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Singular perturbed problems in the zero mass case: asymptotic behavior of spikes

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Abstract We discuss the asymptotic behavior of the least energy solution of a Dirichlet problem in the zero mass case. If Q is a uniformly positive potential having k isolated local minima, then we prove the existence of a positive multi-spike solutions having k peaks concentrating at each local minima of the potential.

Keywords Concentration phenomena · Peak solutions · Morse index · Finite dimensional reduction

Mathematics Subject Classification (2000) 35J10 · 35J65

1 Introduction

There has been considerable interest in understanding the behavior of positive solutions of the elliptic problem

$$\begin{cases} -\varepsilon^2 \Delta u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a parameter, f is a superlinear function and Ω is a smooth bounded domain in $\mathbb{R}^N$. Let $F(x, u) = \int_0^u f(x, t)\mathrm{d}t$. We consider the problems in the zero mass case i.e. when $f(x, 0) = 0$ and $f_u(x, 0) = 0$. Let $f(x, u) = f(u)$. Then problem (1.1) can be viewed as borderline problems because if $f'(0) > 0$, there is no non-trivial solutions for small $\varepsilon > 0$ Berestycki and Lions [2] proved the existence of ground state solutions if $f(u)$ behaves

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like $|u|^p$ for large u and $|u|^q$ for small u where p and q are supercritical and subcritical, respectively.

In this paper we consider the problems,

$$\begin{cases} -\varepsilon^2 \Delta u = u^p - u^q & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

$$\begin{cases} -\varepsilon^2 \Delta u = u^p - Q(x) u^q & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

where $1 < q < p < \frac{N+2}{N-2}$, $N \geq 3$ and $Q(x) \geq b > 0$ for all $x \in \Omega$, Q is bounded and smooth. Let U be a solution of

$$\begin{cases} -\Delta u = u^p - u^q & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u \to 0 & \text{as } |x| \to \infty \\ u \in C^2(\mathbb{R}^N). \end{cases} \tag{1.4}$$

By [12] and [11], U is radial and unique. Locating the points of concentration is important because they provide a concrete way of understanding the geometry of a class of solutions. In this paper, we study problems concerning the asymptotic behavior of the mountain pass solution and existence of multi-peak solutions for $\varepsilon > 0$ sufficiently small. Let $N \geq 3$ and $q^\star := \frac{N}{N-2}$. The exponent $q^\star$ is somewhat critical to the problems considered above. Then

Theorem 1.1 *Consider the problem* (1.2)*. For $q > q^\star$, there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, there exists a least energy positive solution $u_\varepsilon \in H_0^1(\Omega)$ of the problem and u_ε has a unique point of maximum x_ε. Then u_ε concentrates at a minima of $\psi_x(x)$, where ψ_x satisfies,*

$$\begin{cases} -\Delta \psi_x = 0 & \text{in } \Omega \\ \psi_x = \frac{1}{|x-y|^{N-2}} & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Hence u_ε concentrates at a harmonic center of Ω.

Note that in the case $q = 1$, the least energy solution to the problem (1.2) has a unique maxima x_ε; as ε tends to zero u_ε decays exponentially away from x_ε and $d(x_\varepsilon, \partial\Omega) \to \max_{x\in\Omega} d(x, \partial\Omega)$. This implies that the solution concentrates at an interior point furthest from the boundary of Ω. This was studied by Ni and Wei [13]. Later Flucher and Wei [10], proved that if $f(u) = (u-1)_+^p$, then the least energy solution of (1.1) concentrates at the harmonic center of Ω. Note that harmonic center in general is different from the point of maximal distance from the boundary. With a slight modification of our proof we can prove that results of Theorem 1.1 holds for the nonlinearity

$$f(u) = u^p - \sum_{j=1}^{m} c_j u^{q_j}$$

where $1 < q_j < p$, $c_j > 0$ and $m \in \mathbb{N}$.

Let $\alpha = \max\{\frac{2}{q-1}, N-2\}$. We have the following result:

Theorem 1.2 *Consider the problem* (1.3) *and assume $q \neq q^\star$. Let Q has k isolated local minima in Ω say $z_1, z_2, \ldots, z_k$. Then, there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$,* *there exists a positive solution* $u_\varepsilon \in H_0^1(\Omega)$ *to the problem* (1.2) *possessing exactly* k *maxima* $x_{\varepsilon,j} \in \Omega$ *such that* $x_{\varepsilon,j} \to z_j$ *for* $j = 1, 2, \dots, k$ *and there exists a constant* $C > 0$ *independent of* ε, Q *such that*

$$u_\varepsilon(x) \leq C \frac{\varepsilon^\alpha}{|x - x_{\varepsilon,j}|^\alpha}$$

away from z_j.

In the case $q = 1$, the existence of a single spike solution first studied by Floer and Weinstein [8]. When $\Omega = \mathbb{R}$ and $f(u) = u^3$, they constructed a single spike solution concentrating around any given non-degenerate critical point of the potential Q. Later Yong-Geun [16,17], extended the result of Floer and Weinstein in the higher dimensional case. Wang [19] showed that the mountain pass solution concentrate around a global minimum point of Q. When $\Omega = \mathbb{R}^N$, Del Pino and Felmer [5], proved an analogue of Wang's result imposing the condition on Q that there exists a bounded domain Λ with

$$\inf_\Lambda Q < \inf_{\partial\Lambda} Q.$$

They then prove that the above problem has a solution concentrating around a minimum of Q in Λ. Moreover, in [6,7] they proved the existence of multi-peak solutions concentrating near any finite set of local minima of a uniformly positive potential. Problem (1.2) was studied by Dancer [3] in domains having some kind of symmetry. In fact, he proved that for sufficiently small $\varepsilon > 0$, the positive solution is unique. Note that the positive solutions we obtain are concentrating exactly at the local minima of V. Our main contribution is to cover the case where $q > 1$. Before proving the main theorems, we look in to the difficulties associated with the problem.

- The solution of (1.4), $U \in D^{1,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$ and U decays algebraically.
- Since our proof requires nondegeneracy results and $U \in D^{1,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$, we work in the larger space $D^{1,2}(\mathbb{R}^N)$.
- Approximate solution to U may not be positive in Ω in the Dirichlet case. In the case the problem (1.2) with Neumann boundary conditions, the approximate solution to U is positive and satisfy

$$\begin{cases} -\varepsilon^2 \Delta Z_\varepsilon + qU_\varepsilon^{q-1} Z_\varepsilon = U_\varepsilon^p + (q-1)U_\varepsilon^q & \text{in } \Omega \\ \frac{\partial Z_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

  where U_ε is a re-scaled version of U and one expects to obtain similar results to [14] and [15].
- Most surprising fact is the existence of the exponent $q^\star$ such that for all $q \in (1, q^\star]$, the asymptotic behavior of least energy solution of problem (1.1) cannot be studied by our method. The natural question arises, is it possible to obtain a higher order expansion for the case $q \in (1, q^\star]$? This runs into a major problem as U^{q-1} is not integrable at infinity. In fact, for $q = q^\star$, we expect the entire solution U to satisfy $U \sim r^{-(N-2)}(\log r)^{-\frac{N-2}{2}}$ as $r \to \infty$.
- The reduction method could in principle be applied to $Q \equiv 1$, but it seems difficult to determine the location of peaks by our method.
- Finally note that we cannot extend Theorem 1.2 to unbounded domains. The main reason for that is we cannot obtain good boundary estimates as (7.7).

2 Preliminaries

Let us modify the problem (1.2) to

$$\begin{cases} -\varepsilon^2 \Delta u = (u^+)^p - (u^+)^q & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

It is easy to show that any solution of (2.1) is positive and is in fact a positive solution to (1.2). Note that the associated functional to the problem (1.2) is

$$\Phi_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 - \frac{1}{p+1} (u^+)^{p+1} + \frac{1}{q+1} (u^+)^{q+1} \right) \mathrm{d}x$$

Note that Φ_ε satisfies Palais Smale condition and all the conditions of the mountain pass theorem and hence there exist a mountain pass solution $u_\varepsilon > 0$ and a mountain pass critical value

$$0 < c_\varepsilon = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\varepsilon(\gamma(t))$$

where

$$\Gamma = \left\{ \gamma \in C\left([0,1], H_0^1(\Omega)\right) : \gamma(0) = 0, \gamma(1) \neq 0, \Phi_\varepsilon(\gamma(1)) \leq 0 \right\}.$$

With a change of variable the problem (1.2) takes the form

$$\begin{cases} -\Delta u = u^p - u^q & \text{in } \Omega_\varepsilon \\ u > 0 & \text{in } \Omega_\varepsilon \\ u = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \tag{2.2}$$

where Ω_ε is a re-scaled version of Ω. The functional associated to the problem (2.2) is

$$I_\varepsilon(u) = \int_{\Omega_\varepsilon} \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} (u^+)^{p+1} + \frac{1}{q+1} (u^+)^{q+1} \right) \mathrm{d}x$$

Note that $I_\varepsilon(0) = 0$, $I_\varepsilon(tu) \to -\infty$ as $t \to +\infty$ and I_ε satisfies the Palais Smale condition on $H_0^1(\Omega)$. Hence, we obtain a positive solution v_ε for each $\varepsilon > 0$ obtained by the mountain pass theorem. Then the mountain pass critical value b_ε is given by

$$b_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} I_\varepsilon(\gamma(t))$$

where

$$\Gamma_\varepsilon = \left\{ \gamma \in C\left([0,1], H_0^1(\Omega_\varepsilon)\right) : \gamma(0) = 0, \gamma(1) \neq 0, I_\varepsilon(\gamma(1)) \leq 0 \right\}$$

Note that as 0 is a strict local minima of I_ε, $b_\varepsilon > 0$, $\forall \varepsilon > 0$. Also note that $\Phi_\varepsilon(u) = \varepsilon^N I_\varepsilon(u)$ which implies that $c_\varepsilon = \varepsilon^N b_\varepsilon$. Let

$$\mathcal{N}_\varepsilon(\Omega_\varepsilon) = \left\{ u \in H_0^1(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} |\nabla u|^2 + \int_{\Omega_\varepsilon} (u^+)^{q+1} = \int_{\Omega_\varepsilon} (u^+)^{p+1} \right\}.$$

Lemma 2.1 *We have for all* $\varepsilon > 0$

$$b_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} I_\varepsilon(\gamma(t)) = \inf_{u \in \mathcal{N}_\varepsilon(\Omega_\varepsilon)} I_\varepsilon(u) = \inf_{u \in H_0^1(\Omega_\varepsilon), u \neq 0} \max_{t \geq 0} I_\varepsilon(tu).$$

Proof For the sake of completeness we prove this well-known lemma. Let $\varepsilon > 0$ be fixed. First note that

$$\inf_{\gamma\in\Gamma_\varepsilon}\max_{t\in[0,1]} I_\varepsilon(\gamma(t)) \le \inf_{u\in H_0^1(\Omega_\varepsilon)}\max_{t\ge 0} I_\varepsilon(tu) \tag{2.3}$$

We first claim that $\inf_{u\in\mathcal{N}_\varepsilon(\Omega_\varepsilon)} I_\varepsilon(u) = \inf_{u\in H_0^1(\Omega_\varepsilon)}\max_{t\ge0} I_\varepsilon(tu)$. Define $h(t) = I_\varepsilon(tu)$. Then as discussed earlier and due to the nature of the nonlinearity we have $h(0) = 0$, $h(t) > 0$ for small $t > 0$ and $h(t) < 0$ for $t > 0$ sufficiently large. Hence $\max_{t\in[0,+\infty)} h(t)$ is achieved. Also note that $h'(t) = 0$ implies $\|u\|^2_{H_0^1(\Omega_\varepsilon)} = g(t)$ where

$$g(t) = t^{p-1}\int\limits_{\Omega_\varepsilon}\left(u^+\right)^{p+1} - t^{q-1}\int\limits_{\Omega_\varepsilon}\left(u^+\right)^{q+1}.$$

It is easy to see that g is an increasing function of t whenever $g(t) > 0$. Thus there exists a unique t such that $\|u\|_{H_0^1(\Omega)} = g(t)$. Hence there exist a unique point $\theta(u)$ such that $h'(\theta(u)u) = 0$ and $\theta(u)u \in \mathcal{N}_\varepsilon(\Omega_\varepsilon)$. This implies that $\mathcal{N}_\varepsilon(\Omega_\varepsilon)$ is radially homeomorphic to $H_0^1(\Omega_\varepsilon)\backslash\{0\}$ if we prove that $\theta : H_0^1(\Omega_\varepsilon)\backslash\{0\} \to \mathbb{R}^+$ is continuous. In order to do so let $u_n \to u$ in $H_0^1(\Omega_\varepsilon)\backslash\{0\}$. Then $u_n \to u$ in $H_0^1(\Omega_\varepsilon)$ and $u_n \to u$ in $L^r(\Omega_\varepsilon)$ for all $r \le \frac{N+2}{N-2}$ and

$$\int\limits_{\Omega_\varepsilon}|\nabla u_n|^2 = \theta^{p-1}(u_n)\int\limits_{\Omega_\varepsilon}\left(u_n^+\right)^{p+1} - \theta^{q-1}(u_n)\int\limits_{\Omega_\varepsilon}\left(u_n^+\right)^{q+1} \tag{2.4}$$

which proves there exist constants $m > 0$ and $M > 0$ independent of n such that $m \le \theta(u_n) \le M$. By passing to the limit in (2.4) the whole sequence $\{\theta(u_n)\}$ converges as u_n is convergent and hence $\theta(u) = \theta_0$ where $\theta_0 u \in \mathcal{N}_\varepsilon$ which proves our claim.

Next, we claim that $\inf_{\gamma\in\Gamma_\varepsilon}\max_{t\in[0,1]} I_\varepsilon(\gamma(t)) = \inf_{u\in\mathcal{N}_\varepsilon(\Omega_\varepsilon)} I_\varepsilon(u)$. It is easy to see that $\inf_{\gamma\in\Gamma_\varepsilon}\max_{t\in[0,1]} I_\varepsilon(\gamma(t)) \ge \inf_{u\in\mathcal{N}_\varepsilon(\Omega_\varepsilon)} I_\varepsilon(u)$ by (2.3). It is enough to prove that any $\gamma \in \Gamma_\varepsilon$ intersects $\mathcal{N}_\varepsilon$. Note that $I_\varepsilon(u) > 0$ for $\|u\|_{H_0^1(\Omega)}$ sufficiently small and $I_\varepsilon(\gamma(1)) < 0$ which implies the required result. □

Lemma 2.2 *There exists a $C > 0$ independent of ε such that $b_\varepsilon \le C$ for sufficiently small ε. Hence along a subsequence b_ε converges as $\varepsilon \to 0$.*

Proof Let $\varphi_1 > 0$ be the eigenfunction corresponding to the first eigenvalue λ_1 of $-\Delta$ in Ω with respect to the zero Dirichlet boundary conditions. Let $\int_\Omega \varphi_1^2 = 1$. Note that *supp* $\varphi_1 \subset \Omega \subset \Omega_\varepsilon$ for sufficiently small ε. Choose a $t > 0$ such that $I_\varepsilon(t\varphi_1) \le 0$. We claim that in fact t is uniformly bounded. We have

$$\begin{aligned} I_\varepsilon(t\varphi_1) &= \int\limits_{\Omega_\varepsilon}\left(\frac{1}{2}|\nabla t\varphi_1|^2 - \frac{1}{p+1}(t\varphi_1)^{p+1} + \frac{1}{q+1}(t\varphi_1)^{q+1}\right)\mathrm{d}x \\ &= \lambda_1 t^2\frac{1}{2}\int\limits_{\Omega_\varepsilon}\varphi_1^2 - \frac{t^{p+1}}{p+1}\int\limits_{\Omega_\varepsilon}\varphi_1^{p+1} + \frac{t^{q+1}}{q+1}\int\limits_{\Omega_\varepsilon}\varphi_1^{q+1} \\ &= \frac{\lambda_1 t^2}{2}\int\limits_{\Omega}\varphi_1^2 - \frac{t^{p+1}}{p+1}\int\limits_{\Omega}\varphi_1^{p+1} + \frac{t^{q+1}}{q+1}\int\limits_{\Omega}\varphi_1^{q+1} \end{aligned}$$

which implies $t^{p-1} \le C$. Now the right-hand side is independent of ε. Since $p > q > 1$, we can find $\bar{t} > 0$ such that $I_\varepsilon(\bar{t}\varphi_1) < 0$ for all ε small. Now

$$b_\varepsilon = \inf_{\gamma_\varepsilon \in \Gamma_\varepsilon} \max_{t \in [0,1]} I_\varepsilon(\gamma(t)).$$

Define $\gamma_1 : [0, 1] \to H_0^1(\Omega_\varepsilon)$ such that $\gamma_1(t) = t\bar{t}\varphi_1$. Hence we have

$$b_\varepsilon \le \max_{t \in [0,1]} I_\varepsilon(\gamma_1(t)) \le C$$

where $C > 0$ independent of ε, as required. □

Lemma 2.3 *The function $\psi_x(y)$ is positive and continuous in $\Omega \times \Omega$. Also $\psi_x(x) \to +\infty$ as $dist(x, \partial\Omega) \to 0$.*

Proof The result can be found in Bandle and Flucher [1]. □

As a result,

$$h(x) = \psi_x(x)$$

is strictly positive in Ω, locally bounded and $h(x) \to +\infty$ as $x \to \partial\Omega$. Hence it achieves a minimum in the interior of Ω.

Remark 2.4 Since

$$b_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon(\Omega_\varepsilon)} I_\varepsilon(u) = I_\varepsilon(v_\varepsilon)$$

we have

$$b_\varepsilon = I_\varepsilon(v_\varepsilon) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int\limits_{\Omega_\varepsilon} |\nabla v_\varepsilon|^2 + \left(\frac{1}{q+1} - \frac{1}{p+1}\right) \int\limits_{\Omega_\varepsilon} v_\varepsilon^{q+1} \tag{2.5}$$

which implies that $\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^2$, $\int_{\Omega_\varepsilon} v_\varepsilon^{p+1}$ and $\int_{\Omega_\varepsilon} v_\varepsilon^{q+1}$ are uniformly bounded. First note that from (1.2), $\max_{x \in \Omega} u_\varepsilon \ge 1$. Also note that by Gidas-Spruck [9] we obtain $\|v_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \le C$ and from Schauder estimates, it follows that there exists $C > 0$ such that $\|v_\varepsilon\|_{C^{2,\beta}_{loc}(\mathbb{R}^N)} \le C$ for some $0 < \beta \le 1$. Hence by the Ascoli-Arzela's theorem there exists an $U \ne 0$ such that

$$\|v_\varepsilon - U\|_{C^2_{loc}(\mathbb{R}^N)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Blowing up around z_ε (where z_ε is a point of maximum of v_ε) we easily see by a limit argument and the strong maximum principle U satisfies (1.4). (That $U \to 0$ as $|x| \to +\infty$ will be proved in the next section.) The only case we have difficulty is if z_ε is within order 1 of $\partial\Omega_\varepsilon$. In this case, we obtain a non-trivial solution of the half space problem.

$$\begin{cases} -\Delta u = u^p - u^q & \text{in } \mathbb{R}^N_+ \\ \quad u = 0 & \text{on } y_N = 0 \\ \quad u \in C^2\left(\mathbb{R}^N_+\right) \end{cases} \tag{2.6}$$

Suppose $\tilde{U}$ is a solution of (2.6) which achieves its maximum, then by [4] it follows that $\frac{\partial \tilde{U}}{\partial y_N} > 0$ in $\mathbb{R}^N_+$ and hence $\tilde{U}$ cannot achieve a maximum, a contradiction. Using the above argument, it is easy to show that $d(z_\varepsilon, \partial\Omega_\varepsilon) \to +\infty$ as $\varepsilon \to 0$. We call U to be the entire solution.

3 Asymptotics of the entire solution

Lemma 3.1 *Then U satisfies*

$$\nabla U \in L^2(\mathbb{R}^N), \quad U \in L^{p+1}(\mathbb{R}^N) \quad \text{and} \quad U \in L^{q+1}(\mathbb{R}^N).$$

Moreover,

$$\lim_{|x| \to +\infty} U(x) = 0,$$

and U is radially decreasing about the origin, U is the unique positive decaying solution of (1.4). For $q \neq q^\star$,

$$U(r) \sim \frac{1}{r^\alpha}$$

as $r \to +\infty$ where $\alpha = \max\left\{\frac{2}{q-1}, N-2\right\}$.

Proof Note that from (2.5) it follows easily that $\int_{\mathbb{R}^N} |\nabla U|^2$, $\int_{\mathbb{R}^N} U^{p+1}$ and $\int_{\mathbb{R}^N} U^{q+1}$ are finite. Hence applying one sided Harnack inequality [18], we have

$$\max_{B_1(x)} U \leq c \left(\int_{B_2(x)} U^{q+1} \right)^{1/q+1}$$

where $x \in \mathbb{R}^N$ is an arbitrary point and c is a constant depending on N. Hence we have

$$U(x) \to 0 \quad \text{as } |x| \to +\infty$$

Applying the result in [12], we obtain that U is radial. The uniqueness of U follows from [11]. Also note that $-U_{rr} - \frac{N-1}{r} U_r = (U^p - U^q)$, $U(0) > 1$ and hence for large r, $U_{rr} > 0$, which implies that U_r is increasing and hence $\lim_{r\to+\infty} |U_r| = U_r(0) = 0$.

First, we obtain the decay for the case $\alpha = N - 2$. Consider the problem $\Delta u_1 = 0$ in $\mathbb{R}^N \backslash B_R(0)$. Let $u_1 = r^{-(N-2)}$ and hence there exist $C > 0$ such that $U - Cu_1 < 0$ in ∂B_R and

$$-\Delta(U - Cu_1) < 0 \quad \text{in } \mathbb{R}^N \backslash B_R$$

and $U - Cr^{-(N-2)} \to 0$ as $r \to +\infty$. Note that if $U - Cu_1$ is positive somewhere on $\mathbb{R}^N \backslash B_R(0)$, it has a positive maxima which contradicts the fact that $\Delta(U - Cu_1) > 0$ in $\mathbb{R}^N \backslash B_R(0)$. Hence $U \leq Cr^{2-N}$ in $\mathbb{R}^N \backslash B_R$.

In the case $q < \frac{N}{N-2}$, we claim that there exists a $C_1 > 0$ such that $C_1 r^{-\frac{2}{q-1}} \geq U(r)$ for r sufficiently large. Define

$$H(r) = \frac{1}{2}\left(U'\right)^2 + \frac{1}{p+1} U^{p+1} - \frac{1}{q+1} U^{q+1}$$

Then $H(r)$ is a decreasing function. For large r, $U'(r)$ is small and hence it follows that $H(r) \to 0$ as $r \to +\infty$. Note that $H(r) \geq 0$ and hence for large r we have

$$|U'(r)|^2 \geq \left(\frac{2}{q+1} - \epsilon\right) U^{q+1}$$

for some $\epsilon > 0$ small and hence

$$\left|\left(U^{\frac{1-q}{2}}(r)\right)'\right| \geq k$$

Hence we have $U^{\frac{1-q}{2}} \geq kr$ for large r which implies that $U \leq C_1 r^{-\frac{2}{q-1}}$ for large r.

Define $v(r) = U(r)r^{\alpha}$. Then v is bounded and satisfies

$$-v_{rr} - \frac{(N-2\alpha-1)}{r}v_r + \frac{\alpha(N-2\alpha-2)}{r^2}v = r^{\alpha(1-p)}v^p - r^{\alpha(1-q)}v^q \tag{3.1}$$

that is

$$v_{rr} + \frac{|N-2\alpha-1|}{r}v_r = \frac{\alpha(N-2\alpha-2)}{r^2}v - r^{\alpha(1-p)}v^p + r^{\alpha(1-q)}v^q$$

where $\alpha = \max\left\{\frac{2}{q-1}, N-2\right\}$. For $N > 3$ we use the transformations $r = \mathrm{e}^{\frac{t}{|N-2\alpha-1|}}$ and $w(t) = v(r)$ in the above equation, we have

$$\begin{aligned} w''(t) &= \alpha(N-2\alpha-2)(N-2\alpha-1)^{-2}w \\ &\quad -(N-2\alpha-1)^{-2}\mathrm{e}^{\frac{(2+\alpha(1-p)|N-2\alpha-1|)t}{|N-2\alpha-1|}}w^p \\ &\quad +(N-2\alpha-1)^{-2}\mathrm{e}^{\frac{(2+\alpha(1-q)|N-2\alpha-1|)t}{|N-2\alpha-1|}}w^q \end{aligned} \tag{3.2}$$

Let $g(t)$ be the right-hand side of (3.2). Note that $(N-2\alpha-2) < 0$ and $\frac{(2+\alpha(1-q)|N-2\alpha-1|)t}{|N-2\alpha-1|} < 0$, hence w'' has definite sign after a certain stage and hence $\lim_{t\to+\infty} w'(t) = l$ (where l may be $\pm\infty$). For the case $l > 0$ and $l < 0$ we can deduce that $w(t) \to +\infty$ and $w(t) \to -\infty$ respectively as $t \to +\infty$ which contradicts the fact that $w(t)$ is bounded. Therefore, $w'(t) \to 0$ as $t \to +\infty$. Now $g(t)$ is integrable and as a result $w'(t) = -\int_t^{+\infty} g(s)\mathrm{d}s$. Hence $w'(t)$ has definite sign after a certain stage and hence we conclude that there exists $\mu \geq 0$ such that

$$\lim_{t\to+\infty} w(t) = \mu.$$

We claim that when $\alpha = \frac{2}{q-1}$, then $\mu > 0$. If $\mu = 0$, then by (3.2), $w''(t) < 0$ for $t \gg 0$. Thus there exists t_2 large such that $w'(t_2) < 0$. Note that $w(t) > 0$ in $(0, +\infty)$. Hence $w'(t) \leq w'(t_2) < 0$ for $t \geq t_2$ and this implies $w(t) \to -\infty$ as $t \to +\infty$, a contradiction. Hence $\mu > 0$.

For $\alpha = N-2$, and $N > 3$, we use the same technique as above to obtain $\mu > 0$.

For $N = 3$, note that $(N-2\alpha-1) = (N-3) = 0$ and hence (3.1) reduces to

$$v_{rr} + \frac{1}{r^2}v = r^{(1-p)}v^p - r^{(1-q)}v^q.$$

Hence we obtain for $r \gg 0$, $v_{rr} \leq 0$ as $\frac{v}{r^2} \geq 0$. This implies that $\lim_{r\to+\infty} v_r = 0$ by similar argument to above. Hence

$$v_r(r) = -\int_r^{+\infty}\left(\frac{1}{s^2}v(s) + \frac{1}{s^{p-1}}v^p(s) - \frac{1}{s^{q-1}}v^q(s)\right)\mathrm{d}s.$$

As a result v_r has a definite sign and hence $\lim_{r\to+\infty} v(r)$ exists. Applying the same technique as in the case $\alpha = \frac{2}{q-1}$ we obtain $\lim_{r\to+\infty} rU(r) > 0$. □

Corollary 3.2 *As $r \to +\infty$ we have,*

$$|U_r| \sim \begin{cases} \frac{1}{r^{N-1}} & \text{if } \alpha = N-2 \\ \frac{1}{r^{\alpha q-1}} & \text{if } \alpha = \frac{2}{q-1}. \end{cases} \tag{3.3}$$

Proof Since $(r^{N-1}U_r)_r$ is positive after a certain stage, which implies that $(r^{N-1}U_r)$ is increasing after a certain stage $\lim_{r\to+\infty} r^{N-1}|U_r| = l$ exists finitely as the right-hand side is integrable if $q \neq q^\star$; and non-zero when $\alpha = N-2$. (Otherwise it will contradict Lemma 3.1.) Hence $0 < \int_{\mathbb{R}^N}(U^p - U^q)\mathrm{d}x < +\infty$ as $\lim_{r\to+\infty}\int_0^r (U^p - U^q)\, s^{N-1}\mathrm{d}r = \lim_{r\to+\infty} r^{N-1}|U_r| = \int_0^{+\infty}(U^p - U^q)\, r^{N-1}\mathrm{d}r$. As a result $|U_r| \sim r^{-(N-1)}$ as $r \to +\infty$.

When $\alpha = \frac{2}{q-1}$, then $r^{(N-1)}U_r(r) \to 0$. We have as $r \to +\infty$

$$\left(r^{N-1}U_r\right)_r \sim U^q r^{N-1}$$

and note that $\alpha q > N$ and integrating we obtain

$$-r^{N-1}U_r = \int_r^{+\infty}\left(s^{N-1}U_s\right)_s \sim \int_r^{+\infty} U^q s^{N-1} \sim \int_r^{+\infty} s^{-\alpha q+N-1}\mathrm{d}s$$

which implies that

$$|U_r| \sim r^{-\alpha q+1}.$$

□

Remark 3.3 Note that if $q = q^\star$, it is easy to show that in fact $\lim_{r\to+\infty} r^{N-1}|U_r| < +\infty$. Note that in fact the limit is zero since otherwise $U^{q^\star}$ is not integrable at infinity which contradicts the fact that $\lim_{r\to+\infty} r^{N-1}|U_r|$ exists and thus $\lim_{r\to+\infty} r^{N-2}U = 0$. Hence $\int_{\mathbb{R}^N} U^q\mathrm{d}x < +\infty$.

Remark 3.4 Let us define a space $\mathcal{D} = D^{1,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$. Define a norm on $\mathcal{D}$ as

$$\|u\|_{\mathcal{D}} = \left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{1/2} + \left(\int_{\mathbb{R}^N}|u|^{q+1}\right)^{1/q+1} \qquad \forall u \in \mathcal{D}$$

Note that $(\mathcal{D}, \|u\|_{\mathcal{D}})$ is a reflexive Banach space. We claim that $\mathcal{D} \hookrightarrow L^{p+1}(\mathbb{R}^N)$ is a continuous embedding provided $p+1 \le \frac{2N}{N-2}$. In order to prove this first note that there exists $0 < \theta < 1$ such that $\frac{1}{p+1} = \frac{\theta}{q+1} + \frac{1-\theta}{2^*}$ we have by interpolation and Sobolev inequality

$$\begin{aligned} \|u\|_{L^{p+1}} &\le \|u\|^{\theta}_{L^{q+1}}\|u\|^{1-\theta}_{L^{2^*}} \\ &\le C\|u\|^{\theta}_{L^{q+1}}\|u\|^{1-\theta}_{D^{1,2}} \\ &\le C\|u\|^{\theta}_{\mathcal{D}}\|u\|^{1-\theta}_{\mathcal{D}} \\ &= C\|u\|_{\mathcal{D}}. \end{aligned} \tag{3.4}$$

Hence the embedding is continuous. Note that as $1 < q < p < 2^* - 1$, by (3.4) follows that $U \in \mathcal{D}$. Define $I_\infty : \mathcal{D} \to \mathbb{R}$ as

$$I_\infty(u) = \int_{\mathbb{R}^N}\left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1} + \frac{1}{q+1}|u|^{q+1}\right)$$

Now we need to show that I_∞ satisfies Palais Smale condition on $\mathcal{D}$. Let u_n be a sequence in $\mathcal{D}$ such that $I_\infty(u_n) \le C$ and $I'_\infty(u_n)u_n = o(1)\|u_n\|_{\mathcal{D}}$. Then we obtain that u_n satisfies

$$\left(\frac{1}{2} - \frac{1}{p+1}\right) \int\limits_{\mathbb{R}^N} |\nabla u_n|^2 + \left(\frac{1}{q+1} - \frac{1}{p+1}\right) \int\limits_{\mathbb{R}^N} |u_n|^{q+1} = C + o(1)\|u_n\|_{\mathcal{D}}$$

Hence there exists $C_1 > 0$ such that

$$C_1 \left(\int\limits_{\mathbb{R}^N} |\nabla u_n|^2 + \int\limits_{\mathbb{R}^N} |u_n|^{q+1} \right) = C + o(1)\|u_n\|_{\mathcal{D}}$$

which implies that

$$\left(\int\limits_{\mathbb{R}^N} |\nabla u_n|^2 \right) \le C + o(1)\|u_n\|_{\mathcal{D}}$$

$$\left(\int\limits_{\mathbb{R}^N} |u_n|^{q+1} \right) \le C + o(1)\|u_n\|_{\mathcal{D}}.$$

Hence

$$\|u_n\|_{\mathcal{D}} \le \min\left\{(C + o(1)\|u_n\|_{\mathcal{D}})^{1/2}, (C + o(1)\|u_n\|_{\mathcal{D}})^{1/q+1}\right\}$$

which implies that u_n is bounded in $\mathcal{D}$.

In order to prove the Palais Smale condition we prove the following lemma.

Lemma 3.5 *Let $\mathcal{D}_r$ be the subspace of $\mathcal{D}$ consisting of radially symmetric functions. Then $\mathcal{D}_r \hookrightarrow L^{p+1}(\mathbb{R}^N)$ is a compact embedding provided* $2 < p + 1 < \frac{2N}{N-2}$.

Proof Suppose T is a bounded set in $\mathcal{D}_r$. If $u \in T$,

$$u(r) = -\int\limits_r^\infty u'(s)\mathrm{d}s$$

and hence by Cauchy–Schwartz inequality, and the definition of the norm on $\mathcal{D}$

$$|u(r)| \le Cr^{-\frac{N-2}{2}},$$

where $C > 0$ is independent of u. Thus $|u(r)| \le \epsilon$ if $u \in T$ and $r \ge R$. Hence

$$\int\limits_R^\infty |u(r)|^{p+1} r^{N-1} = \int\limits_R^\infty |u(r)|^{p-q} |u(r)|^{q+1} r^{N-1}$$

$$\le \epsilon \int\limits_R^\infty |u|^{q+1} r^{N-1} \le \epsilon \|u\|_{L^{q+1}}$$

Now, we know that bounded sets in $\mathcal{D}_r$ will converge strongly in $L^{p+1}(\mathbb{R}^N)$ on compact subsets and hence we can use the usual diagonalization argument to obtain a strongly convergent subsequence in $L^{p+1}(\mathbb{R}^N)$ from a sequence in T. □

As a matter of fact I_∞ satisfies all the conditions of the mountain pass theorem in $\mathcal{D}_r$. Hence there exists a $c > 0$ such that

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\infty(\gamma(t)) = \inf_{u\in\mathcal{D}_r} \max_{t\geq 0} I_\infty(tu)$$

where

$$\Gamma = \{\gamma \in C([0,1]; \mathcal{D}_r); \gamma(0) = 0, I_\infty(\gamma(1)) \leq 0\}$$

Hence there exists a positive radial solution of (1.4) obtained by the mountain pass theorem. Hence by Lemma 2.2, U is a mountain pass solution of (1.4).

4 Kernel of $\Delta + pU^{p-1} - qU^{q-1}$ in $D^{1,2}\left(\mathbb{R}^N\right)$

Let U be the radial solution to (1.4). In this section, we want to prove that $\Delta+pU^{p-1}-qU^{q-1}$ is Fredholm on $D^{1,2}(\mathbb{R}^N)$. Let us write

$$\phi = \sum_{k=1}^{\infty} \phi_k(r) S_k(\theta)$$

where $r = |x|, \theta = \frac{x}{|x|} \in \mathbb{S}^{N-1}$; and $-\Delta_{\mathbb{S}^{N-1}} S_k = \lambda S_k$ where $\lambda_k = k(N-2+k)$; $k \in \mathbb{Z}^+ \cup \{0\}$ and whose multiplicity is given by $M_k - M_{k-2}$ where $M_k = \frac{(N+k-1)!}{(N-1)!k!}$ for $k \geq 2$. Note that $\lambda_0 = 0$ has algebraic multiplicity one and $\lambda_1 = (N-1)$ has algebraic multiplicity N. Then ϕ_k satisfy an infinite system of ODE given by,

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(pU^{p-1} - qU^{q-1} - \frac{\lambda_k}{r^2}\right)\phi_k = 0, \quad r \in (0,\infty) \tag{4.1}$$

Also note that (4.1) has two linearly independent solutions $z_{1,k}$ and $z_{2,k}$. Let

$$A_k(\phi) = \phi'' + \frac{N-1}{r}\phi' + \left(pU^{p-1} - qU^{q-1} - \frac{\lambda_k}{r^2}\right)\phi$$

Also recall that if one solution $z_{1,k}$ to (4.1) is known, a second linearly independent solution can be found in any interval where $z_{1,k}$ does not vanish as

$$z_{2,k}(r) = z_{1,k}(r) \int z_{1,k}^{-2} r^{1-N} \mathrm{d}r$$

where $\int$ denotes antiderivatives. One can obtain the asymptotic behavior of any solution z as $r \to \infty$ by examining the indicial roots of the associated Euler equation. Note that in the case $\alpha = \frac{2}{q-1}$, the limiting equation becomes

$$r^2\phi'' + (N-1)r\phi' - (q\zeta + \lambda_k)\phi = 0 \tag{4.2}$$

where $r^2U^{q-1} \to \zeta > 0$ as $r \to \infty$ and when $\alpha = N-2$, the limiting equation becomes

$$r^2\phi'' + (N-1)r\phi' - \lambda_k\phi = 0 \tag{4.3}$$

whose indicial roots are given by

$$\mu_k^{\pm} = \begin{cases} \frac{N-2}{2} \pm \frac{\sqrt{(N-2)^2+4(q\zeta+\lambda_k)}}{2} & \text{if } k \neq 0 \\ \frac{N-2}{2} \pm \frac{\sqrt{(N-2)^2+4q\zeta}}{2} & \text{if } k = 0 \end{cases}$$

In this way we see that the asymptotic behavior is ruled by $z(r) \sim r^{-\mu}$ as $r \to +\infty$; where μ satisfies the problem

$$\begin{cases} \mu^2 - (N-2)\mu - (q\zeta + \lambda_k) = 0 & \text{if } \alpha = \frac{2}{q-1} \\ \mu^2 - (N-2)\mu - \lambda_k = 0 & \text{if } \alpha = N-2 \end{cases} \tag{4.4}$$

Lemma 4.1 *If $k = 0$, Eq. (4.1) has no nontrivial solution in $D^{1,2}(\mathbb{R}^N)$.*

Proof This follows exactly as in [11]. □

Lemma 4.2 *If $k = 1$, then all solutions of equation (4.1) are constant multiples of U'.*

Proof In this case $\lambda_1 = N - 1$ and hence we have $z_{1,1}(r) = -U'(r)$ is a solution to the problem (4.1) and is positive $(0, +\infty)$. Hence we define

$$z_{1,2}(r) = z_{1,1}(r) \int_1^r z_{1,1}(s)^{-2} s^{1-N} \mathrm{d}s$$

Let us check how $z_{1,2}(r)$ behaves at infinity. By Corollary 3.2, when $\alpha = N - 2$ then $|U_r| \sim r^{1-N}$ at infinity and hence $z_{1,2}(r) \sim r$ as $r \to \infty$ as a result $z_{1,2}$ does not belong to $D^{1,2}(\mathbb{R}^N)$.

Again when $\alpha = \frac{2}{q-1}$, then $|U_r| \sim r^{-\alpha q+1}$ as $r \to \infty$ and hence $z_{1,2}(r) \sim r^{\alpha q - N+1}$ and as $\alpha q > N$, $z_{1,2} \notin D^{1,2}(\mathbb{R}^N)$. Hence any family of solutions of (4.1) is given by $\phi_1 = cU'(r)$ for some $c \in \mathbb{R}$. □

Lemma 4.3 *If $k \geq 2$, Eq. (4.1) admits only trivial solution in $D^{1,2}(\mathbb{R}^N)$.*

Proof We will show that if $A_k(\phi_k) = 0$, then $\phi_k = 0$. Note that $-U'$ is a positive solution of A_1. Let us study the first eigenvalue of the problem

$$\begin{cases} A_1(\phi) = \lambda\phi & \text{in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} \phi^2 = 1 \end{cases} \tag{4.5}$$

We know from Lemma 3.1 that $U_{rr} > 0$ after a certain stage and when $\alpha = N-2$, $U_{rr} \sim \frac{1}{r^N}$ and when $\alpha = \frac{2}{q-1}$, $U_{rr} \sim \frac{1}{r^{\alpha q}}$ as $r \to \infty$. Note that if $\lambda_1 > 0$, then $\int_{\mathbb{R}^N} \phi_1 U' = 0$ and hence there exists a point in $\mathbb{R}^N$ such that ϕ_1 changes sign. But ϕ_1 is the first eigenfunction corresponding to λ_1 and hence it has a definite sign. Hence $\lambda_1 \leq 0$. Thus A_1 is an operator having no positive eigenvalues. Hence for $k \geq 2$, $c_k = k(N-2+k) - (N-1) > 0$. Now

$$A_k = A_1 - \frac{k(N-2+k) - (N-1)}{r^2} I$$

where I is the identity. Hence $0 = \langle -A_k(\phi_k), \phi_k \rangle \geq c_k \int_{\mathbb{R}^N} \frac{\phi_k^2}{r^2}$ and as $\phi_k \in C(\mathbb{R}^N)$, we have $\phi_k \equiv 0$. □

Lemma 4.4 $\operatorname{Ker}\left(-\Delta - pU^{p-1} + qU^{q-1}\right) = \left\{\frac{\partial U}{\partial x_1}, \ldots, \frac{\partial U}{\partial x_N}\right\}$ *in $D^{1,2}(\mathbb{R}^N)$.*

Proof From the previous lemmas, we deduce that for any $\phi \in \operatorname{Ker}(-\Delta - pU^{p-1} + qU^{q-1})$, then $\phi = U'(r)S_1$ where S_1 satisfies

$$-\Delta_{\mathbb{S}^{N-1}} S_1 = \lambda_1 S_1.$$

Now $\text{Ker}(-\Delta_{\mathbb{S}^{N-1}} - \lambda_1 I)$ is N-dimensional and hence $\text{Ker}(-\Delta_{\mathbb{S}^{N-1}} - \lambda_1 I) = \text{span}\{S_{1,1}, \ldots, S_{1,N}\} \simeq \text{span}\, \mathbb{R}^N$. Hence

$$\begin{aligned}\text{Ker}\left(-\Delta - pU^{p-1} + qU^{q-1}\right) &= \text{span}\left\{U'(r)S_{1,1}, \ldots, U'(r)S_{1,N}\right\}\\ &= \text{span}\left\{\frac{\partial U}{\partial x_1}, \ldots, \frac{\partial U}{\partial x_N}\right\}.\end{aligned}$$

□

Remark 4.5 Also note that there is always a nontrivial bounded radial solution to the linearized equation. As a result, the operator is not nondegenerate in the space of bounded functions.

5 Profile of spikes

Let z be a point of minimum of h in Ω. Let us define $U_{\varepsilon,z}(x) = U\left(\frac{x-z}{\varepsilon}\right)$, then $U_{\varepsilon,z}$ satisfies

$$\begin{cases} -\varepsilon^2 \Delta U_{\varepsilon,z} = U_{\varepsilon,z}^p - U_{\varepsilon,z}^q & \text{in } \mathbb{R}^N \\ U_{\varepsilon,z} > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{5.1}$$

Also let $\hat{V}_{\varepsilon,z}$ be the unique solution of

$$\begin{cases} -\varepsilon^2 \Delta \hat{V}_{\varepsilon,z} = U_{\varepsilon,z}^p - U_{\varepsilon,z}^q & \text{in } \Omega \\ \hat{V}_{\varepsilon,z} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Then by the maximum principle $\hat{V}_{\varepsilon,z} \le U_{\varepsilon,z}$ in Ω. Note that $\hat{V}_{\varepsilon,z}$ may not be a positive solution of (5.2).

Lemma 5.1 *For sufficiently small* $\varepsilon > 0$,

$$U_{\varepsilon,z} - \hat{V}_{\varepsilon,z} = (C + o(1))\varepsilon^{\alpha}\psi_z \tag{5.3}$$

for some constant $C > 0$.

Proof Subtracting (5.1) from (5.2) we have

$$\begin{cases} -\varepsilon^2 \Delta \left(U_{\varepsilon,z} - \hat{V}_{\varepsilon,z}\right) = 0 & \text{in } \Omega \\ U_{\varepsilon,z} - \hat{V}_{\varepsilon,z} = U_{\varepsilon,z} & \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

Now $U_{\varepsilon,z} = \frac{C+o(1)}{|x-z|^{\alpha}}\varepsilon^{\alpha}$ on $\partial\Omega$, by Lemma 3.1. Hence by the maximum principle and the definition of ψ_z, $U_{\varepsilon,z} - \hat{V}_{\varepsilon,z} = (C+o(1))\varepsilon^{\alpha}\psi_z$ and $U - \hat{V}_{\varepsilon,z}(z+\varepsilon y) = (C+o(1))\psi_z(z+\varepsilon y)\varepsilon^{\alpha}$ in $\Omega_{\varepsilon,z}$. □

Remark 5.2 Note that from Lemma 3.1, we have $U_{\varepsilon,z} \sim \varepsilon^{\alpha}|x-z|^{-\alpha}$ when $|x-z|$ is large. For $\alpha q > N$,

$$\begin{aligned}\int_{\mathbb{R}^N} U_{\varepsilon,z}^{q+1} &= \int_{\mathbb{R}^N\setminus\Omega} U_{\varepsilon,z}^{q+1} + \int_{\Omega} U_{\varepsilon,z}^{q+1}\\ &= \int_{\Omega} U_{\varepsilon,z}^{q+1} + O\left(\varepsilon^{\alpha(q+1)}\right)\end{aligned}$$

and $\varepsilon^{\alpha(q+1)} = \varepsilon^{N+\alpha} o(1)$. Hence we have

$$\int_\Omega U_{\varepsilon,z}^{q+1} = \varepsilon^N \int_{\mathbb{R}^N} U^{q+1} + \varepsilon^{N+\alpha} o(1).$$

Lemma 5.3 *Let c be the mountain pass value of* (1.4) *and* $\frac{N}{N-2} < q < \frac{N+2}{N-2}$. *Then, we have*

$$c_\varepsilon \le \varepsilon^N \left(c + \frac{C}{2} \varepsilon^{N-2} \min_\Omega h \int_{\mathbb{R}^N} (U^p - U^q) \mathrm{d}x + o(\varepsilon^{N-2}) \right).$$

Proof First note that by the mean value theorem,

$$\int_\Omega \left(\hat{V}_{\varepsilon,z}\right)_+^{q+1} = \int_\Omega (U_{\varepsilon,z})^{q+1} + (q+1) \int_\Omega (U_{\varepsilon,z})^q \left(\hat{V}_{\varepsilon,z} - U_{\varepsilon,z}\right) + o(1)\varepsilon^{N+N-2} \tag{5.5}$$

Hence, by the equation satisfied by $\hat{V}_{\varepsilon,z}$ and integration by parts,

$$\begin{aligned}
\Phi_\varepsilon\left(\hat{V}_{\varepsilon,z}\right) &= \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla \hat{V}_{\varepsilon,z}|^2 - \frac{1}{p+1} \left(\hat{V}_{\varepsilon,z}\right)_+^{p+1} + \frac{1}{q+1} \left(\hat{V}_{\varepsilon,z}\right)_+^{q+1} \right) \\
&= \int_\Omega \left(\frac{1}{2} \left(U_{\varepsilon,z}^p - U_{\varepsilon,z}^q\right) \hat{V}_{\varepsilon,z} \right. \\
&\quad \left. - \frac{1}{p+1} \left(\hat{V}_{\varepsilon,z}\right)_+^{p+1} + \frac{1}{q+1} \left(\hat{V}_{\varepsilon,z}\right)_+^{q+1} \right) \\
&= \int_\Omega \left(\frac{1}{2} \left(U_{\varepsilon,z}^p - U_{\varepsilon,z}^q\right) \left(U_{\varepsilon,z} - (C + o(1)) \psi_z \varepsilon^{N-2}\right) \right. \\
&\quad \left. - \frac{1}{p+1} \left(\hat{V}_{\varepsilon,z}\right)_+^{p+1} + \frac{1}{q+1} \left(\hat{V}_{\varepsilon,z}\right)_+^{q+1} \right) \\
&= \frac{1}{2} \int_\Omega \left(U_{\varepsilon,z}^{p+1} - U_{\varepsilon,z}^{q+1}\right) - \frac{C + o(1)}{2} \varepsilon^{N-2} \int_\Omega \psi_z \left(U_{\varepsilon,z}^p - U_{\varepsilon,z}^q\right) \\
&\quad - \frac{1}{p+1} \int_\Omega \left(\hat{V}_{\varepsilon,z}\right)_+^{p+1} + \frac{1}{q+1} \int_\Omega \left(\hat{V}_{\varepsilon,z}\right)_+^{q+1}.
\end{aligned} \tag{5.6}$$

Here we have used (5.5), Remark 5.2 and that $U_{\varepsilon,z}$ has algebraic decay. Since $\psi_z(x)$ is bounded on Ω and $\psi_z(z + \varepsilon x)$ converges pointwise to h, we can use the dominated convergence theorem to conclude that $\int_{\Omega_\varepsilon} (U^p - U^q) \psi_z(z + \varepsilon x) = h(z) \int_{\mathbb{R}^N} (U^p - U^q) + o(1)$. Thus we have

$$\Phi_\varepsilon(\hat{V}_{\varepsilon,z}) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int\limits_\Omega U_{\varepsilon,z}^{p+1} - \left(\frac{1}{2} - \frac{1}{q+1}\right)\int\limits_\Omega U_{\varepsilon,z}^{q+1}$$
$$+\left(1-\frac{1}{2}\right)C\varepsilon^{N-2}\int\limits_\Omega (U_{\varepsilon,z}^p - U_{\varepsilon,z}^q)\psi_z \mathrm{d}x$$
$$+\, o(1)\varepsilon^{N-2+N}$$
$$= \left(\frac{1}{2} - \frac{1}{p+1}\right)\varepsilon^N\int\limits_{\mathbb{R}^N} U^{p+1} - \left(\frac{1}{2} - \frac{1}{q+1}\right)\varepsilon^N\int\limits_{\mathbb{R}^N} U^{q+1}$$
$$+\frac{C}{2}\varepsilon^{N+N-2}h(z)\int\limits_{\mathbb{R}^N}\left(U^p - U^q\right) + \varepsilon^{N+N-2}o(1)$$
$$= \varepsilon^N\left(c + \frac{C}{2}\varepsilon^{N-2}\min_\Omega h\int\limits_{\mathbb{R}^N}(U^p - U^q)\mathrm{d}x + o\left(\varepsilon^{N-2}\right)\right) \tag{5.7}$$

Let $t_\varepsilon \in (0, +\infty)$ be the unique constant such that

$$\Phi\left(t_\varepsilon \hat{V}_{\varepsilon,z}\right) = \max_{t\geq 0}\Phi\left(t\hat{V}_{\varepsilon,z}\right)$$

Hence

$$\left\langle \Phi_\varepsilon'\left(t_\varepsilon\hat{V}_{\varepsilon,z}\right), \hat{V}_{\varepsilon,z}\right\rangle = 0 \tag{5.8}$$

We claim that $t_\varepsilon \to 1$ as $\varepsilon \to 0$. By the equation satisfied by $\hat{V}_{\varepsilon,z}$ we have

$$\left\langle \Phi_\varepsilon'\left(\hat{V}_{\varepsilon,z}\right), \hat{V}_{\varepsilon,z}\right\rangle = \int\limits_\Omega\left(\varepsilon^2|\nabla\hat{V}_{\varepsilon,z}|^2 - \left(\hat{V}_{\varepsilon,z}\right)_+^{p+1} + \left(\hat{V}_{\varepsilon,z}\right)_+^{q+1}\right)$$
$$= \int\limits_\Omega\left(U_{\varepsilon,z}^p\hat{V}_{\varepsilon,z} - U_{\varepsilon,z}^q\hat{V}_{\varepsilon,z} - \left(\hat{V}_{\varepsilon,z}\right)_+^{p+1} + \left(\hat{V}_{\varepsilon,z}\right)_+^{q+1}\right)$$
$$= O(1)\varepsilon^{N+N-2} \tag{5.9}$$

and analyzing the higher order terms, and using the fact that

$$\int\limits_{\mathbb{R}^N}|\nabla U|^2 = \int\limits_{\mathbb{R}^N} U^{p+1} - \int\limits_{\mathbb{R}^N} U^{q+1}$$

there exists a $c' > 0$ such that

$$\Phi_\varepsilon''\left(\hat{V}_{\varepsilon,z}\right)\langle\hat{V}_{\varepsilon,z}, \hat{V}_{\varepsilon,z}\rangle = \int\limits_{\Omega_\varepsilon}\left(\varepsilon^2|\nabla\hat{V}_{\varepsilon,z}|^2 - p\left(\hat{V}_{\varepsilon,z}\right)_+^{p+1} + q\left(\hat{V}_{\varepsilon,z}\right)_+^{q+1}\right)$$
$$= \varepsilon^N\int\limits_{\mathbb{R}^N}\left(-(p-1)U^{p+1} + (q-1)U^{q+1}\right) + o(1)\varepsilon^N$$

$$
\begin{aligned}
&= \varepsilon^N \left(-(p-q) \int\limits_{\mathbb{R}^N} U^{p+1} - (q-1) \int\limits_{\mathbb{R}^N} |\nabla U|^2 + o(1) \right) \\
&\le -c' \varepsilon^N
\end{aligned} \tag{5.10}
$$

Since $\left\langle \Phi'_\varepsilon \left(t_\varepsilon \hat{V}_{\varepsilon,z} \right), \hat{V}_{\varepsilon,z} \right\rangle = 0$ and $\left\langle \Phi'_\varepsilon \left(\hat{V}_{\varepsilon,z} \right), \hat{V}_{\varepsilon,z} \right\rangle = o(1)\varepsilon^N$, we have

$$
\left\langle \Phi'_\varepsilon \left(t_\varepsilon \hat{V}_\varepsilon \right) - \Phi'_\varepsilon \left(\hat{V}_\varepsilon \right), \hat{V}_{\varepsilon,z} \right\rangle = o(1)\varepsilon^N
$$

which implies

$$
\left(t_\varepsilon^2 - 1\right) \int\limits_\Omega \varepsilon^2 |\nabla \hat{V}_{\varepsilon,z}|^2 - \left(t_\varepsilon^{p+1} - 1\right) \int\limits_\Omega \left(\hat{V}_{\varepsilon,z} \right)_+^{p+1} + \left(t_\varepsilon^{q+1} - 1\right) \int\limits_\Omega \left(\hat{V}_{\varepsilon,z} \right)_+^{q+1} = o(1)\varepsilon^N
$$

and letting $\tilde{V}_{\varepsilon,z}(x) = \hat{V}_{\varepsilon,z}(\varepsilon x + z)$ in Ω_ε we have

$$
\left(t_\varepsilon^2 - 1\right) \int\limits_{\Omega_\varepsilon} |\nabla \tilde{V}_{\varepsilon,z}|^2 - \left(t_\varepsilon^{p+1} - 1\right) \int\limits_{\Omega_\varepsilon} \left(\tilde{V}_{\varepsilon,z} \right)_+^{p+1} + \left(t_\varepsilon^{q+1} - 1\right) \int\limits_{\Omega_\varepsilon} \left(\tilde{V}_{\varepsilon,z} \right)_+^{q+1} = o(1)
$$

which implies that $t_\varepsilon - 1 = o(1)$.

$$
\begin{aligned}
\Phi_\varepsilon(u_\varepsilon) &\le \max_{t>0} \Phi_\varepsilon \left(t \hat{V}_{\varepsilon,z} \right) = \Phi_\varepsilon(t_\varepsilon \hat{V}_\varepsilon) \\
&= \Phi_\varepsilon \left(\hat{V}_{\varepsilon,z} \right) + (t_\varepsilon - 1) \left\langle \Phi'_\varepsilon \left(\hat{V}_{\varepsilon,z} \right), \hat{V}_{\varepsilon,z} \right\rangle + \frac{1}{2}(t_\varepsilon - 1)^2 \Phi''_\varepsilon \left(\xi_\varepsilon \hat{V}_{\varepsilon,z} \right) \left\langle \hat{V}_{\varepsilon,z}, \hat{V}_{\varepsilon,z} \right\rangle \\
&\le \Phi_\varepsilon \left(\hat{V}_{\varepsilon,z} \right) + o(1)\varepsilon^{N+N-2} \\
&\le \varepsilon^N \left(c + \frac{C}{2} \varepsilon^{N-2} \min_\Omega h \int\limits_{\mathbb{R}^N} (U^p - U^q) \mathrm{d}x + o\left(\varepsilon^{N-2} \right) \right)
\end{aligned}
$$

where ξ_ε lies in between t_ε and 1. Hence we have

$$
c_\varepsilon \le \varepsilon^N \left(c + \frac{C}{2} \varepsilon^{N-2} \min_\Omega h \int\limits_{\mathbb{R}^N} (U^p - U^q) \mathrm{d}x + o\left(\varepsilon^{N-2} \right) \right). \tag{5.11}
$$

□

Lemma 5.4 *For sufficiently small* $\varepsilon > 0$, u_ε *has a unique maximum.*

Proof First note by Lemma 5.3, $\varepsilon^2 \int_\Omega |\nabla u_\varepsilon|^2 \le C$ and hence by Moser iteration, $u_\varepsilon(x)$ is uniformly bounded. Thus applying Schauder estimates we obtain a $C > 0$ such that $\|\varepsilon D u_\varepsilon\|_{L^\infty} \le C$. If possible, let $z_{\varepsilon,1}$ and $z_{\varepsilon,2}$ are two distinct local maxima of u_ε. Then it easily follows that $u_\varepsilon(z_{\varepsilon,1}) \ge 1$ and $u_\varepsilon(z_{\varepsilon,2}) \ge 1$. Suppose $z_\varepsilon = \frac{z_{\varepsilon,1} - z_{\varepsilon,2}}{\varepsilon}$. Suppose along a subsequence $|z_\varepsilon| \to \delta \in [0, +\infty)$. Let $z = \lim_{\varepsilon \to 0} \frac{z_{\varepsilon,1} - z_{\varepsilon,2}}{\varepsilon}$. Then if $\delta > 0$, then define $v_\varepsilon(y) = u_\varepsilon(\varepsilon y + z_{\varepsilon,2})$ then it follows from Remark 2.4, $v_\varepsilon \to U$ in $C^2_{loc}(\mathbb{R}^N)$ and satisfies

$$
\begin{cases}
-\Delta U = U^p - U^q & \text{in } \mathbb{R}^N \\
U(0) = U'(\delta) = 0 & \\
U \to 0 & \text{as } |x| \to \infty
\end{cases}
$$

which is a contradiction as $U'(r) < 0$ for $r \in (0, +\infty)$. Now suppose $\delta = 0$. Then $v_\varepsilon \to U$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$ and U has a unique critical point at 0 (since $U(0) > 1$ and U is a radial). Thus v_ε has a critical point in a neighborhood of zero which is a contradiction. Hence $|z_\varepsilon| \to +\infty$ as $\varepsilon \to 0$.

We claim that u_ε has exactly one maximum for sufficiently small $\varepsilon > 0$. First, note that as u_ε is a mountain pass solution and hence it has Morse index at most one. Let $\tilde{z}_{1,\varepsilon}$ and $\tilde{z}_{2,\varepsilon}$ be two maxima of v_ε. Then by the above result $|\tilde{z}_{1,\varepsilon} - \tilde{z}_{2,\varepsilon}| \to +\infty$ as $\varepsilon \to 0$. Now by [3] p. 145, it was proved that there exist $r < 0$ and h exponentially decreasing such that $-\Delta h - f'(U)h = rh$ and hence $\int_{\mathbb{R}^N} |\nabla h|^2 - f'(U)h^2 < 0$. Now using an appropriate cut off function we can obtain the same property for h with compact support. Now define a two-dimensional space spanned by $h_1(x) = h(x + \tilde{z}_{1,\varepsilon})$ and $h_2(x) = h(x + \tilde{z}_{2,\varepsilon})$ where $x \in \Omega_\varepsilon$. Note that the support $supp\ h_1 \cap supp\ h_2 = \emptyset$ as $|\tilde{z}_{1,\varepsilon} - \tilde{z}_{2,\varepsilon}| \to +\infty$. Hence we obtain a two dimensional space on which $\int_{\Omega_\varepsilon} |\nabla h_i|^2 - f'(v_\varepsilon)h_i^2 = \int_{\mathbb{R}^N} |\nabla h_i|^2 - f'(U)h_i^2 < 0$ for $i = 1, 2$. Note that we are using the fact that $v_\varepsilon \to U$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$ and h_i has compact support. Hence u_ε has Morse index at least two, a contradiction. □

Now we require to obtain the second-order lower bound. To this context, we first note that $U - \hat{V}_{\varepsilon,z_\varepsilon}(z_\varepsilon + \varepsilon y) = (C + o(1))\psi_{z_\varepsilon}(z_\varepsilon + \varepsilon y)\varepsilon^\alpha$ in Ω_ε. Let $\tilde{V}_\varepsilon = \hat{V}_{\varepsilon,z_\varepsilon}(z_\varepsilon + \varepsilon y)$, and $\tilde{u}_\varepsilon = u_\varepsilon(z_\varepsilon + \varepsilon y)$. Then

$$-\Delta\left(\tilde{u}_\varepsilon - \tilde{V}_\varepsilon\right) = f(\tilde{u}_\varepsilon) - f(U) = f'\left(\tilde{W}_\varepsilon\right)(\tilde{u}_\varepsilon - U)$$

where $\tilde{W}_\varepsilon$ is between $\tilde{u}_\varepsilon$ and U. Hence

$$-\Delta\left(\tilde{u}_\varepsilon - \tilde{V}_\varepsilon\right) = f'\left(\tilde{W}_\varepsilon\right)\left(\tilde{u}_\varepsilon - \tilde{V}_\varepsilon\right) + f'\left(\tilde{W}_\varepsilon\right)\left(\tilde{V}_\varepsilon - U\right).$$

Thus

$$\begin{cases} -\Delta\left(\tilde{u}_\varepsilon - \tilde{V}_\varepsilon\right) - f'\left(\tilde{W}_\varepsilon\right)\left(\tilde{u}_\varepsilon - \tilde{V}_\varepsilon\right) = f'\left(\tilde{W}_\varepsilon\right)\left(\tilde{V}_\varepsilon - U\right) & \text{in } \Omega_\varepsilon \\ \left(\tilde{u}_\varepsilon - \tilde{V}_\varepsilon\right) = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \tag{5.12}$$

Define

$$\tilde{\varphi}_\varepsilon = \frac{\tilde{u}_\varepsilon - \tilde{V}_\varepsilon}{C\varepsilon^{N-2}h(z_\varepsilon)}$$

where z_ε is the point of maximum of u_ε. Then

$$\begin{cases} -\Delta\tilde{\varphi}_\varepsilon - f'\left(\tilde{W}_\varepsilon\right)\tilde{\varphi}_\varepsilon = f'\left(\tilde{W}_\varepsilon\right)S_\varepsilon & \text{in } \Omega_\varepsilon \\ \tilde{\varphi}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \tag{5.13}$$

where

$$S_\varepsilon = \frac{\left(\tilde{V}_\varepsilon - U\right)}{C\varepsilon^{N-2}h(z_\varepsilon)}.$$

Lemma 5.5 *For sufficiently small $\varepsilon > 0$, then up to a subsequence*

$$\tilde{\varphi}_\varepsilon \to \varphi_0$$

uniformly as $\varepsilon \to 0$ and φ_0 satisfies

$$\begin{cases} -\Delta\varphi_0 - f'(U)\varphi_0 + f'(U) = 0 & \text{in } \mathbb{R}^N \\ \varphi_0 \to 0 & \text{as } |x| \to \infty \\ \varphi_0 \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) & \end{cases} \tag{5.14}$$

Proof Note that since $\frac{\mathrm{dist}(z_\varepsilon,\partial\Omega)}{\varepsilon} \to \infty$ we have $\frac{\psi_{z_\varepsilon}(z_\varepsilon+\varepsilon x)}{h(z_\varepsilon)}$ is uniformly bounded and hence by Lemma 5.1, S_ε is uniformly bounded. Note that by the decay property of $\tilde{u}_\varepsilon$ and U, $\tilde{W}_\varepsilon \leq \frac{C}{|x|^{N-2}}$ for $|x|$ sufficiently large. Hence $f'(\tilde{W}_\varepsilon) \leq 0$ for $|x| \geq R_0$ and $f'(\tilde{W}_\varepsilon) \leq \frac{k}{|x|^r}$ where $r > 2$. Hence we can choose $\tilde{C}|x|^{2-r}$ as a super-solution of (5.13) for $|x| \geq R_0$ if we choose $\tilde{r} \geq 2$ but close to 2 and $\tilde{C} > 0$ is large. Hence we can bound $\tilde{C} > 0$ if we have a uniform bound $\tilde{\varphi}_\varepsilon$ on $|x| = R_0$. Thus we have a uniform decay for $\tilde{\varphi}_\varepsilon$ if we can bound $\tilde{\varphi}_\varepsilon$ on $|x| = R_0$.

If possible let $\tilde{\varphi}_\varepsilon$ be unbounded. Then $\|\tilde{\varphi}_\varepsilon\|_\infty \to \infty$ (up to a subsequence). Define $\psi_\varepsilon = \frac{\tilde{\varphi}_\varepsilon}{\|\tilde{\varphi}_\varepsilon\|_\infty}$. Then $\|\psi_\varepsilon\|_\infty = 1$. Hence the right-hand term in (5.13) is uniformly small and thus by the argument in the previous paragraph ψ_ε has a uniform decay for large $|x|$. Thus the maximum of ψ_ε must occur at k_ε where $|k_\varepsilon| \leq R$ for sufficiently small ε. Let k be a subsequential limit of k_ε. By Schauder estimates we obtain $\|\psi_\varepsilon\|_{C^{1,\theta}_{\mathrm{loc}}}$ is bounded for some $\theta \in (0, 1]$ and hence by the Arzela-Ascoli's theorem there exists $\psi_0 \in C^1$ such that $\|\psi_\varepsilon - \psi_0\|_{C^1_{\mathrm{loc}}} \to 0$ as $\varepsilon \to 0$. Then ψ_0 satisfies

$$\begin{cases} -\Delta\psi_0 - f'(U)\psi_0 = 0 & \text{in } \mathbb{R}^N \\ \psi_0(k) = 1 & \\ \psi_0(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{5.15}$$

Note that we use the fact that $\mathrm{dist}(k_\varepsilon, \partial\Omega_\varepsilon) \to \infty$ in order to conclude that the above problem is not a half space problem. We can now use $C|x|^{-(N-2)}$ as a super-solution to deduce that $|x|^{N-2}\psi_0$ is bounded. This implies that $\psi_0 \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$. On the other hand we have,

$$\int_{\mathbb{R}^N} |\nabla\psi_0|^2 = \int_{\mathbb{R}^N} f'(U)\psi_0^2 < \infty.$$

As a result, $\psi_0 \in D^{1,2}(\mathbb{R}^N) \cap \ker(-\Delta - f'(U))$. Since $\psi_0 \not\equiv 0$ and since by Lemma 4.4, $\ker(-\Delta - f'(U)) = \left\{\frac{\partial U}{\partial y_1}, \frac{\partial U}{\partial y_2}, \ldots, \frac{\partial U}{\partial y_N}\right\}$, we have

$$\psi_0 = \sum_{j=1}^{N} a_j \frac{\partial U}{\partial y_i}$$

where not all a_j's are zero. Since U is radial, $U'(0) = 0$ and $\Delta U(0) \neq 0$, it follows that $\psi_0(0) = 0$ and $\nabla\psi_0(0) \neq 0$. We obtain a contradiction by proving $\nabla\psi_0(0) = 0$. Note that $\nabla\tilde{u}_\varepsilon(0) = 0$ and $\nabla U(0) = 0$ and hence

$$\nabla\tilde{\psi}_\varepsilon(0) = \frac{\nabla\tilde{\varphi}_\varepsilon(0)}{\varepsilon^{N-2}h(z_\varepsilon)\|\tilde{\varphi}_\varepsilon\|_{L^\infty}} = \frac{\nabla U(0)}{\varepsilon^{N-2}h(z_\varepsilon)\|\tilde{\varphi}_\varepsilon\|_{L^\infty}}$$

Thus $\nabla\tilde{\psi}_\varepsilon(0) = 0$ and by C^1_{loc} convergence we have $\nabla\psi_0(0) = 0$. This gives a contradiction. Hence $\tilde{\varphi}_\varepsilon$ is uniformly bounded.

By our earlier argument with a super-solution, we obtain that $\tilde{\varphi}_\varepsilon$ decays uniformly, while by elliptic regularity theory applied to (5.13) we have $\tilde{\varphi}_\varepsilon$ converges uniformly to φ_0 in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ where φ_0 satisfies the problem (5.14). By uniform decay of $\tilde{\varphi}_\varepsilon$, we can conclude that $\varphi_0 \to 0$ as $|x| \to \infty$. Hence $\tilde{\varphi}_\varepsilon \to \varphi_0$ as $\varepsilon \to 0$ uniformly. This completes the proof. □

Remark 5.6 Hence we have $u_\varepsilon = U_{\varepsilon,z_\varepsilon} - C\varepsilon^{N-2}(\psi_{z_\varepsilon} - \varphi_0 h(z_\varepsilon) + o(1))$ in Ω and by using the fact that z_ε is the only maximum of u_ε, we have

$$\max_{\Omega\setminus\Omega\cap B_{\varepsilon R}(z_\varepsilon)} u_\varepsilon \leq C\varepsilon^{N-2}$$

Lemma 5.7 *We have,*

$$c_\varepsilon \geq \varepsilon^N \left(c + \frac{C}{2}\varepsilon^{N-2} h(z_\varepsilon) \int_{\mathbb{R}^N} (U^p - U^q)\mathrm{d}x + o\left(\varepsilon^{N-2}\right) \right).$$

Proof Multiplying both sides of (5.14) by $U \in D^{1,2}(\mathbb{R}^N)$ and integrating by parts we obtain,

$$(p-1)\int_{\mathbb{R}^N} U^p \varphi_0 - (q-1)\int_{\mathbb{R}^N} U^q \varphi_0 = p\int_{\mathbb{R}^N} U^p - q\int_{\mathbb{R}^N} U^q. \tag{5.16}$$

Also note that $u_\varepsilon = U_{\varepsilon,z_\varepsilon} - C\varepsilon^{N-2}(\psi_{z_\varepsilon} - \varphi_0 h(z_\varepsilon) + o(1))$ in Ω. Choose a $R > 0$ sufficiently large such that $U(r) < 1$ for $r > R$, and by using Taylors expansion,

$$\begin{aligned}
\int_{\Omega\cap B_{\varepsilon R}(z_\varepsilon)} u_\varepsilon^{p+1} = &\int_{\Omega\cap B_{\varepsilon R}(z_\varepsilon)} U_{\varepsilon,z_\varepsilon}^{p+1} \\
&- (p+1)C\varepsilon^{N-2} \int_{\Omega\cap B_{\varepsilon R}(z_\varepsilon)} U_{\varepsilon,z_\varepsilon}^{p} \left(\psi_{z_\varepsilon} - \varphi_0 h(z_\varepsilon)\right) \\
&+ o(1)\varepsilon^{N+N-2}.
\end{aligned}$$

Then by Remark 5.6 we have,

$$\begin{aligned}
c_\varepsilon = \Phi_\varepsilon(u_\varepsilon) &= \int_\Omega \left(\frac{\varepsilon^2}{2}|\nabla u_\varepsilon|^2 - \frac{1}{p+1}(u_\varepsilon)_+^{p+1} + \frac{1}{q+1}(u_\varepsilon)_+^{q+1} \right) \\
&= \int_{\Omega\cap B_{\varepsilon R}(z_\varepsilon)} \left(\frac{1}{2} f(u_\varepsilon)u_\varepsilon - F(u_\varepsilon) \right) + \int_{\Omega\setminus\Omega\cap B_{\varepsilon R}(z_\varepsilon)} \left(\frac{1}{2} f(u_\varepsilon)u_\varepsilon - F(u_\varepsilon) \right) \\
&= \int_{\Omega\cap B_{\varepsilon R}(z_\varepsilon)} \left(\left(\frac{1}{2} - \frac{1}{p+1} \right) u_\varepsilon^{p+1} - \left(\frac{1}{2} - \frac{1}{q+1} \right) u_\varepsilon^{q+1} \right) + o(1)\varepsilon^{N+N-2} \\
&= \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\Omega\cap B_{\varepsilon R}(z_\varepsilon)} U_{\varepsilon,z_\varepsilon}^{p+1} - \left(\frac{1}{2} - \frac{1}{q+1} \right) \int_{\Omega\cap B_{\varepsilon R}(z_\varepsilon)} U_{\varepsilon,z_\varepsilon}^{q+1} \\
&\quad - \frac{p-1}{2} C\varepsilon^{N-2} \int_{\Omega\cap B_{\varepsilon R}(z_\varepsilon)} U_{\varepsilon,z_\varepsilon}^{p} \psi_{z_\varepsilon} \\
&\quad + \frac{q-1}{2} C\varepsilon^{N-2} \int_{\Omega\cap B_{\varepsilon R}(z_\varepsilon)} U_{\varepsilon,z_\varepsilon}^{q} \psi_{z_\varepsilon} \\
&\quad + \frac{p-1}{2} C\varepsilon^{N-2} h(z_\varepsilon) \int_{\Omega\cap B_{\varepsilon R}(z_\varepsilon)} U_{\varepsilon,z_\varepsilon}^{p} \varphi_0 \\
&\quad - \frac{q-1}{2} C\varepsilon^{N-2} h(z_\varepsilon) \int_{\Omega\cap B_{\varepsilon R}(z_\varepsilon)} U_{\varepsilon,z_\varepsilon}^{q} \varphi_0 + o(1)\varepsilon^{N+N-2}.
\end{aligned}$$

By our decay estimates and Remark 5.2, we have

$$\int\limits_{\Omega\cap B_{\varepsilon R}(z_\varepsilon)} U_{\varepsilon,z_\varepsilon}^{p+1} = \int\limits_{\mathbb{R}^N} U_{\varepsilon,z_\varepsilon}^{p+1} - \int\limits_{\mathbb{R}^N\setminus\Omega\cap B_{\varepsilon R}(z_\varepsilon)} U_{\varepsilon,z_\varepsilon}^{p+1}$$

$$= \varepsilon^N \int\limits_{\mathbb{R}^N} U^{p+1} + o(1)\varepsilon^{N+N-2}.$$

Also by Taylors expansion in $B_{\varepsilon R}(z_\varepsilon)$, we have $\psi_{z_\varepsilon}(z) - h(z_\varepsilon) = o(1)$

$$\int\limits_{\Omega\cap B_{\varepsilon R}(z_\varepsilon)} U_{\varepsilon,z_\varepsilon}^{p}\psi_{z_\varepsilon} = h(z_\varepsilon)\int\limits_{\Omega\cap B_{\varepsilon R}(z_\varepsilon)} U_{\varepsilon,z_\varepsilon}^{p} + o(1)\varepsilon^N$$

$$= h(z_\varepsilon)\varepsilon^N \int\limits_{\mathbb{R}^N} U^p + o(1)\varepsilon^N$$

$$= h(z_\varepsilon)\varepsilon^N \int\limits_{\mathbb{R}^N} U^p + o(1)\varepsilon^N.$$

Hence we have

$$c_\varepsilon = \left(\frac{1}{2} - \frac{1}{p+1}\right)\varepsilon^N \int\limits_{\mathbb{R}^N} U^{p+1} - \left(\frac{1}{2} - \frac{1}{q+1}\right)\varepsilon^N \int\limits_{\mathbb{R}^N} U^{q+1}$$

$$-\frac{p-1}{2}C\varepsilon^{N+N-2}h(z_\varepsilon)\int\limits_{\mathbb{R}^N} U^p + \frac{q-1}{2}C\varepsilon^{N+N-2}h(z_\varepsilon)\int\limits_{\mathbb{R}^N} U^q$$

$$+\frac{p-1}{2}C\varepsilon^{N+N-2}h(z_\varepsilon)\int\limits_{\mathbb{R}^N} U^p\varphi_0$$

$$-\frac{q-1}{2}C\varepsilon^{N+N-2}h(z_\varepsilon)\int\limits_{\mathbb{R}^N} U^q\varphi_0 + o(1)\varepsilon^{N+N-2}.$$

using (5.16) we deduce

$$c_\varepsilon \geq \varepsilon^N\left(c + \frac{C}{2}\varepsilon^{N-2}h(z_\varepsilon)\int\limits_{\mathbb{R}^N}(U^p - U^q) + o(\varepsilon^{N-2})\right).$$

□

Remark 5.8 As a result of Lemmas 5.3 and 5.5, we obtain $h(z_\varepsilon) \to \min_\Omega h$. Hence Theorem 1.1 is proved. Note that for $\alpha = \frac{2}{q-1}$, from Corollary 3.2 we have $\int_{\mathbb{R}^N}(U^p - U^q)dx = 0$ and as a result we cannot obtain any information on the point of concentration of spikes.

6 Multi-peak solutions

We modify the problem (1.3) to

$$\begin{cases} -\varepsilon^2\Delta u = (u^+)^p - Q(x)(u^+)^q & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Choose $\delta > 0$ such that $Q(x) > Q(z_j)$ for all $x \in B_\delta(z_j)\backslash\{z_j\}$ and $B_\delta(z_i) \cap B_\delta(z_j) = \emptyset$ for $i \neq j$. Let $Q(z_j) = b_j > 0$. Then for any $b > 0$, let W be the unique radial solution

$$\begin{cases} -\Delta W = W^p - bW^q & \text{in } \mathbb{R}^N \\ W > 0 & \text{in } \mathbb{R}^N \\ W \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{6.2}$$

Define the transformation, $W(x) = b^{\frac{1}{p-q}} U\left(b^{\frac{p-1}{2(p-q)}} x\right)$. Then U satisfies the problem (1.4). We can assume that $Q(z_j)$ are all equal. This is not needed but it simplifies the notation. In this case, we can re-scale so that $b_j = 1$ for all j. Let $\gamma > 0$ be small and $\tau > 0$ is defined in Lemma 7.1. For $x = (x_1, \ldots, x_k)$, define

$$D_{k,\varepsilon} = \left\{ x \in \Omega^k,\ j = 1, \ldots, k;\ x_j \in B_\delta(z_j),\ |Q(x_j) - 1| \leq \varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}}}, \right.$$
$$\left. U\left(\frac{x_i - x_j}{\varepsilon}\right) \leq \varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}}},\ i \neq j \right\}.$$

Also let $\hat{V}_{\varepsilon,z}$ be the unique solution of

$$\begin{cases} -\varepsilon^2\Delta \hat{V}_{\varepsilon,z} = U^p_{\varepsilon,z} - U^q_{\varepsilon,z} & \text{in } \Omega \\ \hat{V}_{\varepsilon,z} = 0 & \text{on } \partial\Omega \end{cases} \tag{6.3}$$

Define a norm on $H_0^1(\Omega)$

$$\|v\|_\varepsilon^2 = \varepsilon^2 \int_\Omega |\nabla v|^2 \mathrm{d}x \tag{6.4}$$

For any $x \in D_{k,\varepsilon}$, let

$$E_{\varepsilon,x,k} = \left\{ \omega \in H_0^1(\Omega), \left\langle \omega, \frac{\partial \hat{V}_{\varepsilon,x_j}}{\partial x_{jl}} \right\rangle_\varepsilon = 0;\ l = 1, \ldots, N,\ j = 1, \ldots, k \right\}$$

where $x_j = (x_{j1}, \ldots, x_{jN}) \in \mathbb{R}^N$.

Choose $R > 0$ sufficiently large such that $U(x) < 1$ for $|x| \geq R$.

Remark 6.1 Let $2^* = \frac{2N}{N-2}$. We derive an important inequality which we will use in the later stage of our proof. We have by the Sobolev and Hölder inequalities,

$$\begin{aligned}\int_{B_{\varepsilon R}} |\omega| &\le |B_{\varepsilon R}|^{\frac{1}{2}} \left(\int_{B_{\varepsilon R}} |\omega|^2 \right)^{\frac{1}{2}} \\ &\le C\varepsilon^{\frac{N}{2}} \left(\int_{B_{\varepsilon R}} |\omega|^2 \right)^{\frac{1}{2}} \\ &\le C\varepsilon^{\frac{N}{2}} |B_{\varepsilon R}|^{\frac{1}{2}-\frac{1}{2^*}} \left(\int_{B_{\varepsilon R}} |\omega|^{2^*} \right)^{\frac{1}{2^*}} \\ &\le C\varepsilon^{\frac{N}{2}} \left(\varepsilon^2 \int_{\Omega} |D\omega|^2 \right)^{\frac{1}{2}} \\ &\le C\varepsilon^{\frac{N}{2}} \|\omega\|_{\varepsilon} \end{aligned} \tag{6.5}$$

for some constant $C > 0$ independent of ε.

Lemma 6.2 *For any $\omega \in H_0^1(\Omega)$ and $\varepsilon > 0$ sufficiently small, there exists a $C > 1$ independent of ε such that*

$$\|\omega\|_{\varepsilon} \le \left(\varepsilon^2 \int_{\Omega} |\nabla\omega|^2 \mathrm{d}x + q \left(\sum_{j=1}^{k} U_{\varepsilon,x_j} \right)^{q-1} \omega^2 \right)^{\frac{1}{2}} \le C\|\omega\|_{\varepsilon}.$$

Proof Note that the left hand side of the inequality follows trivially. Now let us estimate the term

$$\begin{aligned}\int_{\Omega} \left(\sum_{j=1}^{k} U_{\varepsilon,x_j} \right)^{q-1} \omega^2 &= \int_{\cup B_{\varepsilon R}(x_i)} \left(\sum_{j=1}^{k} U_{\varepsilon,x_j} \right)^{q-1} \omega^2 \\ &\quad + \int_{\Omega\setminus\cup B_{\varepsilon R}(x_i)} \left(\sum_{j=1}^{k} U_{\varepsilon,x_j} \right)^{q-1} \omega^2 \\ &\le C \int_{B_{\varepsilon R}(x_i)} \omega^2 + C\varepsilon^{\alpha(q-1)} \int_{\Omega\setminus\cup B_{\varepsilon R}(x_i)} \omega^2. \end{aligned} \tag{6.6}$$

Note that $\varepsilon^{\alpha(q-1)} \int_{\Omega\setminus\cup B_{\varepsilon R}(x_i)} \omega^2 \le \varepsilon^2 \int_{\Omega} |\nabla\omega|^2$ and by (6.5) we obtain that the above inequality holds. □

7 The reduction

In this section, we will reduce the proof of Theorem 1.2 to find a solution of the form $\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \omega$ for (6.1) to a finite dimensional problem. We will prove that for each $x \in D_{k,\varepsilon}$, there is a unique $\omega_{\varepsilon,x} \in E_{\varepsilon,x,k}$ such that

$$\left\langle I'_\varepsilon \left(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \omega_{\varepsilon,x} \right), \eta \right\rangle_\varepsilon = 0 \quad \forall \eta \in E_{\varepsilon,x,k}.$$

Let

$$k(x,\omega) = I_\varepsilon \left(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \omega_{\varepsilon,x} \right).$$

If we expand $k(x,\omega)$ near $\omega = 0$ as

$$k(x,\omega) = k(x,0) + l_{\varepsilon,x}(\omega) + \frac{1}{2} Q_{\varepsilon,x}(\omega,\omega) + R_\varepsilon(\omega)$$

where

$$l_{\varepsilon,x}(\omega) = \sum_{j=1}^{k} \int_\Omega \varepsilon^2 D\hat{V}_{\varepsilon,x_j} D\omega - \int_\Omega \left(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} \right)_+^p \omega + \int_\Omega Q \left(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} \right)_+^q \omega, \tag{7.1}$$

$$Q_{\varepsilon,x}(\omega,\eta) = \int_\Omega \varepsilon^2 D\omega D\eta - p \int_\Omega \left(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} \right)_+^{p-1} \omega\eta + q \int_\Omega Q \left(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} \right)_+^{q-1} \omega\eta, \tag{7.2}$$

and

$$R_\varepsilon(\omega) = J_{1,\varepsilon}(\omega) + J_{2,\varepsilon}(\omega). \tag{7.3}$$

Here

$$J_{1,\varepsilon}(\omega) = \frac{1}{p+1} \int_\Omega \left(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \omega \right)_+^{p+1} - \frac{1}{p+1} \int_\Omega \left(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} \right)_+^{p+1} - \int_\Omega \left(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \omega \right)_+^{p} - \frac{p}{2} \int_\Omega \left(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} \right)_+^{p-1} \omega^2 \tag{7.4}$$

and

$$J_{2,\varepsilon}(\omega) = \frac{1}{q+1}\int\limits_{\Omega} Q\left(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}+\omega\right)_{+}^{q+1} - \frac{1}{q+1}\int\limits_{\Omega} Q\left(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}\right)_{+}^{q+1}$$
$$-\int\limits_{\Omega} Q\left(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}+\omega\right)_{+}^{q} - \frac{q}{2}\int\limits_{\Omega} Q\left(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}\right)_{+}^{q-1}\omega^2. \tag{7.5}$$

We will prove in Lemma 7.1 that $l_{\varepsilon,x}(\omega)$ is a bounded linear functional in $E_{\varepsilon,x,k}$. Hence it will follow by the Riesz representation theorem, that there exists $l_{\varepsilon,x} \in E_{\varepsilon,x,k}$ such that

$$\langle l_{\varepsilon,x}, \omega\rangle_{\varepsilon} = l_{\varepsilon,x}(\omega) \quad \forall\, \omega \in E_{\varepsilon,x,k}.$$

In Lemma 7.2 we will prove that $Q_{\varepsilon,x}(\omega,\eta)$ is a bounded linear operator from $E_{\varepsilon,x,k}$ to $E_{\varepsilon,x,k}$ such that

$$\langle Q_{\varepsilon,x}\omega, \eta\rangle_{\varepsilon} = Q_{\varepsilon,x}(\omega,\eta) \quad \forall\, \omega,\eta \in E_{\varepsilon,x,k}.$$

Thus finding a critical point of $k(x,\omega)$ is equivalent to solving the problem in $E_{\varepsilon,x,k}$:

$$l_{\varepsilon,x} + Q_{\varepsilon,x}\omega + R'_{\varepsilon}(\omega) = 0. \tag{7.6}$$

We will prove in Lemma 7.3 that the operator $Q_{\varepsilon,x}$ is invertible in $E_{\varepsilon,x,k}$. In Lemma 7.4, we will prove that if ω belongs to a suitable set, $R'_{\varepsilon}(\omega)$ is a small perturbation term in (7.6). Thus we can use the contraction mapping theorem to prove that (7.6) has a unique solution for each fixed $x \in D_{k,\varepsilon}$.

Lemma 7.1 *The functional $l_{\varepsilon,x} : H_0^1(\Omega) \to \mathbb{R}$ defined in (7.1) is a bounded linear functional. Moreover,*

$$\|l_{\varepsilon,x}\|_{\varepsilon} = \varepsilon^{\frac{N}{2}}\, O\left(\sum_{j=1}^{k}|Q(x_j)-1| + \sum_{i<j} U\left(\frac{|x_i-x_j|}{\varepsilon}\right) + \varepsilon^{\tau}\right)$$

where $\tau = \min\{\alpha,\sigma\} > 0$.

Proof We have

$$l_{\varepsilon,x}(\omega) = \sum_{j=1}^{k}\int\limits_{\Omega}\varepsilon^2 D\hat{V}_{\varepsilon,x_j} D\omega - \int\limits_{\Omega}\left(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}\right)_{+}^{p}\omega + \int\limits_{\Omega} Q\left(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}\right)_{+}^{q}\omega$$
$$= \sum_{j=1}^{k}\int\limits_{\Omega}\left(U_{\varepsilon,x_j}^{p} - U_{\varepsilon,x_j}^{q}\right)\omega - \int\limits_{\Omega}\left(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}\right)_{+}^{p}\omega + \int\limits_{\Omega} Q\left(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}\right)_{+}^{q}\omega$$
$$= \sum_{j=1}^{k}\int\limits_{\Omega}\left(U_{\varepsilon,x_j}^{p} - U_{\varepsilon,x_j}^{q}\right)\omega - \int\limits_{\Omega}\left(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}\right)_{+}^{p}\omega + \int\limits_{\Omega}\left(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}\right)_{+}^{q}\omega$$
$$+ \int\limits_{\Omega}(Q-1)\left(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}\right)_{+}^{q}\omega$$

In order to estimate the last term we decompose the domain into $\Omega = (\Omega\setminus \cup B_{\varepsilon R}(x_i)) \cup (\cup B_{\varepsilon R}(x_i))$. Since Q is bounded we have

$$\begin{aligned}
\int_\Omega (Q-1)\left(\sum_{j=1}^k \hat V_{\varepsilon,x_j}\right)_+^q \omega &= \int_{\cup B_{\varepsilon R}(x_i)} (Q-1)\left(\sum_{j=1}^k \hat V_{\varepsilon,x_j}\right)_+^q \omega \\
&\quad + \int_{\Omega\setminus\cup B_{\varepsilon R}(x_i)} (Q-1)\left(\sum_{j=1}^k \hat V_{\varepsilon,x_j}\right)_+^q \omega \\
&\leq \int_{\cup B_{\varepsilon R}(x_i)} (Q-1)\left(\sum_{j=1}^k \hat V_{\varepsilon,x_j}\right)_+^q \omega + \varepsilon^{\alpha q} \int_{\Omega\setminus\cup B_{\varepsilon R}(x_i)} |\omega| \\
&\leq \sum_{i=1}^k \int_{B_{\varepsilon R}(x_i)} (Q-1)\left(\sum_{j=1}^k \hat V_{\varepsilon,x_j}\right)_+^q \omega + C\varepsilon^{\alpha q} \int_\Omega |D\omega|^2
\end{aligned}$$

Here we have used the decay estimates of $\hat V$. On the other hand using Taylors theorem on Q in $B_{\varepsilon R}(x_i)$ and using (6.5) we have

$$Q(x) = Q(x_i) + \langle DQ(x_i), x - x_i\rangle + O(\varepsilon^2).$$

Hence

$$\begin{aligned}
\int_{B_{\varepsilon R}(x_i)} (Q-1)\left(\sum_{j=1}^k \hat V_{\varepsilon,x_j}\right)_+^q \omega &\leq C|Q(x_i)-1| \int_{B_{\varepsilon R}(x_i)} |\omega| + \varepsilon^{\frac{N}{2}} O\left(\varepsilon^{\frac{N}{2}+1}\right) \|\omega\|_\varepsilon \\
&= \varepsilon^{\frac{N}{2}} O\left(|Q(x_i)-1| + \varepsilon^{\frac{N}{2}+1}\right) \|\omega\|_\varepsilon
\end{aligned}$$

Using Taylors theorem and our estimate for $U_{\varepsilon,x_j} - \hat V_{\varepsilon,x_j}$,

$$\begin{aligned}
&\int_\Omega \left(\sum_{j=1}^k U_{\varepsilon,x_j} + \sum_{j=1}^k (\hat V_{\varepsilon,x_j} - U_{\varepsilon,x_j})\right)_+^q \omega \\
&= \int_\Omega \left(\sum_{j=1}^k U_{\varepsilon,x_j}\right)^q \omega + O(1)\varepsilon^\alpha \int_\Omega \left(\sum_{j=1}^k U_{\varepsilon,x_j}\right)^{q-1} \omega
\end{aligned}$$

In order to estimate the second term we decompose the domain into $\Omega = (\Omega\setminus \cup B_{\varepsilon R}(x_i)) \cup (\cup B_{\varepsilon R}(x_i))$ and we have from (6.5)

$$\varepsilon^\alpha \int_{B_{\varepsilon R}(x_i)} \left(\sum_{j=1}^k U_{\varepsilon,x_j}\right)^{q-1} \omega \leq C\varepsilon^{\frac{N}{2}+\alpha} \|\omega\|_\varepsilon$$

and by decay estimates,

$$\varepsilon^{\alpha} \int\limits_{\Omega \setminus \cup B_{\varepsilon R}(x_i)} \left(\sum_{j=1}^{k} U_{\varepsilon, x_j} \right)^{q-1} \omega \leq C \varepsilon^{\alpha q} \int\limits_{\Omega} |\omega| = C \varepsilon^{\frac{N}{2} + \sigma} \|\omega\|_{\varepsilon}$$

where $\sigma = \frac{N}{2} - 1$. We will use the following basic facts, in our proof

$$|a + b|^q - |a|^q - |b|^q = O(1) \left(|a|^{\frac{q}{2}} |b|^{\frac{q}{2}} \right) \quad \text{if} \quad 1 < q < 2$$
$$|a + b|^q - |a|^q - |b|^q = O(1) |a|^{q-1} |b| \quad \text{if} \quad q \geq 2.$$

For the case $q \geq 2$, we have

$$\int\limits_{\Omega} \left(\sum_{j=1}^{k} U_{\varepsilon, x_j} \right)^q \omega = \sum_{j=1}^{k} \int\limits_{\Omega} U_{\varepsilon, x_j}^q \omega + O \left(\sum_{j \neq i} \int\limits_{\Omega} U_{\varepsilon, x_j}^{q-1} U_{\varepsilon, x_i} |\omega| \right)$$

In order to estimate the second term we decompose the domain into $\Omega = (\Omega \setminus \cup B_{\varepsilon R}(x_i)) \cup (\cup B_{\varepsilon R}(x_i))$ and we have

$$\int\limits_{\Omega} U_{\varepsilon, x_j}^{q-1} U_{\varepsilon, x_i} |\omega| = \int\limits_{\Omega \setminus \cup B_{\varepsilon R}(x_i))} U_{\varepsilon, x_j}^{q-1} U_{\varepsilon, x_i} |\omega| + \int\limits_{\cup B_{\varepsilon R}(x_i)} U_{\varepsilon, x_j}^{q-1} U_{\varepsilon, x_i} |\omega|$$

Now from (6.5) we have

$$\begin{aligned}
\int\limits_{B_{\varepsilon R}(x_i)} U_{\varepsilon, x_j}^{q-1} U_{\varepsilon, x_i} |\omega| &\leq \left(\int\limits_{B_{\varepsilon R}(x_i)} U_{\varepsilon, x_j}^{2(q-1)} U_{\varepsilon, x_i}^2 \right)^{\frac{1}{2}} \left(\int\limits_{B_{\varepsilon R}(x_i)} |\omega|^2 \right)^{\frac{1}{2}} \\
&\leq \left(\int\limits_{B_{\varepsilon R}(x_i)} U_{\varepsilon, x_j}^{2(q-1)} U_{\varepsilon, x_i}^2 \right)^{\frac{1}{2}} \|\omega\|_{\varepsilon} \\
&\leq \varepsilon^{\frac{N}{2}} \left(\int\limits_{B_R} U_{1, \frac{x_i - x_j}{\varepsilon}}^{2(q-1)} U^2 \right)^{\frac{1}{2}} \|\omega\|_{\varepsilon} \\
&= \varepsilon^{\frac{N}{2}} O \left(U \left(\frac{x_i - x_j}{\varepsilon} \right) \right) \|\omega\|_{\varepsilon}.
\end{aligned}$$

On the boundary we have from decay estimates and since $\alpha q > N$,

$$\begin{aligned}\int\limits_{\Omega\setminus\cup B_{\varepsilon R}(x_i))} U_{\varepsilon,x_j}^{q-1} U_{\varepsilon,x_i}|\omega| &\leq C\varepsilon^{\alpha q} \int\limits_{\Omega\setminus\cup B_{\varepsilon R}(x_i))} |\omega| \\ &\leq C\varepsilon^{\alpha q} \int\limits_{\Omega} |\omega| \qquad (7.7) \\ &\leq C\varepsilon^{\alpha q} \left(\int\limits_{\Omega} |D\omega|^2\right)^{\frac{1}{2}} \\ &\leq C\varepsilon^{\frac{N}{2}} \varepsilon^{\frac{N}{2}-1} \left(\int\limits_{\Omega} \varepsilon^2 |D\omega|^2\right)^{\frac{1}{2}} \\ &\leq C\varepsilon^{\frac{N}{2}} \varepsilon^{\sigma} \|\omega\|_{\varepsilon} \qquad (7.8)\end{aligned}$$

In the case when $1 < q < 2$,

$$\int\limits_{\Omega} \left(\sum_{j=1}^{k} U_{\varepsilon,x_j}\right)^q \omega = \sum_{j=1}^{k} \int\limits_{\Omega} U_{\varepsilon,x_j}^q \omega + O\left(\sum_{j\neq i} \int\limits_{\Omega} U_{\varepsilon,x_j}^{\frac{q}{2}} U_{\varepsilon,x_i}^{\frac{q}{2}} |\omega|\right)$$

and we proceed as in the case $q \geq 2$.

$$\begin{aligned}\int\limits_{B_{\varepsilon R}(x_i)} U_{\varepsilon,x_j}^{\frac{q}{2}} U_{\varepsilon,x_i}^{\frac{q}{2}} |\omega| \leq C \int\limits_{B_{\varepsilon R}(x_i)} U_{\varepsilon,x_j}^{\frac{q}{2}} |\omega| &\leq C\varepsilon^{\frac{N}{2}} U\left(\frac{|x_i - x_j|}{\varepsilon}\right)^{\frac{q}{2}} \|\omega\|_{\varepsilon} \\ &\leq C\varepsilon^{\frac{N}{2}} U\left(\frac{|x_i - x_j|}{\varepsilon}\right) \|\omega\|_{\varepsilon}\end{aligned}$$

as $U\left(\frac{|x_i - x_j|}{\varepsilon}\right)$ is small. Hence we obtain

$$\begin{aligned}l_{\varepsilon,x}(\omega) &= \sum_{j=1}^{k} \int\limits_{\Omega} (U_{\varepsilon,x_j}^p - U_{\varepsilon,x_j}^q)\omega - \int\limits_{\Omega} \left(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j}\right)_+^p \omega + \int\limits_{\Omega} Q \left(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j}\right)_+^q \omega \\ &= \varepsilon^{\frac{N}{2}} O\left(\sum_{j=1}^{k} |Q(x_j) - 1| + \sum_{j\neq i} U\left(\frac{|x_i - x_j|}{\varepsilon}\right) + \varepsilon^{\tau}\right) \|\omega\|_{\varepsilon}.\end{aligned}$$

□

Lemma 7.2 *The bilinear form $Q_{\varepsilon,x}(\omega)$ defined in (7.2) is a bounded linear. Moreover*

$$|Q_{\varepsilon,x}(\omega,\eta)| \leq C\|\omega\|_{\varepsilon}\|\eta\|_{\varepsilon}$$

where C is independent of ε.

Proof Note that there exists a $C > 0$, such that

$$\int\limits_{\Omega} \left(\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j}\right)_+^{p-1} \omega\eta \leq C \int\limits_{\Omega} \left(\sum_{j=1}^{k} U_{\varepsilon,x_j}\right)^{q-1} |\omega||\eta| \leq C\|\omega\|_{\varepsilon}\|\eta\|_{\varepsilon}$$

and

$$\left|\varepsilon^2 \int_\Omega D\omega D\eta + q \int_\Omega Q \left(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j}\right)_+^{q-1} \omega\eta\right| \le C\|\omega\|_\varepsilon \|\eta\|_\varepsilon$$

□

Lemma 7.3 *There exists* $\rho > 0$ *independent of* ε*, such that*

$$\|Q_{\varepsilon,x}\omega\|_\varepsilon \ge \rho\|\omega\|_\varepsilon \qquad \forall\, \omega \in E_{\varepsilon,x,k},\ x \in D_{k,\varepsilon}$$

Proof Note that Q is uniformly positive and bounded. Purely for simplicity, we assume $Q \equiv 1$. Suppose there exists a sequence $\varepsilon_n \to 0$, $x_{j,n} \in D_{k,\varepsilon_n}$, with $x_{j,n} \to z_j$, $\omega_n \in E_{\varepsilon_n,x_n,k}$ such that $\|\omega_n\|_{\varepsilon_n} = \varepsilon_n^{\frac{N}{2}}$ and

$$\|Q_{\varepsilon_n}\omega_n\|_{\varepsilon_n} = o\left(\varepsilon_n^{\frac{N}{2}}\right)$$

Let $\tilde{\omega}_{i,n} = \omega_n(\varepsilon_n y + x_{i,n})$ and $\Omega_n = \{y : \varepsilon_n y + x_{i,n} \in \Omega\}$ such that

$$\int_{\Omega_n} |D\tilde{\omega}_{i,n}|^2 = \varepsilon_n^{-N}\left(\varepsilon_n^2 \int_\Omega |D\omega_n|^2\right) = 1 \tag{7.9}$$

Hence there exists $\omega_i \in D^{1,2}(\mathbb{R}^N)$ such that $\tilde{\omega}_{i,n} \rightharpoonup \omega_i \in D^{1,2}(\mathbb{R}^N)$ and hence $\tilde{\omega}_{i,n} \to \omega_i \in L^2_{\text{loc}}(\mathbb{R}^N)$. We claim that

$$-\Delta\omega_i = pU^{p-1}\omega_i - qU^{q-1}\omega_i \quad \text{in } \mathbb{R}^N$$

that is for all $\eta \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} D\omega_i D\eta = p\int_{\mathbb{R}^N} U^{p-1}\omega_i\eta - q\int_{\mathbb{R}^N} U^{q-1}\omega_i\eta. \tag{7.10}$$

Now

$$\begin{aligned}
&\int_\Omega \varepsilon_n^2 D\omega_n D\eta - p\int_\Omega \left(\sum_{j=1}^k \hat{V}_{\varepsilon_n,x_{j,n}}\right)_+^{p-1} \omega_n\eta + q\int_\Omega \left(\sum_{j=1}^k \hat{V}_{\varepsilon_n,x_{j,n}}\right)_+^{q-1} \omega_n\eta \\
&= \langle Q_{\varepsilon_n,x_n}\omega_n, \eta\rangle_\varepsilon \\
&= o\left(\varepsilon_n^{\frac{N}{2}}\right)\|\eta\|_{\varepsilon_n}
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_{\Omega_n} D\tilde{\omega}_{i,n} D\tilde{\eta} - p\int_{\Omega_n} \left(\sum_{j=1}^k \tilde{V}_{\varepsilon_n,x_{j,n}}\right)_+^{p-1} \tilde{\omega}_{i,n}\tilde{\eta} + q\int_{\Omega_n} \left(\sum_{j=1}^k \tilde{V}_{\varepsilon_n,x_{j,n}}\right)_+^{q-1} \tilde{\omega}_{i,n}\tilde{\eta} \\
&= o(1)\|\tilde{\eta}\|,
\end{aligned} \tag{7.11}$$

where

$$\tilde{V}_{\varepsilon_n,x_{j,n}} = \hat{V}_{\varepsilon_n,x_{j,n}}(\varepsilon_n y + x_{i,n}),$$
$$\|\tilde{\eta}\|^2 = \int_{\Omega_n} |D\tilde{\eta}|^2,$$
$$\tilde{E}_{\varepsilon_n,x_n,k} = \left\{\tilde{\eta} : \int_{\Omega_n} D\tilde{\eta} D\tilde{W}_{n,j,l} = 0\right\},$$

and $\tilde{W}_{n,j,l} = \varepsilon_n \frac{\partial \hat{V}_{\varepsilon_n,x_{j,n}}(\varepsilon_n y + x_{i,n})}{\partial x_{jl}}$. Let $\eta \in C_0^\infty(\mathbb{R}^N)$. Then we can choose $a_{jln} \in \mathbb{R}$ such that

$$\tilde{\eta}_n = \eta - \sum_{j=1}^{k}\sum_{l=1}^{N} a_{jln}\tilde{W}_{n,j,l}.$$

Note that $\tilde{W}_{n,j,l}$ satisfies the problem

$$\begin{cases} -\Delta \tilde{W}_{n,j,l} = \left(pU^{p-1}\left(y - \frac{x_{i,n}-x_{j,n}}{\varepsilon_n}\right) - qU^{q-1}\left(y - \frac{x_{i,n}-x_{j,n}}{\varepsilon_n}\right)\right)\frac{\partial U}{\partial x_l} & \text{in } \Omega_n \\ \tilde{W}_{n,j,l} = 0 & \text{on } \partial\Omega_n \end{cases} \tag{7.12}$$

Let $\alpha = \frac{2}{q-1}$. Then we claim that $\tilde{W}_{n,j,l}$ is bounded in $D^{1,2}(\Omega_n)$. Now using Hölder's and Hardy's inequality we have

$$\int_{\Omega_n} |\nabla \tilde{W}_{n,j,l}|^2 = \int_{\Omega_n} \left(pU^{p-1} - qU^{q-1}\right)\frac{\partial U}{\partial x_l}\tilde{W}_{n,j,l}$$
$$\leq C\left(\int_{\Omega_n} U^{q-1}\tilde{W}^2_{n,j,l}\right)^{\frac{1}{2}} \leq C\left(\int_{\Omega_n} |\nabla \tilde{W}_{n,j,l}|^2\right)^{\frac{1}{2}} \tag{7.13}$$

Hence $\int_{\Omega_n} |\nabla \tilde{W}_{n,j,l}|^2$ is uniformly bounded and as a result there exists W such that

$$\tilde{W}_{n,j,l} \rightharpoonup W \quad \text{in } D^{1,2}$$

at least for a subsequence. Hence

$$\tilde{W}_{n,j,l} \to W \quad \text{in } L^2_{\text{loc}}.$$

Note that W satisfies the problem,

$$\begin{cases} -\Delta W = \left(pU^{p-1} - qU^{q-1}\right)\frac{\partial U}{\partial x_l} & \text{in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} |\nabla W|^2 = \int_{\mathbb{R}^N}\left(pU^{p-1} - qU^{q-1}\right)\frac{\partial U}{\partial x_l} W. \end{cases} \tag{7.14}$$

We claim that $\tilde{W}_{n,j,l} \to W$ in $D^{1,2}$. First note that

$$\int_{\Omega_n} |U^{p-1}\frac{\partial U}{\partial x_l}\tilde{W}_{n,j,l}| \leq C\int_{\Omega_n} |U^{q-1}\frac{\partial U}{\partial x_l}\tilde{W}_{n,j,l}|$$
$$\int_{\Omega_n} |\nabla \tilde{W}_{n,j,l}|^2 = p\int_{\Omega_n} U^{p-1}\frac{\partial U}{\partial x_l}\tilde{W}_{n,j,l}$$

$$
\begin{aligned}
&- q \int\limits_{\Omega_n} U^{q-1} \frac{\partial U}{\partial x_l} \tilde{W}_{n,j,l} \\
&\to p \int\limits_{\mathbb{R}^N} U^{p-1} \frac{\partial U}{\partial x_l} W - q \int\limits_{\mathbb{R}^N} U^{q-1} \frac{\partial U}{\partial x_l} W \\
&= \int\limits_{\mathbb{R}^N} |\nabla W|^2. \qquad (7.15)
\end{aligned}
$$

Here we have used that $\tilde{W}_{n,j,l}$ converges weakly in $L^{2^\star}$. Hence $\tilde{W}_{n,j,l} \to W = \frac{\partial U}{\partial x_l}$ in $D^{1,2}$ strongly. Now for $i \neq j$, we have

$$
\begin{aligned}
\left\langle \eta, \tilde{W}_{n,j,l} \right\rangle &= \int\limits_{\Omega_n \cap supp\ \eta} \left\{ pU\left(y - \frac{x_{i,n} - x_{j,n}}{\varepsilon_n}\right)^{p-1} - qU\left(y - \frac{x_{i,n} - x_{j,n}}{\varepsilon_n}\right)^{q-1} \right\} \frac{\partial U}{\partial x_l} \eta \\
&= o(1)
\end{aligned}
$$

For $i = j$ we have

$$
\left| \langle \eta, \tilde{W}_{n,j,l} \rangle \right| \leq C
$$

Hence using a coordinate transformation we obtain $a_{jln} = (I + O(1))^{-1} \langle \eta, \tilde{W}_{n,j,l} \rangle$ where I is the identity matrix and $O(1)$ has small off diagonal elements. Hence $a_{jln} \to 0$ as $n \to \infty$ for $i \neq j$. Putting the value of η_n in (7.11) and letting $n \to \infty$, we have

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^N} D\omega_i D\eta - p \int\limits_{\mathbb{R}^N} U^{p-1} \omega_i \eta + q \int\limits_{\mathbb{R}^N} U^{q-1} \omega_i \eta \\
&= \sum_{l=1}^{N} a_l \left(\int\limits_{\mathbb{R}^N} D\omega_i D \frac{\partial U}{\partial x_l} - p \int\limits_{\mathbb{R}^N} U^{p-1} \omega_i \frac{\partial U}{\partial x_l} + q \int\limits_{\mathbb{R}^N} U^{q-1} \omega_i \frac{\partial U}{\partial x_l} \right)
\end{aligned}
$$

where $a_l = \lim_{n\to\infty} a_{jln}$. Using Lemma 4.4, we have

$$
\int\limits_{\mathbb{R}^N} D\omega_i D \frac{\partial U}{\partial x_l} - p \int\limits_{\mathbb{R}^N} U^{p-1} \omega_i \frac{\partial U}{\partial x_l} + q \int\limits_{\mathbb{R}^N} U^{q-1} \omega_i \frac{\partial U}{\partial x_l} = 0
$$

and

$$
\int\limits_{\mathbb{R}^N} D\omega_i D\eta - p \int\limits_{\mathbb{R}^N} U^{p-1} \omega_i \eta + q \int\limits_{\mathbb{R}^N} U^{q-1} \omega_i \eta = 0
$$

Hence we have (7.10).

Since $\omega_i \in D^{1,2}(\mathbb{R}^N)$, it follows by nondegeneracy

$$
\omega_i = \sum_{l=1}^{N} b_l \frac{\partial U}{\partial x_l}
$$

Since $\tilde{\omega}_{i,n} \in \tilde{E}_{\varepsilon_n, x_n, k}$, letting $n \to \infty$ in (7.11), we have

$$
\int\limits_{\mathbb{R}^N} D\omega_i D \frac{\partial U}{\partial x_l} = 0
$$

which implies $b_l = 0$ for all $l = 1, 2, \ldots, N$. Thus $\omega_i = 0$. Hence for any $R > 0$ we have

$$\int\limits_{B_{\varepsilon_n R}(x_{i,n})} |\omega_n|^2 = o(\varepsilon_n^N).$$

Now

$$\begin{aligned}
\int\limits_{\Omega} \left(\sum_{j=1}^{k} \hat{V}_{\varepsilon, x_{j,n}} \right)_+^{p-1} \omega_n^2 &= \int\limits_{\cup B_{\varepsilon_n R}(x_{i,n})} \left(\sum_{j=1}^{k} \hat{V}_{\varepsilon, x_{j,n}} \right)_+^{p-1} \omega_n^2 + \int\limits_{\Omega \setminus \cup B_{\varepsilon_n R}(x_{i,n})} \left(\sum_{j=1}^{k} \hat{V}_{\varepsilon, x_{j,n}} \right)_+^{p-1} \omega_n^2 \\
&\le \int\limits_{\cup B_{\varepsilon_n R}(x_{i,n})} \omega_n^2 + \int\limits_{\Omega \setminus \cup B_{\varepsilon_n R}(x_{i,n})} \left(\sum_{j=1}^{k} U_{\varepsilon, x_{j,n}} \right)_+^{p-1} \omega_n^2 \\
&\le o(1)\varepsilon_n^N + \varepsilon_n^{\alpha(p-q)} \int\limits_{\Omega \setminus \cup B_{\varepsilon_n R}(x_{i,n})} \left(\sum_{j=1}^{k} U_{\varepsilon, x_{j,n}} \right)_+^{q-1} \omega_n^2 \\
&\le o(1)\varepsilon_n^N + \varepsilon_n^{\alpha(p-q)} \|\omega_n\|_{\varepsilon_n}^2.
\end{aligned}$$

Hence

$$\begin{aligned}
o\left(\varepsilon_n^N\right) \ge \langle Q_{\varepsilon_n, x_n}(\omega_n), \omega_n \rangle_{\varepsilon_n} &\ge \|\omega_n\|_{\varepsilon_n}^2 - p \int\limits_{\Omega} \left(\sum_{j=1}^{k} \hat{V}_{\varepsilon, x_{j,n}} \right)_+^{p-1} \omega_n^2 \\
&\ge \varepsilon_n^N - o(1)\varepsilon_n^N
\end{aligned} \tag{7.16}$$

which implies a contradiction.

For the case $\alpha = N - 2$. We claim that $\tilde{W}_{n,j,l}$ is bounded in $D^{1,2}(\Omega_n)$. As $\frac{\partial U}{\partial x_l} \in L^2$ and $N(N-2)(q-1) > N$, we have

$$\begin{aligned}
\int\limits_{\Omega_n} |\nabla \tilde{W}_{n,j,l}|^2 &= \int\limits_{\Omega_n} \left(pU^{p-1} - qU^{q-1} \right) \frac{\partial U}{\partial x_l} \tilde{W}_{n,j,l} \\
&\le C \left(\int\limits_{\Omega_n} U^{2(q-1)} \tilde{W}_{n,j,l}^2 \right)^{\frac{1}{2}} \\
&\le C \left(\int\limits_{\Omega_n} U^{\frac{2^*(2q-2)}{2^*-2}} \right)^{\frac{1}{2}(1-\frac{2}{2^*})} \left(\int\limits_{\Omega_n} |\tilde{W}_{n,j,l}|^{2^*} \right)^{\frac{1}{2^*}} \\
&\le \left(\int\limits_{\mathbb{R}^N} U^{N(q-1)} \right)^{\frac{1}{2}(1-\frac{2}{2^*})} \left(\int\limits_{\Omega_n} |\tilde{W}_{n,j,l}|^{2^*} \right)^{\frac{1}{2^*}} \\
&\le C \left(\int\limits_{\Omega_n} |\nabla \tilde{W}_{n,j,l}|^2 \right)^{\frac{1}{2}}
\end{aligned} \tag{7.17}$$

as $\int_1^\infty \frac{1}{r^{N(N-2)(q-1)-(N-1)}} < \infty$, which implies that $\tilde{W}_{n,j,l}$ is bounded in $D^{1,2}(\Omega_n)$. there exists W such that

$$\tilde{W}_{n,j,l} \rightharpoonup W \text{ in } D^{1,2}$$

and hence

$$\tilde{W}_{n,j,l} \to W \text{ in } L^2_{loc}.$$

Note that W satisfies the problem,

$$\begin{cases} -\Delta W = (pU^{p-1} - qU^{q-1})\frac{\partial U}{\partial x_l} & \text{in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} |\nabla W|^2 = \int_{\mathbb{R}^N} (pU^{p-1} - qU^{q-1})\frac{\partial U}{\partial x_l} W. \end{cases} \tag{7.18}$$

We claim that $\tilde{W}_{n,j,l} \to W$ in $D^{1,2}$. First note that for any compact subset $\Omega' \subset \Omega_n$ we have

$$\int\limits_{\Omega_n} U^{q-1}\frac{\partial U}{\partial x_l}\tilde{W}_{n,j,l} = \int\limits_{\Omega'} U^{q-1}\frac{\partial U}{\partial x_l}\tilde{W}_{n,j,l} + \int\limits_{\Omega_n \setminus \Omega'} U^{q-1}\frac{\partial U}{\partial x_l}\tilde{W}_{n,j,l}.$$

Hence the first integral

$$\int\limits_{\Omega'} U^{q-1}\frac{\partial U}{\partial x_l}\tilde{W}_{n,j,l} \to \int\limits_{\Omega'} U^{q-1}\frac{\partial U}{\partial x_l}W$$

Using the fact that $(N-2)(q-1) > 2$ and Hardy inequality, we obtain

$$\begin{aligned} \int\limits_{\Omega_n \setminus \Omega'} U^{q-1}\tilde{W}^2_{n,j,l} &\le C \int\limits_{\Omega_n \setminus \Omega'} |x|^{-(N-2)(q-1)}\tilde{W}^2_{n,j,l} \\ &\le C \int\limits_{\Omega_n \setminus \Omega'} |x|^{-2}\tilde{W}^2_{n,j,l} \\ &\le C \int\limits_{\Omega_n \setminus \Omega'} |\nabla \tilde{W}_{n,j,l}|^2. \end{aligned} \tag{7.19}$$

As a result we obtain

$$\int\limits_{\Omega_n \setminus \Omega'} U^{q-1}\frac{\partial U}{\partial x_l}\tilde{W}_{n,j,l} \to \int\limits_{\mathbb{R}^N \setminus \Omega'} U^{q-1}\frac{\partial U}{\partial x_l}W.$$

Hence

$$\begin{aligned} \int\limits_{\Omega_n} |\nabla \tilde{W}_{n,j,l}|^2 &= p\int\limits_{\Omega_n} U^{p-1}\frac{\partial U}{\partial x_l}\tilde{W}_{n,j,l} \\ &\quad - q\int\limits_{\Omega_n} U^{q-1}\frac{\partial U}{\partial x_l}\tilde{W}_{n,j,l} \\ &\to p\int\limits_{\mathbb{R}^N} U^{p-1}\frac{\partial U}{\partial x_l}W - q\int\limits_{\mathbb{R}^N} U^{q-1}\frac{\partial U}{\partial x_l}W \\ &= \int\limits_{\mathbb{R}^N} |\nabla W|^2. \end{aligned} \tag{7.20}$$

Hence $\tilde{W}_{n,j,l} \to W = \frac{\partial U}{\partial x_l}$ in $D^{1,2}$ strongly. The remainder of the proof follows exactly as above. □

Lemma 7.4 *Let $R_\varepsilon(\omega)$ be the functional defined by (7.3). Let $\omega \in H_0^1(\Omega)$, then*

$$|R_\varepsilon(\omega)| \leq C\varepsilon^{N(1-\frac{\min\{p+1,3\}}{2})}\|\omega\|_\varepsilon^{\frac{\min\{p+1,3\}}{2^*}} + C\varepsilon^{N\left(1-\frac{\min\{q+1,3\}}{2}\right)}\|\omega\|_\varepsilon^{\frac{\min\{q+1,3\}}{2^*}} + o(1)\|\omega\|_\varepsilon^2 \tag{7.21}$$

and

$$\|R'_\varepsilon(\omega)\|_\varepsilon \leq C\varepsilon^{N(1-\frac{\min\{p,2\}}{2})}\|\omega\|_\varepsilon^{\frac{\min\{p,2\}}{2^*}} + C\varepsilon^{N\left(1-\frac{\min\{q,2\}}{2}\right)}\|\omega\|_\varepsilon^{\frac{\min\{q,2\}}{2^*}} + o(1)\|\omega\|_\varepsilon. \tag{7.22}$$

Proof As before we have $R_\varepsilon(\omega) = J_{1,\varepsilon}(\omega) + J_{2,\varepsilon}(\omega)$. Then

$$\begin{aligned} |J_{1,\varepsilon}(\omega)| &\leq \int\limits_{\cup B_{\varepsilon R}(x_i)} |J_{1,\varepsilon}(\omega)| + \int\limits_{\Omega\setminus\cup B_{\varepsilon R}(x_i)} |J_{1,\varepsilon}(\omega)| \\ &\leq \int\limits_{\cup B_{\varepsilon R}(x_i)} |\omega|^{\min\{p+1,3\}} + p\, o\left(\int\limits_{\Omega\setminus\cup B_{\varepsilon R}(x_i)} \left(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j}\right)_+^{p-1} \omega^2\right) \end{aligned}$$

Here we have used (7.4). However,

$$\begin{aligned} \int\limits_{\cup B_{\varepsilon R}(x_i)} |\omega|^{\min\{p+1,3\}} &\leq C\varepsilon^{N(1-\frac{\min\{p+1,3\}}{2})}\left(\int\limits_{B_{\varepsilon R}(x_i)} |\omega|^{2^*}\right)^{\frac{\min\{p+1,3\}}{2^*}} \\ &\leq C\varepsilon^{N\left(1-\frac{\min\{p+1,3\}}{2}\right)}\|\omega\|_\varepsilon^{\frac{\min\{p+1,3\}}{2^*}}. \end{aligned}$$

Moreover, by the algebraic decay of $\hat{V}_{\varepsilon,x_j}$ we obtain,

$$o\left(\int\limits_{\Omega\setminus\cup B_{\varepsilon R}(x_i)} \left(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j}\right)_+^{p-1} \omega^2\right) \leq Co(1)\varepsilon^{\alpha(p-1)}\int\limits_\Omega \omega^2 \leq Co(1)\varepsilon^2\int\limits_\Omega |\nabla\omega|^2$$

Hence the result follows. □

Lemma 7.5 *There exists an $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$, there exists a C^1 map $\omega_{\varepsilon,x} : D_{k,\varepsilon} \to H$, such that $\omega_{\varepsilon,x} \in E_{\varepsilon,x,k}$ we have*

$$\left\langle I'_\varepsilon\left(\sum_{j=1}^k \hat{V}_{\varepsilon,x_j} + \omega_{\varepsilon,x}\right), \eta\right\rangle_\varepsilon = 0, \quad \forall\eta \in E_{\varepsilon,x,k}.$$

Moreover, we have

$$\|\omega_{\varepsilon,x}\|_\varepsilon \leq C\varepsilon^{\frac{N}{2}}\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}+\kappa}$$

where $\kappa > 0$ is sufficiently small.

Proof We have $l_{\varepsilon,x} + Q_{\varepsilon,x}\omega + R'_\varepsilon(\omega) = 0$. As $Q^{-1}_{\varepsilon,x}$ exists, the above equation is equivalent to solving

$$Q^{-1}_{\varepsilon,x} l_{\varepsilon,x} + \omega + Q^{-1}_{\varepsilon,x} R'_\varepsilon(\omega) = 0.$$

Define

$$G(\omega) = -Q^{-1}_{\varepsilon,x} l_{\varepsilon,x} - Q^{-1}_{\varepsilon,x} R'_\varepsilon(\omega) \quad \forall \omega \in E_{\varepsilon,x,k}.$$

Hence the problem is reduced to finding a fixed point of the map G.

Choose $\gamma > 0$ small. For any $\omega_1 \in E_{\varepsilon,x,k}$ and $\omega_2 \in E_{\varepsilon,x,k}$ with $\|\omega_1\|_\varepsilon \le \varepsilon^{\frac{N}{2}} \varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}}$, $\|\omega_2\|_\varepsilon \le \varepsilon^{\frac{N}{2}} \varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}}$

$$\|G(\omega_1) - G(\omega_2)\|_\varepsilon \le C\|R'_\varepsilon(\omega_1) - R'_\varepsilon(\omega_2)\|_\varepsilon.$$

Note that

$$\left\langle R'_\varepsilon(\omega_1) - R'_\varepsilon(\omega_2), \eta\right\rangle_\varepsilon = \left\langle J'_{1,\varepsilon}(\omega_1) - J'_{1,\varepsilon}(\omega_2), \eta\right\rangle_\varepsilon + \left\langle J'_{2,\varepsilon}(\omega_1) - J'_{2,\varepsilon}(\omega_2), \eta\right\rangle_\varepsilon$$

From Lemma 7.4, we have

$$\begin{aligned}\left\langle R'_\varepsilon(\omega_1) - R'_\varepsilon(\omega_2), \eta\right\rangle_\varepsilon \le\ & C\varepsilon^{N(1-\frac{\min\{p,2\}}{2})}\|\omega_1 - \omega_2\|_\varepsilon^{\min\{p,2\}}\|\eta\|_\varepsilon \\ & + C\varepsilon^{N(1-\frac{\min\{q,2\}}{2})}\|\omega_1 - \omega_2\|_\varepsilon^{\min\{q,2\}}\|\eta\|_\varepsilon \\ & + o(1)\|\omega_1 - \omega_2\|_\varepsilon\|\eta\|_\varepsilon.\end{aligned}$$

Hence we have

$$\begin{aligned}\|R'_\varepsilon(\omega_1) - R'_\varepsilon(\omega_2)\|_\varepsilon &\le C\varepsilon^{N(1-\frac{\min\{p,2\}}{2})}\|\omega_1 - \omega_2\|_\varepsilon^{\min\{p,2\}} \\ &\quad + C\varepsilon^{N(1-\frac{\min\{q,2\}}{2})}\|\omega_1 - \omega_2\|_\varepsilon^{\min\{q,2\}} + o(1)\|\omega_1 - \omega_2\|_\varepsilon \\ &\le o(1)\|\omega_1 - \omega_2\|_\varepsilon.\end{aligned}$$

Hence G is a contraction as

$$\|G(\omega_1) - G(\omega_2)\|_\varepsilon \le Co(1)\|\omega_1 - \omega_2\|_\varepsilon.$$

Also for $\omega \in E_{\varepsilon,x,k}$ with $\|\omega\|_\varepsilon \le \varepsilon^{\frac{N}{2}} \varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}}$, and $\kappa > 0$ sufficiently small

$$\begin{aligned}\|G(\omega)\|_\varepsilon &\le C\|l_{\varepsilon,x}\|_\varepsilon + C\|R'_\varepsilon(\omega)\|_\varepsilon \\ &\le C\varepsilon^{\frac{N}{2}}\varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}}} + \varepsilon^{\frac{N}{2}}\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}+\kappa} \\ &\le C\varepsilon^{\frac{N}{2}}\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}+\kappa} \\ &\le \varepsilon^{\frac{N}{2}}\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}}\end{aligned} \tag{7.23}$$

if $\|l_\varepsilon\|_\varepsilon \le \varepsilon^{\frac{N}{2}}\varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}}}$. Hence

$$G : E_{\varepsilon,x,k} \cap B_{\varepsilon^{\frac{N}{2}}\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}}}(0) \to E_{\varepsilon,x,k} \cap B_{\varepsilon^{\frac{N}{2}}\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}}}(0)$$

is a contraction map if $\|l_\varepsilon\|_\varepsilon \le \varepsilon^{\frac{N}{2}}\varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}}}$. Hence by the contraction mapping principle there exists a unique $\omega \in E_{\varepsilon,x,k} \cap B_{\varepsilon^{\frac{N}{2}}\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}}}(0)$ such that $\omega = G(\omega)$ and

$$\|\omega_{\varepsilon,x}\|_\varepsilon = \|G(\omega_{\varepsilon,x})\|_\varepsilon \le C\varepsilon^{\frac{N}{2}}\varepsilon^{\frac{\gamma\tau}{\min\{q,2\}}+\kappa}.$$

□

8 Existence of interior peaks

Lemma 8.1 *For any positive integer k, we have*

$$I_\varepsilon\left(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}\right)=k\varepsilon^N c-c_1\varepsilon^N\sum_{i<j}U\left(\frac{|x_i-x_j|}{\varepsilon}\right)+c_2\varepsilon^N\sum_{j=1}^{k}(Q(x_j)-1)$$
$$+\varepsilon^N O\left(\sum_{i=1}^{k}|Q(x_i)-1|^2+\sum_{i<j}U^{1+\lambda}\left(\frac{|x_i-x_j|}{\varepsilon}\right)+\varepsilon^{\min\{1,\alpha\}}\right) \tag{8.1}$$

where $c_1, c_2, \lambda > 0$, *and* c *is the mountain pass critical value of the limiting problem.*

Proof We have

$$I_\varepsilon\left(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}\right)=\sum_{j=1}^{k}I_\varepsilon\left(\hat{V}_{\varepsilon,x_j}\right)+\frac{1}{2}\sum_{i\neq j}\int_\Omega\varepsilon^2 D\hat{V}_{\varepsilon,x_i}D\hat{V}_{\varepsilon,x_j}$$
$$-\int_\Omega F\left(x,\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}\right)+\int_\Omega\sum_{j=1}^{k}F\left(x,\hat{V}_{\varepsilon,x_j}\right).$$

From Remark 5.2 we have

$$\begin{aligned}\frac{\varepsilon^2}{2}\int_\Omega|D\hat{V}_{\varepsilon,x_j}|^2&=\frac{1}{2}\int_\Omega U^p_{\varepsilon,x_j}\hat{V}_{\varepsilon,x_j}-\frac{1}{2}\int_\Omega U^q_{\varepsilon,x_j}\hat{V}_{\varepsilon,x_j}\\&=\frac{1}{2}\int_\Omega U^p_{\varepsilon,x_j}(U_{\varepsilon,x_j}-C\varepsilon^\alpha)-\frac{1}{2}\int_\Omega U^q_{\varepsilon,x_j}(U_{\varepsilon,x_j}-C\varepsilon^\alpha)\\&=\frac{1}{2}\int_\Omega U^{p+1}_{\varepsilon,x_j}-\frac{1}{2}\int_\Omega U^{q+1}_{\varepsilon,x_j}+O(\varepsilon^{N+\alpha})\\&=\frac{1}{2}\varepsilon^N\int_{\mathbb{R}^N}(U^{p+1}-U^{q+1})+O(\varepsilon^{N+\alpha}).\end{aligned}$$

Similarly we have

$$\begin{aligned}\frac{1}{p+1}\int_\Omega(\hat{V}_{\varepsilon,x_j})_+^{p+1}&=\frac{1}{p+1}\int_\Omega U^{p+1}_{\varepsilon,x_j}+O\left(\varepsilon^\alpha\int_\Omega U^p_{\varepsilon,x_j}\right)\\&=\frac{1}{p+1}\varepsilon^N\int_{\mathbb{R}^N}U^{p+1}+O\left(\varepsilon^{N+\alpha}\right),\end{aligned}$$
$$\frac{1}{q+1}\int_\Omega\left(\hat{V}_{\varepsilon,x_j}\right)_+^{q+1}=\frac{1}{q+1}\varepsilon^N\int_{\mathbb{R}^N}U^{q+1}+O\left(\varepsilon^{N+\alpha}\right),$$

and

$$\frac{1}{q+1}\int_{\Omega}(Q-1)\left(\hat{V}_{\varepsilon,x_j}\right)_+^{q+1}=\frac{1}{q+1}\int_{\Omega}(Q(x)-Q(x_j))\left(\hat{V}_{\varepsilon,x_j}\right)_+^{q+1}$$
$$+\frac{1}{q+1}(Q(x_j)-1)\int_{\Omega}\left(\hat{V}_{\varepsilon,x_j}\right)_+^{q+1}. \qquad (8.2)$$

To estimate the first term, we decompose $\Omega = B_{\varepsilon R}(x_j)\cup\left(\Omega\backslash B_{\varepsilon R}(x_j)\right)$ and using Taylor's theorem on Q we have,

$$\begin{aligned}\int_{\Omega}(Q(x)-Q(x_j))(\hat{V}_{\varepsilon,x_j})_+^{q+1}&=\int_{B_{\varepsilon R}(x_j)}(Q(x)-Q(x_j))(\hat{V}_{\varepsilon,x_j})_+^{q+1}\\&\quad+\int_{\Omega\backslash B_{\varepsilon R}(x_j)}(Q(x)-Q(x_j))\left(\hat{V}_{\varepsilon,x_j}\right)_+^{q+1}\\&\le C\varepsilon^{N+1}+C\varepsilon^{\alpha(q+1)}.\end{aligned}$$

To estimate the second term in (8.2) we use

$$(Q(x_j)-1)\int_{\Omega}\left(\hat{V}_{\varepsilon,x_j}\right)_+^{q+1}=(Q(x_j)-1)\varepsilon^N\int_{\mathbb{R}^N}U^{q+1}+O\left(\varepsilon^{N+\alpha}\right)$$

Hence we have

$$\begin{aligned}I_\varepsilon\left(\hat{V}_{\varepsilon,x_j}\right)&=\frac{1}{2}\varepsilon^N\int_{\mathbb{R}^N}(U^{p+1}-U^{q+1})-\frac{1}{p+1}\varepsilon^N\int_{\mathbb{R}^N}U^{p+1}\\&\quad+\frac{1}{q+1}\varepsilon^N\int_{\mathbb{R}^N}U^{q+1}+(Q(x_j)-1)\frac{1}{q+1}\varepsilon^N\int_{\mathbb{R}^N}U^{q+1}+O\left(\varepsilon^{N+\min\{1,\alpha\}}\right)\\&=\varepsilon^N\left[\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^N}U^{p+1}-\left(\frac{1}{2}-\frac{1}{q+1}\right)\int_{\mathbb{R}^N}U^{q+1}\right]\\&\quad+(Q(x_j)-1)\frac{1}{q+1}\varepsilon^N\int_{\mathbb{R}^N}U^{q+1}+O\left(\varepsilon^{N+\min\{1,\alpha\}}\right).\end{aligned}$$

On the other hand, we know that for $i\neq j$

$$U_{1,\frac{x_i-x_j}{\varepsilon}}=U\left(\frac{|x_i-x_j|}{\varepsilon}\right)+O\left(\varepsilon^\alpha\right)$$

and using Remark 5.2,

$$\begin{aligned}
\frac{\varepsilon^2}{2}\sum_{i\neq j}\int_\Omega D\hat{V}_{\varepsilon,x_i}D\hat{V}_{\varepsilon,x_j} &= \frac{1}{2}\sum_{i\neq j}\int_\Omega \left(U^p_{\varepsilon,x_j}-U^q_{\varepsilon,x_j}\right)\hat{V}_{\varepsilon,x_i}\\
&= \frac{1}{2}\sum_{i\neq j}\int_\Omega \left(U^p_{\varepsilon,x_j}-U^q_{\varepsilon,x_j}\right)U_{\varepsilon,x_i}+O(\varepsilon^{N+\alpha})\\
&= \frac{\varepsilon^N}{2}\sum_{i\neq j}\int_{\mathbb{R}^N}\left(U^p-U^q\right)U_{1,\frac{x_i-x_j}{\varepsilon}}+O\left(\varepsilon^{N+\alpha}\right)\\
&= \frac{\varepsilon^N}{2}\sum_{i\neq j}\int_{\mathbb{R}^N}\left(U^p-U^q\right)U_{1,\frac{x_i-x_j}{\varepsilon}}+O\left(\varepsilon^{N+\alpha}\right)\\
&= C\varepsilon^N\sum_{i<j}U\left(\frac{|x_i-x_j|}{\varepsilon}\right)+O\left(\varepsilon^{N+\alpha}\right).
\end{aligned}$$

Similarly

$$\begin{aligned}
\int_\Omega\left(\sum_{j=1}^k\hat{V}_{\varepsilon,x_j}\right)^{q+1} &= \int_\Omega\left(\sum_{j=1}^k U_{\varepsilon,x_j}\right)^{q+1}+O\left(\int_\Omega\left(\sum_{j=1}^k\hat{V}_{\varepsilon,x_j}\right)^q\varepsilon^\alpha\right)\\
&= \int_\Omega\left(\sum_{j=1}^k U_{\varepsilon,x_j}\right)^{q+1}+O\left(\int_\Omega\left(\sum_{j=1}^k U_{\varepsilon,x_j}\right)^q\varepsilon^\alpha\right)\\
&= \int_\Omega\left(\sum_{j=1}^k U_{\varepsilon,x_j}\right)^{q+1}+O(\varepsilon^{N+\alpha}).
\end{aligned}$$

If we note that

$$\begin{aligned}
&||a+b|^{q+1}-|a|^{q+1}-|b|^{q+1}-(q+1)a^qb-(q+1)ab^q|\\
&\quad=O(1)a^{\frac{q+1}{2}}b^{\frac{q+1}{2}}\qquad\text{if}\quad 1<q<2\\
&||a+b|^{q+1}-|a|^{q+1}-|b|^{q+1}-(q+1)a^qb-(q+1)ab^q|\\
&\quad=O(1)|a|^q|b|+O(1)|a||b|^q\quad\text{if}\quad q\geq 2
\end{aligned}$$

and the decomposition technique used in Lemma 7.1, we find that

$$\begin{aligned}
&\int_\Omega\left(\sum_{j=1}^k U_{\varepsilon,x_j}\right)^{q+1}-\sum_{j=1}^k\int_\Omega U^{q+1}_{\varepsilon,x_j}\\
&=\int_\Omega\left(\sum_{j=2}^k U_{\varepsilon,x_j}\right)^{q+1}-\sum_{j=2}^k\int_\Omega U^{q+1}_{\varepsilon,x_j}+(q+1)\int_\Omega\left(\sum_{j=2}^k U_{\varepsilon,x_j}\right)^q U_{\varepsilon,x_1}
\end{aligned}$$

$$+(q+1)\int\limits_{\Omega} U^{q}_{\varepsilon,x_1}\sum_{j=2}^{k}U_{\varepsilon,x_j}+O\left(\varepsilon^{N+\alpha}\right)$$
$$=(q+1)\sum_{i<j}\int\limits_{\Omega} U^{q}_{\varepsilon,x_j}U_{\varepsilon,x_i}+\varepsilon^{N}O\left(U^{1+\lambda}\left(\frac{|x_i-x_j|}{\varepsilon}\right)+\varepsilon^{\alpha}\right).$$

As a result we obtain

$$\begin{aligned}\int\limits_{\Omega}F\left(\sum_{j=1}^{k}U_{\varepsilon,x_j}\right)-\int\limits_{\Omega}\sum_{j=1}^{k}F(U_{\varepsilon,x_j}) &= \left\{\int\limits_{\Omega}F\left(\sum_{j=1}^{k}U_{\varepsilon,x_j}\right)-\int\limits_{\Omega}\sum_{j=1}^{k}F(U_{\varepsilon,x_j})\right.\\ &\quad\left.-\sum_{i\neq j}f(U_{\varepsilon,x_j})U_{\varepsilon,x_i}\right\}+\sum_{i\neq j}f(U_{\varepsilon,x_j})U_{\varepsilon,x_i}\\ &=\sum_{i\neq j}f(U_{\varepsilon,x_j})U_{\varepsilon,x_i}+O\left(\varepsilon^{N+\alpha}\right)\\ &\quad+\varepsilon^{N}O\left(U^{1+\lambda}\left(\frac{|x_i-x_j|}{\varepsilon}\right)+\varepsilon^{\alpha}\right).\end{aligned}\tag{8.3}$$

where $f(u)=u^p-u^q$ and $\lambda>0$. Now let us estimate

$$\begin{aligned}&\int\limits_{\Omega}(Q-1)\left\{(\sum_{j=1}^{k}U_{\varepsilon,x_j})^{q+1}-\sum_{j=1}^{k}U^{q+1}_{\varepsilon,x_j}\right\}\\ &=\int\limits_{\Omega}(Q(x)-Q(x_i))\left\{\left(\sum_{j=1}^{k}U_{\varepsilon,x_j}\right)^{q+1}-\sum_{j=1}^{k}U^{q+1}_{\varepsilon,x_j}\right\}\\ &\quad+(Q(x_i)-1)\int\limits_{\Omega}\left\{\left(\sum_{j=1}^{k}U_{\varepsilon,x_j}\right)^{q+1}-\sum_{j=1}^{k}U^{q+1}_{\varepsilon,x_j}\right\}\\ &=\varepsilon^{N}O\left(\sum_{i=1}^{k}|Q(x_i)-1|^2+\sum_{i<j}U^{1+\lambda}\left(\frac{|x_i-x_j|}{\varepsilon}\right)+\varepsilon^{\min\{1,\alpha\}}\right).\end{aligned}$$

We have used the fact that

$$\begin{aligned}&(Q(x_i)-1)\int\limits_{\Omega}\left\{\left(\sum_{j=1}^{k}U_{\varepsilon,x_j}\right)^{q+1}-\sum_{j=1}^{k}U^{q+1}_{\varepsilon,x_j}\right\}\\ &=\varepsilon^{N}O\left(|Q(x_i)-1|+\varepsilon\right)\sum_{i<j}U\left(\frac{|x_i-x_j|}{\varepsilon}\right)\\ &=\varepsilon^{N}O\left(|Q(x_i)-1|^2+\sum_{i<j}U^{2}\left(\frac{|x_i-x_j|}{\varepsilon}\right)+\varepsilon\right).\end{aligned}\tag{8.4}$$

This proves the result. □

Proof [Proof of Theorem 1.2] Define

$$G_\varepsilon(x) = I_\varepsilon\left(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j} + \omega_{\varepsilon,x}\right)$$

and consider the problem

$$\min_{x\in D_{k,\varepsilon}} G_\varepsilon(x).$$

To prove that $\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j} + \omega_{\varepsilon,x}$ is a solution of (6.1), we need to prove that x is a critical point of $G_\varepsilon(x)$.

For any $x \in D_{k,\varepsilon}$, we have from Lemma 8.1,

$$\begin{aligned}
G_\varepsilon(x) &= I_\varepsilon\left(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}\right) + O(\|l_{\varepsilon,x}\|_\varepsilon\|\omega_{\varepsilon,x}\|_\varepsilon + \|\omega_{\varepsilon,x}\|_\varepsilon^2 + R_\varepsilon(\omega_{\varepsilon,x}))\\
&= I_\varepsilon\left(\sum_{j=1}^{k}\hat{V}_{\varepsilon,x_j}\right) + \varepsilon^N O\left(\varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}}+\kappa}\right)\\
&= k\varepsilon^N c - c_1\varepsilon^N\sum_{i<j}U\left(\frac{|x_i-x_j|}{\varepsilon}\right) + c_2\varepsilon^N\sum_{i=1}^{k}(Q(x_i)-1)\\
&\quad + \varepsilon^N O\left(|Q(x_i)-1|^2 + U^{1+\lambda}\left(\frac{|x_i-x_j|}{\varepsilon}\right) + \varepsilon^{\min\{\alpha,1\}}\right)\\
&\quad + \varepsilon^N O\left(\varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}}+\kappa}\right).
\end{aligned} \tag{8.5}$$

Let $x_\varepsilon \in D_{k,\varepsilon}$ be a point of minimum of G_ε in $D_{k,\varepsilon}$. Choose $\tilde{x}_\varepsilon = \left(\tilde{x}_{\varepsilon,1},\ldots,\tilde{x}_{\varepsilon,k}\right)$ such that

$$|\tilde{x}_{\varepsilon,j} - z_j| \le \varepsilon^{\frac{1}{2}} \qquad j = 1,2,\ldots,k$$

and

$$|\tilde{x}_{\varepsilon,i} - \tilde{x}_{\varepsilon,j}| \ge \frac{1}{2k}\sqrt{\varepsilon} \qquad i \ne j.$$

Then we have $U\left(\frac{|\tilde{x}_{\varepsilon,i}-\tilde{x}_{\varepsilon,j}|}{\varepsilon}\right) \le C\varepsilon^{\frac{\alpha}{2}}$ for $i \ne j$ and the mean value theorem yields

$$|Q\left(\tilde{x}_{\varepsilon,i}\right) - 1| \le C|\tilde{x}_{\varepsilon,i} - z_i|^2 \le C\varepsilon \quad i = 1,2,\ldots,k.$$

Thus $\tilde{x}_\varepsilon \in D_{k,\varepsilon}$.

Hence it follows from (8.5) that

$$G_\varepsilon\left(\tilde{x}_\varepsilon\right) = ck\varepsilon^N + \varepsilon^N O\left(\varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}}+\kappa}\right). \tag{8.6}$$

But since $G_\varepsilon\left(\tilde{x}_\varepsilon\right) \ge G_\varepsilon(x_\varepsilon)$ we have from (8.5) and (8.6)

$$-c_1\sum_{i<j}U\left(\frac{|x_{\varepsilon,i}-x_{\varepsilon,j}|}{\varepsilon}\right) + c_2\sum_{i=1}^{k}(Q(x_{\varepsilon,i})-1) \le O\left(\varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}}+\kappa}\right).$$

Thus we have

$$0 \leq Q(x_{\varepsilon,i}) - 1 \leq O\left(\varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}}+\kappa}\right) \qquad i = 1, 2, \ldots, k$$

and

$$-U\left(\frac{|x_i - x_j|}{\varepsilon}\right) \leq O\left(\varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}}+\kappa}\right) \qquad i \neq j.$$

This implies

$$U\left(\frac{|x_i - x_j|}{\varepsilon}\right) \leq O\left(\varepsilon^{\frac{2\gamma\tau}{\min\{q,2\}}+\kappa}\right) \qquad i \neq j.$$

Hence x_ε is an interior point of $D_{k,\varepsilon}$ and hence is a critical point as required. It easily follows $\sum_{j=1}^{k} \hat{V}_{\varepsilon,x_j} + \omega_{\varepsilon,x}$ is a positive solution of (1.3). This finishes the proof. □

Remark 8.2 Consider the problem,

$$\begin{cases} -\varepsilon^2 \text{div}\,(a(x)\nabla u) = u^p - Q(x)u^q & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{8.7}$$

where a is a smooth function satisfying $a(x) \geq \mu > 0$ in Ω. Note that for some $x_0 \in \mathbb{R}^N$, the limiting problem to (8.7) is

$$\begin{cases} -a(x_0)\Delta u = u^p - Q(x_0)u^q & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to +\infty \end{cases} \tag{8.8}$$

By a change of variable of the form $u(x) = Q^{\frac{1}{p-q}}(x_0)v\left(\frac{Q^{\frac{p-1}{2(p-q)}}(x_0)}{a^{1/2}(x_0)}x\right)$, then v satisfies the problem (1.4). Define $\zeta : \Omega \to \mathbb{R}$ by

$$\zeta(x) = \frac{Q^{\frac{N(p-1)+2(p+1)}{2(p-q)}}(x)}{a^{\frac{N}{2}}(x)}$$

in Ω. Let ζ has k isolated local minima. Then using the results of Theorem 1.2 it seems likely that one can show that for sufficiently small $\varepsilon > 0$, there exists a positive solution u_ε having k peaks with each peak concentrating at a local minima of ζ.

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