Annali di Matematica (2010) 189:227–240
DOI 10.1007/s10231-009-0106-9

On a singular elliptic system at resonance

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Received: 19 December 2008 / Accepted: 10 June 2009 / Published online: 1 July 2009

Abstract This paper is devoted to the study of an elliptic system with singular coefficients. Existence and multiplicity results at resonance are obtained via variational methods.

Keywords Brézis-Nirenberg problem · Singular coefficients · Elliptic system at resonance · Variational methods

Mathematics Subject Classification (2000) 35J20 · 35J60 · 35B33

1 Introduction and main results

In this paper, we will study the existence of nontrivial solutions to the following elliptic system

$$(S_A)\begin{cases} -L_{\mu,a}u = (a_{11}u + a_{12}v)\frac{1}{|x|^{\gamma}} + (\alpha+1)\,\frac{u|u|^{\alpha-1}|v|^{\beta+1}}{|x|^{2_* b}} & \text{in } \Omega\backslash\{0\} \\ -L_{\mu,a}v = (a_{12}u + a_{22}v)\frac{1}{|x|^{\gamma}} + (\beta+1)\,\frac{|u|^{\alpha+1}v|v|^{\beta-1}}{|x|^{2_* b}} & \text{in } \Omega\backslash\{0\} \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $L_{\mu,a}u := \operatorname{div}\left(|x|^{-2a}\,\nabla u\right) + \frac{\mu u}{|x|^{-2(a+1)}}$, $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded smooth domain containing 0 in its interior, $-\infty < a < \frac{N-2}{2}$, $a \leq b < a+1$, $c > 2|b|$, $\gamma := 2(a+1) - c$, $-\infty < \mu < \overline{\mu}_a := \left(\frac{N-2-2a}{2}\right)^2$, $\alpha, \beta \geq 0$ such that $\alpha + \beta = 2_* - 2$ where $2_* := 2_*\,(a,b) = \frac{2N}{N-2\eta}$ with $\eta := a + 1 - b$; a_{11}, a_{12}, a_{22} are real parameters.

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The range of a, b and the definition of 2_* are related to the well-known Caffarelli–Kohn–Nirenberg inequalities [5],

$$\left(\int_{\mathbb{R}^N} |x|^{-2_* b} |u|^{2_*} \,\mathrm{d}x\right)^{\frac{2}{2_*}} \leq C_{a,b} \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \,\mathrm{d}x \tag{1.1}$$

for all $u \in C_0^\infty\left(\mathbb{R}^N\right)$. For sharp constants and extremal functions we refer to [7,9]. As $b = a + 1$ then $2_* = 2$ in (1.1), we have

$$\int_{\mathbb{R}^N} |x|^{-2(a+1)} |u|^2 \,\mathrm{d}x \leq \left(\frac{N - 2a - 2}{2}\right)^2 \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \,\mathrm{d}x \tag{1.2}$$

for all $u \in C_0^\infty\left(\mathbb{R}^N\right)$.

In order to state our main results, we introduce the weighted Sobolev space $\mathcal{D}_a^{1,2}(\Omega)$, which is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \left(\int_\Omega |x|^{-2a} |\nabla u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}},$$

by the weighted Hardy inequality this norm is equivalent to the following

$$\|u\|_{\mu,a} = \left(\int_\Omega (|x|^{-2a} |\nabla u|^2 - \mu |x|^{-2(a+1)} u^2) \mathrm{d}x\right)^{\frac{1}{2}}.$$

As a consequence of (1.2), the operator $L_{\mu,a}$ is symmetric, uniformly positive and has a discrete spectrum $\sigma_{\mu,a}$ in $\mathcal{D}_a^{1,2}(\Omega)$ for all $\mu < \overline{\mu}_a$ (see [14] for more details).

It is convenient to rewrite system (S_A) as

$$(S_A) \begin{cases} -L_{\mu,a} U = \frac{1}{|x|^\gamma} AU + \nabla H & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega, \end{cases}$$

where $U = \begin{pmatrix} u \\ v \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$ and $H(x, u, v) = |x|^{-2_* b} |u|^{\alpha+1} |v|^{\beta+1}$.

We denote by λ_1 and λ_2 the real eigenvalues of the matrix A and assume that $\lambda_1 \leq \lambda_2$.

The study of this type of problems is motivated by its various applications, for example, it has been introduced as a model for several physical phenomena related to the equilibrium of anisotropic media (see [13]). The mathematical interest lies in the fact that these problems are doubly critical due to the presence of the exponent of the nonlinearities (which is critical in the sense of the weighted Sobolev embeddings) and the singularities.

The case of single equations has been deeply investigated in literature (see [7,9,15,18,20] and the references therein). Some of them have considered the resonant and non resonant case, many techniques are developed in this sense to solve these problems; we cite for example [6,10,16,17,20].

Regular systems have been the subject of many papers; we can quote, among others [1,11,21]. For a survey on the system case, we recommend de Figueiredo's paper [12] and the references therein.

However, as far as we know, there are few results on critical systems with singular coefficients.

For $a = b = 0$ and $c = 2$ in (S_A), the existence of solutions are discussed in [3] where the authors have taken into account the position of the eigenvalues of the matrix A on $\mathbb{R}^+$.

We mention that in [19] existence results are obtained for a class of singular quasilinear elliptic systems using topological methods.

Costa and Magalhães [11] have introduced the notion of resonance for a system as follows: the kernel of $-\left(L_{\mu,a} + A\right)$ is nonzero if and only if $A - \mu_j I$ is singular for some μ_j eigenvalue of the operator $-L_{\mu,a}$ with zero Dirichlet boundary condition. If $x = (x_1, x_2) \neq (0, 0)$ is such that $Ax = \mu_j x$ then $\left(x_1\phi_j, x_2\phi_j\right) \in \text{Ker}\left(-L_{\mu,a} - A\right)$ where ϕ_j is an eigenfunction associated to μ_j.

The purpose of this paper is to further the study of problem (S_A). First of all, we intend to complete the results obtained in [3] for the resonant case: we consider the case where $A - \mu_j I$ is singular and the case where $A - \mu_j I$ is regular and there exist $k, k' \in \mathbb{N}^*, k \leq k'$ such that $a_{11} = \mu_k$ and $a_{22} = \mu_{k'}$. After that, we prove the multiplicity result of nontrivial solutions when λ_2 belongs to a suitable left neighborhood of an arbitrary eigenvalue of the operator $-L_{\mu,a}$.

Our study deals with the case when λ_1 or λ_2 is equal to the eigenvalues of the operator $L_{\mu,a}$. In our results we required $\lambda_2 > 0$ and no assumption on the sign of λ_1 was asked (it may be negative) i.e. they can have opposite signs. This case appears only in a system. When $\lambda_2 \leq 0$, system (S_A) has no nontrivial solution if Ω is a star shaped domain with respect to the origin. This is obtained by a Pohozaev type identity adapted for a system (see [3]).

Our main results in the present paper are the following ones:

Theorem 1 *Suppose* $N > \frac{c+2+\sqrt{(c+2)^2+8c\eta}}{2}$ *and* $\overline{\mu}_a - \left(\frac{N-2}{2}\right)^2 \leq \mu < \overline{\mu}_a - \left(\frac{c(N+2\eta)}{2N}\right)^2$. *Assume one of the following conditions holds*:

i) *If there exists* $k, k' \in \mathbb{N}^*$, $k \leq k'$ *such that* $a_{11} = \mu_k$, $a_{22} = \mu_{k'}$ *and* $a_{12} < \min(\mu_{k+1} - \mu_k, \mu_{k'+1} - \mu_{k'})$.
ii) *If there exists* $k \in \mathbb{N}^*$ *such that* $\lambda_2 = \mu_k$.
Then system (S_A) *has at least one solution.*

Theorem 2 *Assume* $N > c + 2$, $\overline{\mu}_a - \frac{c^2}{4} < \mu < \overline{\mu}_a - b^2$ *and* $\lambda_2 > 0$. *Let* $\mu_+ = \min\{\mu_k / \lambda_2 < \mu_k\}$, $\theta := \left(\frac{\alpha+1}{\beta+1}\right)^{-\frac{1}{2}}$ *and suppose*

$$(\mu_+ - \lambda_1) < \widetilde{\mathcal{K}}_{a,b} \frac{\theta^{(\beta+1)\left(\frac{N-2\eta}{N}\right)}}{\left(1+\theta^2\right)} \left(\int_\Omega |x|^{\left(\frac{Nc}{2\eta} - N\right)} \mathrm{d}x\right)^{\frac{-2\eta}{N}}.$$

Let m *be the multiplicity of* μ_+. *Then system* (S_A) *admits at least* $2m$ *pairs of solutions.*

The proof of our results is obtained with the critical point theory, however, standard variational arguments do not apply because of a lack of compactness of some weighted Sobolev embedding, the action functional does not satisfy the Palais−Smale condition (PS condition in the sequel). We follow Brézis and Nirenberg's argument [4] to verify that the associated functional to the system (S_A) satisfies the (PS) condition on a suitable "compactness range". Then, by employing the techniques introduced in [10,17,20], we get some results on Brézis–Nirenberg type problems for a system of elliptic equation involving nonlinearities with critical growth and singular coefficients.

The paper is organized as follows: in Sect. 2, we recall some preliminary results and asymptotic estimates, in Sect. 3, we establish some lemmas to prove our results, Sect. 4 is devoted to the proof of our results.

2 Preliminaries

Throughout this paper, we denote by C, C_1, $C_2, \ldots$ generic positive constants; B_R is the ball centered at 0 with radius R; supp φ denotes the support of the function φ; $\langle .,. \rangle$ denotes the $\mathbb{R}^2$ usual inner product, we use $L^p\left(\Omega, |x|^{-q}\,\mathrm{d}x\right)$ to denote the weighted $L^p(\Omega)$ space with the weight $|x|^{-q}$. We endow the space $E := \mathcal{D}_a^{1,2}(\Omega) \times \mathcal{D}_a^{1,2}(\Omega)$ with the norm $\|(u,v)\|_{\mu,a} = \left(\|u\|_{\mu,a}^2 + \|v\|_{\mu,a}^2\right)^{\frac{1}{2}}$.

Let

$$C_{a,b}^{-1} := \mathcal{K}_{a,b} = \inf_{u \in \mathcal{D}_a^{1,2}(\Omega)\setminus\{0\}} \frac{\|u\|_{\mu,a}^2}{\left(\int_\Omega \frac{|u|^{2_*}}{|x|^{2_* b}}\mathrm{d}x\right)^{\frac{2}{2_*}}}$$

and

$$\widetilde{\mathcal{K}}_{a,b} := \inf_{(u,v)\in E\setminus\{(0,0)\}} \frac{\|(u,v)\|_{\mu,a}^2}{\left(\int_\Omega \frac{|u|^{\alpha+1}|v|^{\beta+1}}{|x|^{2_* b}}\mathrm{d}x\right)^{\frac{2}{2_*}}}.$$

From [15,20], we know that $\mathcal{K}_{a,b}$ is achieved on $\mathbb{R}^N$ by the family of functions

$$\omega_\varepsilon^*(x) = \frac{C_0 \varepsilon^{\frac{2}{2_*-2}}}{\left(\varepsilon^{\frac{2\sqrt{\overline{\mu}_a-\mu}}{\sqrt{\overline{\mu}_a-\mu-b}}} |x|^{\frac{(2_*-2)}{2}\left(\sqrt{\overline{\mu}_a}-\sqrt{\overline{\mu}_a-\mu}\right)} + |x|^{\frac{(2_*-2)}{2}\left(\sqrt{\overline{\mu}_a}+\sqrt{\overline{\mu}_a-\mu}\right)}\right)^{\frac{2}{2_*-2}}}$$

for $\varepsilon > 0$, $\frac{N-2}{2} - \sqrt{\overline{\mu}_a - \mu} + a < \frac{N-2}{2}$, $\mu < \overline{\mu}_a - b^2$ and C_0 is a positive constant such that $\omega_\varepsilon^*(x)$ is a weak solution of the problem

$$-L_{\mu,a}u = \delta \frac{u\,|u|^{2_*-2}}{|x|^{2_* b}} \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

for $\delta = 1$ and satisfies

$$\int_{\mathbb{R}^N} \left(|x|^{-2a}\left|\nabla\omega_\varepsilon^*(x)\right|^2 - \mu\left|\omega_\varepsilon^*(x)\right|^2 |x|^{-2(a+1)}\right)\mathrm{d}x = \int_{\mathbb{R}^N} \left|\omega_\varepsilon^*(x)\right|^{2_*}|x|^{-2_* b}\mathrm{d}x = \left(\mathcal{K}_{a,b}\right)^{\frac{2_*}{2_*-2}}.$$

Lemma 1 *Let Ω be a domain (not necessarily bounded), $\mu < \overline{\mu}_a - b^2$ and $\alpha + \beta \le 2_* - 2$. Then, we have*

$$\widetilde{\mathcal{K}}_{a,b} = \left[\left(\frac{\alpha+1}{\beta+1}\right)^{\frac{\beta+1}{\alpha+\beta+2}} + \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{-\alpha-1}{\alpha+\beta+2}}\right]\mathcal{K}_{a,b}.$$

Moreover, if ω_0 realizes $\mathcal{K}_{a,b}$ then $(u_0, v_0) = (B\omega_0, D\omega_0)$ realizes $\widetilde{\mathcal{K}}_{a,b}$ for any positive constants B and D such that $\frac{B}{D} = \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{1}{2}}$.

Proof The proof is essentially given in [1] with minor modifications. □

From Lemma 1, we conclude that the following system

$$\begin{cases} -L_{\mu,a}u = (\alpha+1)\,\frac{u|u|^{\alpha-1}|v|^{\beta+1}}{|x|^{2_*b}} & \text{in } \mathbb{R}^N\setminus\{0\} \\ -L_{\mu,a}v = (\beta+1)\,\frac{|u|^{\alpha+1}v|v|^{\beta-1}}{|x|^{2_*b}} & \text{in } \mathbb{R}^N\setminus\{0\} \\ u(x)\to 0,\, v(x)\to 0 & \text{as } |x|\to\infty, \end{cases}$$

has a solution in the form $(B\omega_\varepsilon^*, D\omega_\varepsilon^*)$ with B and D are positive constants satisfying $\frac{B}{D} = \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{1}{2}}$.

For proving our results we may recall some technical asymptotic estimates. The ideas are essentially given in [10,16,17].

Fix $k\in\mathbb{N}^*$ and for all $i\in\mathbb{N}^*$ denote by e_i an L^2 normalized eigenfunction relative to $\mu_i\in\sigma_{\mu,a}$. Let X_k denote the space spanned by the eigenfunctions corresponding to the eigenvalues $\mu_1,\mu_2,\ldots,\mu_k$ and $Y_k=(X_k)^\perp$.

Take always $m\in\mathbb{N}$ large enough so that $B_{\frac{1}{m}}\subset\Omega$ and define the function $\zeta_m:\Omega\to\mathbb{R}$ by

$$\zeta_m(x) = \begin{cases} 0 & \text{if } x\in B_{\frac{1}{m}} \\ m\,|x|-1 & \text{if } x\in B_{\frac{2}{m}}\setminus B_{\frac{1}{m}} \\ 1 & \text{if } x\in\Omega\setminus B_{\frac{2}{m}}, \end{cases}$$

the approximating eigenfunction $e_i^m = e_i\zeta_m$ and the space $X_k^m := \text{span}\left\{e_i^m,\ i=1,\ldots,k\right\}$.

For all $\varepsilon>0$, consider the shifted functions

$$\omega_\varepsilon^m(x) = \begin{cases} \omega_\varepsilon^*(x) - \omega_\varepsilon^*\left(\frac{1}{m}\right) & \text{if } x\in B_{\frac{1}{m}}\setminus\{0\} \\ 0 & \text{if } x\in\Omega\setminus B_{\frac{1}{m}}. \end{cases}$$

We need the following lemmas:

Lemma 2 [20] *Assuming that* $\mu<\overline{\mu}_a$, *we have*

i) *For any* $i\in\mathbb{N}^*$, $\left\{e_i^m\right\}$ *converges to* e_i *in* $\mathcal{D}_a^{1,2}(\Omega)$ *as* $m\to\infty$.

ii) *Moreover, we have the following estimates* $\|e_i\|_{\mu,a}^2 \le \mu_k + Cm^{-2\sqrt{\overline{\mu}_a-\mu}}$
$\left|\left(e_i,e_j\right)_{\mathcal{D}_a^{1,2}(\Omega)}\right| \le Cm^{-2\sqrt{\overline{\mu}_a-\mu}}$ *for* $i\ne j$
$\left|\left(e_i,e_j\right)_2\right| \le Cm^{-c-2\sqrt{\overline{\mu}_a-\mu}}$ *for* $i\ne j$ *where* $\left(e_i,e_j\right)_2 := \int_\Omega |x|^{-\gamma} e_ie_j\mathrm{d}x$

iii) $\max_{\{u\in X_k^m,\|u\|_2=1\}}\|u\|_{\mu,a} \le \mu_k + Cm^{-2\sqrt{\overline{\mu}_a-\mu}}$ *where* $\|u\|_2^2 := \int_\Omega |x|^{-\gamma}u^2\mathrm{d}x$.

Lemma 3 [20] *For m large enough and* ε *small enough, we have*

$$\|\omega_\varepsilon^m\|_{\mu,a}^2 \le \left(\mathcal{K}_{a,b}\right)^{\frac{N}{2\eta}} + C\varepsilon^{\frac{4}{2_*-2}}m^{2\sqrt{\overline{\mu}_a-\mu}}$$

and

$$\int_\Omega \frac{|\omega_\varepsilon^m|^{2_*}}{|x|^{2_*b}}\mathrm{d}x \ge \left(\mathcal{K}_{a,b}\right)^{\frac{N}{2\eta}} + C\varepsilon^{\frac{22_*}{2_*-2}}m^{2_*\sqrt{\overline{\mu}_a-\mu}},$$

Furthermore, let $\varepsilon = m^{-h}$ *and* $h \geq \frac{2_*-2}{2}(1+l)\left(\sqrt{\overline{\mu}_a - \mu}\right)$, $l \geq 0$; *we have the following estimate*

$$\int_\Omega \left|\omega_\varepsilon^m\right|^2 |x|^{-\gamma}\,\mathrm{d}x \geq C\varepsilon^{\frac{4}{2_*-2}} m^{(1+l)\left(2\sqrt{\overline{\mu}_a-\mu}-c\right)}, \quad \text{as } m \to \infty.$$

The corresponding energy functional to (S_A) is:

$$J(u,v) = \frac{1}{2}\|(u,v)\|^2_{\mu,a} - \frac{1}{2}\int_\Omega \frac{\langle AU, U\rangle}{|x|^\gamma}\mathrm{d}x - \int_\Omega \frac{|u|^{\alpha+1}|v|^{\beta+1}}{|x|^{2_* b}}\mathrm{d}x,$$

then $J \in C^1(E,\mathbb{R})$ and the critical points of the functional J correspond to the solutions of (S_A).

3 Variational characterization

The variational characterization is based on a linking argument. To do this, we need to use the following lemma.

Lemma 4 *Assume that* $-\infty < a < \frac{N-2}{2}$, $a \leq b < a+1$, $\alpha, \beta \geq 0$ *such that* $\alpha + \beta = \frac{4\eta}{N-2\eta}$ *and* $c, \lambda > 0$. *Then there exists* $C_\lambda(x) := C(x, N, \alpha, \beta, \lambda) > 0$ *such that*

$$G_\lambda(x,u,v) := \frac{|u|^{\alpha+1}|v|^{\beta+1}}{|x|^{2_* b}} - \frac{\lambda}{|x|^\gamma}\left(u^2+v^2\right) \geq -C_\lambda(x),$$

for all $(u,v) \in \mathbb{R}^2 \backslash \{\mathbb{R}\times\{0\} \cup \{0\}\times\mathbb{R}\}$, *with* $C_\lambda(x) := \frac{2_*\eta}{N}\left(\frac{2\lambda}{(\alpha+1)|x|^{(2\eta-c)}}\right)^{\frac{N}{2\eta}} \theta^{\frac{-2(\beta+1)}{\alpha+\beta}}$ *and* $\theta := \left(\frac{\alpha+1}{\beta+1}\right)^{-\frac{1}{2}}$.

Proof (u,v) is an extremum point of G_λ if

$$(\alpha+1)\frac{u|u|^{\alpha-1}|v|^{\beta+1}}{|x|^{2_* b}} - 2\lambda\frac{u}{|x|^\gamma} = 0 \tag{3.1}$$

and

$$(\beta+1)\frac{|u|^{\alpha+1}v|v|^{\beta-1}}{|x|^{2_* b}} - 2\lambda\frac{v}{|x|^\gamma} = 0. \tag{3.2}$$

Multiplying (3.1) and (3.2) by $(\beta+1)u$ and $(\alpha+1)v$ respectively and subtracting them, we get

$$|v| = \theta|u|.$$

Put

$$g(u) := G_\lambda(|u|, \theta|u|) = \theta^{\beta+1}\frac{|u|^{2_*}}{|x|^{2_* b}} - \lambda(1+\theta^2)\frac{u^2}{|x|^\gamma}$$

$g(u)$ attains its minimum $-C_\lambda(x)$ at

$$u_0 = \left(\frac{2\lambda|x|^{2_* b}}{(\alpha+1)|x|^\gamma\theta^{\beta+1}}\right)^{\frac{1}{\alpha+\beta}},$$

the conclusion follows. □

We recall that the sequence $\{(u_n, v_n)\} \subset E$ is called a (PS) sequence for J at level c if $J(u_n, v_n) \to c$ and $J'(u_n, v_n) \to 0$ in E' (dual of E) *as* $n \to +\infty$.

Lemma 5 *Let* $(u_n, v_n) \subset E$ *be a* (PS) *sequence for* J; *then there exists* (u, v) *such that* $(u_n, v_n) \rightharpoonup (u, v)$, *up to subsequence, and* $J'(u, v) = 0$. *Moreover, if* $J(u_n, v_n) \to c$ *with* $0 < c < c_0 := \frac{\eta}{N} 2_*^{-\frac{N-2\eta}{2\eta}} \left(\widetilde{\mathcal{K}}_{a,b}\right)^{\frac{N}{2\eta}}$ *then* $(u, v) \neq (0, 0)$ *and hence* (u, v) *is a nontrivial solution of* (S_A).

Proof Let (u_n, v_n) be a $(PS)_c$ sequence, i.e.

$$\frac{1}{2}\|(u_n, v_n)\|_{\mu,a}^2 - \frac{1}{2}\int_\Omega \frac{\langle AU_n, U_n\rangle}{|x|^\gamma} \mathrm{d}x - \int_\Omega \frac{|u_n|^{\alpha+1}|v_n|^{\beta+1}}{|x|^{2_* b}} \mathrm{d}x = c + \circ(1) \tag{3.3}$$

and

$$\|(u_n, v_n)\|_{\mu,a}^2 - \int_\Omega \frac{\langle AU_n, U_n\rangle}{|x|^\gamma} \mathrm{d}x - 2_* \int_\Omega \frac{|u_n|^{\alpha+1}|v_n|^{\beta+1}}{|x|^{2_* b}} \mathrm{d}x = \circ(1) \tag{3.4}$$

Using (3.3) and (3.4) we obtain

$$2J(u_n, v_n) - \left\langle J'(u_n, v_n), (u_n, v_n)\right\rangle = (2_* - 2)\int_\Omega \frac{|u_n|^{\alpha+1}|v_n|^{\beta+1}}{|x|^{2_* b}} \mathrm{d}x \le 2c + \circ(1). \tag{3.5}$$

(3.3), (3.5) and Lemma 4 with $\lambda := \lambda_2$ yield that

$$\begin{aligned}\|(u_n, v_n)\|_{\mu,a}^2 &= 2J(u_n, v_n) + \int_\Omega \frac{\langle AU_n, U_n\rangle}{|x|^\gamma} \mathrm{d}x + 2\int_\Omega \frac{|u_n|^{\alpha+1}|v_n|^{\beta+1}}{|x|^{2_* b}} \mathrm{d}x \\ &\le 2J(u_n, v_n) + \lambda_2 \int_\Omega \frac{(u_n^2 + v_n^2)}{|x|^\gamma} \mathrm{d}x + 2\int_\Omega \frac{|u_n|^{\alpha+1}|v_n|^{\beta+1}}{|x|^{2_* b}} \mathrm{d}x \\ &\le C,\end{aligned}$$

and therefore we conclude that the sequence (u_n, v_n) is bounded in E.

Then, there exists a subsequence again denoted by (u_n, v_n) such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in E.

We claim that $(u, v) \neq (0, 0)$. Assume by contradiction that $(u, v) = (0, 0)$.

We know that

$$\left\langle J'(u_n, v_n), (u_n, v_n)\right\rangle = \circ(1).$$

Since the embedding $\mathcal{D}_a^{1,2}(\Omega) \hookrightarrow L^2\left(\Omega, |x|^{-\gamma}\,\mathrm{d}x\right)$ is compact, it follows that

$$\|(u_n, v_n)\|_{\mu,a}^2 - 2_* \int_\Omega \frac{|u_n|^{\alpha+1}|v_n|^{\beta+1}}{|x|^{2_* b}} \mathrm{d}x = \circ(1), \tag{3.6}$$

using the definition of $\widetilde{\mathcal{K}}_{a,b}$, we obtain

$$\|(u_n, v_n)\|_{\mu,a}^2 \ge \widetilde{\mathcal{K}}_{a,b}\left(\int_\Omega \frac{|u_n|^{\alpha+1}|v_n|^{\beta+1}}{|x|^{2_* b}} \mathrm{d}x\right)^{\frac{2}{2_*}}.$$

Hence

$$\|(u_n, v_n)\|_{\mu,a}^2 - 2_* \widetilde{\mathcal{K}}_{a,b}^{\frac{-2_*}{2}} \|(u_n, v_n)\|_{\mu,a}^{2_*} \leq \circ(1),$$

thus

$$\|(u_n, v_n)\|_{\mu,a}^2 \left(1 - 2_* \widetilde{\mathcal{K}}_{a,b}^{\frac{-2_*}{2}} \|(u_n, v_n)\|_{\mu,a}^{2_*-2}\right) \leq \circ(1).$$

If $\|(u_n, v_n)\|_{\mu,a}^2 \to 0$, passing in the limit in (3.3) we reach a contradiction with the assumption $c > 0$.

Therefore, we deduce

$$\left(1 - 2_* \widetilde{\mathcal{K}}_{a,b}^{-\frac{2_*}{2}} \|(u_n, v_n)\|_{\mu}^{2_*-2}\right) \leq 0$$

i.e.

$$\|(u_n, v_n)\|_{\mu,a}^2 \geq 2_*^{-\frac{2}{\alpha+\beta}} \widetilde{\mathcal{K}}_{a,b}^{\frac{2_*}{\alpha+\beta}}.$$

From (3.6), we have that

$$\begin{aligned} J(u_n, v_n) &= \left(\frac{1}{2} - \frac{1}{2_*}\right) \|(u_n, v_n)\|_{\mu,a}^2 \\ &\quad + \frac{1}{2_*}\left(\|(u_n, v_n)\|_{\mu,a}^2 - 2_* \int_\Omega \frac{|u_n|^{\alpha+1} |v_n|^{\beta+1}}{|x|^{2_* b}} \mathrm{d}x + \circ(1)\right) \\ &= \frac{\eta}{N} \|(u_n, v_n)\|_{\mu,a}^2 + \circ(1) \\ &\geq \frac{\eta}{N} \left(\frac{\widetilde{\mathcal{K}}_{a,b}^{\frac{2_*}{2}}}{2_*}\right)^{\frac{2}{\alpha+\beta}} + \circ(1), \end{aligned}$$

which contradicts the fact that $c < \frac{\eta}{N} 2_*^{-\frac{N-2\eta}{2\eta}} \widetilde{\mathcal{K}}_{a,b}^{\frac{N}{2\eta}}$. Thus $(u, v) \neq (0, 0)$ and (u, v) is a nontrivial solution of (S_A). □

Now, we prove that the functional J has linking geometry.

Proposition 1 *Suppose there exist* $k, k' \in \mathbb{N}^*$ *such that* $a_{11} = \mu_k$ *and* $a_{22} = \mu_{k'}$ *with* $k \leq k'$ *and* $a_{12} < \min\left(\mu_{k+1} - \mu_k, \mu_{k'+1} - \mu_{k'}\right)$. *Then*

1) *there exist* $\rho, \sigma > 0$ *such that* $J(u, v) \geq \sigma$ *for all* $(u, v) \in \left(\partial B_\rho \cap Y_k\right) \times \left(\partial B_\rho \cap Y_{k'}\right)$.
2) *there exists* $R > \rho$ *such that* $J\,|_{\partial Q_\varepsilon^m} \leq p(m)$ *with* $p(m) \to 0$ *as* $m \to +\infty$ *where*

$$Q_\varepsilon^m = \left(\left(\overline{B}_R \cap X_k^m\right) \oplus \left\{Br\omega_\varepsilon^m / 0 \leq r < R\right\}\right) \times \left(\left(\overline{B}_R \cap X_{k'}^m\right) \oplus \left\{Dr\omega_\varepsilon^m / 0 \leq r < R\right\}\right).$$

Proof For any $(u, v) \in Y_k \times Y_{k'}$, we have

$$\|u\|_{\mu,a}^2 \geq \mu_{k+1} \int_\Omega u^2 |x|^{-\gamma} \mathrm{d}x \quad \text{and} \quad \|v\|_{\mu,a}^2 \geq \mu_{k'+1} \int_\Omega v^2 |x|^{-\gamma} \mathrm{d}x.$$

From above relations, Young's and Caffarelli–Kohn–Nirenberg's inequalities, we obtain

$$J(u, v) \geq \frac{1}{2}\left(1 - \frac{\mu_k + a_{12}}{\mu_{k+1}}\right) \|u\|_{\mu,a}^2 + \frac{1}{2}\left(1 - \frac{\mu_{k'} + a_{12}}{\mu_{k'+1}}\right) \|v\|_{\mu,a}^2 - C\left(\|u\|_{\mu,a}^{2_*} + \|v\|_{\mu,a}^{2_*}\right).$$

Hence, we can choose $\|(u,v)\|_{\mu,a} = \rho$ small enough, $\sigma > 0$ and under the assumption that $a_{12} < \min\left(\mu_{k+1} - \mu_k, \mu_{k'+1} - \mu_{k'}\right)$ we obtain

$$J\mid_{(\partial B_\rho \cap Y_k)\times(\partial B_\rho \cap Y_{k'})} \geq \sigma.$$

From Lemmas 2 and 4, we have for $(u,v) \in X_k^m \times X_{k'}^m$,

$$\begin{aligned} J(u,v) &\leq \frac{1}{2} C_1 m^{-2\sqrt{\overline{\mu}_a - \mu}} \int_\Omega \frac{\left(u^2+v^2\right)}{|x|^\gamma} \mathrm{d}x - \int_\Omega \frac{|u|^{\alpha+1}|v|^{\beta+1}}{|x|^{2_* b}} \mathrm{d}x \\ &\leq -\int_\Omega G_\lambda(u,v)\,\mathrm{d}x \quad \text{with } \lambda := C_1 m^{-2\sqrt{\overline{\mu}_a - \mu}}, \end{aligned}$$

thus

$$J(u,v) \leq C_2 m^{-\frac{N\sqrt{\overline{\mu}_a - \mu}}{\eta}},$$

hence there results that

$$\lim_{m\to\infty} \max_{(u,v)\in X_k^m \times X_{k'}^m} J(u,v) = 0.$$

On the other hand, we have

$$J\left(Br\omega_\varepsilon^m, Dr\omega_\varepsilon^m\right) \leq \frac{r^2}{2}\left(B^2 + D^2\right)\|\omega_\varepsilon^m\|_{\mu,a}^2 - r^{2_*} B^{\alpha+1} D^{\beta+1} \int_\Omega \frac{|\omega_\varepsilon^m|^{2_*}}{|x|^{2_* b}} \mathrm{d}x,$$

then $J\left(Br\omega_\varepsilon^m, Dr\omega_\varepsilon^m\right)$ becomes negative if $r = R$ and R is large enough.

Therefore

$$J(u,v) \leq C_2 m^{-\frac{N\sqrt{\overline{\mu}_a - \mu}}{\eta}}$$

for all $(u,v) \in \left(X_k^m \cup \left(X_k^m \oplus R\left\{B\omega_\varepsilon^m\right\}\right)\right) \times \left(X_{k'}^m \cup \left(X_{k'}^m \oplus R\left\{D\omega_\varepsilon^m\right\}\right)\right)$. Since

$$\max_{0\leq r\leq R} J\left(Br\omega_\varepsilon^m, Dr\omega_\varepsilon^m\right) < +\infty$$

as $(u,v) \in \left(X_k^m \oplus \mathbb{R}^+\left\{B\omega_\varepsilon^m\right\}\right) \times \left(X_{k'}^m \oplus \mathbb{R}^+\left\{D\omega_\varepsilon^m\right\}\right)$ we may write $u = w_1 + tB\omega_\varepsilon^m$ and $v = w_2 + tD\omega_\varepsilon^m$, hence

$$\operatorname{meas}\left(\operatorname{supp}\left(\omega_\varepsilon^m\right) \cap \operatorname{supp}(w_i)\right) = 0, \quad i = 1, 2$$

then for large R,

$$J\mid_{\partial Q_\varepsilon^m} \leq 0,$$

where $Q_\varepsilon^m = \left(\left(\overline{B}_R \cap X_k^m\right) \oplus \left\{Br\omega_\varepsilon^m / 0 \leq r < R\right\}\right) \times \left(\left(\overline{B}_R \cap X_k^m\right) \oplus \left\{Dr\omega_\varepsilon^m / 0 \leq r < R\right\}\right)$.

□

Proposition 2 *Suppose there exists $k \in \mathbb{N}^*$ such that $\lambda_2 = \mu_k$. Then*

1) *there exist $\rho, \sigma > 0$ such that $J(u,v) \geq \sigma$ for all $(u,v) \in \left(\partial B_\rho \cap Y_k\right) \times \left(\partial B_\rho \cap Y_k\right)$.*
2) *there exists $R > \rho$ such that $J\mid_{\partial Q_\varepsilon^m} \leq p(m)$ with $p(m) \to 0$ as $m \to +\infty$ where*

$$Q_\varepsilon^m = \left(\left(\overline{B}_R \cap X_k^m\right) \oplus \left\{Br\omega_\varepsilon^m / 0 \leq r < R\right\}\right) \times \left(\left(\overline{B}_R \cap X_k^m\right) \oplus \left\{Dr\omega_\varepsilon^m / 0 \leq r < R\right\}\right).$$

Proof For any $(u, v) \in Y_k \times Y_k$, we have

$$\|(u, v)\|_{\mu,a}^2 \geq \mu_{k+1} \int_\Omega \frac{\left(u^2 + v^2\right)}{|x|^\gamma} \mathrm{d}x, \tag{3.7}$$

using (3.7) and Caffarelli–Kohn–Nirenberg's inequality we obtain

$$J(u, v) \geq \frac{1}{2}\left(1 - \frac{\lambda_2}{\mu_{k+1}}\right) \|(u, v)\|_{\mu,a}^2 - C \|(u, v)\|_{\mu,a}^{2_*}.$$

Then there exists $(u, v) \in Y_k \times Y_k$ such that $J|_{(\partial B_\rho \cap Y_k)^2} \geq \sigma > 0$ with $\|(u, v)\|_{\mu,a} = \rho$ for ρ sufficiently small. For any $(u, v) \in X_k^m \times X_k^m$, we obtain from the estimates of Lemmas 2 and 4 that

$$J(u, v) \leq \frac{C_1 m^{-2\sqrt{\overline{\mu}_a - \mu}}}{2} \int_\Omega \frac{\left(u^2 + v^2\right)}{|x|^\gamma} \mathrm{d}x - \int_\Omega \frac{|u|^{\alpha+1} |v|^{\beta+1}}{|x|^{2_* b}} \mathrm{d}x$$
$$\leq C_2 m^{-\frac{N\sqrt{\overline{\mu}_a - \mu}}{\eta}}.$$

Therefore

$$J|_{\partial Q_\varepsilon^m} \leq C_2 m^{-\frac{N\sqrt{\overline{\mu}_a - \mu}}{\eta}} \quad \text{with} \quad C_2 m^{-\frac{N\sqrt{\overline{\mu}_a - \mu}}{\eta}} \to 0 \text{ as } m \to +\infty.$$

The rest of the proof is the same as in Proposition 1. □

4 Proof of Theorems 1 and 2

Set

$$c_\varepsilon = \inf_{h \in \Gamma_{\varepsilon,m}} \max_{U \in Q_\varepsilon^m} J(h(U))$$
$$\Gamma_{\varepsilon,m} = \left\{h \in C\left(Q_\varepsilon^m, E\right) / h(U) = U, \quad \text{for all U} \in Q_\varepsilon^m\right\}$$

and

$$Q_\varepsilon^m = \left(\left(\overline{B}_R \cap X_k^m\right) \oplus \left\{Br\omega_\varepsilon^m / 0 \leq r < R\right\}\right) \times \left(\left(\overline{B}_R \cap X_{k'}^m\right) \oplus \left\{Dr\omega_\varepsilon^m / 0 \leq r < R\right\}\right)$$

if $a_{11} = \mu_k$ and $a_{22} = \mu_{k'}$, or

$$Q_\varepsilon^m = \left(\left(\overline{B}_R \cap X_k^m\right) \oplus \left\{Br\omega_\varepsilon^m / 0 \leq r < R\right\}\right) \times \left(\left(\overline{B}_R \cap X_k^m\right) \oplus \left\{Dr\omega_\varepsilon^m / 0 \leq r < R\right\}\right)$$

if $\lambda_2 = \mu_k$

Lemma 6 *Let* $N > \frac{c+2+\sqrt{(c+2)^2+8c\eta}}{2}$ *and* $\overline{\mu}_a - \left(\frac{N-2}{2}\right)^2 \leq \mu < \overline{\mu}_a - \left(\frac{c(N+2\eta)}{2N}\right)^2$. *Assume one of the following conditions holds*:

a) *there exists* $k, k' \in \mathbb{N}^*$, $k \leq k'$ *such that* $a_{11} = \mu_k$ *and* $a_{22} = \mu_{k'}$.
b) *there exists* $k \in \mathbb{N}^*$ *such that* $\lambda_2 = \mu_k$. *Then, we have*

$$c_\varepsilon < \frac{\eta}{N} 2_*^{-\frac{N-2\eta}{2\eta}} \left(\widetilde{\mathcal{K}}_{a,b}\right)^{\frac{N}{2\eta}}.$$

Proof Let

$$\max_{(u,v)\in Q_\varepsilon^m} J(u,v) = J\left(y_m + t_\varepsilon^m B\omega_\varepsilon^m, z_m + t_\varepsilon^m D\omega_\varepsilon^m\right)$$

where $B, D > 0$ such that $\frac{B}{D} = \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{1}{2}}$, and

$$(y_m, z_m) \in X_k^m \times X_{k'}^m \quad \text{if } a_{11} = \mu_k \text{ and } a_{22} = \mu_{k'}$$

or

$$(y_m, z_m) \in X_k^m \times X_k^m \text{ if } \lambda_2 = \mu_k.$$

From Propositions 1 and 2, we have

$$J(y_m, z_m) \le Cm^{-\frac{N}{\eta}\sqrt{\overline{\mu}_a - \mu}}$$

for m large enough. Taking $\varepsilon = m^{-h}$, $h = (2_* - 1)\sqrt{\overline{\mu}_a - \mu} \ge (1+l)\sqrt{\overline{\mu}_a - \mu}\left(\frac{2_*-2}{2}\right)$, we get

$$J(y_m, z_m) \le C_1 \varepsilon^{\frac{4}{2_*-2}} m^{2\sqrt{\overline{\mu}_a - \mu}}.$$

Since meas $\left(\text{supp}\left(\omega_\varepsilon^m\right) \cap \text{supp}\,(y_m)\right) = 0$ and meas $(\text{supp}\left(\omega_\varepsilon^m\right) \cap \text{supp}(z_m)) = 0$, we conclude that

$$\begin{aligned} c_\varepsilon &\le \max_{(u,v)\in Q_\varepsilon^m} J(u,v) = J(y_m, z_m) + J\left(t_\varepsilon^m B\omega_\varepsilon^m, t_\varepsilon^m D\omega_\varepsilon^m\right) \\ &\le C_1 \varepsilon^{\frac{4}{2_*-2}} m^{2\sqrt{\overline{\mu}_a - \mu}} + \frac{\left(t_\varepsilon^m\right)^2}{2}\left(B^2 + D^2\right)\left(\left\|\omega_\varepsilon^m\right\|_{\mu,a}^2 - C_2 \int_\Omega \frac{\left|\omega_\varepsilon^m\right|^2}{|x|^\gamma} \mathrm{d}x\right) \\ &\quad - B^{\alpha+1} D^{\beta+1}\left(t_\varepsilon^m\right)^{2_*} \int_\Omega \frac{\left|\omega_\varepsilon^m\right|^{2_*}}{|x|^{2_* b}} \mathrm{d}x. \end{aligned}$$

Using Lemma 3, we obtain

$$\begin{aligned} c_\varepsilon &\le C_1 \varepsilon^{\frac{4}{2_*-2}} m^{2\sqrt{\overline{\mu}_a - \mu}} \\ &\quad + \frac{\left(t_\varepsilon^m\right)^2}{2}\left(B^2 + D^2\right)\left(\left(\mathcal{K}_{a,b}\right)^{\frac{N}{2\eta}} + C_2 \varepsilon^{\frac{4}{2_*-2}} m^{2\sqrt{\overline{\mu}_a - \mu}} - C_3 \varepsilon^{\frac{4}{2_*-2}} m^{(1+l)\left(2\sqrt{\overline{\mu}_a - \mu} - c\right)}\right) \\ &\quad - B^{\alpha+1} D^{\beta+1}\left(t_\varepsilon^m\right)^{2_*}\left(\left(\mathcal{K}_{a,b}\right)^{\frac{N}{2\eta}} - C_4 \varepsilon^{\frac{22_*}{2_*-2}} m^{2_*\sqrt{\overline{\mu}_a - \mu}}\right). \end{aligned}$$

Put

$$\begin{aligned} g\left(t_\varepsilon^m\right) :=& \frac{\left(t_\varepsilon^m\right)^2}{2}\left(B^2 + D^2\right)\left(\left(\mathcal{K}_{a,b}\right)^{\frac{N}{2\eta}} + C_2 \varepsilon^{\frac{4}{2_*-2}} m^{2\sqrt{\overline{\mu}_a - \mu}} - C_3 \varepsilon^{\frac{4}{2_*-2}} m^{(1+l)\left(2\sqrt{\overline{\mu}_a - \mu} - c\right)}\right) \\ &- B^{\alpha+1} D^{\beta+1}\left(t_\varepsilon^m\right)^{2_*}\left(\left(\mathcal{K}_{a,b}\right)^{\frac{N}{2\eta}} - C_4 \varepsilon^{\frac{22_*}{2_*-2}} m^{2_*\sqrt{\overline{\mu}_a - \mu}}\right), \end{aligned}$$

then

$$\max_{t_\varepsilon^m>0} g\left(t_\varepsilon^m\right) \leq \frac{\eta}{N}\left(\frac{\left(\widetilde{\mathcal{K}}_{a,b}\right)^{\frac{2_*}{2}}}{2_*}\right)^{\frac{2}{\alpha+\beta}} + C_1\varepsilon^{\frac{4}{2_*-2}}m^{2\sqrt{\overline{\mu}_a-\mu}}$$
$$- C_2\varepsilon^{\frac{4}{2_*-2}}m^{(1+l)\left(2\sqrt{\overline{\mu}_a-\mu}-c\right)} + C_3\varepsilon^{\frac{22_*}{2_*-2}}m^{2_*\sqrt{\overline{\mu}_a-\mu}}.$$

Consequently,

$$c_\varepsilon \leq \max_{(u,v)\in Q_\varepsilon^m} J(u,v) \leq \frac{\eta}{N}\left(\frac{\left(\widetilde{\mathcal{K}}_{a,b}\right)^{\frac{2_*}{2}}}{2_*}\right)^{\frac{2}{\alpha+\beta}} - C\varepsilon^{\frac{4}{2_*-2}}m^{(1+l)\left(2\sqrt{\overline{\mu}_a-\mu}-c\right)},$$

where we used the fact that $c(l+1) < 2_*l\left(\sqrt{\overline{\mu}_a-\mu}\right)$ and $2\sqrt{\overline{\mu}_a-\mu} < (l+1)\left(2\sqrt{\overline{\mu}_a-\mu}-c\right)$ in which these inequalities are consequence of the assumption

$$\frac{c(2\eta+N)(N-2\eta)}{2N^2} < \frac{c(2\eta+N)}{2N} < \sqrt{\overline{\mu}_a-\mu}.$$

Then for m large enough, $\overline{\mu}_a - \left(\frac{N-2}{2}\right)^2 \leq \mu < \overline{\mu}_a - \left(\frac{c(2\eta+N)}{2N}\right)^2$, hence there results that

$$c_\varepsilon < \frac{\eta}{N}2_*^{-\frac{N-2\eta}{2\eta}}\left(\widetilde{\mathcal{K}}_{a,b}\right)^{\frac{N}{2\eta}}.$$

□

Set

$$\mu_+ = \min\left\{\mu_k \in \sigma_{\mu,a}, \lambda_2 < \mu_k\right\}.$$

We denote by M^+ and M^- the following subspaces:

$$M^+ = \overline{\oplus_{\mu_k\geq\mu_+}M(\mu_k)},\ \ M^- = \oplus_{\mu_k\leq\mu_+}M(\mu_k)$$

where the closure is taken over $\mathcal{D}_a^{1,2}(\Omega)$ and $M(\mu_k)$ denotes the eigenspace corresponding to the eigenvalue μ_k.

Lemma 7 *We have*

$$\beta_\lambda = \sup_{(u,v)\in M^-\times M^-} J(u,v) \leq \frac{\eta}{N}\left[\frac{\left(\left(1+\theta^2\right)(\mu_+-\lambda_1)\right)^{2_*}}{\left(2_*\theta^{\beta+1}\right)^2}\right]^{\frac{1}{2_*-2}}\int_\Omega |x|^{\frac{cN}{2\eta}-N}\,\mathrm{d}x.$$

Moreover, there exist constants ρ_{λ_2} *and* $\delta_{\lambda_2} \in (0,\beta_\lambda)$ *such that*

$$J(u,v) \geq \delta_{\lambda_2}\ \ for\ \ any\ \ (u,v)\in M^+\times M^+,\ \ \|(u,v)\|_{\mu,a} = \rho_{\lambda_2}.$$

Proof Using the definition of the subspaces M^+ and M^-, we have the following inequalities

$$\|(u,v)\|_{\mu,a}^2 \leq \mu_+\int_\Omega \frac{\left(u^2+v^2\right)}{|x|^\gamma}\mathrm{d}x \quad \text{for any } (u,v)\in M^-\times M^-, \tag{4.3}$$

$$\|(u,v)\|_{\mu,a}^2 \geq \mu_+\int_\Omega \frac{\left(u^2+v^2\right)}{|x|^\gamma}\mathrm{d}x \quad \text{for any } (u,v)\in M^+\times M^+. \tag{4.4}$$

Taking into account (4.3) and Lemma 4, we have for all $(u, v) \in M^- \times M^-$

$$J(u,v) \leq \frac{1}{2}(\mu_+ - \lambda_1) \int_\Omega \frac{(u^2+v^2)}{|x|^\gamma} dx - \int_\Omega \frac{|u|^{\alpha+1}|v|^{\beta+1}}{|x|^{2_* b}} dx$$

$$\leq \frac{\eta}{N} \left[\frac{\left((1+\theta^2)(\mu_+ - \lambda_1)\right)^{2_*}}{\left(2_* \theta^{\beta+1}\right)^2} \right]^{\frac{1}{2_*-2}} \int_\Omega |x|^{\frac{cN}{2\eta} - N} dx.$$

Let $(u, v) \in M^+ \times M^+$, from the definition of $\widetilde{\mathcal{K}}_{a,b}$ and (3.9) we get

$$J(u,v) \geq \frac{1}{2}\left(1 - \frac{\lambda_2}{\mu_+}\right) \|(u,v)\|^2_{\mu,a} - \left(\widetilde{\mathcal{K}}_{a,b}\right)^{\frac{-2_*}{2}} \|(u,v)\|^{2_*}_{\mu,a},$$

if we take

$\|(u,v)\|_{\mu,a} = \rho_{\lambda_2} := \left(\frac{\mu_+ - \lambda_2}{\mu_+ 2_* (\widetilde{\mathcal{K}}_{a,b})^{\frac{-2_*}{2}}} \right)^{\frac{1}{2_*-2}}$ and $\delta_{\lambda_2} < \frac{\eta}{N}\left(1 - \frac{\lambda_2}{\mu_+}\right)^{\frac{2_*}{2_*-2}} \left(\frac{\widetilde{\mathcal{K}}_{a,b}}{2_*}\right)^{\frac{1}{2_*-2}}$, then we obtain

$$J(u,v) \geq \delta_{\lambda_2} \quad \text{for all } (u,v) \in \left(M^+ \times M^+\right) \cap \left(\partial B_{\rho_{\lambda_2}}\right)^2.$$

Since $M^+ \cap M^- = M(\mu_+)$, we have $M^+ \cap M^- \cap \partial B_{\rho_{\lambda_2}} \neq \varnothing$ for any $(u, v) \in \left(M^+ \cap M^- \cap \partial B_{\rho_{\lambda_2}}\right)^2$ satisfying $\delta_{\lambda_2} < J(u,v) \leq \sup_{(u,v)\in M^- \times M^-} J(u,v) \leq \beta_\lambda$. □

Now, we are ready to prove our results.

Proof of Theorem 1 Propositions 1 and 2, Lemmas 5 and 6 allow us to use Linking Theorem in [2]. As a consequence the functional J has a critical value $c \in \left(0, \frac{\eta}{N}\left(\frac{(\widetilde{\mathcal{K}}_{a,b})^{\frac{2_*}{2}}}{2_*}\right)^{\frac{2}{\alpha+\beta}}\right)$ □

Proof of Theorem 2 It suffices to apply Theorem 2.5 in [8] with $H = E$, $W = \left(M^-\right)^2$ and $V = \left(M^+\right)^2$, $\beta = \frac{\eta}{N}\left(\frac{(\widetilde{\mathcal{K}}_{a,b})^{\frac{2_*}{2}}}{2_*}\right)^{\frac{2}{\alpha+\beta}}$, $\delta = \delta_{\lambda_2}$, $\beta' = \beta_\lambda$, $\rho = \rho_{\lambda_2}$. □

Acknowledgments The authors would like to thank the referees for several helpful comments.

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