and $\tau \geq 1$. From the continuous dependence on the data, we get $\lim_{\tau\to\infty}\eta(t) = \eta_\infty(t)$ uniformly on compact sets in $[\bar{T},\infty)$. Hence,

$$\lim_{\tau\to\infty}\varphi_\tau(t) = \lim_{\tau\to\infty} -\,\mathrm{e}^{\frac{\eta(t)}{2\tau}+\theta_\tau^-}\sin\eta(t) = -\sin\eta_\infty(t) \tag{31}$$

uniformly on compact sets in $[\bar{T},\infty)$. Consequently, because of (29), (30) and (31), we can apply the Lebesque dominated convergence theorem to obtain (see also F_1))

$$\left|\tau\Psi_\tau(\omega) + \int_{\bar{T}}^{\infty}\sin\eta_\infty(t)\,\mathrm{e}^{\imath\omega t}\,dt\right| \leq \int_{\bar{T}}^{\infty}|\sin\eta_\infty(t) + \varphi_\tau(t)|\,dt \to 0$$

as $\tau \to \infty$. Some computations show

$$\sin\eta_\infty(t) = \frac{4\,\mathrm{e}^{t-\bar{T}}\cot\frac{\pi}{8}\left(\mathrm{e}^{2(t-\bar{T})} - \cot^2\frac{\pi}{8}\right)}{\left(\mathrm{e}^{2(t-\bar{T})} + \cot^2\frac{\pi}{8}\right)^2}$$
$$= \frac{4\,\mathrm{e}^{t-\bar{T}}(1+\sqrt{2})\left(\mathrm{e}^{2(t-\bar{T})} - 3 - 2\sqrt{2}\right)}{\left(\mathrm{e}^{2(t-\bar{T})} + 3 + 2\sqrt{2}\right)^2}.$$

Then,

$$\int_{\bar{T}}^{\infty} \sin \eta_\infty(t)\, \mathrm{e}^{\imath \omega t}\, dt = \mathrm{e}^{\imath \omega \bar{T}} \int_0^{\infty} \frac{4\,\mathrm{e}^t (1+\sqrt{2}) \left(\mathrm{e}^{2t} - 3 - 2\sqrt{2}\right)}{\left(\mathrm{e}^{2t} + 3 + 2\sqrt{2}\right)^2}\, \mathrm{e}^{\imath \omega t}\, dt.$$

Next, we need the following result.

Lemma 1 *Consider a function* $\Upsilon : [0, \infty) \to [0, \infty)$ *defined as*

$$\Upsilon(\omega) := \left| \int_{\bar{T}}^{\infty} \sin \eta_\infty(t)\, e^{\imath \omega t}\, dt \right|.$$

Then, $\Upsilon(\omega) > 0$ *for any* $\omega \geq 0$.

Proof Let $a = 1 + \sqrt{2}$ so that $a^2 = 3 + 2\sqrt{2}$. Integration by parts gives

$$\int_0^{\infty} \frac{\mathrm{e}^t \left(\mathrm{e}^{2t} - a^2\right)}{\left(\mathrm{e}^{2t} + a^2\right)^2}\, \mathrm{e}^{\imath \omega t}\, dt = -\frac{\mathrm{e}^t}{\mathrm{e}^{2t} + a^2}\, \mathrm{e}^{\imath \omega t}\Big|_0^{\infty} + \imath \omega \int_0^{\infty} \frac{\mathrm{e}^{t(1+\omega \imath)}}{\mathrm{e}^{2t} + a^2} dt$$

$$= \frac{1}{1 + a^2} + \imath \omega \int_0^{\infty} \frac{\mathrm{e}^{t(1+\imath \omega)}}{\mathrm{e}^{2t} + a^2}\, dt.$$

Now, applying Cauchy theorem to the meromorphic function $F(z) = \frac{\mathrm{e}^{z(1+\imath\omega)}}{\mathrm{e}^{2z}+a^2}$ in the rectangle $R_\rho := \{z \in \mathbb{C} \mid 0 \leq \Re z \leq \rho,\ 0 \leq \Im z \leq \pi\}$ for ρ large, we derive

$$\int_0^{\rho} \frac{\mathrm{e}^{t(1+\imath\omega)}}{\mathrm{e}^{2t} + a^2} dt + \imath \int_0^{\pi} \frac{\mathrm{e}^{(\rho+\imath t)(1+\imath\omega)}}{\mathrm{e}^{2(\rho+\imath t)} + a^2}\, dt - \int_0^{\rho} \frac{\mathrm{e}^{(t+i\pi)(1+\imath\omega)}}{\mathrm{e}^{2(t+i\pi)} + a^2}\, dt$$
$$- \imath \int_0^{\pi} \frac{\mathrm{e}^{\imath t(1+\imath\omega)}}{\mathrm{e}^{2\imath t} + a^2} dt = 2\pi \imath \operatorname{Res}\left(F(z), \ln a + \imath \frac{\pi}{2}\right).$$

Now, we observe that

$$\int_0^{\pi} \frac{\mathrm{e}^{(\rho+\imath t)(1+\imath\omega)}}{\mathrm{e}^{2(\rho+\imath t)} + a^2}\, dt = \frac{\mathrm{e}^{\imath\omega\rho}}{\mathrm{e}^{\rho}} \int_0^{\pi} \frac{\mathrm{e}^{\imath t(1+\imath\omega)}}{\mathrm{e}^{2\imath t} + a^2\, \mathrm{e}^{-2\rho}}\, dt$$

and then

$$\left| \int_0^{\pi} \frac{\mathrm{e}^{(\rho+\imath t)(1+\imath\omega)}}{\mathrm{e}^{2(\rho+\imath t)} + a^2}\, dt \right| \leq 2\, \mathrm{e}^{-\rho} \int_0^{\pi} \mathrm{e}^{-\omega t}\, dt$$

provided $\rho > \frac{\log(2a^2)}{2}$. So, using also $\mathrm{e}^{\imath\pi(1+\imath\omega)} = -\,\mathrm{e}^{-\pi\omega}$ and taking $\lim_{\rho\to\infty}$, we get

$$(1 + \mathrm{e}^{-\pi\omega}) \int_0^{\infty} \frac{\mathrm{e}^{t(1+\imath\omega)}}{\mathrm{e}^{2t} + a^2}\, dt = \imath \left[2\pi \operatorname{Res}\left(F(z), \ln a + \imath \frac{\pi}{2}\right) + \int_0^{\pi} \frac{\mathrm{e}^{\imath t}}{\mathrm{e}^{2\imath t} + a^2}\, \mathrm{e}^{-\omega t}\, dt \right].$$

Setting for simplicity $z^* = \log a + \imath \frac{\pi}{2}$ we get:

$$\operatorname{Res}\left(F(z), z^*\right) = \lim_{z \to z^*} \mathrm{e}^{(1+\imath\omega)z} \frac{z - z^*}{\mathrm{e}^{2z} + a^2} = \frac{\mathrm{e}^{(1+\imath\omega)z^*}}{2\,\mathrm{e}^{2z^*}}$$
$$= -\frac{\imath}{2a}\,\mathrm{e}^{-\pi\omega/2}\left[\cos(\omega \log a) + \imath \sin(\omega \log a)\right].$$

Moreover,

$$\int_0^\pi \frac{\mathrm{e}^{\imath t}}{\mathrm{e}^{2\imath t} + a^2}\,\mathrm{e}^{-\omega t}\,dt = \frac{1}{2}\left[\Lambda(a) + \Lambda(-a)\right]$$

where

$$\Lambda(a) = \int_0^\pi \frac{\mathrm{e}^{-\omega t}}{\mathrm{e}^{\imath t} + \imath a}\,dt.$$

It is easy to verify

$$\int \frac{\mathrm{e}^{-\omega t}}{\mathrm{e}^{\imath t} + \imath a}\,dt = \frac{\imath\,\mathrm{e}^{-\omega t}}{a\omega} F(1, \imath\omega, 1 + \imath\omega, \imath\,\mathrm{e}^{\imath t}/a) + c,$$

where $F(a_1, a_2, a_3, z)$ is the hypergeometric function [43]. Note $|\imath\,\mathrm{e}^{\imath t}/a| = 1/a = (\sqrt{2} - 1) < 1$, so $F(1, \imath\omega, 1 + \imath\omega, \imath\,\mathrm{e}^{\imath t}/a)$ is well-defined uniformly for $t \in \mathbb{R}$. Hence,

$$\Lambda(a) = \frac{\imath\,\mathrm{e}^{-\omega\pi}}{a\omega} F(1, \imath\omega, 1 + \imath\omega, -\imath/a) - \frac{\imath}{a\omega} F(1, \imath\omega, 1 + \imath\omega, \imath/a),$$

and so

$$\begin{aligned}\int_0^\pi \frac{\mathrm{e}^{\imath t}}{\mathrm{e}^{2\imath t} + a^2}\,\mathrm{e}^{-\omega t}\,dt &= \frac{1}{2}\left[\Lambda(a) + \Lambda(-a)\right]\\
&= \frac{\imath}{2a\omega}\left(1 + \mathrm{e}^{-\omega\pi}\right)\left(F(1, \imath\omega, 1 + \imath\omega, -\imath/a) - F(1, \imath\omega, 1 + \imath\omega, \imath/a)\right)\\
&= \frac{\imath}{2a\omega}\left(1 + \mathrm{e}^{-\omega\pi}\right)\sum_{k=0}^\infty \frac{\imath\omega}{k + \imath\omega}\left(\left(-\frac{\imath}{a}\right)^k - \left(\frac{\imath}{a}\right)^k\right)\\
&= \imath\left(1 + \mathrm{e}^{-\omega\pi}\right)\sum_{k=0}^\infty \frac{(-1)^k}{(2k + 1 + \imath\omega)a^{2k+2}}.\end{aligned}$$

Hence,

$$\int_0^\infty \frac{\mathrm{e}^{t(1+\imath\omega)}}{\mathrm{e}^{2t} + a^2}\,dt = \frac{\pi}{2a}\,\mathrm{sech}\frac{\pi\omega}{2}\left[\cos(\omega \log a) + \imath \sin(\omega \log a)\right] + \sum_{k=0}^\infty \frac{(-a^{-2})^{k+1}}{2k + 1 + \imath\omega}.$$

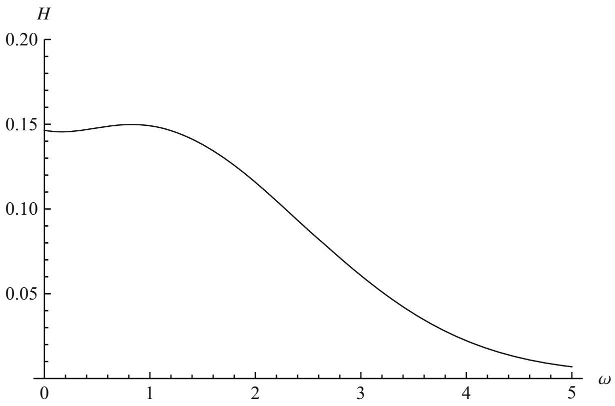

Fig. 5 The graph of $H(\omega)$ over the interval [0, 5]

Summarizing, we obtain

$$\begin{aligned}\int_0^\infty \frac{e^t\left(e^{2t}-a^2\right)}{\left(e^{2t}+a^2\right)^2}e^{\imath\omega t}dt &= \frac{1}{1+a^2}+\frac{\omega\pi}{2a}\operatorname{sech}\frac{\pi\omega}{2}\left[-\sin(\omega\log a)+\imath\cos(\omega\log a)\right]\\ &\quad+\sum_{k=0}^\infty\frac{\imath\omega(-a^{-2})^{k+1}}{2k+1+\imath\omega}\\ &= \frac{1}{1+a^2}+\frac{\omega\pi}{2a}\operatorname{sech}\frac{\pi\omega}{2}\left[-\sin(\omega\log a)+\imath\cos(\omega\log a)\right]\\ &\quad-\frac{\imath\omega}{(1+\imath\omega)a^2}+\sum_{k=1}^\infty\frac{\imath\omega(-a^{-2})^{k+1}}{2k+1+\imath\omega},\end{aligned}$$

and using

$$\left|\sum_{k=1}^\infty\frac{\imath\omega(-a^{-2})^{k+1}}{2k+1+\imath\omega}\right|\le\frac{\omega}{\sqrt{9+\omega^2}}\sum_{k=1}^\infty a^{-2(k+1)}=\frac{\omega}{\sqrt{9+\omega^2}}\frac{1}{a^2(a^2-1)},$$

we derive

$$\begin{aligned}\left|\int_0^\infty \frac{e^t\left(e^{2t}-a^2\right)}{\left(e^{2t}+a^2\right)^2}e^{\imath\omega t}dt\right| &\ge \left|\frac{1}{1+a^2}+\frac{\omega\pi}{2a}\operatorname{sech}\frac{\pi\omega}{2}\left[-\sin(\omega\log a)+\imath\cos(\omega\log a)\right]\right.\\ &\quad\left.-\frac{\imath\omega}{(1+\imath\omega)a^2}\right|-\frac{\omega}{\sqrt{9+\omega^2}}\frac{1}{a^2(a^2-1)}:=H(\omega).\end{aligned}$$

By plotting the graph of $H(\omega)$ over [0, 5] on Fig. 5, we see that $H(\omega)>0$ for any $\omega\in[0,5]$.

On the other hand, setting $g(t):=\frac{e^t\left(e^{2t}-a^2\right)}{\left(e^{2t}+a^2\right)^2}$, we derive

$$\begin{aligned}g(0)&=-\frac{a^2-1}{(a^2+1)^2},\quad g'(0)=-\frac{(a^2-2a-1)(a^2+2a-1)}{(a^2+1)^3},\\ g''(0)&=-\frac{(a^2-1)(a^4-22a^2+1)}{(a^2+1)^4},\quad g'''(t)=\frac{e^t}{e^{2t}+a^2}\widetilde{g}(e^{2t})\end{aligned}$$

with

$$\widetilde{g}(\kappa) = -\frac{\kappa^4 - 76a^2\kappa^3 + 230a^4\kappa^2 - 76a^6\kappa + a^8}{(\kappa + a^2)^4}.$$

Hence, using $a = \sqrt{2} + 1$,

$$g(0) = -\frac{1}{4}\left(\sqrt{2} - 1\right), \quad g'(0) = 0, \quad g''(0) = \frac{1}{2}\left(\sqrt{2} - 1\right).$$

Next,

$$\widetilde{g}'(\kappa) = \frac{16a^2(a^2 - \kappa)\left(5\kappa^2 - 38a^2\kappa + 5a^4\right)}{(\kappa + a^2)^5}.$$

Solving $\widetilde{g}'(\kappa) = 0$, we get $\kappa_1 = a^2, \kappa_{2,3} = \frac{19 \pm 4\sqrt{21}}{5}a^2$. Inserting the solutions into $\widetilde{g}(\kappa)$, we obtain $\widetilde{g}(\kappa_1) = -5, \widetilde{g}(\kappa_2) = \widetilde{g}(\kappa_3) = \frac{19}{6}$ and since $\lim_{\kappa \to \pm\infty} \widetilde{g}(\kappa) = 1$, we see that $|\widetilde{g}(\kappa)| \leq 5$. Consequently,

$$\int_0^\infty \left|g'''(t)\right| dt \leq 5 \int_0^\infty \frac{e^t}{e^{2t} + a^2} dt = 5\frac{\pi - 2\arctan\left(\frac{1}{a}\right)}{2a} = \frac{15}{8}\pi\left(\sqrt{2} - 1\right) < 6\left(\sqrt{2} - 1\right)$$

(having used $a = \sqrt{2} + 1$). Finally, integrating by parts, we derive

$$\begin{aligned}
\left|\int_0^\infty \frac{e^t\left(e^{2t} - a^2\right)}{\left(e^{2t} + a^2\right)^2} e^{\imath\omega t} dt\right| &= \left| -\frac{g(0)}{\imath\omega} - \frac{g'(0)}{\omega^2} + \frac{g''(0)}{\imath\omega^3} + \frac{1}{\imath\omega^3}\int_0^\infty g'''(t)\, e^{\imath\omega t} dt\right| \\
&\geq \left|\frac{1}{\imath\omega}\frac{\sqrt{2} - 1}{4} + \frac{1}{\imath\omega^3}\frac{\sqrt{2} - 1}{2}\right| - \frac{1}{\omega^3}\int_0^\infty \left|g'''(t)\right| dt \\
&\geq \frac{\sqrt{2} - 1}{4\omega} + \frac{\sqrt{2} - 1}{2\omega^3} - \frac{6\left(\sqrt{2} - 1\right)}{\omega^3} = \frac{\sqrt{2} - 1}{4\omega}\left(1 - \frac{22}{\omega^2}\right) > 0
\end{aligned}$$

for any $\omega > \sqrt{22}$. Note $\sqrt{22} < 5$. Summarizing, $\Upsilon(\omega) > 0$ for any $\omega \geq 0$. The proof is finished.

Consequently, using Lemma 1, the statement of Theorem 3 can be extended as follows.

Theorem 4 *There exists a continuous function $G(\omega) : (0, \infty) \to \left[\frac{1}{8}, \infty\right)$ such that for any given constants $\omega \in (0, \infty)$, $\beta > G(\omega)$, $\omega_1 > 0$ and an almost periodic C^2–function $q_1(t)$ with bounded derivatives and such that its second-order derivative is uniformly continuous, there exists $\varepsilon_0 = \varepsilon_0(\beta, \omega, \omega_1, q_1(\cdot))$ such that for $0 < |\varepsilon| < \varepsilon_0$ and*

$$g(t) = \begin{pmatrix} \cos(\omega t) \\ 0 \\ q_1(\omega_1 t) \end{pmatrix}$$

system (14), (15) is chaotic.

Certainly, a lower bound $G(\omega)$ for β could be numerically estimated, but we do not carry out these awkward computations in this paper. From Theorem 3, it would be enough to estimate $G(\omega)$ on the interval $[0.2, 9.3]$.

Remark 3 We can directly apply the above results to a 2DDE

$$\begin{aligned} \dot{y} &= z \\ \dot{z} &= y - \frac{1}{2}y^3 + yz \quad \text{for } z < \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}} \\ \dot{y} &= z \\ \dot{z} &= y - \frac{1}{2}y^3 + (y-q)z \quad \text{for } z > \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}} \end{aligned} \tag{32}$$

possessing a sliding homoclinic orbit to a saddle (0,0) for any $q \geq 36.1$. Indeed, we start from (6) with $\beta = 1/2$. Then, we get $\tau = \sqrt{3}/2$ (cf. (10)), $\Omega_\tau = \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}}$ (cf. (12)) and $y_+(\bar{T}) = \sqrt{2 + 2\,\mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}}}$ (cf. (13)). The interval

$$\left\{ \left(y,\ \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}} \right) \in \mathbb{R}^2 \mid 0 \leq y \leq y_+(\bar{T}) \right\}$$

is attractive from above for (32), if

$$y_+(\bar{T}) + \frac{1}{2}y_+(\bar{T})^3 + \big(y_+(\bar{T}) - q\big)\,\Omega_\tau < 0$$

so

$$q > \frac{y_+(\bar{T}) + \frac{1}{2}y_+(\bar{T})^3 + y_+(\bar{T})\Omega_\tau}{\Omega_\tau} \doteq 36.093.$$

Hence, we could take $q \geq 36.1$.

Next, we add a periodic perturbation to (32)

$$\begin{aligned} \dot{y} &= z \\ \dot{z} &= y - \frac{1}{2}y^3 + yz + \varepsilon \cos \omega t \quad \text{for } z < \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}} \\ \dot{y} &= z \\ \dot{z} &= y - \frac{1}{2}y^3 + (y-q)z + \varepsilon \cos \omega t \quad \text{for } z > \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}}. \end{aligned} \tag{33}$$

The Melnikov function is the same as in Sect. 3, and we can apply Theorem 3 with $F(1/2) \doteq 0.00228$ and $D(1/2) \doteq 25.3974$. As a consequence, if either $0 < \omega < 0.0022$ or $\omega > 25.3975$, then (33) is chaotic.

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