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Existence results for periodic solutions of integro-dynamic equations on time scales

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Abstract Using the topological degree method and Schaefer's fixed point theorem, we deduce the existence of periodic solutions of nonlinear system of integro-dynamic equations on periodic time scales. Furthermore, we provide several applications to scalar equations, in which we develop a time scale analog of Lyapunov's direct method and prove an analog of Sobolev's inequality on time scales to arrive at a priori bound on all periodic solutions. Therefore, we improve and generalize the corresponding results in Burton et al. (Ann Mat Pura Appl 161:271–283, 1992)

Keywords Periodic time scale · Dynamic equation · Volterra integral equation · Sobolev's inequality · Schaefer · Lyapunov · Periodic solution

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1 Introduction and preliminaries

Existence of periodic solutions of Volterra-type nonlinear integro-differential and summation equations has been intensively investigated in the literature including [7–9], and references therein. In recent years, time scales (closed nonempty subset of the real numbers $\mathbb{R}$) and time scale versions of well-known equations have taken prominent attention (e.g., [1–4, 12, 13]) since the introduction of the new derivative concept by Stefan Hilger. This derivative (called Δ-derivative) gives the ordinary derivative if the time scale (denoted as $\mathbb{T}$) is the set of reals $\mathbb{R}$, and the forward difference operator if $\mathbb{T} = \mathbb{Z}$. Thus, the need for obtaining separate results for discrete and continuous cases is avoided by unification of them under the umbrella of time

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scale calculus (for a comprehensive review of this topic, we direct the reader to the monograph [4]). Since there are many more time scales other than $\mathbb{R}$ and $\mathbb{Z}$, the investigation of dynamic equations on time scales yields a general theory. Among time scales, periodic ones deserve a special interest since they enable researchers to develop a theory for the existence of periodic solutions of dynamic equations on time scales (see for example [2,6,10,11]).

For clarity, we restate the following definitions and introductory examples which can be found in [2,11].

Definition 1 A time scale $\mathbb{T}$ is said to be *periodic* if there exists a $P > 0$ such that $t \pm P \in \mathbb{T}$ for all $t \in \mathbb{T}$. If $\mathbb{T} \neq \mathbb{R}$, the smallest positive P is called the *period* of the time scale.

Example 1 The following time scales are periodic:

(i) $\mathbb{T} = \mathbb{Z}$ has period $P = 1$,
(ii) $\mathbb{T} = h\mathbb{Z}$ has period $P = h$,
(iii) $\mathbb{T} = \mathbb{R}$,
(iv) $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [(2i-1)h, 2ih],\ h > 0$ has period $P = 2h$,
(v) $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z}, m \in \mathbb{N}_0\}$, where $0 < q < 1$ has period $P = 1$,

Remark 1 All periodic time scales are unbounded above and below.

Definition 2 Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period P. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = nP$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

Define the *forward jump operator* σ by

$$\sigma(t) = \inf \{s > t : s \in \mathbb{T}\},$$

and the *graininess function* μ by $\mu(t) = \sigma(t) - t$. A point t of a time scale is called right scattered if $\sigma(t) > t$. Hereafter, we denote by x^{σ} the composite function $x \circ \sigma$. Note that in a periodic time scale $\mathbb{T}$ with period P, the inequality $0 \leq \mu(t) \leq P$ holds for all $t \in \mathbb{T}$.

Remark 2 If $\mathbb{T}$ is a periodic time scale with period P, then $\sigma(t \pm nP) = \sigma(t) \pm nP$. Consequently, the graininess function μ satisfies $\mu(t \pm nP) = \sigma(t \pm nP) - (t \pm nP) = \sigma(t) - t = \mu(t)$ and so is a periodic function with period P.

For the sake of brevity, we assume familiarity with the basic properties of Δ-derivative and Δ-integral. For further details one may consult [4].

Let $\mathbb{T}$ be a periodic time scale with period P. Let $T > 0$ be fixed and if $\mathbb{T} \neq \mathbb{R}$, $T = nP$ for some $n \in \mathbb{N}$. This paper focuses on nonlinear system of infinite delay integro-dynamic equations of the form

$$x^{\Delta}(t) = Dx(t) + f(x(t)) + \int_{-\infty}^{t} K(t,s)\, g(x(s))\, \Delta s + p(t), \tag{1}$$

where $x^{\Delta}(t)$ is $n \times 1$ column vector determined by Δ-derivative of components of $x(t)$, D is an $n \times n$ constant matrix with real entries, $f, g : \mathbb{R}^n \to \mathbb{R}^n$, $p : \mathbb{T} \to \mathbb{R}^n$, and K is an $n \times n$ matrix valued function with real entries. We shall assume throughout this paper that the following assumptions hold:

A 1. f and g are continuous functions, and p is T periodic,
A.2. the kernel $K(t,s)$ is continuous in (t,s) for $s \leq t$, $K(t,s) = 0$ for $s > t$,

$$K(t+T, s+T) = K(t,s) \quad \text{for all } (t,s) \in \mathbb{T} \times \mathbb{T}, \tag{2}$$

and

$$\sup_{t \in \mathbb{T}} \int_{-\infty}^{t} |K(t,s)| \, \Delta s < \infty, \tag{3}$$

where $|\cdot|$ denotes the matrix norm induced by a norm, also denoted by $|\cdot|$, in $\mathbb{R}^n$.

In this paper, we prove an existence theorem for periodic solutions of the system of integro-dynamic equations (1) whose special cases include the systems of integro-differential equations ($\mathbb{T} = \mathbb{R}$) held in [7] and Volterra summation equations ($\mathbb{T} = \mathbb{Z}$) treated by [13]. Moreover, we apply our existence theorem to the scalar equations on periodic time scales. Some of our results in this paper are new even for the mentioned special cases.

In the following, we present some preliminary material that we will need through the remainder of the paper.

Theorem 1 ([4, Theorems 1.16,1.20]) *Let $f : \mathbb{T} \to \mathbb{R}$ and $g : \mathbb{T} \to \mathbb{R}$ be two Δ-differentiable functions. Then:*

(i) $f^{\Delta}(t) = \frac{f^{\sigma}(t) - f(t)}{\mu(t)}$ *if* $\sigma(t) > t$.
(ii) $f^{\sigma}(t) = f(t) + \mu(t) f^{\Delta}(t)$ *for all* $t \in \mathbb{T}$.
(iii) *The product $fg : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t) g(t) + f^{\sigma}(t) g^{\Delta}(t) = f^{\Delta}(t) g^{\sigma}(t) + f(t) g^{\Delta}(t).$$

(iv) *If $g(t) g^{\sigma}(t) \neq 0$, then $\frac{f}{g}$ is Δ-differentiable with*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t) g(t) - f(t) g^{\Delta}(t)}{g(t) g^{\sigma}(t)}.$$

Theorem 2 ([4, Theorems 1.75, 1.77, 1.98]) *Let $f, g : \mathbb{T} \to \mathbb{R}$ be two Δ-differentiable and rd-continuous functions. Then we have*

(i)

$$\int_{t}^{\sigma(t)} f(s) \mathrm{d}s = \mu(t) f(t) \quad \text{for } t \in \mathbb{T}.$$

(ii) *If $\nu : \mathbb{T} \to \mathbb{R}$ is a strictly increasing function with $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ and $\nu^{\tilde{\Delta}} \in C_{rd}$, then*

$$\int_{a}^{b} f(t) \nu^{\Delta}(t) \, \Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s) \, \tilde{\Delta} s.$$

(iii) *For $a, b \in \mathbb{T}$*

$$\int_{a}^{b} f(\sigma(t)) g^{\Delta}(t) \Delta t = (fg)(b) - (fg)(a) - \int_{a}^{b} f^{\Delta}(t) g(t) \Delta t.$$

To differentiate the Lyapunov functionals given in further sections we will employ the next lemma which can be proved similar to [2, Lemma 2.9].

Lemma 1 *Let* $\mathbb{T}$ *be a periodic time scale with period* P. *Suppose that* $f : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ *satisfies the assumptions of* ([4, Theorem 1.117])*, then*

$$\left[\int_{t-T}^{t} f(t,s)\Delta s\right]^{\Delta} = f(\sigma(t),t) - f(\sigma(t), t-T) + \int_{t-T}^{t} f^{\Delta}(t,s)\Delta s,$$

where $T = n_0 P$ *and* $n_0 \in \mathbb{N}$ *is a positive constant.*

We shall invoke Jensen's inequality ([4, Theorem 6.17]) in the proof of Theorem 8.

Theorem 3 (Jensen's inequality) *Let* $a, b \in \mathbb{T}$ *and* $c, d \in \mathbb{R}$. *If* $g : [a,b] \to [c,d]$ *is rd-continuous and* $F : (c,d) \to \mathbb{R}$ *is continuous and convex, then*

$$F\left(\frac{\int_a^b g(t)\Delta t}{b-a}\right) \le \frac{\int_a^b F(g(t))\Delta t}{b-a}.$$

Definition 3 A function $h : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* provided $1 + \mu(t)h(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$. The set of all regressive rd-continuous functions $h : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^{+}$ is given by $\mathcal{R}^{+} = \{h \in \mathcal{R} : 1 + \mu(t)h(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $h \in \mathcal{R}$ and $\mu(t) > 0$ for all $t \in \mathbb{T}$. The *exponential function* on $\mathbb{T}$ is defined by

$$e_h(t,s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \mathrm{Log}(1 + \mu(z)h(z))\,\Delta z\right).$$

It is well known that if $p \in \mathcal{R}^{+}$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^{\Delta} = p(t)y,\ y(s) = 1$. Other properties of the exponential function are given in the following lemma.

Lemma 2 ([4, Theorem 2.36]) *Let* $p, q \in \mathcal{R}$. *Then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$;
(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$ *where*, $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;
(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(vi) $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

To prove existence of periodic solutions of Eq. (1) we will use the following theorem.

Theorem 4 (Schaefer [14]) *Let* $(\mathbf{B}, |\cdot|)$ *be a normed linear space,* H *a continuous mapping of* $\mathbf{B}$ *into* $\mathbf{B}$ *which is compact on each bounded subset of* $\mathbf{B}$. *Then either*

(i) *the equation* $x = \lambda H x$ *has a solution for* $\lambda = 1$, *or*
(ii) *the set of all solutions* x, *for* $0 < \lambda < 1$, *is unbounded.*

2 Existence of periodic solutions

We will state and prove our main result in this section. By the notation $[a, b]_{\mathbb{T}}$ we mean

$$[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}.$$

Define $P_T = \{\varphi \in C(\mathbb{T}, \mathbb{R}^k) : \varphi(t + T) = \varphi(t)\}$, where, $C(\mathbb{T}, \mathbb{R}^k)$ is the space of all vector valued continuous functions on $\mathbb{T}$. Then P_T is a Banach space with the norm

$$|x|_0 = \max_{i=1,2,\ldots,k} \left\{ \sup_{t \in [0,T]_{\mathbb{T}}} |x_i(t)| \right\}.$$

Consider Eq. (1), and corresponding family of systems of equations

$$\begin{aligned} x^{\Delta}(t) &= \lambda\left[-\gamma I + D\right] x(t) + \gamma x(t) \\ &\quad + \lambda \left\{ f(x(t)) + \int_{-\infty}^{t} K(t, s)\, g(x(s))\, \Delta s + p(t) \right\}, \end{aligned} \tag{4}$$

where $0 \le \lambda \le 1$ and $\gamma \in \mathcal{R}$.

Lemma 3 *If $x \in P_T$, then x is a solution of Eq. (4) if, and only if,*

$$x(t) = \lambda \left(1 - e_{\gamma}(t, t - T)\right)^{-1} \int_{t-T}^{t} A(s, x(s))\, e_{\gamma}(t, \sigma(s))\, \Delta s, \tag{5}$$

where

$$A(s, x(s)) = \{-\gamma I + D\} x(s) + f(x(s)) + \int_{-\infty}^{s} K(s, r)\, g(x(r))\, \Delta r + p(s).$$

Proof Let $x \in P_T$ be a solution of (4). Equation (4) can be expressed as

$$x^{\Delta}(t) = \lambda A(t, x(t)) + \gamma x(t). \tag{6}$$

Multiplying both sides of (6) by $e_{\ominus\gamma}(\sigma(t), t_0)$ we get

$$x^{\Delta}(t)\, e_{\ominus\gamma}(\sigma(t), t_0) - \gamma x(t)\, e_{\ominus\gamma}(\sigma(t), t_0)\, x(t) = \lambda A(t, x(t))\, e_{\ominus\gamma}(\sigma(t), t_0).$$

From (vi) in Lemma 2 we have

$$\left[x(t)\, e_{\ominus\gamma}(t, t_0)\right]^{\Delta} = \lambda A(t, x(t))\, e_{\ominus\gamma}(\sigma(t), t_0),$$

Taking integral from $t - T$ to t, we arrive at

$$x(t)\, e_{\ominus\gamma}(t, t_0) - x(t - T)\, e_{\ominus\gamma}(t - T, t_0) = \lambda \int_{t-T}^{t} A(s, x(s))\, e_{\ominus\gamma}(\sigma(s), t_0)\, \Delta s.$$

By $x(t) = x(t - T)$ for $x \in P_T$, and

$$\frac{e_{\ominus\gamma}(t - T, t_0)}{e_{\ominus\gamma}(t, t_0)} = e_{\gamma}(t, t - T), \quad \frac{e_{\ominus\gamma}(\sigma(s), t_0)}{e_{\ominus\gamma}(t, t_0)} = e_{\gamma}(t, \sigma(s)),$$

we find (5). Since each step in the above work is reversible, the proof is complete. □

It is worth mentioning that in the special case $\mathbb{T} = \mathbb{R}$, Lemma 3 provides a solution different than the one given in [7, Theorem 2.3, (2_λ)].

Using periodicity of the kernel $K(t, s)$, we obtain

$$A(s, x(s)) = A(s + T, x(s + T)) \tag{7}$$

for $x \in P_T$.

Define the operator H by

$$(Hx)(t) = \left(1 - e_\gamma(t, t - T)\right)^{-1} \int_{t-T}^{t} A(s, x(s))\, e_\gamma(t, \sigma(s))\, \Delta s. \tag{8}$$

Then (5) is equivalent to the equation

$$\lambda Hx = x. \tag{9}$$

Moreover, it can be also shown by making use of the substitution $u = s + T$, Theorem 2 (ii), and the equalities

$$e_\gamma(t + T, \sigma(s + T)) = e_\gamma(t, \sigma(s)),\ \ e_\gamma(t + T, t) = e_\gamma(t, t - T)$$

that

$$\begin{aligned}
(Hx)(t + T) &= \left(1 - e_\gamma(t + T, t)\right)^{-1} \int_{t}^{t+T} A(s, x(s))\, e_\gamma(t + T, \sigma(s))\, \Delta s \\
&= \left(1 - e_\gamma(t, t - T)\right)^{-1} \int_{t-T}^{t} A(u + T, x(u + T))\, e_\gamma(t + T, \sigma(u + T))\, \Delta u \\
&= \left(1 - e_\gamma(t, t - T)\right)^{-1} \int_{t-T}^{t} A(u, x(u))\, e_\gamma(t, \sigma(u))\, \Delta u \\
&= (Hx)(t).
\end{aligned}$$

Thus,

$$H : P_T \to P_T.$$

On the other hand, (7) and Lemma 1 imply that

$$\begin{aligned}
H^\Delta x(t) &= (1 - e_\gamma(\sigma(t), \sigma(t) - T))^{-1} \left(\int_{t-t}^{t} A(s, x(s)) e_\gamma(t, \sigma(s)) \Delta s \right)^\Delta \\
&\quad + \left((1 - e_\gamma(t, t - T))^{-1}\right)^\Delta \int_{t-t}^{t} A(s, x(s)) e_\gamma(t, \sigma(s)) \Delta s \\
&= (1 - e_\gamma(\sigma(t), \sigma(t) - T))^{-1} \Bigg\{ A(t, x(t)) e_\gamma(\sigma(t), \sigma(t))
\end{aligned}$$

$$- A(t-T, x(t-T))e_\gamma(\sigma(t), \sigma(t-T)) + \gamma \int_{t-t}^{t} A(s, x(s))e_\gamma(t, \sigma(s))\Delta s \Bigg\}$$

$$+ \frac{\gamma e_\gamma(t, t-T)}{(1-e_\gamma(t, t-T))(1-e_\gamma(\sigma(t), \sigma(t)-T))} \int_{t-t}^{t} A(s, x(s))e_\gamma(t, \sigma(s))\Delta s$$

$$= A(t, x(t)) + \gamma(1-e_\gamma(t, t-T))^{-1} Hx(t), \tag{10}$$

where $k_\gamma = 1 - e_\gamma(t, t-T)$ does not depend on $t \in \mathbb{T}$.

The following result concerns the compactness of the operator H.

Lemma 4 *The operator $H : P_T \to P_T$, as defined by (8), is continuous and compact.*

Proof Define the set $X := \{x \in P_T : |x|_0 \leq B\}$. Obviously X is closed and bounded in P_T.

For $\phi_1, \phi_2 \in X$, we obtain

$$\begin{aligned}
|H\phi_1(t) - H\phi_2(t)| &\leq E_1 E_2 \int_{t-T}^{t} |A(s, \phi_1(s)) - A(s, \phi_2(s))| \Delta s \\
&\leq E_1 E_2 \int_{t-T}^{t} |-\gamma I + D| \, |\phi_1(s) - \phi_2(s)| \Delta s \\
&\quad + E_1 E_2 \int_{t-T}^{t} |f(\phi_1(s)) - f(\phi_2(s))| \Delta s \\
&\quad + E_1 E_2 \int_{t-T}^{t} \int_{-\infty}^{s} |K(s, r)| \, |g(\phi_1(r)) - g(\phi_2(r))| \Delta r \Delta s,
\end{aligned}$$

where

$$E_1 = \max_{t\in[0,T]} \left| \left(1 - e_\gamma(t, t-T)\right)^{-1} \right|,$$
$$E_2 = \max_{s\in[t-T,t]} \left| e_\gamma(t, \sigma(s)) \right|.$$

Since f and g are continuous and $\phi_1, \phi_2 \in X$, f and g are uniformly continuous on $[-B, B]^n$. That is, H is continuous in ϕ. We wish to prove compactness of the operator H. To do so, we shall employ the Arzela–Ascoli theorem. We need to show that the set $W = \{H\phi_n(t) : \phi_n \in X\} \subset \mathbb{R}^n$ is relatively compact and the sequence $\{H\phi_n\}_{n\in\mathbb{N}}$ is equicontinuous. It follows from the compactness of $[-B, B]^n$ in $\mathbb{R}^n$ and the continuity of the functions $f, g : \mathbb{R}^n \to \mathbb{R}^n$ that there exists a positive constant C such that

$$|A(s, x(s))| \leq C \quad \text{for all } x \in X \quad \text{and} \quad s \in [0, T]_\mathbb{T}. \tag{11}$$

Let $t \in [0, T]_\mathbb{T}$, $\phi_n \in X$ for all $n \in \mathbb{N}$. Then

$$|H\phi_n(t)| \leq E_1 E_2 \int_{t-T}^{t} |A(s, \phi_n(s))| \Delta s \leq E_1 E_2 T C = N. \tag{12}$$

Hence, $\{H\phi_n\}_{n\in\mathbb{N}}$ is uniformly bounded. Finally, we get by (10) and (11–12) that $\left|H^{\Delta}\phi_n(t)\right|$ is bounded for $\phi_n \in X$. Thus, $\{H\phi_n\}_{n\in\mathbb{N}}$ is equicontinuous. Consequently, the Arzela–Ascoli theorem yields compactness of the operator H. □

Now we are in a position that we can state and prove our main result.

Theorem 5 *If there exists a positive number B such that for any T-periodic solution of* (4), $0 < \lambda < 1$ *satisfies $|x|_0 \leq B$, then the nonlinear system* (1) *has a solution in P_T.*

Proof Let H be defined by (8). Then, by Lemma 4, T-periodic operator H is continuous and compact. The hypothesis $|x| \leq B$ rules out part (ii) of Schaefer's fixed point theorem and thus $x = \lambda Hx$ has a solution for $\lambda = 1$, which solves Eq. (1). This concludes the proof. □

3 Applications to scalar equations

In this section, we are concerned with the scalar integro-dynamic equation

$$x^{\Delta}(t) = ax(t) + f(x(t)) + \int_{-\infty}^{t} K(t,s)\, g(x(s))\, \Delta s + p(t), \quad t \in \mathbb{T} \tag{13}$$

and the corresponding family of equations

$$x_{\lambda}^{\Delta}(t) = [\lambda(\alpha + a) - \alpha]\, x_{\lambda}(t) + \lambda f(x_{\lambda}(t)) + \lambda \int_{-\infty}^{t} K(t,s)\, g(x(s))\, \Delta s + \lambda p(t) \tag{14}$$

for $t \in \mathbb{T}$, where $\mathbb{T}$ is a periodic time scale with period P and $0 < \lambda < 1$.

Besides A.1 and A.2, we also suppose that there exist a function $J : \mathbb{R}_+ \to \mathbb{R}_+$ and a positive constant Q such that

$$|K(t, t-u)| \leq J(u) \quad \text{with} \int_{0}^{\infty} J(u)\, \Delta u = Q \tag{15}$$

and

$$\sup_{t\in\mathbb{T}} \int_{-\infty}^{t} \int_{t}^{\infty} |K(u,s)|\, \Delta u \Delta s < \infty. \tag{16}$$

Note that (15) implies

$$\sup_{t\in\mathbb{T}} \int_{t}^{\infty} |K(u,t)|\, \Delta u \leq Q.$$

We handle Eq. (13) for the cases $a > 0$, $a < 0$, and $a = 0$, and provide sufficient conditions guaranteeing the existence of periodic solutions. The main steps of the existence proofs can be summarized as follows. First, we differentiate the Lyapunov functionals V_1 and V_2, where

$$V_1(t, x_{\lambda}(.)) = |x_{\lambda}(t)| + \lambda \int_{-\infty}^{t} \int_{t}^{\infty} |K(u,s)|\, |g(x_{\lambda}(s))|\, \Delta u \Delta s \tag{17}$$

and

$$V_2(t, x_\lambda(.)) = |x_\lambda(t)| - \lambda \int_{-\infty}^{t} \int_{t}^{\infty} |K(u, s)| |g(x_\lambda(s))| \Delta u \Delta s. \tag{18}$$

Then to obtain the priori bounds for solutions of Eq. (14), we use periodicity of the functionals V_1 and V_2 and time scale analog of Sobolev's inequality. Given the priori bounds and Theorem 5, the proofs are completed.

Different than the analysis of the integro-differential equation for the cases $a > 0$, $a < 0$, and $a = 0$ in [7], finding the estimates for Δ-derivatives of V_1 and V_2 makes our analysis quite challenging. The main problem arises when we attempt to differentiate $|x_\lambda(t)|$. While one can easily find

$$\frac{\mathrm{d}}{\mathrm{d}t} |x(t)| = \frac{x(t)}{|x(t)|} x'(t)$$

by using the equation $x^2(t) = |x(t)|^2$ and the product rule in real case, $|x|^\Delta$ is obtained as

$$|x|^\Delta = \frac{x + x^\sigma}{|x| + |x^\sigma|} x^\Delta \quad \text{for } x \neq 0 \tag{19}$$

since the product rule is changed to $(fg)^\Delta = f^\Delta g^\sigma + f g^\Delta$ in time scale calculus (for the proof of (19) see [5]). That is, the coefficient of x^Δ in (19) depends not only on the sign of $x(t)$ but also on that of $x^\sigma(t)$. Therefore, the equality $|x|^\Delta = \frac{x}{|x|} x^\Delta$ holds only if $x x^\sigma \geq 0$ and $x \neq 0$. For a fixed x let us keep this case distinct from the case $x x^\sigma < 0$ by separating the time scale $\mathbb{T}$ into two disjoint parts as follows:

$$\mathbb{T}_- := \left\{ s \in \mathbb{T} : x(s) x^\sigma(s) < 0 \right\},$$
$$\mathbb{T}_+ := \left\{ s \in \mathbb{T} : x(s) x^\sigma(s) \geq 0 \right\}.$$

Note that the set $\mathbb{T}_-$ consists only of right scattered points of $\mathbb{T}$. To see the relation between $|x|^\Delta$ and $\frac{x}{|x|} x^\Delta$, we prove the next result.

Lemma 5 *Let $x \neq 0$ be Δ-differentiable, then*

$$|x(t)|^\Delta = \begin{cases} \frac{x(t)}{|x(t)|} x^\Delta(t) & \text{if } t \in \mathbb{T}_+, \\ -\frac{2}{\mu(t)} |x(t)| - \frac{x(t)}{|x(t)|} x^\Delta(t) & \text{if } t \in \mathbb{T}_-. \end{cases} \tag{20}$$

Proof If $t \in \mathbb{T}_+$, then $x x^\sigma \geq 0$, and from (19) we obtain

$$|x|^\Delta = \frac{x}{|x|} x^\Delta,$$

since $x \neq 0$. Let $t \in \mathbb{T}_-$. Then t is right scattered (i.e., $\mu(t) > 0$) and we get by (i) of Theorem 1 that

$$\left[\frac{x}{|x|} \right]^\Delta = \frac{1}{\mu} \left[\frac{x^\sigma}{|x^\sigma|} - \frac{x}{|x|} \right] = -\frac{2}{\mu} \frac{x}{|x|},$$

since $\frac{x^\sigma}{|x^\sigma|} = -\frac{x}{|x|}$ for $t \in \mathbb{T}_-$. Applying product rule, the equality $|x| = \frac{x}{|x|} x$ gives

$$|x|^\Delta = \left[\frac{x}{|x|} \right]^\Delta x + \left[\frac{x}{|x|} \right]^\sigma x^\Delta = -\frac{2}{\mu} \frac{x}{|x|} x - \frac{x}{|x|} x^\Delta = -\frac{2}{\mu} |x| - \frac{x}{|x|} x^\Delta.$$

□

From (20) we obtain a fruitful inequality as follows

Remark 3 If $xx^{\sigma} \neq 0$, then

$$\frac{x}{|x|}x^{\Delta} \leq |x|^{\Delta} \leq \frac{x^{\sigma}}{|x^{\sigma}|}x^{\Delta} \quad \text{for } t \in \mathbb{T}. \tag{21}$$

Proof If $t \in \mathbb{T}_{+}$, then $\frac{x^{\sigma}}{|x^{\sigma}|} = \frac{x}{|x|}$ and (21) is immediate from (20). If $t \in \mathbb{T}_{-}$, then $\frac{x^{\sigma}}{|x^{\sigma}|} = -\frac{x}{|x|}$ and (20) gives

$$|x|^{\Delta} = -\frac{2}{\mu}|x| - \frac{x}{|x|}x^{\Delta} \leq -\frac{x}{|x|}x^{\Delta} = \frac{x^{\sigma}}{|x^{\sigma}|}x^{\Delta}.$$

On the other hand, we have

$$|x|^{\Delta} = \frac{x}{|x|\,\mu}\left(-2x - \mu x^{\Delta}\right) = \frac{x}{|x|\,\mu}(-x^{\sigma} - x) \geq \frac{x}{|x|\,\mu}(x^{\sigma} - x) = \frac{x}{|x|}x^{\Delta},$$

since $xx^{\sigma} < 0$ for $t \in \mathbb{T}_{-}$. This completes the proof. □

Next, we prove an analog of Sobolev's inequality on an arbitrary (not necessarily periodic) time scale. The inequality is essential when proving the existence of a priori bound on all possible periodic solutions of (14).

Corollary 1 (Sobolev's inequality) *If $x \in C_{rd}$, then*

$$|x|_1 + \sigma(T)\left|x^{\Delta}\right|_1 \geq T\,|x|_0\,,$$

where

$$|x|_1 = \int_0^T |x(s)|\,\Delta s.$$

Proof From (21), we have

$$|x(t)|^{\Delta} \leq |x(t)|^{\Delta} \quad \text{for all } t \in \mathbb{T}.$$

Integration by parts (Lemma 2 (iii)) yields the following:

$$\int_0^t |x(s)|\,\Delta s = t\,|x(t)| - \int_0^t \sigma(s)\,|x(s)|^{\Delta}\,\Delta s \geq t\,|x(t)| - \sigma(t)\int_0^t \left|x^{\Delta}(s)\right|\Delta s.$$

Taking supremum over the interval $[0, T]_{\mathbb{T}}$, we obtain the desired inequality. □

In the following theorem, we let $\phi(x) = -f(x)$ and show that there is a priori bound for the solution x_{λ} of (14) for $0 < \lambda < 1$. Then we use Theorem 5 to infer the existence of periodic solutions of Eq. (13) whenever $a < 0$.

Theorem 6 *Let $a < 0$ and $x^{\sigma}(t)\phi(x(t)) > 0$ for all $x \neq 0$ and $t \in \mathbb{T}$. Suppose that there exist positive constants η, β, and M such that*

$$|1 + \mu(t)(a - \eta)| \leq 1 \tag{22}$$

and

$$-\,|\phi(x(t))| + Q\,|g(x(t))| \leq -\beta\,|g(x(t))| + M \tag{23}$$

for all $t \in \mathbb{T}$ and $x \in P_T$, where Q is given by (15). Then Eq. (13) has a solution in P_T.

Proof Let $x_\lambda \in P_T$ be a solution of (14). Setting $\alpha = -a$ Eq. (14) can be rewritten as

$$x_\lambda^\Delta(t) = a x_\lambda(t) - \lambda\phi(x_\lambda(t)) + \lambda \int_{-\infty}^{t} K(t,s)\, g(x_\lambda(s))\,\Delta s + \lambda p(t). \tag{24}$$

If $t \in \mathbb{T}_-$, then t is right scattered, so $xx^\sigma < 0$ implies $\frac{x_\lambda^\sigma(t)}{|x_\lambda^\sigma(t)|} = -\frac{x_\lambda(t)}{|x_\lambda(t)|}$, and hence, $x\phi(x) < 0$ since $x^\sigma\phi(x) > 0$. On the other hand, the condition $|1+\mu(t)(a-\eta)| \le 1$ guarantees that $-\frac{2}{\mu(t)} - a \le -\eta$ for all $t \in \mathbb{T}_-$. It follows from (20) and (24) that

$$\begin{aligned} |x_\lambda(t)|^\Delta &= -\frac{2}{\mu(t)}|x_\lambda(t)| - \frac{x_\lambda(t)}{|x_\lambda(t)|}x_\lambda^\Delta(t) \\ &\le \left(-\frac{2}{\mu(t)} - a\right)|x_\lambda(t)| - \lambda|\phi(x_\lambda(t))| \\ &\quad + \lambda\int_{-\infty}^{t}|K(t,s)|\,|g(x_\lambda(s))|\,\Delta s + |p|_0 \end{aligned} \tag{25}$$

$$\le -\eta|x_\lambda(t)| - \lambda|\phi(x_\lambda(t))| + \lambda\int_{-\infty}^{t}|K(t,s)|\,|g(x_\lambda(s))|\,\Delta s + |p|_0 \tag{26}$$

for all $t \in \mathbb{T}_-$. Similarly, if $t \in \mathbb{T}_+$, then $x\phi(x) > 0$, and from (20) and (24) we find

$$\begin{aligned} |x_\lambda(t)|^\Delta &= \frac{x_\lambda(t)}{|x_\lambda(t)|}x_\lambda^\Delta(t) \\ &\le a|x_\lambda(t)| - \lambda|\phi(x_\lambda(t))| + \lambda\int_{-\infty}^{t}|K(t,s)|\,|g(x_\lambda(s))|\,\Delta s + |p|_0. \end{aligned} \tag{27}$$

Combining (26) and (27) we get that

$$|x_\lambda(t)|^\Delta \le -\eta^*|x_\lambda(t)| - \lambda|\phi(x_\lambda(t))| + \lambda\int_{-\infty}^{t}|K(t,s)|\,|g(x_\lambda(s))|\,\Delta s + |p|_0 \tag{28}$$

for all $t \in \mathbb{T}$, where $\eta^* > 0$ is a constant given by

$$\eta^* = \min\{\eta, -a\}. \tag{29}$$

Define the Lyapunov functional V_1 by (17). It is obvious from (2) and (ii) in Theorem 2 that

$$V_1(t+T, x_\lambda(.)) = V_1(t, x_\lambda(.)),$$

i.e., V_1 is periodic for $x_\lambda \in P_T$. We get by (17) and (28) that

$$\begin{aligned} V_1^\Delta(t, x_\lambda(.)) &= |x_\lambda(t)|^\Delta + \lambda|g(x_\lambda(t))|\int_{\sigma(t)}^{\infty}|K(u,t)|\,\Delta u \\ &\quad - \lambda\int_{-\infty}^{t}|K(t,s)|\,|g(x_\lambda(s))|\,\Delta s \end{aligned}$$

$$\begin{aligned} &\leq -\eta^* |x_\lambda(t)| - \lambda |\phi(x_\lambda(t))| + \lambda Q |g(x_\lambda(t))| + |p|_0 \\ &\leq -\eta^* |x_\lambda(t)| - \lambda\beta |g(x_\lambda(t))| + L \\ &\leq -\eta^* |x_\lambda(t)| + L \end{aligned} \tag{30}$$

for all $t \in \mathbb{T}$, where $L = M + |p|_0$. Hence, we obtain

$$0 = \int_0^T V_1^\Delta(s, x_\lambda(.))\,\Delta s \leq -\eta^* \int_0^T |x_\lambda(s)|\,\Delta s + TL,$$

which gives a priori bound $R = TL/\eta^*$ for $|x_\lambda|_1$. The second inequality in (30) yields the priori bound $L_1 = LT/\beta$ for $\lambda \int_0^T |g(x_\lambda(t))|\,\Delta t$. Similarly, from the first inequality in (30), we find the priori bound $L_2 = QL_1 + T|p|_0$ for the integral $\lambda \int_0^T |\phi(x_\lambda(t))|\,\Delta t$. Thus, using (15) and (24) we arrive at

$$\begin{aligned} \int_t^{t+T} \left|x_\lambda^\Delta(s)\right| \Delta s &\leq -aR + L_2 + \lambda \int_0^T \int_{-\infty}^u |K(u,s)|\,|g(x_\lambda(s))|\,\Delta s \Delta u + T|p|_0 \\ &\leq -aR + L_2 + \lambda \int_0^T \int_0^\infty |K(u,u-r)|\,|g(x_\lambda(u-r))|\,\Delta r \Delta u + T|p|_0 \\ &\leq -aR + L_2 + \lambda \int_0^\infty J(r)\,\Delta r \int_0^T |g(x_\lambda(u-r))|\,\Delta u + T|p|_0 \\ &\leq -aR + L_2 + \lambda L_1 \int_0^\infty J(r)\,\Delta r + T|p|_0 = R^*. \end{aligned}$$

Here, [3, Remark 2.17] allows us to interchange the order of integration. Hence, Corollary 1 provides a priori bound $B = \frac{1}{T}(R + \sigma(T) R^*)$ for $|x|_0$. Consequently, from Theorem 5, we deduce existence of a periodic solution of Eq. (13) in P_T. □

Example 2 Let $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z}, m \in \mathbb{N}_0\}$, where $0 < q < 1$. Evidently, $\mu(t) = q^m(1-q)$ for $t = k - q^m$. This shows that

$$0 < \mu(t) < 1 - q \quad \text{for all } t \in \mathbb{T}.$$

Choose $\eta = (1-q)^{-1}$. One may easily verify that the condition (22) holds if $(q-1)^{-1} < a < 0$. If we can guarantee that the functions f and g satisfy the conditions of Theorem 6, and the kernel K obey the assumptions given in Sects. 1 and 3, then the existence of a solution $x \in P_T$ of Eq. (13) follows. Note that the Δ-derivative of a function $\varphi \in C^1(\mathbb{T}, \mathbb{R})$ defined on this time scale is given by

$$\varphi^\Delta(t) = \frac{\varphi(k - q^{m+1}) - \varphi(k - q^m)}{q^m(1-q)} \quad \text{for } t = k - q^m.$$

The following theorem covers the case $a > 0$.

Theorem 7 *Let $a > 0$ and $xf(x) > 0$ for $x \neq 0$. Suppose that there exist positive constants, β and M, such that*

$$|f(x)| - Q|g(x)| \geq \beta |g(x)| - M,$$

where Q is given by (15). Then Eq. (13) has a solution in P_T.

Proof The proof is similar to that of Theorem 6. We only outline the details. Set $\alpha = -a$ and rewrite Eq. (14) as in (24). (20) and (21) imply

$$|x_\lambda|^\Delta \geq \frac{x_\lambda}{|x_\lambda|} x_\lambda^\Delta \quad \text{for all } t \in \mathbb{T}. \tag{31}$$

Then from (18), (24), and (31) we have

$$\begin{aligned} V_2^\Delta(t, x_\lambda(.)) &= |x_\lambda(t)|^\Delta - \lambda |g(x_\lambda(t))| \int_{\sigma(t)}^{\infty} |K(u,t)| \Delta u \\ &\quad + \lambda \int_{-\infty}^{t} |K(t,s)| |g(x_\lambda(s))| \Delta s \\ &\geq a|x_\lambda(t)| + \lambda |f(x_\lambda(t))| - \lambda Q |g(x_\lambda(t))| - |p|_0 \\ &\geq a|x_\lambda(t)| + \lambda\beta |g(x_\lambda(t))| - M - |p|_0 \\ &\geq a|x_\lambda(t)| - L, \end{aligned} \tag{32}$$

where $L = M + |p|_0$. That is, there exists an $R = TL/a > 0$ such that $|x_\lambda|_1 \leq R$. Therefore, the priori bounds for the integrals $\lambda \int_0^T |g(x_\lambda(t))| \Delta t$ and $\lambda \int_0^T |f(x_\lambda(t))| \Delta t$ can be obtained from the second and first inequalities in (32), respectively. The proof is completed in a similar way to that of Theorem 6. □

In the case $a = 0$, existence of periodic solutions is guaranteed by the next theorem.

Theorem 8 *Assume that $a = 0$ and that all remaining hypothesis of Theorem 7. In addition, suppose that there exists $\theta > 0$ such that $|g(x)| \geq \theta g(|x|) \geq 0$, $g(|x|)$ is convex downward, and $g(u) \to \infty$ as $u \to \infty$. Then Eq. (13) has a solution in P_T.*

Proof Let $a = 0$. Set $\alpha = -1$ and rewrite (14) as

$$x_\lambda^\Delta(t) = (1-\lambda) x_\lambda(t) + \lambda f(x_\lambda(t)) + \lambda \int_{-\infty}^{t} K(t,s) g(x(s)) \Delta s + \lambda p(t).$$

Consider the Lyapunov functional (18) to obtain

$$\begin{aligned} V_2^\Delta(t, x_\lambda(.)) &\geq (1-\lambda)|x_\lambda(t)| + \lambda |f(x_\lambda(t))| - \lambda Q |g(x_\lambda(t))| - |p|_0 \\ &\geq (1-\lambda)|x_\lambda(t)| + \lambda\beta |g(x_\lambda(t))| - M - |p|_0 . \end{aligned} \tag{33}$$

If $0 < \lambda \leq \frac{1}{2}$, then from (33), $V_2^\Delta(t, x_\lambda(.)) \geq \frac{1}{2}|x_\lambda(t)| - L$, and so there exists $R_1 > 0$ such that $|x_\lambda|_1 \leq R_1$. Let $\frac{1}{2} < \lambda < 1$, then we get by (33) that

$$V_2^\Delta(t, x_\lambda(.)) \geq \frac{1}{2}\beta |g(x_\lambda(t))| - L \geq \frac{1}{2}\beta\theta g(|x_\lambda(t)|) - L.$$

Therefore, from Jensen's inequality (see Theorem 3), we find

$$0 = \int_0^T V_2^{\Delta}(t, x_\lambda(t))\,\Delta t \geq \frac{1}{2}\beta\theta \int_0^T g(|x_\lambda(t)|) - LT$$
$$\geq \frac{1}{2}\beta\theta T g\left(\frac{1}{T}\int_0^T |x_\lambda(t)|\,\Delta t\right) - LT. \tag{34}$$

Since $g(u) \to \infty$ as $u \to \infty$, it follows from (34) that there exists $R_2 > 0$ such that $|x_\lambda|_1 \leq R_2$. Let $R = \max\{R_1, R_2\}$, so $|x_\lambda|_1 \leq R$ for all $0 < \lambda < 1$. Priori bounds for the integrals $\lambda \int_0^T |g(x_\lambda(t))|\,\Delta s$ and $\lambda \int_0^T |f(x_\lambda(t))|\,\Delta s$ follow from the second and first inequalities in (33), respectively. The remaining part of the proof is same as that of Theorem 6. □

Example 3 In the special case $\mathbb{T} = \mathbb{R}$, Eq. (13) becomes the integro-differential equation

$$x'(t) = ax(t) + f(x(t)) + \int_{-\infty}^{t} K(t,s)g(x(s))ds + p(t), \quad t \in \mathbb{R}$$

for which the existence of periodic solutions has been investigated in [7] under the following conditions:

(i) $a < 0$ and $-|f(x)| + Q|g(x)| \leq -\beta|g(x)| + M$ for some $\beta > 0$ and $M > 0$,
(ii) $a \geq 0$ and $|f(x)| - Q|g(x)| \geq \beta|g(x)| - M$ for some $\beta > 0$ and $M > 0$.

Since the real line $\mathbb{R}$ contains no right scattered points, i.e., $\mu(t) = 0$ for all $t \in \mathbb{R}$, we can rule out the condition "$|1 + \mu(a - \eta)| \leq 1$" in Theorem 6. Hence, results of [7] can be obtained from Theorems 6–8 in the particular case $\mathbb{T} = \mathbb{R}$.

Henceforth, we provide alternative conditions that guarantee the existence of periodic solutions.

Theorem 9 *Suppose that $a > 0$ and there exists a positive constant β such that $\beta < a$ and*

$$|f(x)| + Q|g(x)| \leq \beta|x|, \tag{35}$$

hold. Then Eq. (13) has a solution in P_T.

Proof Set $\alpha = -a$ and rewrite (14) as in (24). As we did in (32) we obtain

$$V_2^{\Delta}(t, x_\lambda(.)) \geq a|x_\lambda(t)| - \lambda|f(x_\lambda(t))| - \lambda Q|g(x_\lambda(t))| - |p|_0$$
$$\geq (a - \beta)|x_\lambda(t)| - |p|_0.$$

Hence, we can find a priori bound for $|x_\lambda|_1$. On the other hand, priori bounds for the integrals $\lambda \int_0^T |g(x_\lambda(t))|\,\Delta t$ and $\lambda \int_0^T |f(x_\lambda(t))|\,\Delta t$ can be easily obtained from the condition (35). The proof is completed as it is done in the proof of Theorem 6. □

Theorem 10 *Assume that $a < 0$. Also, we assume that there exist positive constants $\beta < -a$ and $\eta > \beta$ such that (35) and*

$$|1 + \mu(t)(a - \eta)| \leq 1 \tag{36}$$

hold for all $t \in \mathbb{T}$. Then Eq. (13) has a solution in P_T.

Proof We will proceed with a proof similar to that of Theorem 6. Set $\alpha = -a$ in (14) and rewrite it in the form of (24). Applying similar arguments in (26) and (27) we get that

$$|x_\lambda (t)|^\Delta \leq \zeta(t) |x_\lambda (t)| + \lambda |f (x_\lambda (t))| + \int_{-\infty}^{t} |K (t, s)| |g (x_\lambda (s))| \Delta s + \lambda + |p|_0 ,$$

where

$$\zeta(t) = \begin{cases} -\eta & \text{for } t \in \mathbb{T}_- \\ a & \text{for } t \in \mathbb{T}_+ \end{cases} .$$

Taking the Lyapunov functional (17) and the condition (36) into account we find

$$\begin{aligned} V_1^\Delta (t, x_\lambda (.)) = |x_\lambda (t)|^\Delta + \lambda |g (x_\lambda (t))| \int_{\sigma(t)}^{\infty} |K (u, t)| \Delta u \\ -\lambda \int_{-\infty}^{t} |K (t, s)| |g (x_\lambda (s))| \Delta s \\ \leq \zeta(t) |x_\lambda (t)| + \lambda |f (x_\lambda (t))| + \lambda Q |g (x_\lambda (t))| + |p|_0 \\ \leq \zeta^* |x_\lambda (t)| + |p|_0 , \end{aligned}$$

where $\zeta^* < 0$ is given by

$$\zeta^* = \max_{t \in \mathbb{T}} \{\zeta(t) + \beta\} .$$

This gives a priori bound for $|x_\lambda|_1$. Priori bounds for the integrals $\lambda \int_0^T |g (x_\lambda (t))| \Delta t$ and $\lambda \int_0^T |f (x_\lambda (t))| \Delta t$ can be easily obtained from the condition (35). The proof is completed in a similar way to that of Theorem 6. □

Theorem 11 *Let the periodic time scale* $\mathbb{T}$ *consists only of right scattered points. Assume that* $a < 0$. *We also assume that there exist positive constants* β *and* $\eta > \beta$ *such that* (35) *and*

$$|1 + \mu(t)a| \geq 1 + \eta\mu(t) \tag{37}$$

hold for all $t \in \mathbb{T}$. *Then Eq.* (13) *has a solution in* P_T.

Proof Letting $\alpha = -a$ we rewrite Eq. (14) as in (24). From Theorem 1 (ii.) (24), and (37) we arrive at

$$\begin{aligned} |x_\lambda (t)|^\Delta &= \frac{\left|x_\lambda^\sigma (t)\right| - |x_\lambda(t)|}{\mu(t)} \\ &= \frac{\left|x_\lambda(t) + \mu(t)x^\Delta\right| - |x_\lambda(t)|}{\mu(t)} \end{aligned}$$

$$\geq \left(\frac{|1+a\mu(t)|-1}{\mu(t)}\right)|x_\lambda(t)| - \lambda\,|f(x_\lambda(t))|$$
$$-\lambda \int_{-\infty}^{t} |K(t,s)|\,|g(x_\lambda(s))|\,\Delta s - |p|_0$$
$$\geq \eta\,|x_\lambda(t)| - \lambda\,|f(x_\lambda(t))| - \lambda \int_{-\infty}^{t} |K(t,s)|\,|g(x_\lambda(s))|\,\Delta s - |p|_0 .$$

It follows from (35) that

$$V_2^\Delta(t, x_\lambda(.)) \geq \eta\,|x_\lambda(t)| - \lambda\,|f(x_\lambda(t))| - \lambda Q\,|g(x_\lambda(t))| - |p|_0$$
$$\geq (\eta-\beta)\,|x_\lambda(t)| - |p|_0 .$$

The proof is completed as in Theorem 6. □

Example 4 Let $\mathbb{T}=\mathbb{Z}$. Then Eq. (13) turns into the familiar Volterra difference equation

$$x(t+1) = bx(t) + f(x(t)) + \sum_{j=-\infty}^{t-1} K(t,j)g(x(j)) + p(t), \tag{38}$$

where $b=a+1$. In [13], the author assumed that the terms f, g, K, and p obey the same conditions as that of Eq. (13) and proved the existence of periodic solutions of Eq. (38) in the following cases:

(i) $|b|<1$ and there exists a positive constant β such that

$$|f(x)| + Q\,|g(x)| \leq \beta\,|x|,$$

and

$$|b|-1 < -\beta,$$

(ii) $|b|>1$ and there exists a positive constant β such that

$$|f(x)| + Q\,|g(x)| \leq \beta\,|x|,$$

and

$$|b|-1 > \beta.$$

Theorems 9, 10, and 11 imply these results in the particular case when $\mathbb{T}=\mathbb{Z}$.

Remark 4 In [13] the author was not able to prove existence of periodic solutions of Eq. (38) in the case $b=1$. Theorem 8 not only helps us to get over this constraint but also gives more general result since there are many periodic time scales other than $\mathbb{R}$ and $\mathbb{Z}$. On the other hand, Theorems 6 and 7 assume weaker conditions than the conditions supposed to hold in Theorems 9 and 10.

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