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C^k-solvability near the characteristic set for a class of planar complex vector fields of infinite type

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Abstract This paper deals with semi-global C^k-solvability of complex vector fields of the form $\mathsf{L} = \partial/\partial t + x^r(a(x) + ib(x))\partial/\partial x$, $r \geq 1$, defined on $\Omega_\epsilon = (-\epsilon, \epsilon) \times S^1$, $\epsilon > 0$, where a and b are C^∞ real-valued functions in $(-\epsilon, \epsilon)$. It is shown that the interplay between the order of vanishing of the functions a and b at $x = 0$ influences the C^k-solvability at $\Sigma = \{0\} \times S^1$. When $r = 1$, it is permitted that the functions a and b of L depend on the x and t variables, that is, $\mathsf{L} = \partial/\partial t + x(a(x,t) + ib(x,t))\partial/\partial x$, where $(x,t) \in \Omega_\epsilon$.

Keywords Solvability near the characteristic set · Complex vector fields · Normalization · Condition $(\mathcal{P})$

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1 Introduction

Let

$$\mathsf{L} = \partial/\partial t + (a(x) + ib(x))\partial/\partial x, \quad a, b \in C^\infty(S^1), \tag{1}$$

be a complex vector field, defined on $\Omega_\epsilon = (-\epsilon, \epsilon) \times S^1$, $\epsilon > 0$. Assume that $\Sigma = \{0\} \times S^1$ is the characteristic set of the structure associated with L and that L is of infinity type along Σ. Hence, L is elliptic on $\Omega_\epsilon \backslash \Sigma$ and $a(0) = b(0) = 0$. In particular, $b(x) \neq 0$ if $x \neq 0$.

We recall the concept of infinity type: a point p in the characteristic set $\mathcal{C}(\mathsf{L})$ is said to be of finite type ν ($\nu \in \mathbb{Z}_+$) if there exists a Lie bracket of L and $\overline{\mathsf{L}}$ of length ν which is nonzero at p. If $p \in \Sigma$ is not of finite type then it is said to be of infinite type.

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We say that L is locally solvable on Ω_ϵ if given $p \in \Omega_\epsilon$ and $f \in C^\infty(\mathcal{U})$, where $\mathcal{U} \subset \Omega_\epsilon$ is an open neighborhood of p, there exists a distribution u such that $\mathsf{L}u = f$ in $\mathcal{V}$, where $\mathcal{V} \subset \mathcal{U}$ is some neighborhood of p.

The local solvability of linear partial differential equations (see, for instance, [12,13] and [15]) is characterized by the well-known *Nirenberg–Treves* condition $(\mathcal{P})$. For operators given by (1), condition $(\mathcal{P})$ has a simple statement: L satisfies condition $(\mathcal{P})$ if and only if the function b does not change sign on any integral curve of $\Re(\mathsf{L}) = \partial/\partial t + a(x)\partial/\partial x$ (see [12]).

The operator L, given by (1), satisfies condition $(\mathcal{P})$ under our hypotheses, and the local solvability is well understood.

The problem is interesting if we are concerned with solvability in a full neighborhood of Σ. Hörmander (see [13]) gives us the following concept of semi-global solvability:

Let P be a linear partial differential operator defined on a smooth manifold $\mathcal{M}$, and let K be a compact subset of $\mathcal{M}$. We say that P is solvable at K if for every f in a subspace of $C^\infty(\mathcal{M})$ of finite codimension there is a distribution u in $\mathcal{M}$ such that $Pu = f$ in a neighborhood of K.

Condition $(\mathcal{P})$ is necessary for solvability at K of P (see [13]); moreover, for operators of principal type on $\mathcal{M}$, condition (P) is sufficient for solvability at K (see [13]). The operator L defined by (1) is not of principal type on Ω_ϵ. Let us recall the definition of principal type: as in [13] we say that a point $(x, \xi) \in T^*\mathcal{M}\backslash 0$ is a characteristic point of P if $p(x, \xi) = 0$, where p is the principal symbol of operator P. When P satisfies condition $(\mathcal{P})$ we say that P is of principal type if P satisfies the following geometric condition (see [12] and [13]), for each K compact subset of $\mathcal{M}$: (GC) every characteristic point of P over K lies on a compact interval of a bicharacteristic of $\Re(pq)$, on which $q \neq 0$, with no characteristic endpoint over K.

A bicharacteristic of $\Re(pq)$ is an integral curve of the Hamilton field of $\Re(pq)$ over which $\Re(pq)$ vanishes. Note that the principal symbol of our operator L is $\ell(x, t, \xi, \tau) = \tau + (a(x) + ib(x))\xi$, which implies that a characteristic point of L is of the form $(0, t, \xi, 0)$, with $\xi \neq 0$. Hence, (GC) is not satisfied.

The solvability at Σ was studied for operators given by (1), in papers such as [4] and [11]; moreover, [3] and [5] studied this problem for a more general class of complex vector fields, for instance, when the functions a and b of L depend on x and t variables. See, for instance, [1,2,7–9] and [13], for more related papers. In [7] $b \equiv 0$ was permitted.

It follows from [11] (see also [4]):

Theorem 1 (Theorem 2.1, [11]) *Consider the complex vector field*

$$L = \partial/\partial t + (a(x) + ib(x))\partial/\partial x, \quad b \not\equiv 0,$$

defined on Ω_ϵ, where a and b are real-valued smooth functions. Assume that the characteristic set of L is equal to $\Sigma = \{0\} \times S^1$. If we have $(a + ib)(x) = x^n a_0(x) + ix^m b_0(x)$, where $b_0(0) \neq 0$ and $2 \leq m < 2n - 1$, then given f belonging to $C^\infty(\Omega_\epsilon)$ and satisfying

$$\int_0^{2\pi} \frac{\partial^j f}{\partial x^j}(0, t)\mathrm{d}t = 0, \quad j = 0, \ldots, r - 1, \quad r = \min\{m, n\}, \tag{2}$$

there exists a C^∞ function u solving the equation $Lu = f$ in a neighborhood of Σ.

Theorem 2 (Theorem 2.2, [11]) *Consider the complex vector field*

$$L = \partial/\partial t + (a(x) + ib(x))\partial/\partial x, \quad b \not\equiv 0,$$

defined on Ω_ϵ, where a and b are real-valued smooth functions. Assume that the characteristic set of L is equal to $\Sigma = \{0\} \times S^1$. If we have $(a + ib)(x) = x^n a_0(x) + i x^m b_0(x)$ and one of the following conditions is satisfied

1⋆. $a_0(0) \neq 0$ *and* $m > 2n - 1$;
2⋆. $b_0(0) \neq 0$ *and* $m = 1$;
3⋆. $a_0(0) \neq 0, b_0(0) \neq 0, m = 2n - 1$ *and* $n \geq 2$;

then there exists a function f belonging to $C^\infty(\Omega_\epsilon)$ and satisfying (2), for which the equation $Lu = f$ does not have C^∞ solution in any neighborhood of Σ.

Theorem 2 leaves us the following question:

Under the hypotheses of Theorem 2, does the equation $\mathsf{L}u = f$ have C^k solution in a neighborhood of Σ, for some $k \geq 1$?

The goal of this paper is to answer the question above. One of the motivations for this paper was [14], where the existence of C^k solutions was studied in a full neighborhood of Σ, considering the model operator

$$T_\lambda = \lambda\partial/\partial t - ix\partial/\partial x; \tag{3}$$

moreover, [14] proved theorems about the normalization of complex vector fields, and the results about solvability at Σ of (3) can be used to study the solvability of more general complex vector fields. Indeed, [14] looked for C^k solutions, in a neigborhood of Σ, for equation $T_\lambda u = f$, when $f \in C^\infty(\Omega_\epsilon)$ satisfies

$$\int_0^{2\pi} f(0, t)\mathrm{d}t = 0.$$

An important means of finding C^k solutions was to use results reported by Vekua (see [16]) on generalized analytic functions. In this paper, we have employed Vekua's results to find C^k solutions for other classes of complex vector fields. Let us give some information about these results: let $f, g \in L^1(G)$, where G is a bounded domain. If f and g satisfy the relation

$$\int\int_G g\frac{\partial\phi}{\partial\bar{z}}\mathrm{d}x\mathrm{d}y + \int\int_G f\phi\mathrm{d}x\mathrm{d}y = 0, \quad \forall\, \phi \in C_0^1(G),$$

the function f is said to be the *generalized derivative* of g with respect to $\bar{z}$. We will denote by $D_{\bar{z}}(G)$ the linear manifold constituted by the functions having generalized derivatives. The function $w(z)$ is said to satisfy the equation

$$\mathsf{X}(w) \equiv \frac{\partial w}{\partial\bar{z}} + Aw + B\bar{w} = F, \quad \text{where } A, B, F \in L^p(G), \quad p > 2,$$

in a neighborhood G_0 of a point z_0 if $w \in D_{\bar{z}}(G_0)$ and $\mathsf{X}(w) = F$ almost everywhere in G_0. If $\mathsf{X}(w) = F$ in a neighborhood of every point of G then w is called *regular solution*. As in [16], we will denote by $\mathcal{U}(A, B, F, G)$ the class of regular solutions of $\mathsf{X}u = F$ in G. Theorem 3.3 of Chapter III of [16] states that: if $A, B, F \in C^{m,\alpha}(G)$, $m \geq 0$ and $0 < \alpha < 1$, then the function $w(z)$ of the classe $\mathcal{U}(A, B, F, G)$ belongs to the class $C^{m+1,\alpha}(G)$.

The present paper is organized as follows:

In Sect. 2, we will consider that L satisfies 2⋆ of Theorem 2; in fact, complex vector fields will be considered in a more general form, that is,

$$\mathcal{L} = \partial/\partial t + x(a(x,t) + ib(x,t))\partial/\partial x, \quad a, b \in C^\infty(\Omega_\epsilon),$$

where $t \mapsto b(0,t) \neq 0$, for all $t \in S^1$.

In Sect. 3, we will consider that 1⋆ and 3⋆ of Theorem 2 are satisfied. It will be shown that the interplay between the order of vanishing of the functions a and b at $x = 0$ influences the existence of C^k solutions of $\mathsf{L}u = f$ in a neighborhood of Σ.

2 Assuming 2⋆ of Theorem 2

Let

$$\mathcal{L} = \partial/\partial t - ix(b(x,t) + ia(x,t))\partial/\partial x, \tag{4}$$

be a complex vector field defined on $\Omega_\epsilon = (-\epsilon, \epsilon) \times S^1, \epsilon > 0$, where a and b are C^∞ real-valued functions. Suppose that

1. $\Sigma = \{0\} \times S^1$ is the characteristic set of $\mathcal{L}$;
2. $t \mapsto b(0,t) > 0$, for all $t \in S^1$.

It was proved in [14] that

$$\lambda = \frac{1}{2\pi} \int_0^{2\pi} (b + ia)(0,t)\mathrm{d}t \tag{5}$$

is an invariant that characterizes $\mathcal{L}$ and also that if $\lambda \in \mathbb{C}\backslash\mathbb{R}$ then, for any $p \in \mathbb{Z}_+$, there exists a C^p diffeomorphism defined near Σ that transforms $\mathcal{L}$ into a multiple of the vector field

$$T_\lambda = \lambda\partial/\partial t - ix\partial/\partial x.$$

The case $\lambda \in \mathbb{C}\backslash\{0\}$ was treated in [10].

Results from [10] and [14] allow us to state that:

Theorem 3 *Let $\mathcal{L}$ be given by (4) and λ be given by (5). Assume that $\lambda \in \mathbb{C}\backslash\mathbb{Q}$. Then, for any $p \geq 1$, there exists a C^p diffeomorphism sending $\mathcal{L}$ into a multiple of $T_\lambda = \lambda\partial/\partial t - ix\partial/\partial x$.*

Theorem 4 *Let $\mathcal{L}$ be given by (4) and λ be given by (5). Assume that $\lambda \in \mathbb{Q}$. Let q_λ be equal to 1 if $\lambda \in \mathbb{Z}$ and let q_λ be the largest positive integer such that $j\lambda \notin \mathbb{Z}$ for $1 < j \leq q_\lambda$, if $\lambda \notin \mathbb{Z}$. Define*

$$s = \begin{cases} \lambda q_\lambda - 1, & \text{if } \lambda \leq 1 \\ q_\lambda - 1, & \text{if } \lambda > 1 \end{cases}$$

Then there exist $0 < \sigma < 1$ and a C^p diffeomorphism, where $p = \max\{1 + \sigma, s\}$, sending $\mathcal{L}$ into a multiple of $T_\lambda = \lambda\partial/\partial t - ix\partial/\partial x$.

Theorem 3 follows from Theorem 4.4 of [10]. Theorem 4 follows from Theorem 4.4 of [10] and Theorem 2.2 of [14].

It is worth mentioning that [10] proved that, for each $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$, there exists a complex vector field $\mathcal{L}$, given by (4), such that the function $a + ib$ satisfies (5), which is not equivalent to a multiple of T_λ by any C^∞-diffeomorphism. Moreover, the optimal regularity of the class of the diffeomorphism given by Theorem 4 is unknown.

Now, let us study the equation

$$T_\lambda u = f \tag{6}$$

in a neighborhood of Σ. Using the same arguments of the proof of Theorem 5.2 of [14], we can state that:

Theorem 5 *Let*

$$T_\lambda = \lambda\partial/\partial t - ix\partial/\partial x$$

be a complex vector field defined on Ω_ϵ. *We have*:

I. *If* $\lambda = \alpha + i\beta \in \mathbb{C}\backslash\mathbb{Q}$ *then fixed* $k \geq 1$ *there exists* $p = p(k, \lambda)$ *such that for all* $f \in C_0^p(\Omega_\epsilon)$, *satisfying*

$$\int_0^{2\pi} f(0,t)\mathrm{d}t = 0, \tag{7}$$

the equation $T_\lambda u = f$ *has a* C^k *solution* u *in a neighborhood of* Σ.

II. *If* $\lambda \in \mathbb{Q}$ *then fixed* $k \geq 1$ *there are* $p = p(k, \lambda)$ *and* $n = n(k, \lambda)$ *such that for all* $f \in C_0^p(\Omega_\epsilon)$, *satisfying*

$$\int_0^{2\pi} \frac{\partial^j f}{\partial x^j}(0,t)\mathrm{d}t = 0, \quad j = 0, \ldots, n, \tag{8}$$

the equation $T_\lambda u = f$ *has a* C^k *solution* u *in a neighborhood of* Σ.

The next result is an immediate consequence of Theorem 3 and Theorem 5:

Theorem 6 *Let* $\mathcal{L}$ *be given by* (4) *and* λ *be given by* (5). *Assume that* $\lambda \in \mathbb{C}\backslash\mathbb{Q}$. *Let* $f \in C_0^\infty(\Omega_\epsilon)$. *If* f *satisfies* (7), *then for every* $k \geq 1$ *there exists* $u \in C^k$ *solution of* $\mathcal{L}u = f$ *in a neighborhood of* Σ.

Proof Fixed $k \geq 1$, take $p = p(k, \lambda)$ given by Theorem 5. By Theorem 3, there exists a C^p diffeomorphism sending $\mathcal{L}$ into a multiple of $T_\lambda = \lambda\partial/\partial t - ix\partial/\partial x$; hence, the equation $\mathcal{L}u = f$ on Ω_ϵ is equivalent to $T_\lambda\tilde{u} = \tilde{f}$ on $\tilde{\Omega}_\epsilon$, where $\tilde{u}$ and $\tilde{f}$ are the pushforward of u and f, respectively. Now, applying Theorem 5, we can find the wanted solution. □

Remark 1 In conclusion, we can write:

1. As mentioned in [10], we have no example showing that regularity results of normalization given by Theorem 4 are optimal.
2. Let $\mathcal{L}$ be given by (4) and λ be given by (5). Assume that $\lambda \in \mathbb{Q}$. Let $f \in C_0^\infty(\Omega_\epsilon)$. If f satisfies (7), then, it follows from [14] that there exists $u \in C^\alpha$, for some $0 < \alpha < 1$, solution of $\mathcal{L}u = f$ in a neighborhood of Σ.
3. Let $\mathcal{L}$ be given by (4) and λ be given by (5). Assume that $\lambda \in \mathbb{Q}$. Let $f \in C_0^\infty(\Omega_\epsilon)$ and assume that f satisfies (7). Finding C^1 solution for the equation $\mathcal{L}u = f$ in a neighborhood of Σ is still an open problem which the author hopes to consider in a future paper.

3 Assuming 1⋆ and 3⋆ of Theorem 2

Let

$$\mathsf{L} = \partial/\partial t + (a(x) + ib(x))\partial/\partial x, \quad a, b \in C^\infty(-\epsilon, \epsilon),$$

be a complex vector field defined on $\Omega_\epsilon = (-\epsilon, \epsilon) \times S^1$, for some $\epsilon > 0$, where a and b are real-valued functions.

Suppose that $\Sigma = \{0\} \times S^1$ is the characteristic set of L and that L is tangent along Σ. Hence, L is elliptic on $\Omega_\epsilon \backslash \Sigma$, $a(0) = b(0) = 0$, and L satisfies condition $(\mathcal{P})$.

Assume that we can write $a(x) + ib(x) = x^n a_0(x) + ix^m b_0(x)$ with $(a_0 + ib_0)(0) \neq 0$ and that some of the conditions below are satisfied:

1. $a_0(0) \neq 0$ and $m > 2n - 1$;
2. $a_0(0) \neq 0$, $b_0(0) \neq 0$, $m = 2n - 1$ and $n \geq 2$.

We start considering that 2. above holds, that is, $m = 2n - 1$ and $n \geq 2$.

Theorem 7 *Let*

$$\mathsf{L} = -i\partial/\partial t + (b_0(x)x^{2n-1} - ia_0(x)x^n)\partial/\partial x, \quad n \geq 2,$$

where a_0, b_0 are C^∞ real-valued functions, with $a_0(x) \neq 0$ and $b_0(x) \neq 0$ for all x, be a complex vector field defined on Ω_ϵ. Fixed $k \in \mathbb{Z}_+$, given $f \in C^\infty(\Omega_\epsilon)$, satisfying

$$\int_0^{2\pi} \frac{\partial^j f}{\partial x^j}(0, t)\mathrm{d}t = 0, \quad j = 0, \ldots, n-1, \tag{9}$$

there exists $u \in C^k$ solution for $\mathsf{L}u = f$, in a neighborhood of Σ.

Proof It follows from [4] that given $f \in C^\infty(\Omega_\epsilon)$, satisfying (9), there exists a function $v \in C^\infty$ such that $\mathsf{L}v - f$ is a flat function along Σ. Hence, to solve the equation $\mathsf{L}u = f$ in a neighborhood of Σ, when f satisfies (9), it is enough to assume that f is flat along Σ. From now on, we will assume that f is flat along Σ.

Without loss of generality, we can assume that $b_0(x) > 0$ and $a_0, b_0 \in L^\infty([-\epsilon, \epsilon])$.

Consider

$$\Omega_\epsilon^+ = \{(x, t) \in \Omega_\epsilon; x > 0\} \quad \text{and} \quad \Omega_\epsilon^- = \{(x, t) \in \Omega_\epsilon; x < 0\}.$$

Define

$$Z(x,t) = \begin{cases} \mathrm{e}^{-\int_x^\epsilon \frac{b_0(y)y^{2n-1}}{a_0^2(y)y^{2n} + b_0^2(y)y^{4n-2}}\mathrm{d}y} \cdot \mathrm{e}^{-i\left(t + \int_x^\epsilon \frac{a_0(y)y^n}{a_0(y)^2y^{2n} + b_0(y)^2y^{4n-2}}\mathrm{d}y\right)}, & x > 0 \\ 0, & x = 0 \\ \mathrm{e}^{\int_{-\epsilon}^x \frac{b_0(y)y^{2n-1}}{a_0^2(y)y^{2n} + b_0^2(y)y^{4n-2}}\mathrm{d}y} \cdot \mathrm{e}^{-i\left(t - \int_{-\epsilon}^x \frac{a_0(y)y^n}{a_0(y)^2y^{2n} + b_0(y)^2y^{4n-2}}\mathrm{d}y\right)}, & x < 0 \end{cases}.$$

Evidently, $Z \in C^\infty(\Omega_\epsilon^\pm)$ and, by a simple computation, we have

$$\mathsf{L}Z = 0 \quad \text{and} \quad \mathsf{L}\overline{Z} = \frac{2b_0(x)x^{n-1}}{b_0(x)x^{n-1} + ia_0(x)}\overline{Z}.$$

Define $F(x) = |Z(x,t)|$. As shown in [11], $F \in C^\infty((-\epsilon, \epsilon)\backslash\{0\})$ and it is continuous in $(-\epsilon, \epsilon)$. Moreover, F is injective in $(-\epsilon, \epsilon)\backslash\{0\}$. Thus, if $x \neq 0$ then $x = F^{-1}(|z|)$, for some $z \in D(0; 1)$.

The pushforward of the equations

$$\mathsf{L}w = f \quad \text{in} \quad \Omega_\epsilon^\pm \tag{10}$$

via the map Z are given by

$$\frac{2b_0(F^{-1}(|z|))[F^{-1}(|z|)]^{n-1}}{b_0(F^{-1}(|z|))[F^{-1}(|z|)]^{n-1} + ia_0(F^{-1}(|z|))}\overline{z}\frac{\partial \tilde{w}^\pm}{\partial \overline{z}} = \tilde{f}^\pm \quad \text{in} \quad D(0;1)\backslash\{0\},$$

where $\tilde{w}^\pm$ and $\tilde{f}^\pm$ are the pushforward of functions w and f in Ω_ϵ^+ and Ω_ϵ^-, respectively. Thus, we have (using that $z = |z|e^{i\theta}$)

$$\frac{\partial \tilde{w}^\pm}{\partial \overline{z}} = \frac{\left(b_0(F^{-1}(|z|))[F^{-1}(|z|)]^{n-1} + ia_0(F^{-1}(|z|))\right) \mathrm{e}^{i\theta} \tilde{f}^\pm}{2b_0(F^{-1}(|z|))|z|(F^{-1}(|z|))^{n-1}}. \tag{11}$$

Since $\tilde{f}^\pm$ are flat at $z = 0$, it is easy to see that

$$\frac{\left(b_0(F^{-1}(|z|))[F^{-1}(|z|)]^{n-1} + ia_0(F^{-1}(|z|))\right) \mathrm{e}^{i\theta} \tilde{f}^\pm}{2b_0(F^{-1}(|z|))[F^{-1}(|z|)]^{n-1}|z|} \in C^\infty(D(0;1));$$

hence we have that the solutions

$$\tilde{w}^\pm(z) = \frac{1}{2\pi i}\iint\limits_{D(0;1)} \frac{\left(b_0(F^{-1}(|\zeta|))[F^{-1}(|\zeta|)]^{n-1} + ia_0(F^{-1}(|\zeta|))\right) \mathrm{e}^{i\theta} \tilde{f}^\pm}{2b_0(F^{-1}(|\zeta|))(F^{-1}(|\zeta|))^{n-1}|\zeta|(\zeta - z)} \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}$$

belong to $C^\infty(D(0;1))$ (see, for example, [16]). Thus, for any fixed $\ell \in \mathbb{Z}_+$, we can write

$$\tilde{w}^\pm(z) = \sum_{0 \le j \le \ell-1} c_j^\pm z^j + |z|^\ell \tilde{v}^\pm(z),$$

where $\tilde{v}^\pm(z)$ belong to $C^\infty(D(0;1))$. Note that $|z|^\ell \tilde{v}^\pm(z)$ are also solutions of the equations (11).

Define $\beta = \frac{b_0(0)}{a_0^2(0)}$. Using Taylor's formula we have

$$\frac{b_0(x)x^{2n-1}}{a_0^2(x)x^{2n} + b_0^2(x)x^{4n-2}} = \frac{1}{x}\left(\beta + O(|x|)\right);$$

consequently,

$$|Z(x,t)| = \begin{cases} \left(\frac{x}{\epsilon}\right)^\beta \mathrm{e}^{-\int_x^\epsilon \frac{O(|y|)}{y}\mathrm{d}y}, & x > 0 \\ 0, & x = 0. \\ \left(\frac{-x}{\epsilon}\right)^\beta \mathrm{e}^{\int_{-\epsilon}^x \frac{O(|y|)}{y}\mathrm{d}y}, & x < 0 \end{cases}$$

Now, define

$$u(x,t) = \begin{cases} \left(\frac{x}{\epsilon}\right)^{\ell\beta} \mathrm{e}^{-\ell\int_x^\epsilon \frac{O(|y|)}{y}\mathrm{d}y} \tilde{v}^+(Z(x,t)), & x > 0 \\ 0, & x = 0. \\ \left(\frac{-x}{\epsilon}\right)^{\ell\beta} \mathrm{e}^{\ell\int_{-\epsilon}^x \frac{O(|y|)}{y}\mathrm{d}y} \tilde{v}^-(Z(x,t)), & x < 0 \end{cases}$$

We have that u satisfies $\mathsf{L}u = f$ in a neighborhood of Σ. Therefore, it is enough to choose an ℓ sufficiently large to obtain $u \in C^k$. □

Example 1 Consider the complex vector field

$$\mathsf{L} = -i\partial/\partial t + (x^{2n-1} - ix^n)\partial/\partial x, \quad n \geq 2,$$

defined on Ω_ϵ. Define

$$Z(x,t) = \frac{x}{(1+x^{2n-2})^{\frac{1}{2n-2}}} \mathrm{e}^{-i\left[t+\frac{1}{n-1}\left(\frac{1}{x^{n-1}}+\arctan x^{n-1}\right)\right]}.$$

By a simple computation, we have

$$\mathsf{L}Z = 0 \quad \text{and} \quad \mathsf{L}\overline{Z} = \frac{2x^{n-1}}{x^{n-1}+i}\overline{Z}.$$

Let $f \in C^\infty(\Omega_\epsilon)$ and be flat along Σ. Since $x = \pm\frac{|Z|}{(1-|Z|^{2n-2})^{\frac{1}{2n-2}}}$, the pushforward of the equations

$$\mathsf{L}w = f \quad \text{in} \quad \Omega_\epsilon^\pm$$

via the map Z yields

$$\frac{2|Z|^{n-1}}{|Z|^{n-1} + i\sqrt{1-|Z|^{2n-2}}}\,\overline{Z}\,\frac{\partial \tilde{w}^+}{\partial \overline{z}} = \tilde{f}^+ \quad \text{in} \quad D(0;\epsilon/(1+\epsilon^{2n-2})^{\frac{1}{2n-2}})\backslash\{0\}$$

and

$$\frac{2|Z|^{n-1}}{|Z|^{n-1} + i(-1)^{n-1}\sqrt{1-|Z|^{2n-2}}}\,\overline{Z}\,\frac{\partial \tilde{w}^-}{\partial \overline{z}} = \tilde{f}^- \quad \text{in} \quad D(0;\epsilon/(1+\epsilon^{2n-2})^{\frac{1}{2n-2}})\backslash\{0\},$$

where $\tilde{w}^\pm$ and $\tilde{f}^\pm$ are the pushforward of functions u and f in Ω_ϵ^+ and Ω_ϵ^-, respectively. Denote $D^\epsilon = D(0;\epsilon/(1+\epsilon^{2n-2})^{\frac{1}{2n-2}})$.

Hence, we have

$$\frac{\partial \tilde{w}^+}{\partial \overline{z}} = \frac{|Z|^{n-1} + i\sqrt{1-|Z|^{2n-2}}}{2|Z|^n}\mathrm{e}^{i\theta}\tilde{f}^+ \quad \text{in} \quad D^\epsilon\backslash\{0\} \tag{12}$$

and

$$\frac{\partial \tilde{w}^-}{\partial \overline{z}} = \frac{|Z|^{n-1} + i(-1)^{n-1}\sqrt{1-|Z|^{2n-2}}}{2|Z|^n}\mathrm{e}^{i\theta}\tilde{f}^- \quad \text{in} \quad D^\epsilon\backslash\{0\}. \tag{13}$$

The solutions

$$\tilde{w}^+(z) = \frac{1}{2\pi i}\iint\limits_{D^\epsilon} \frac{|\zeta|^{n-1} + i\sqrt{1-|\zeta|^{2n-2}}}{2|\zeta|^n(\zeta - z)}\mathrm{e}^{i\theta}\tilde{f}^+(\zeta)\mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}$$

and

$$\tilde{w}^-(z) = \frac{1}{2\pi i}\iint\limits_{D^\epsilon} \frac{|\zeta|^{n-1} + i(-1)^{n-1}\sqrt{1-|\zeta|^{2n-2}}}{2|\zeta|^n(\zeta - z)}\mathrm{e}^{i\theta}\tilde{f}^-(\zeta)\mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}$$

belong to $C^\infty(D^\epsilon)$ (see, for example, [16]). Thus, for any fixed $\ell \in \mathbb{Z}_+$, we can write

$$\tilde{w}^\pm(z) = \sum_{0\leq j\leq \ell-1} c_j^\pm z^j + |z|^\ell \tilde{v}^\pm(z),$$

where $\tilde{v}^{\pm}(z)$ belongs to $C^\infty(D^\epsilon)$; note that $|z|^\ell \tilde{v}^+(z)$ and $|z|^\ell \tilde{v}^-(z)$ also satisfy Eqs. (12) and (13), respectively. Hence, the function u defined by

$$u(x,t) = \frac{|x|^\ell}{(1+|x|^{2n-2})^{\frac{\ell}{(2n-2)}}} \tilde{v}^+\left(\frac{x}{(1+x^{2n-2})^{\frac{1}{2n-2}}} \mathrm{e}^{-i\left[t+\frac{1}{n-1}\left(\frac{1}{x^{n-1}}+\arctan x^{n-1}\right)\right]} \right),$$

if $x \neq 0$ and $u(0,t) = 0$ for all $t \in S^1$, satisfies $\mathsf{L}u = f$ in a neighborhood of Σ. Therefore, it is enough to choose an ℓ sufficiently large to obtain $u \in C^k$.

In the next, we will suppose that 1. holds, that is, $a_0(0) \neq 0$, $m \geq 2n$ and $n \geq 1$.

Theorem 8 *Let*

$$\mathsf{L} = \partial/\partial t + (x^n a_0(x) + i x^m b_0(x))\partial/\partial x, \quad n \geq 1 \quad \textit{and} \quad m \geq 2n,$$

be a complex vector field defined on Ω_ϵ, where a_0, b_0 are C^∞ real-valued functions, with $a_0(x) \neq 0$ for all x. There exists $f \in C^\infty(\Omega_\epsilon)$, satisfying (9), such that the equation $\mathsf{L}u = f$ does not have a C^1 solution in any neighborhood of Σ.

Proof Assume that for each function $f \in C^\infty(\Omega_\epsilon)$, satisfying (9), there exists $u \in C^1$ solution of the equation $\mathsf{L}u = f$ in a neighborhood of Σ. As in [3] (see also [6]), an argument using Baire's theorem and the open mapping theorem implies that there exists $\delta_0 > 0$ such that given $f \in C^\infty([-\delta_0, \delta_0] \times S^1)$, satisfying (9), there exists $u \in C^1([-\delta_0, \delta_0] \times S^1)$ solution of $\mathsf{L}u = f$ in some neighborhood of $[-\delta_0, \delta_0] \times S^1$. Indeed, under the assumptions above, let

$$\mathcal{F} = \text{span} < 1 \otimes \delta^{(j)} >, \quad j = 0, \ldots, n-1,$$

and let $\mathcal{F}^\circ$ denote the annihilator of $\mathcal{F}$. Define the Fréchet space

$$G_\ell = \{(f,u) \in (C^\infty([-\epsilon/2, \epsilon/2] \times S^1) \cap \mathcal{F}^\circ) \times C^1(\overline{\mathcal{U}_\ell});\ \mathsf{L}u = f \text{ on } \mathcal{U}_\ell\},$$

where $\mathcal{U}_\ell = (-\delta_\ell, \delta_\ell) \times S^1$ and $\delta_\ell \downarrow 0$ with $\delta_1 = \epsilon/2$. Consider

$$\begin{array}{ll} \pi_\ell : G_\ell & \to C^\infty([-\epsilon/2, \epsilon/2] \times S^1) \cap \mathcal{F}^\circ \\ \quad (f,u) & \mapsto f \end{array},$$

the projection on the first factor.

Under our assumptions

$$\bigcup_\ell \pi_\ell(G_\ell) = C^\infty([-\epsilon/2, \epsilon/2] \times S^1) \cap \mathcal{F}^\circ.$$

It follows from Baire's theorem and the open mapping theorem that

$$\pi_{\ell_0}(G_{\ell_0}) = C^\infty([-\epsilon/2, \epsilon/2] \times S^1) \cap \mathcal{F}^\circ,$$

for some ℓ_0. In other words, for $\delta_0 = \delta_{\ell_0}$, we have that

$$\mathsf{L} : C^1([-\delta_0, \delta_0] \times S^1) \to C^\infty([-\delta_0, \delta_0] \times S^1) \cap \mathcal{F}^\circ \tag{14}$$

is surjective.

Next, we will go to contradict (14) by exhibiting a function f belonging to $C^\infty([-\delta_0, \delta_0] \times S^1) \cap \mathcal{F}^\circ$ for which there is no solution $u \in C^1([-\delta_0, \delta_0] \times S^1)$ of the equation $\mathsf{L}u = f$ in any neighborhood of $[-\delta_0, \delta_0] \times S^1$. The argument is the same as in [4], which we will recall.

We may assume, without loss of generality, that $b(x) > 0$ if $-\delta_0 < x < 0$.

Fixed $x_0 \in (-\delta_0, 0)$, define $(\hat{f}_k)$, $k \in \mathbb{Z}$, by

$$\hat{f}_k(x) = g(x) \cdot (a + ib)(x) \cdot c_k \cdot \, \mathrm{e}^{-ik \int_{x_0}^{x} \frac{a}{a^2+b^2}(y)\mathrm{d}y},$$

where (c_k) is a rapidly decreasing sequence of positive real numbers (to be chosen later) and

$$g(x) = \begin{cases} -\mathrm{e}^{\frac{1}{x}}, & x < 0 \\ 0, & x \geq 0 \end{cases}$$

It follows from [4] that the sequence $(\hat{f}_k)$ defines a function $f \in C^\infty([-\delta_0, \delta_0] \times S^1)$, which is flat along $\Sigma = \{0\} \times S^1$, and evidently belongs to $\mathcal{F}^\circ$.

We will prove that there is no $u \in C^1([-\delta_0, \delta_0] \times S^1)$ solving $\mathsf{L}u = f$ in a neighborhood of $[-\delta_0, \delta_0] \times S^1$. In [4] it was shown that for a similar function f, the equation $\mathsf{L}u = f$ does not have C^∞ solution in any neighborhood of $[-\epsilon_0, \epsilon_0] \times S^1$, for some $0 < \epsilon_0 \leq \epsilon/2$.

Using Fourier series, the equation $\mathsf{L}u = f$ can be rewritten as

$$\sum_{k\in\mathbb{Z}} (\mathsf{L}_k \hat{u}_k)\mathrm{e}^{ikt} = \sum_{k\in\mathbb{Z}} \hat{f}_k \mathrm{e}^{ikt},$$

where $\mathsf{L}_k \hat{u}_k(x) = ik\hat{u}_k(x) + (a + ib)(x)(\hat{u}_k)'(x)$; moreover, the series $\sum_{k\in\mathbb{Z}} \hat{u}_k(x)e^{ikt}$ converges uniformly in a neighborhood of $\{0\} \times S^1$.

It follows from (14) that for f defined above there exists a C^1 function $u(x, t)$ satisfying $\mathsf{L}u(x, t) = f(x, t)$ in a neighborhood of $[-\delta_0, \delta_0] \times S^1$. Hence, we have $\mathsf{L}_k \hat{u}_k(x) = \hat{f}_k(x)$ with $\hat{u}_k(x) = v_k(x) + h_k(x)$, for each k, where h_k is an arbitrary solution of the homogeneous equation $\mathsf{L}_k h_k = 0$ and v_k is a solution of the nonhomogeneous equation.

As in [4], we will look for a solution, v_k, to $\mathsf{L}_k v_k(x) = \hat{f}_k(x)$, for each $k \neq 0$. If $x > 0$ then we obtain $v_k(x) \equiv 0$. If $x < 0$ then we obtain

$$v_k(x) = -\int_x^0 \mathrm{e}^{-ik(C(x)-C(y))} \frac{\hat{f}_k}{a + ib}(y)\mathrm{d}y, \quad \text{where} \quad C(x) = \int_{-\delta_0}^{x} \frac{1}{a + ib}(y)\mathrm{d}y.$$

It is clear that each v_k is smooth in a neighborhood of $[-\delta_0, \delta_0] \times S^1$, and a simple calculation (see [4]) shows that for $k > 0$ we have

$$\begin{aligned} v_k(x_0) &= -\int_{x_0}^{0} g(y)c_k \mathrm{e}^{k \int_{x_0}^{y} \frac{b}{a^2+b^2}(s)\mathrm{d}s}\mathrm{d}y \\ &\geq \mathrm{e}^{kM} M' c_k, \end{aligned}$$

where $M = \int_{x_0}^{\frac{x_0}{2}} \frac{b}{a^2+b^2}(s)\mathrm{d}s > 0$ and $M' = -\int_{\frac{x_0}{2}}^{0} g(y)\mathrm{d}y > 0$.

If in the definition of $\hat{f}_k$ we choose $c_k = \mathrm{e}^{-Mk}$, for each $k > 0$, then we will have that $v_k(x_0) \geq M'$. Hence, the sequence (v_k) cannot be the coefficients of a C^1 function at $x = x_0$.

It follows from the proof of Theorem 4.1 of [6] that there is no $h_k \in C^1$ solution of $\mathsf{L}_k h_k = 0$ in a neighborhood of $x = 0$.

Hence, for each $k > 0$, $\hat{u}_k(x) = v_k(x)$ is the only C^1 solution to the equation $\mathsf{L}_k \hat{u}_k(x) = \hat{f}_k(x)$ and, consequently, $(\hat{u}_k)$ cannot be the sequence of Fourier coefficients of any C^1 periodic function; from (14), we have a contradiction. The proof is complete. □

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