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Rectifiable oscillations in second-order half-linear differential equations

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Abstract Second-order half-linear differential equation (H): $(\Phi(y'))' + f(x)\Phi(y) = 0$ on the finite interval $I = (0, 1]$ will be studied, where $\Phi(u) = |u|^{p-2}u$, $p > 1$ and the coefficient $f(x) > 0$ on I, $f \in C^2((0, 1])$, and $\lim_{x\to 0} f(x) = \infty$. In case when $p = 2$, the equation (H) reduces to the harmonic oscillator equation (P): $y'' + f(x)y = 0$. In this paper, we study the oscillations of solutions of (H) with special attention to some geometric and fractal properties of the graph $G(y) = \{(x, y(x)) : 0 \le x \le 1\} \subseteq \mathbf{R}^2$. We establish integral criteria necessary and sufficient for oscillatory solutions with graphs having finite and infinite arclength. In case when $f(x) \sim \lambda x^{-\alpha}$, $\lambda > 0$, $\alpha > p$, we also determine the fractal dimension of the graph $G(y)$ of the solution $y(x)$. Finally, we study the L^p nonintegrability of the derivative of all solutions of the equation (H).

Keywords Oscillations · Nonlinear equations · Graph · Rectifiability · Fractal dimension · Minkowski content · Asymptotics · Perturbation

Mathematics Subject Classification (2000) 26A30 · 28A12 · 28A75 · 34A26 · 34C10 · 34C20 · 34E10

1 Introduction

We are concerned with the half-linear differential equation on the finite interval $I = (0, 1]$:

$$(\Phi(y'))' + f(x)\Phi(y) = 0, \quad x \in I, \tag{1}$$

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where $y = y(x)$ is a real function, $y \in C^2(I) \cap C(\bar{I})$, $\Phi(u) = |u|^{p-2}u$, where $p > 1$ and $q = p/(p-1)$ is the exponent conjugate to p, and

$$f(x) > 0 \text{ on } I, \quad f \in C^2((0,1]), \quad \text{and} \quad \lim_{x \to 0} f(x) = \infty. \tag{2}$$

Under these conditions imposed on $f(x)$, it is known that every solution $y(x)$ of (1) with prescribed initial conditions at some point $x_0 \in I$ exists on every closed sub-interval of I and is unique, see [5, p. 170, Theorem 1.1] and [6].

As usual, a continuous function $y(x)$ is said to be oscillatory on I if it has an infinite number of zeros in I, see [2,21,23]. We are interested in the non-trivial oscillatory solutions of (1) with special attention to some geometric properties of the graph $G(y)$ of $y(x)$ defined by $G(y) = \{(t, y(t)) : 0 \le t \le 1\} \subseteq \mathbf{R}^2$. We say an oscillatory function $y(x)$ is rectifiable oscillatory on I if the arclength of $G(y)$ is finite and unrectifiable oscillatory on I if the arclength of $G(y)$ is infinite, see Sect. 3.

When the coefficient $f(x)$ is singular at $x = 0$, it can happen that oscillatory solutions of (1) possess graphs with infinite arclength. A simple example is the equation (E): $y'' + x^{-4}y = 0$ where the general solution is given by $y(x) = c_1 x \sin \frac{1}{x} + c_2 x \cos \frac{1}{x}$, and any non-trivial solution of (E) has graph of infinite arclength, see [15,16,25]. Functions of infinite arclength on a finite interval are called fractal functions. A typical example is the Weierstrass' example of a continuous but nowhere differentiable function see [8, p. 162].

In an early paper [11], we have studied the harmonic oscillator equation (P): $y'' + f(x)y = 0$, i.e. Eq. (1) when $p = 2$, with regard to rectifiable and unrectifiable oscillations. In case of unrectifiable oscillation, solutions are fractal functions. In some special cases such as Euler-type equations, i.e. $f(x) = \lambda x^{-\alpha}$, $\lambda > 0$, unrectifiable solutions have fractional dimension $s = 3/2 - 2/\alpha$ which is greater than 1 when $\alpha > 4$. The proofs in the harmonic oscillator case depend heavily on the linear nature of the solutions via asymptotic integration formula developed by Wintner and Hartman [9, p. 371–372], see also [4].

Half-linear equations arise as one-dimensional p-Laplacian nonlinear elliptic equations. These partial differential equations appear in mathematical models of the so-called electro-rheological fluids, see [22].

For the half-linear equation (1), it is known that the solution space of (1) only preserves homogeneity and not additivity, half of the characteristics of linear equations. The purpose of this paper is to generalize our early results from [11] to the more general half-linear equation (1). It turns out that for the study of rectifiable oscillations, one does not need to use the additive property of solution-space as in the case of linear equations. Here we used, instead of Wintner–Hartman's result on asymptotic representation of the harmonic oscillator equation, the nonlinear analogue as developed by Kiguradze and Chanturia [10] and Mirzov [13]. Their method concentrates on the estimate of the energy function defined in terms of a given solution. In some way, it resembles the Gradient method for partial differential equation, see [1].

Contents of this paper are organized as follows. In Sect. 2, we discuss the nonlinear analogue of asymptotic representation of solutions of Eq. (1). Section 3 devotes to the study of integral criteria necessary and sufficient for rectifiable and unrectifiable oscillations. We also prove results on perturbed and forced equations of (1) which showed certain stability of rectifiable oscillations. In Sect. 4, we describe the geometric aspects of the graphs $G(y)$ of solutions curves in terms of its arclength, the box-dimension (also known as the upper Minkowski–Bouligand dimension) and the upper Minkowski content. In Sect. 5, we estimate from below the order of growth for singular behaviour of L^p-norm of the derivative $y'(x)$ of all solutions $y(x)$ of Eq. (1) near the boundary point $x = 0$.

2 On the transformation $(y, y') \to (\varphi, V)$

In this section, a kind of asymptotic behaviour near $x = 0$ of all non-trivial solutions $y(x)$ of Eq. (1) is studied, which plays an essential role in the proof of main results. At the first, we study a transformation $(y, y') \to (\varphi, V)$ given by

$$\begin{cases} y(x) = (p-1)^{\frac{1}{pq}} f^{-\frac{1}{pq}}(x) V^{\frac{1}{p}}(x) w(\varphi(x)), \\ \Phi(y'(x)) = -(p-1)^{-\frac{1}{pq}} f^{\frac{1}{pq}}(x) V^{\frac{1}{q}}(x) \Phi(w'(\varphi(x))), \end{cases} \tag{3}$$

where $x \in I$ and the functions $\varphi(x)$ and $V(x)$ satisfy corresponding differential equations, namely (15) and (26), which will be determined in the process below. Here and in the sequel, the function $w = w(t)$, $t > 0$, is the so-called generalized sine function which is a solution of half-linear differential equation,

$$(\Phi(w'))' + (p-1)\Phi(w) = 0, \quad w(0) = 0, \quad w'(0) = 1. \tag{4}$$

It is known that $w(t)$ satisfies (see [6]):

$$|w'(t)|^p + |w(t)|^p \equiv 1 \quad \text{for all } t > 0, \tag{5}$$

$$w(T_k) = 0, \quad \text{where } T_k = \frac{2k\pi}{p} \frac{1}{\sin(\pi/p)} \quad \text{for all } k \in \mathbf{N}, \tag{6}$$

$$w'(S_k) = 0 \quad \text{and} \quad |w(S_k)| = 1, \quad \text{where} \quad S_k = \frac{1}{2} T_k \quad \text{for all } k \in \mathbf{N}. \tag{7}$$

In our first main result, we give some asymptotic behaviours near $x = 0$ of the functions $\varphi(x)$ and $V(x)$ given in (3), which will be important to establish the oscillatory property of Eq. (1) as well as the finiteness and infiniteness of the graph $G(y)$ of all non-trivial solutions $y(x)$ of Eq. (1).

Theorem 1 *Let $\varphi(x)$ and $V(x)$ be from (3), where $y(x)$ is a non-trivial solution of Eq. (1). Let $f(x)$ satisfy (2) and the following asymptotic condition at $x = 0$,*

$$f^{-\theta} \left[f^{-\gamma}\right]'' \in L^1(I), \tag{8}$$

where θ and γ are two arbitrarily given real numbers such that $\theta + \gamma = \frac{1}{p}$. Then we have:

$$\varphi'(x) < 0 \quad \text{for all } x \in I \quad \text{and} \quad \lim_{x \to 0+} \varphi(x) = \infty, \tag{9}$$

$$0 < \lim_{x \to 0+} V(x) < +\infty. \tag{10}$$

As we will see, the hypothesis (8) is a principal asymptotic condition on the function $f(x)$. It can be represented by

$$f^{-\frac{1}{2p}} \left[f^{-\frac{1}{2p}}\right]'' \in L^1(I) \quad \text{or} \quad f^{-\frac{1}{pq}} \left[f^{-\frac{1}{p^2}}\right]'' \in L^1(I). \tag{11}$$

It is worth to remark that each of two conditions in (11) generalizes the so-called Hartman–Wintner asymptotic condition: $f^{-\frac{1}{4}} \left[f^{-\frac{1}{4}}\right]'' \in L^1(I)$. Analogously to (8), it is a principal hypothesis on $f(x)$ when $p = 2$, see Theorem A in Sect. 3. Furthermore, the condition (8) is equivalent to (11) in the following sense.

Lemma 1 *Let $f(x)$ satisfy (2). Let θ_1, γ_1, θ_2, and γ_2 be four arbitrarily given real positive numbers such that $\theta_1 + \gamma_1 = \theta_2 + \gamma_2$. Then,*

$$f^{-\theta_1}\left[f^{-\gamma_1}\right]'' \in L^1(I) \quad \text{if and only if} \quad f^{-\theta_2}\left[f^{-\gamma_2}\right]'' \in L^1(I).$$

Therefore, in order to check that a function $f(x)$ satisfies the condition (8), it is enough to find only one pair (θ, γ), where $\theta + \gamma = \frac{1}{p}$, for which (8) holds true.

In the following lemma, we show that all functions $f(x)$ which satisfy (8) must possess some other essential asymptotic properties near $x = 0$.

Lemma 2 *Let $f(x)$ satisfy (2) and (8). Then we have:*

$$f^{\frac{1}{p}} \notin L^1(I), \tag{12}$$

$$\lim_{x\to 0} f^{-\frac{1}{p}-1}(x) f'(x) = 0, \tag{13}$$

$$\left[f^{-\frac{1}{p}-1} f'\right]' \in L^1(I). \tag{14}$$

The importance of the properties (12), (13), and (14) can be realized by the following two propositions, which will be proved at the end of this section.

Proposition 1 *Let $\varphi(x)$ be from (3), where $y(x)$ is a non-trivial solution of Eq. (1). Let $f(x)$ satisfy (2), (12), and (13). Then $\varphi'(x) \sim -(p-1)^{-\frac{1}{p}} f^{\frac{1}{p}}(x)$ and $\varphi'(x) < 0$ for all x sufficiently small, and $\lim_{x\to 0+} \varphi(x) = \infty$.*

Proposition 2 *Let $V(x)$ be from (3), where $y(x)$ is a non-trivial solution of Eq. (1). Let $f(x)$ satisfy (2), (13), and (14). Then $0 < \lim_{x\to 0+} V(x) < +\infty$.*

In the sequel, we give some remarks concerning to the transformation (3), and the functions $\varphi(x)$ and $V(x)$. In Appendix of the paper, it will be shown that $\varphi(x)$ satisfies the following differential equation,

$$\varphi'(x) = \frac{-1}{(p-1)^{\frac{1}{p}}} f^{\frac{1}{p}}(x) + \frac{1}{p}\frac{f'(x)}{f(x)} \Phi(w'(\varphi(x))) w(\varphi(x)). \tag{15}$$

This equation does not depend on the specific transformation (3) as the following remark shows.

Remark 1 The transformation (3) could be considered in a slightly general form,

$$\begin{cases} y(x) = a f^{-A}(x) V^{\frac{1}{p}}(x) w(\varphi(x)), \\ \Phi(y'(x)) = -b f^{B}(x) V^{\frac{1}{q}}(x) \Phi(w'(\varphi(x))), \end{cases} \tag{16}$$

where $x \in I$ and the constants $a > 0$, $b > 0$, $A > 0$, and $B > 0$ satisfy

$$a^p = (p-1) b^q \quad \text{and} \quad Ap + Bq = 1. \tag{17}$$

It is clear that (3) is a particular case of (16)–(17) when $a = (p-1)^{\frac{1}{p^2}}$, $b = 1/a$, and $A = B = 1/(pq)$. In Appendix of the paper, it will be shown that the function $\varphi(x)$ arising from (16)–(17) also satisfies the differential equation (15), that is, the function $\varphi(x)$ does not depend on the constants a, b, A, and B, rather it depends only on the constant p and of course on the function $f(x)$. In this sense, instead of (3), we can also use the following transformation,

$$\begin{cases} y(x) = (p-1)^{\frac{1}{p^2}} f^{-\frac{1}{p^2}}(x) V^{\frac{1}{p}}(x) w(\varphi(x)), \quad x > 0, \\ \Phi(y'(x)) = -(p-1)^{-\frac{1}{q^2}} f^{\frac{1}{q^2}}(x) V^{\frac{1}{q}}(x) \Phi(w'(\varphi(x))), \quad x > 0, \end{cases} \tag{18}$$

which is also a special case of (16)–(17).

Next, it is worth to mention that $\lim_{x\to 0+} \varphi(x) = \infty$ and $\lim_{x\to 0+} V(x) > 0$ can result the oscillations of all solutions of (1) as in the following remark.

Remark 2 Under the assumptions (2), (12), (13), and (14), by Proposition 1 and Proposition 2 we know that $\lim_{x\to 0+} \varphi(x) = \infty$ and $\lim_{x\to 0+} V(x) > 0$. Therefore, any non-trivial solution $y(x)$ of (1) is oscillatory on I. Indeed, let T_k be from (6) and S_k from (7). Now, according to (3) and (6), we observe that $a_k = \varphi^{-1}(T_k)$ is a decreasing sequence of consecutive zeros of $y(x)$ such that $a_k \searrow 0$ and $a_k \in I$ for all $k > k_0$ and some $k_0 \in \mathbf{N}$. The existence of the inverse function $\varphi^{-1}(x)$ of $\varphi(x)$ follows from $\varphi'(x) < 0$ for all x near 0 (see Proposition 1 above). Also, $s_k = \varphi^{-1}(S_k)$ is a sequence of consecutive zeros of $y'(x)$ such that $s_k \in (a_{k+1}, a_k)$, $w'(\varphi(s_k)) = 0$ and $|w(\varphi(s_k))| = 1$ for all $k \in \mathbf{N}$.

The finiteness of $\lim_{x\to 0+} V(x)$ will imply an apriori bound of $y'(x)$ for all solutions $y(x)$ of (1) in the following way.

Remark 3 Let $V(x)$ be from (3), let $x \approx 0$, and $C_0 > 0$ and $C_1 > 0$. If $V(x) \le C_0$, then

$$|y'(x)| \le C_1 f^{\frac{1}{p^2}}(x) \quad \text{and} \quad |y'(x)| \le C_1 f^{\frac{1}{pq}}(x),$$

where the last inequality holds provided $p \ge 2$. Indeed, from (5) we obtain $|w'(\varphi(x))| \le 1$ for all $x \in I$. Therefore, from (3) immediately follows

$$|y'(x)| = (p-1)^{-\frac{1}{p^2}} f^{\frac{1}{p^2}}(x) V^{\frac{1}{p}}(x) |w'(\varphi(x))| \le C_1 f^{\frac{1}{p^2}}(x).$$

It is clear that in the case when $p \ge 2$, we have $f^{\frac{1}{p^2}}(x) \le f^{\frac{1}{pq}}(x)$.

To the end of this section, we give the proofs of Lemma 1, Lemma 2, Proposition 1, Proposition 2, and Theorem 1 which are stated above. In this direction, the proof of Lemma 1 as well as Lemma 2 is based on the following useful proposition which will be proved in Appendix of the paper.

Proposition 3 *Let $\psi = \psi(x)$ be a real function such that $\psi \in C^2((0,1])$, $\psi(x) > 0$ on I, and $\psi(0) = 0$. If $\psi^{A-1}\psi'' \in L^1(I)$, where $A > 1$, then we have:*

(i) $\psi^{-A} \notin L^1(I)$,
(ii) $\psi^{A-2}(\psi')^2 \in L^1(I)$ *and* $\lim_{x\to 0} \psi^{A-1}(x)\psi'(x) = 0$,
(iii) $\left[\psi^{A-1}\psi'\right]' \in L^1(I)$.

Proof of Lemma 1 Let us suppose that

$$f^{-\theta_1}\left[f^{-\gamma_1}\right]'' \in L^1(I). \tag{19}$$

Since $\theta_1 + \gamma_1 = \theta_2 + \gamma_2$, we may set $\sigma = \theta_i + \gamma_i$ for $i = 1, 2$. By an easy calculation we derive that

$$f^{-\theta_i}(x)\left[f^{-\gamma_i}(x)\right]'' = \gamma_i(1+\gamma_i) f^{-\sigma-2}(x)(f'(x))^2 - \gamma_i f^{-\sigma-1}(x) f''(x), \tag{20}$$

where $i = 1, 2$. It is crucial that all powers appearing on the right-hand side of the equality (20) does not depend on the numbers θ_i and γ_i but only on σ. Next, from (19) in particular for $\psi(x) = f^{-\gamma_1}(x)$ and $A = \frac{\theta_1}{\gamma_1} + 1$, we obtain that $\psi^{A-1}\psi'' = f^{-\theta_1}\left[f^{-\gamma_1}\right]'' \in L^1(I)$. Hence, we may use Proposition 3 for this choice of $\psi(x)$ and A. In this way, the conclusion (ii) of Proposition 3 implies that

$$\psi^{A-2}(\psi')^2 \in L^1(I). \tag{21}$$

Since $\psi(x) = f^{-\gamma_1}(x)$, $A = \frac{\theta_1}{\gamma_1} + 1$, and $\gamma_1 A = \theta_1 + \gamma_1 = \sigma$, from (21) we get

$$f^{-\sigma-2}(f')^2 \in L^1(I). \tag{22}$$

Also, by means of (19), (20) for $i = 1$ and (22) we observe that

$$f^{-\sigma-1}f'' \in L^1(I). \tag{23}$$

Now, the equality (20) for $i = 2$, and the statements (22) and (23) show that $f^{-\theta_2}\left[f^{-\gamma_2}\right]'' \in L^1(I)$ too. Thus, this lemma is proved. □

Proof of Lemma 2 The proof of this lemma is also an easy consequence of Proposition 3. Indeed, from the assumption $f^{-\theta}\left[f^{-\gamma}\right]'' \in L^1(I)$ in particular for $\psi(x) = f^{-\gamma}(x)$ and $A = \frac{\theta}{\gamma} + 1$, we have that $\psi^{A-1}\psi'' \in L^1(I)$ and hence, we may use Proposition 3. Since $\psi(x) = f^{-\gamma}(x)$, $A = \frac{\theta}{\gamma} + 1$, and $\gamma A = \theta + \gamma = 1/p$, we observe that

$$\psi^{-A}(x) = f^{\frac{1}{p}}(x) \quad \text{and} \quad \psi^{A-1}(x)\psi'(x) = -\gamma f^{-\frac{1}{p}-1}(x)f'(x).$$

Hence, from the conclusions (i), (ii), and (iii) of Proposition 3, we obviously obtain (12), (13) and (14). It proves this lemma. □

Proof of Proposition 1 Multiplying (15) by $1/f^{\frac{1}{p}}(x)$, we obtain

$$\frac{\varphi'(x)}{f^{\frac{1}{p}}(x)} = \frac{-1}{(p-1)^{\frac{1}{p}}} + \frac{1}{p}\frac{f'(x)}{f^{\frac{1}{p}+1}(x)}\Phi(w'(\varphi(x)))w(\varphi(x)). \tag{24}$$

Substituting (5) and (13) into (24), we observe that

$$\lim_{x\to 0}\frac{\varphi'(x)}{f^{\frac{1}{p}}(x)} = \frac{-1}{(p-1)^{\frac{1}{p}}} < 0,$$

and hence, $-\varphi'(x) \sim f^{\frac{1}{p}}(x)$ when $x \approx 0$. It gives a constant $c_1 > 0$ and an $x_1 \in I$ such that $\varphi'(x) \leq -c_1 f^{\frac{1}{p}}(x) < 0$ for all $x \in (0, x_1)$. Integrating this inequality over (x, x_1) for any $x \in (0, x_1)$, we obtain by (12) that

$$\varphi(x) \geq \varphi(x_1) + c_1\int_x^{x_1} f^{\frac{1}{p}}(s)\mathrm{d}s \to +\infty \quad \text{when} \quad x \to 0.$$

This prove Proposition 1. □

Proof of Proposition 2 At the first, in Appendix of the paper, the following two equalities will be shown,

$$V(x) = (p-1)^{\frac{1}{p}} f^{-\frac{1}{p}}(x)|y'|^p + (p-1)^{-\frac{1}{q}} f^{\frac{1}{q}}(x)|y|^p, \tag{25}$$

and

$$V'(x)=\left[(p-1)^{\frac{1}{p}}f^{-\frac{1}{p}}(x)\right]'|y'|^p+\left[(p-1)^{-\frac{1}{q}}f^{\frac{1}{q}}(x)\right]'|y|^p. \tag{26}$$

For any nontrivial solution $y(x)$ of (1), we have $V(x)>0$ on I because of existence and uniqueness of initial value problems. Let $g(x)$ be defined by

$$g(x)=f^{-\frac{1}{p}-1}(x)f'(x). \tag{27}$$

Since $|y'|^p=|\Phi(y')|^q$, the equalities (25) and (26) can be rewritten in the forms

$$V(x)=(p-1)^{\frac{1}{p}}\left[f^{-\frac{1}{p}}(x)|\Phi(y')|^q+(p-1)^{-1}f^{\frac{1}{q}}(x)|y|^p\right], \tag{28}$$

and

$$V'(x)=-p^{-1}(p-1)^{\frac{1}{p}}g(x)\left[|\Phi(y')|^q-f(x)|y|^p\right]. \tag{29}$$

Next, with the help of Eq. (1), one can derive the following identity

$$\begin{aligned}\big(g(x)\Phi(y')y\big)'&=g'(x)\Phi(y')y+g(x)(\Phi(y'))'y+g(x)\Phi(y')y'\\&=g'(x)\Phi(y')y+g(x)\left[-f(x)|y|^p+|\Phi(y')|^q\right].\end{aligned} \tag{30}$$

Now, from (29) and (30), we derive that

$$\left[V(x)+p^{-1}(p-1)^{\frac{1}{p}}g(x)\Phi(y')y\right]'=p^{-1}(p-1)^{\frac{1}{p}}g'(x)\Phi(y')y.$$

Integrating this equality over (x,x_1), we have

$$V(x)=M_1-c_pg(x)\Phi(y')y-c_p\int_x^{x_1}g'(s)\Phi(y'(s))y(s)\mathrm{d}s, \tag{31}$$

where

$$c_p=p^{-1}(p-1)^{\frac{1}{p}}\quad\text{and}\quad M_1=V(x_1)+c_pg(x_1)\Phi(y'(x_1))y(x_1).$$

Note that from (3) and (5), we have $|\Phi(y')y|\le V(x)$ which together with (31) implies that:

$$V(x)\le M_1+c_p|g(x)|V(x)+c_p\int_x^{x_1}|g'(s)|V(s)\mathrm{d}s, \tag{32}$$

and

$$V(x)\ge M_1-c_p|g(x)|V(x)-c_p\int_x^{x_1}|g'(s)|V(s)\mathrm{d}s. \tag{33}$$

For every nontrivial solution $y(x)$ of (1) we must have $V(x_1)>0$ by the uniqueness of initial value problem for half-linear equations. Because of (13) and (27), we can choose x_1 sufficiently small such that for all $x\in(0,x_1)$ we have $c_p|g(x)|\le 1/2$ and $2c_pe^{2c_pG}G\le 1$, where $G=\int_0^{x_1}|g'(x)|\mathrm{d}x$. Thus for all $x\in(0,x_1)$ we have from (32) and (33) that

$$V(x)\le 2M_1+2c_p\int_x^{x_1}|g'(s)|V(s)\mathrm{d}s, \tag{34}$$

and

$$V(x) \geq 2M_1 - 2c_p \int_x^{x_1} |g'(s)|V(s)\mathrm{d}s. \tag{35}$$

Applying Gronwall's inequality to (34) we obtain

$$V(x) \leq 2M_1 e^{2c_p \int_x^{x_1} |g'(s)|\mathrm{d}s} \leq 2M_1 e^{2c_p G}.$$

Using this upper bound on $V(x)$, we note that the right-hand side of (35) is non-negative. Hence, we can use the reverse Gronwall's inequality to obtain

$$2M_1 e^{-2c_p \int_x^{x_1} |g'(s)|\mathrm{d}s} \leq V(x) \leq 2M_1 e^{2c_p \int_x^{x_1} |g'(s)|\mathrm{d}s},$$

which by (14) and (27) shows that $\lim_{x\to 0+} V(x) = c_0$ exists and $0 < c_0 < \infty$. □

Now we are able to give the proof of Theorem 1.

Proof of Theorem 1 By an easy combination of Lemma 2, Proposition 1, and Proposition 2, the desired proof of Theorem 1 follows immediately. □

3 Rectifiable and unrectifiable oscillations

In this section, we impose on the function $f(x)$ an additional asymptotic condition near $x = 0$ which can characterize the finiteness of arclength of graph $G(y)$ of all solutions of (1). At the first, we recall some definitions about rectifiable and unrectifiable oscillations of continuous functions on a finite interval, which have been appeared for the first time in recent papers [15,17,25]. In this sense, the arclength of the graph $G(y)$ is defined as usual by,

$$\text{length}(G(y)) = \sup \sum_{i=1}^{m} ||(t_i, y(t_i)) - (t_{i-1}, y(t_{i-1}))||_2,$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \cdots < t_m = 1$ of the interval $\bar{I}$, and $||\ ||_2$ denotes the norm in $\mathbf{R}^2$.

Definition 1 If $\text{length}(G(y)) < \infty$, then the graph $G(y)$ is said to be *rectifiable* curve in $\mathbf{R}^2$. Otherwise, $G(y)$ is said to be *unrectifiable* curve in $\mathbf{R}^2$. An oscillatory function y on I is said to be *rectifiable* (resp., *unrectifiable*) *oscillatory* on I, if its graph $G(y)$ is a rectifiable (resp., unrectifiable) curve in $\mathbf{R}^2$. Equation (1) is said to be *rectifiable* (resp., *unrectifiable*) *oscillatory* on I, if all its non-trivial solutions are rectifiable (resp., unrectifiable) oscillatory on I.

It is clear that the graph $G(y)$ of the function $y(x) = x^c$, $c > 0$, is a rectifiable curve in $\mathbf{R}^2$. However, rectifiability of the graph $G(y)$ of $y(x) = x^c \sin x^{-d}$, $x \in I$, depends on the positive powers c and d, in the sense that $G(y)$ is a rectifiable (resp., unrectifiable) curve in $\mathbf{R}^2$ provided $c \geq d$ (resp., $c < d$), see [24, Chapter 10]. Furthermore, rectifiability of the oscillations of a linear differential equation of Euler type $y'' + \lambda x^{-\alpha} y = 0$, $x \in I$, ($\lambda > 0$ for $\alpha > 2$ and $\lambda > 1/4$ for $\alpha = 2$), depends on the parameter α in the sense that this equation is rectifiable (resp., unrectifiable) oscillatory on I provided $2 \leq \alpha \leq 4$ (resp., $\alpha > 4$), see [15,25]. In more general setting, one can study the rectifiable oscillations for the linear differential equation $y'' + f(x)y = 0$, $x \in I$, where the coefficient $f(x)$ is positive and smooth in I, singular at $x = 0$, and satisfies the so-called Hartman–Wintner conditions near $x = 0$ as in the following result, see Theorem 1.4 in [11].

Theorem A *Let $f \in C^2((0,1])$, $f(x) > 0$ on I, and let $f(x)$ satisfy the Hartman–Wintner asymptotic condition at $x = 0$,*

$$f^{-\frac{1}{4}} \left[f^{-\frac{1}{4}} \right]'' \in L^1(I). \tag{36}$$

The linear differential equation $y'' + f(x)y = 0$, $x \in I$, is rectifiable oscillatory on I provided $\sqrt[4]{f(x)} \in L^1(0,1)$ and unrectifiable oscillatory on I provided $\sqrt[4]{f(x)} \notin L^1(0,1)$.

In the first main result of the paper, we give a generalization of Theorem A from linear to the half-linear differential equations. More precisely, we will show that the condition $\sqrt[4]{f(x)} \in L^1(0,1)$ when $p = 2$ will be generalized to the corresponding one $f^{1/p^2}(x) \in L^1(0,1)$ when $p > 1$.

Theorem 2 *Let $f(x)$ satisfy (2) and (8).*

(i) *Equation (1) is rectifiable oscillatory on I if*

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} f^{\frac{1}{p^2}}(x) \mathrm{d}x < +\infty. \tag{37}$$

(ii) *Equation (1) is unrectifiable oscillatory on I if*

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} f^{\frac{1}{p^2}}(x) \mathrm{d}x = +\infty. \tag{38}$$

Proof Firstly, we know that the rectifiability of the graph $G(y)$ of a smooth function $y(x)$ defined on I, is equivalent to $y' \in L^1(0,1)$, see [7, Theorem 1, p. 217]. Now, let $y(x)$ be any solution of Eq. (1). From Proposition 1, Proposition 2, and Remark 2, we have that $y(x)$ is oscillatory on I. Moreover, according to Proposition 2, Remark 3, and assumption (37), we obtain that $y' \in L^1(0,1)$, which ensures that $y(x)$ is rectifiable oscillatory on I.

Next, let $y(x)$ be a non-trivial solution of Eq. (1) and let $t = \varphi(x) \to \infty$ when $x \to 0$. Let s_n and S_n be two sequences of consecutive zeros of $y'(x)$ and $w'(t)$ respectively, see (7) and Remark 2. Recall $S_n = \frac{n\pi}{p}(\sin\frac{\pi}{p})^{-1}$, $w(S_n) = 1$, and $s_n = \varphi^{-1}(S_n)$ for all n. In order to prove that $y(x)$ is unrectifiable oscillatory on I, it is enough to show that the series $\sum_n |y(s_n)|$ is divergent, see [15, Proposition 4.2]. Denote $F(x) = f^{-\frac{1}{pq}}(x)$. Note that by (3) and Proposition 2,

$$\sum_n |y(\varphi^{-1}(S_n))| \geq \frac{2c_0}{3} \sum_n F(\varphi^{-1}(S_n)). \tag{39}$$

Denote $J_n = [S_n, S_{n+1}]$, having length $L = \frac{2\pi}{p}(\sin\frac{\pi}{p})^{-1}$. Let $\sigma_n \in J_n$ be chosen so that $F(\varphi^{-1}(\sigma_n)) \geq F(\varphi^{-1}(t))$ for all $t \in J_n$. Observe that

$$F(\varphi^{-1}(S_n)) \geq F(\varphi^{-1}(\sigma_n)) - \max_{t \in J_n} |F'(\varphi^{-1}(t))||S_n - \sigma_n|,$$

so

$$F(\varphi^{-1}(S_n)) \geq F(\varphi^{-1}(\sigma_n)) - L \max_{t \in J_n} |F'(\varphi^{-1}(t))|. \tag{40}$$

From Proposition 1, we get a small enough $x_1 \in I$ such that

$$|\varphi'(x)| \geq \frac{1}{2}(p-1)^{-\frac{1}{p}} f^{\frac{1}{p}}(x) \quad \text{for all } x \in (0, x_1). \tag{41}$$

Also, from (13) and (27), we get a small enough $x_2 \in I$ such that

$$\frac{2L(p-1)^{1/p}}{pq}|g(x)| < \frac{1}{4} \quad \text{for all } x \in (0, x_2). \tag{42}$$

Let $x_0 = \min\{x_1, x_2\}$. Since $\lim_{n\to\infty} S_n = \infty$, we can choose n_0 sufficiently large so that $\varphi^{-1}(t) \in (0, x_0)$ for all $t \geq S_{n_0}$. Now we use (41) and (42) to estimate $F'(\varphi^{-1}(t))$ for all t such that $x = \varphi^{-1}(t) \in (0, x_0)$ as follows:

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}F(\varphi^{-1}(t))\right| &= \frac{1}{pq}\left|\frac{g(x) f^{\frac{1}{p^2}}(x)}{\varphi'(x)}\right| \leq \frac{2(p-1)^{1/p}}{pq}\left|g(x) f^{\frac{1}{p^2}}(x) f^{-\frac{1}{p}}(x)\right| \\ &\leq \frac{2(p-1)^{1/p}}{pq}|g(x)| F(\varphi^{-1}(t)) < \frac{1}{4L}F(\varphi^{-1}(t)),\end{aligned}$$

thus

$$\max_{t\in J_n} |F'(\varphi^{-1}(t))| \leq \frac{1}{4L}F(\varphi^{-1}(\sigma_n)), \quad n \geq n_0. \tag{43}$$

Using (43) in (40), we obtain

$$F(\varphi^{-1}(S_n)) \geq \frac{3}{4}F(\varphi^{-1}(\sigma_n)), \quad n \geq n_0. \tag{44}$$

Using (44) in (39) we obtain

$$\begin{aligned}\sum_n |y(s_n)| &= \sum_n |y(\varphi^{-1}(S_n))| \geq \frac{c_0}{2}\sum_{n\geq n_0} F(\varphi^{-1}(\sigma_n)) \\ &= \frac{c_0}{2}\sum_{n>n_0} f^{-\frac{1}{pq}}(\varphi^{-1}(\sigma_n)) \geq \frac{p}{4\pi}c_0\left(\sin\frac{\pi}{p}\right)\sum_{n\geq n_0}\int_{S_n}^{S_{n+1}} f^{-\frac{1}{pq}}(\varphi^{-1}(t))\mathrm{d}t \\ &= \frac{p}{4\pi}c_0\left(\sin\frac{\pi}{p}\right)\int_{\tau_0}^{\infty} f^{-\frac{1}{pq}}(\varphi^{-1}(t))\mathrm{d}t,\end{aligned} \tag{45}$$

where $\tau_0 = S_{n_0}$. Now by (41) we estimate from below the last integral in (45),

$$\begin{aligned}\int_{\tau_0}^{\infty} f^{-\frac{1}{pq}}(\varphi^{-1}(t))\mathrm{d}t &= -\int_0^{\varphi^{-1}(\tau_0)} f^{-\frac{1}{pq}}(x)\varphi'(x)\mathrm{d}x \\ &\geq c_1\int_0^{\varphi^{-1}(\tau_0)} f^{-\frac{1}{pq}}(x) f^{\frac{1}{p}}(x)\mathrm{d}x = \int_0^{\varphi^{-1}(\tau_0)} f^{\frac{1}{p^2}}(x)\mathrm{d}x.\end{aligned}$$

Hence, according to (38) and (45), from the last inequality follows that the series $\sum_n |y(s_n)|$ is divergent which implies that $y(x)$ is unrectifiable oscillatory on I. □

As a consequence of the previous criterion for the rectifiable and unrectifiable oscillations of Eq. (1) on I, we can derive some sufficient conditions for rectifiable and unrectifiable oscillations of (1) on I related to (37) and (38).

Corollary 1 *Let $f(x)$ satisfy (2) and (8).*

(i) *Equation (1) is rectifiable oscillatory on I if $p \geq 2$ and*

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1} f^{\frac{1}{pq}}(x)\mathrm{d}x < +\infty. \tag{46}$$

(ii) *Equation (1) is unrectifiable oscillatory on I if $1 < p \leq 2$ and*

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1} f^{\frac{1}{pq}}(x)\mathrm{d}x = +\infty. \tag{47}$$

Proof The condition $p \geq 2$ gives $q \leq p$ and so, $f^{\frac{1}{p^2}}(x) \leq f^{\frac{1}{pq}}(x)$ for $x \approx 0$, which together with (46) ensures (37). Hence, the conclusion (i) of this corollary follows from the same conclusion (i) of Theorem 2. Also, the condition $1 < p < 2$ implies that $p < q$ and so, $f^{\frac{1}{p^2}}(x) \geq f^{\frac{1}{pq}}(x)$ for $x \approx 0$, which together with (47) gives (38). Therefore, the conclusion (ii) of this corollary follows from the same conclusion (ii) of Theorem 2. Finally, in the case $p = 2$, the condition (47) becomes $\sqrt[4]{f(x)} \notin L^1(0, 1)$ and the unrectifiable oscillations of (1) immediately follows from Theorem 1.4 in [11]. □

Remark 4 We note that conditions (37), (38), and also conditions (46), (47) are mutually exclusive to one another. Therefore conditions (37) and (46) are necessary and sufficient conditions for rectifiable oscillations. Likewise, conditions (38) and (47) are necessary and sufficient conditions for unrectifiable oscillations.

As the first application of Theorem 2, we show that the number $\alpha = p^2$ could be taken as the so-called critical value for rectifiable oscillations of the linear differential equations of Euler type in the following way.

Theorem 3 *Let $f(x)$ satisfy (2) and (8), and let $f(x) \sim \lambda x^{-\alpha}$ near $x = 0$, where $\alpha > p$. Then Eq. (1) is rectifiable oscillatory on I provided $p < \alpha < p^2$ and unrectifiable oscillatory on I provided $\alpha \geq p^2$.*

Proof It is enough to check that for $f(x) \sim \lambda x^{-\alpha}$ near $x = 0$, we have: if $\alpha < p^2$ (resp., $\alpha \geq p^2$), then the condition (37) (resp., 38) is satisfied. □

In the following examples, we give some applications of Theorem 3 and Theorem 2.

Example 1 We consider the p-generalized Euler type differential equation,

$$(\Phi(y'))' + \lambda x^{-\alpha}\Phi(y) = 0, \quad x \in I, \tag{48}$$

where $\alpha > p$ and $\lambda > 0$. It is easy to check that the function $f(x) = \lambda x^{-\alpha}$ satisfies the condition (11) and also, $f^{1/p^2} \in L^1(I)$ if and only if $p < \alpha < p^2$. Therefore, by means of Theorem 3 and Lemma 1, we observe that (48) is rectifiable oscillatory on I if $p < \alpha < p^2$ and unrectifiable oscillatory on I if $\alpha \geq p^2$.

Example 2 Let $\lambda > 0$, $\beta > 0$, and $\gamma \in \mathbf{R}$. The half-linear differential equation,

$$(\Phi(y'))' + \lambda x^{-\gamma} e^{\beta/x} \Phi(y) = 0, \quad x \in I, \tag{49}$$

is unrectifiable oscillatory on I by Theorem 2. Indeed, since $x^A e^{-B/x} \in L^1(I)$ and $x^A e^{B/x} \notin L^1(I)$ for any $A \in \mathbf{R}$ and $B > 0$, in particular for $f(x) = \lambda x^{-\gamma} e^{\beta/x}$ we have:

$$f^{-\frac{1}{2p}}(x) \left[f^{-\frac{1}{2p}}(x) \right]'' = \frac{\lambda^{-\frac{1}{p}}}{4p^2} \left[(\gamma^2 - 2p\gamma) x^{\frac{\gamma}{p}-2} + 2\beta(\gamma - 2p) x^{\frac{\gamma}{p}-3} + \beta^2 x^{\frac{\gamma}{p}-4} \right] e^{\frac{-\beta}{px}} \in L^1(I)$$

and

$$f^{1/p^2}(x) = \lambda^{-\frac{1}{p}} x^{-\frac{\gamma}{p^2}} e^{\frac{\beta}{p^2 x}} \notin L^1(I).$$

Now, Theorem 2 ensures the unrectifiable oscillations of Eq. (49).

Next, we can prove similar results for the perturbed differential equation

$$(\Phi(y'))' + (f(x) + h(x))\Phi(y) = 0, \quad x \in I, \tag{50}$$

as the linear case in [11]. The following theorem is an extension of Theorem 3.2 which also improves Theorem 1.8 in [11].

Theorem 4 *Let $f(x)$ satisfy (2) and (8). If $h(x)$ satisfies $f^{-2} f' h \in L^1(I)$ then we have:*

(i) *Equation (50) is rectifiable oscillatory on I if (37) holds,*
(ii) *Equation (50) is unrectifiable oscillatory on I if (38) holds.*

Proof Let $y(x)$ be a non-trivial solution of (50) and let $V(x)$ be defined by (25). Then $y(x)$ satisfies the following identity which is similar to (25),

$$V(x) = M_1 - c_p g(x) \Phi(y') y - c_p \int_x^{x_1} g'(s) \Phi(y'(s)) y(s) \mathrm{d}s + c_p \int_x^{x_1} g(s) h(s) |y(s)|^p \mathrm{d}s, \tag{51}$$

where $c_p = p^{-1}(p-1)^{\frac{1}{p}}$. The last integral in (51) can be estimated by

$$c_p \int_x^{x_1} g(s) h(s) |y(s)|^p \mathrm{d}s \leq \frac{1}{q} \int_x^{x_1} |g(s) h(s)| f^{-\frac{1}{q}}(s) V(s) \mathrm{d}s \leq \frac{1}{q} \int_x^{x_1} |f^{-2}(s) f'(s) h(s)| V(s) \mathrm{d}s. \tag{52}$$

Using (52), we can derive upper and lower bounds for $V(x)$ similar to (34) and (35),

$$V(x) \leq M_2 + M_3 \int_x^{x_1} \left[|g'(s)| + |f^{-2}(s) f'(s) h(s)| \right] V(s) \mathrm{d}s, \tag{53}$$

and

$$V(x) \geq m_2 - m_3 \int_x^{x_1} \left[|g'(s)| + |f^{-2}(s) f'(s) h(s)|\right] V(s) \mathrm{d}s. \tag{54}$$

Using (53) and (54), we can repeat a similar argument leading to (34) and (35), and show that $V(x)$ is bounded from above and below by positive constants in the right neighbourhood of 0. Hence $\lim_{x \to 0} V(x)$ exists as a positive finite number. The remaining proof is the same as that of Theorem 2. □

Example 3 Let $\lambda > 0$, $\beta \in (0, 1)$, and $\alpha > p$. We consider the half-linear Euler–Weber type differential equation,

$$(\Phi(y'))' + \frac{1}{x^\alpha} \left(\lambda + \frac{\delta \sin x}{|\ln x|^\beta} \right) \Phi(y) = 0, \quad x \in I. \tag{55}$$

This equation is a logarithm perturbation of (48). According to Theorem 4, Eq. (55) is rectifiable oscillatory on I if $p < \alpha < p^2$ and unrectifiable oscillatory on I if $\alpha \geq p^2$. In order to show that, it is enough to check that the functions $f(x) = \lambda x^{-\alpha}$ and $h(x) = \delta x^{-\alpha} |\ln x|^{-\beta} \sin x$ satisfy $f^{-2} f' h \in L^1(I)$ and also that, $f^{1/p^2} \in L^1(I)$ if and only if $p < \alpha < p^2$.

Remark 5 Let $f(x) = \lambda x^{-\alpha}, \lambda > 0, \alpha > 0$, and $h(x) = \mu x^{-\beta} \sin x, \beta > 0$. The requirement that $f^{-2} f' h \in L^1(I)$ amounts to $\alpha > \beta$. In particular, the differential equation

$$y'' + (\lambda x^{-4} + \mu x^{-3} \sin x) y = 0, \tag{56}$$

where $\lambda > 0$, is unrectifiable oscillatory which cannot be concluded from [11]. This shows that Theorem 4 improves upon our earlier result even when $p = 2$. In other words, the method of proof is superior than that based upon the usual Wintner–Hartman asymptotic formula in our earlier paper [11].

In case of rectifiable oscillation, the conclusion (i) of Theorem 4 can be further improved to include equations with a forcing term. In this connection, let us consider the equation,

$$(\Phi(y'))' + (f(x) + h(x))\Phi(y) = e(x), \quad x \in I, \tag{57}$$

where $f(x)$ and $h(x)$ are same as before, and the forcing term $e(x)$ satisfies $e \in C((0, 1])$.

Theorem 5 *Let $f(x)$ and $h(x)$ satisfy the conditions from Theorem 4, and let $E(x) = f^{-2-\frac{1}{p^2}}(x) f'(x) e(x)$. If*

$$E \in L^1(I), \tag{58}$$

then all solutions of (57) are rectifiable oscillatory if (37) holds.

Proof We proceed in the same manner as in the proof of Theorem 4. We find instead of (53) the following upper bound for $V(x)$:

$$V(x) \leq M_2 + M_3 \int_x^{x_1} \left[|g'(s)| + |f^{-2}(s) f'(s) h(s)|\right] V(s) \mathrm{d}s$$
$$+ \frac{1}{q} \int_x^{x_1} |g(s) e(s)| f^{-\frac{1}{pq}}(s) V^{\frac{1}{p}}(s) \mathrm{d}s. \tag{59}$$

Note that $p > 1$ implies $(f^{-\frac{1}{q}}V)^{\frac{1}{p}} \le 1+\frac{1}{p}f^{-\frac{1}{q}}V$ and condition (58) implies that $G \in L^1(I)$, where $G(x) = g(x)e(x)f^{-\frac{1}{pq}}(x)$. Using this in (59), one can conclude that the function $V(x)$ is bounded by a constant depending on initial conditions of $y(x)$ at $x = x_1$. Thus, rectifiable oscillation of $y(x)$ follows from (25) with an application of (37). □

Remark 6 Let $f(x) = \lambda x^{-\alpha}$, $\lambda > 0$, $\alpha > 0$, and $h(x) = \mu x^{-\beta}\sin x$, $0 < \beta < \alpha$. We can conclude from Theorem 5 that the forced linear differential equation

$$y'' + (\lambda x^{-4} + \mu x^{-3}\sin x)y = x^{-\gamma}\cos x, \tag{60}$$

where $\frac{4}{3}\gamma < \alpha < 4$, is rectifiable oscillatory. This gives further improvement of our early result for the harmonic oscillator. Note that γ in (60) can be zero which corresponds to periodic forcing and γ can also be negative. Rectifiable oscillations are preserved under periodic forcing when the forcing term $e(x)$ and perturbing term $h(x)$ become singular as long as they are dominated by $f(x)$.

4 The s-dimensional fractal oscillations

In this section, we give some sufficient conditions on the function $f(x)$ such that for all solutions of Eq. (1), the graph $G(y)$ is a fractal and smooth curve in $\mathbf{R}^2$. It will be established that the following kind of fractal dimension of $G(y)$, denoted by $\dim_M G(y)$ and called by Minkowski–Bouligand dimension (or the box-counting dimension), see [12,18,24], satisfies: $\dim_M G(y) = s \in (1,2)$, where the fractional value s only depend on the asymptotic behaviour of $f(x)$ near $x = 0$. As usual, it is defined by

$$\dim_M G(y) = \limsup_{\varepsilon\to 0}\left(2 - \frac{\log|G_\varepsilon(y)|}{\log\varepsilon}\right),$$

where $|G_\varepsilon(y)|$ denotes the Lebesgue measure of the ε-neighbourhood $G_\varepsilon(y)$ of the graph $G(y)$ which is defined by,

$$G_\varepsilon(y) = \{(t_1,t_2) \in \mathbf{R}^2 : d((t_1,t_2), G(y)) \le \varepsilon\}, \quad \varepsilon > 0,$$

and $d((t_1,t_2), G(y))$ denotes the distance from (t_1,t_2) to $G(y)$.

Furthermore, it is known (see [24, Chapter 9]) that the so-called one dimensional upper Minkowski content of a rectifiable graph $G(y)$, denoted by $M^1(G(y))$, can characterize the arclength of $G(y)$ in the sense that $M^1(G(y)) = \text{length}(G(y)) < \infty$. Moreover, since $y(x)$ is a non-trivial continuous function on I, we have that $0 < M^1(G(y)) < \infty$. Here, the s-dimensional upper Minkowski content of $G(y)$, $s \in [1,2)$, is as usual defined by

$$M^s(G(y)) = \limsup_{\varepsilon\to 0}(2\varepsilon)^{s-2}|G_\varepsilon(y)|, \quad s \in [1,2).$$

Hence, if $\dim_M G(y) = s > 1$, it is worth to know whether $0 < M^s(G(y)) < \infty$? In this direction, we recall the following definition which appears for the first time in [16], see also in [11].

Definition 2 Let $s \in [1,2)$. A graph $G(y)$ is said to be an *s-set* in $\mathbf{R}^2$ if $\dim_M G(y) = s$ and $0 < M^s(G(y)) < \infty$. An oscillatory function $y(x)$ on I is said to be the *s-dimensional fractal oscillatory* on I if its graph $G(y)$ is an s-set in $\mathbf{R}^2$. An oscillatory linear differential equation $y'' + f(x)y = 0$ on I is said to be the *s-dimensional fractal oscillatory* on I if all its nontrivial solutions $y(x)$ are the s-dimensional fractal oscillatory on I.

It is clear that, if $G(y)$ is an s-set in $\mathbf{R}^2$, $s \in [1,2)$, then for the asymptotic behaviour of $|G_\varepsilon(y)|$ we have $c_0\varepsilon^{2-s} \le |G_\varepsilon(y)| \le c_1\varepsilon^{2-s}$, where $c_0 > 0$ and $c_1 > 0$ are independent of $\varepsilon > 0$.

Let us remark that in [16], it is proved that the linear differential equation of Euler type $(P)_\alpha$: $y'' + \lambda x^{-\alpha} y = 0$, $x \in I$, is the 1-dimensional fractal oscillatory on I if $2 < \alpha < 4$ and the s-dimensional fractal oscillatory on I if $\alpha > 4$, where $s = 3/2 - 2/\alpha$. Moreover, in Theorem 1.12 from [11], this result has been enlarged to linear differential equations as in the following theorem.

Theorem B *Let $f \in C^2((0,1])$, $f(x) > 0$ on I and let $f(x)$ satisfy the Hartman–Wintner condition* (36)*. Let $f(x) \sim \lambda x^{-\alpha}$ near $x = 0$, where $\alpha > 2$. Then the linear differential equation $y'' + f(x)y = 0$, $x \in I$, is the 1-dimensional fractal oscillatory on I provided $2 < \alpha < 4$ and the s-dimensional fractal oscillatory on I provided $\alpha > 4$, where $s = 3/2 - 2/\alpha$.*

In a case when $\alpha = 4$ and $f(x) = \lambda x^{-\alpha}$, the graph $G(y)$ is degenerated in the sense of fractal oscillations. More precisely, for all solutions $y(x)$ of the equation $y'' + \lambda x^{-4} y = 0$, $x \in I$, in [16, Theorem 1.5] has been proved that $\dim_M G(y) = 1$ and $M^1(G(y)) = \infty$.

In the following second main result of the paper, Theorem B will be generalized to the case of Eq. (1) for any $p > 1$. Also, the unrectifiable oscillations which is presented in Theorem 3 above is described here from the fractal geometry point of view.

Theorem 6 *Let $f(x)$ satisfy* (2) *and* (8)*, and let $f(x) \sim \lambda x^{-\alpha}$ near $x = 0$, where $\alpha > p$. Then Eq.* (1) *is the 1-dimensional fractal oscillatory on I provided $p < \alpha < p^2$ and the s-dimensional fractal oscillatory on I provided $\alpha > p^2$, where $s = 2 - \frac{1}{q} - \frac{p}{\alpha}$.*

The proof of this theorem is mainly based on a zero-points analysis given in the following lemma.

Lemma 3 *Let $f(x)$ be as in Theorem 6. Let $y(x)$ be a solution of Eq.* (1) *and let $a_k \in I$ and $s_k \in (a_{k+1}, a_k)$ be decreasing sequences of consecutive zeros of $y(x)$ and $y'(x)$ respectively, obtained as in Remark 2. Then there are $k_0 \in \mathbf{N}$, $\varepsilon_0 > 0$, and positive constants c_i, $i = 0,1,2,3,4$, such that for all $k \in \mathbf{N}$, $k > k_0$ and $\varepsilon \in (0, \varepsilon_0)$ there hold true:*

$$c_1 a_{k+1}^{\frac{\alpha}{p}} \le a_k - a_{k+1} \le c_2 a_k^{\frac{\alpha}{p}}, \tag{61}$$

$$c_3\left(\frac{1}{k+k_0}\right)^{\frac{p}{\alpha-p}} \le a_k \le c_4\left(\frac{1}{k-k_0}\right)^{\frac{p}{\alpha-p}}, \tag{62}$$

and

$$|y(s_k)| \ge c_0 s_k^{\frac{\alpha}{pq}} \ge c_0 a_{k+1}^{\frac{\alpha}{pq}}. \tag{63}$$

Furthermore, for any $\varepsilon \in (0, \varepsilon_0)$ there is an $k(\varepsilon) \in \mathbf{N}$ such that $k(\varepsilon) > k_0$ and

$$a_k - a_{k+1} \le \frac{\varepsilon}{2} \quad \textit{for each } k > k(\varepsilon). \tag{64}$$

Proof The proofs of the statements (61) and (62) will be presented in Appendix of the paper. These results in the case for $p = 2$ have been already shown in [16, Lemmas 3.3 and 3.5]. See also Lemmas 4.1 and 4.3 from [11].

Next, let $y(x)$ be a non-trivial solution of Eq. (1). According to (3), Proposition 2 and Remark 2, for all sufficiently large $k \in \mathbf{N}$, we have that $y'(s_k) = 0$, $V(s_k) \ge c$ and $|w(\varphi(s_k))| = 1$, and hence,

$$|y(s_k)| = (p-1)^{\frac{1}{pq}} f^{-\frac{1}{pq}}(s_k) V^{\frac{1}{p}}(s_k) |w(\varphi(s_k))| \ge c f^{-\frac{1}{pq}}(s_k). \tag{65}$$

Now, the inequality (63) follows from (65) and from assumption that $f(x) \sim \lambda x^{-\alpha}$ near $x = 0$.

Next, it is clear that for any $\varepsilon \in (0, \varepsilon_0)$ there is a $k(\varepsilon) \in \mathbf{N}$ such that $k(\varepsilon) > k_0$ and

$$d_0 \varepsilon^{-\frac{\alpha-p}{\alpha}} + k_0 < k(\varepsilon) < 2d_0 \varepsilon^{-\frac{\alpha-p}{\alpha}} - k_0 - 1, \tag{66}$$

where d_0 and ε_0 are defined by

$$d_0 = \left(2c_2 c_4^{\frac{\alpha}{p}}\right)^{\frac{\alpha-p}{\alpha}} \quad \text{and} \quad \varepsilon_0 = \left(\frac{d_0}{2k_0+2}\right)^{\frac{\alpha}{\alpha-p}}. \tag{67}$$

Indeed, $(2d_0\varepsilon^{-\frac{\alpha-p}{\alpha}} - k_0 - 1) - (d_0\varepsilon^{-\frac{\alpha-p}{\alpha}} + k_0) = d_0\varepsilon^{-\frac{\alpha-p}{\alpha}} - 2k_0 - 1 > 1$ for all $\varepsilon \in (0, \varepsilon_0)$. Furthermore, (61), (62), (66), and (67), show that for $k > k(\varepsilon)$,

$$|a_k - a_{k+1}| \le c_2 a_k^{\frac{\alpha}{p}} \le c_2 c_4^{\frac{\alpha}{p}} \left(\frac{1}{k-k_0}\right)^{\frac{\alpha}{p}\frac{p}{\alpha-p}} \le c_2 c_4^{\frac{\alpha}{p}} d_0^{-\frac{\alpha}{\alpha-p}} \varepsilon^{\frac{\alpha-p}{\alpha}\frac{\alpha}{\alpha-p}} = \frac{\varepsilon}{2},$$

which proves the desired inequality (64). Thus, this lemma is shown. □

Proof of Theorem 6 Let $s = 2 - \frac{1}{q} - \frac{p}{\alpha}$. It is clear that $\alpha > p^2$ implies $s > 1$. Let $y(x)$ be a nontrivial solution of Eq. (1) and $a_k \in I$ be a decreasing sequence of its consecutive zeros. In order to prove this theorem, we need to show that

$$M^s(G(y)) > 0 \quad \text{and} \quad \dim_M G(y) \ge s, \tag{68}$$

and

$$M^s(G(y)) < \infty \quad \text{and} \quad \dim_M G(y) \le s. \tag{69}$$

At the first, from (61), (62), (63), and (66), we obtain

$$\begin{aligned} \sum_{k\ge k(\varepsilon)} |y(s_k)|(a_k - a_{k+1}) &\ge c \sum_{k\ge k(\varepsilon)} a_{k+1}^{\frac{\alpha}{pq}} a_{k+1}^{\frac{\alpha}{p}} = c \sum_{k\ge k(\varepsilon)} a_{k+1}^{\frac{\alpha}{p}(\frac{1}{q}+1)} \\ &\ge c \sum_{k\ge k(\varepsilon)} \left(\frac{1}{k+1+k_0}\right)^{\frac{\alpha}{\alpha-p}(\frac{1}{q}+1)} \ge c\left(\frac{1}{k(\varepsilon)+1+k_0}\right)^{\frac{\alpha}{\alpha-p}(\frac{1}{q}+1)-1} \ge c\varepsilon^{\frac{1}{q}+\frac{p}{\alpha}}. \end{aligned} \tag{70}$$

According to [14, Lemma 2.1], see also [16, Lemma 4.1], we know that: for a continuous function $y(x)$ on $\bar{I}$ and decreasing sequence $a_k \in I$ of consecutive zeros of $y(x)$ such that $a_k \searrow 0$, if there is a natural number $k(\varepsilon)$ and an $\varepsilon_0 > 0$ such that $|a_k - a_{k+1}| \le \varepsilon$ for all $\varepsilon \in (0, \varepsilon_0)$ and for all $k \ge k(\varepsilon)$, then

$$|G_\varepsilon(y)| \ge \sum_{k\ge k(\varepsilon)} |y(s_k)|(a_k - a_{k+1}) \quad \text{for all } \varepsilon \in (0, \varepsilon_0). \tag{71}$$

According to Lemma 3 we may apply this fact to solution $y(x)$ and its sequence a_k. Hence, (64), (70), and (71) implies that

$$|G_\varepsilon(y)| \ge c\varepsilon^{\frac{1}{q}+\frac{p}{\alpha}} \quad \text{for all} \varepsilon \in (0, \varepsilon_0), \tag{72}$$

where ε_0 is given in (67). Multiplying (72) by $(2\varepsilon)^{s-2}$, and passing to the limit superior when ε tends to 0, we obtain that

$$M^s(G(y)) = \limsup_{\varepsilon\to 0}(2\varepsilon)^{s-2}|G_\varepsilon(y)| \ge c_0 \limsup_{\varepsilon\to 0} \varepsilon^{s-2+\frac{1}{q}+\frac{p}{\alpha}} = c_0 > 0, \tag{73}$$

where $c_0 = c2^{s-2}$. Also, from (72) we derive that

$$\dim_M G(y) = \limsup_{\varepsilon\to 0}\left(2 - \frac{\log|G_\varepsilon(y)|}{\log\varepsilon}\right),$$

$$\geq \limsup_{\varepsilon\to 0}\left(2 - \frac{\log c}{\log\varepsilon} - \frac{\log\varepsilon^{\frac{1}{q}+\frac{p}{\alpha}}}{\log\varepsilon}\right) = 2 - \frac{1}{q} - \frac{p}{\alpha} = s. \tag{74}$$

Thus, the desired statement (68) immediately follows from (73) and (74).

Next, we are going to prove the statement (69), that is, we need to estimate $G_\varepsilon(y)$ from above. At the first, $G_\varepsilon(y) = G_\varepsilon(y_{I_1})\cup G_\varepsilon(y_{I_2})\cup G_\varepsilon(y_{I_3})$, where $y_{I_i}(x)$ denotes the restriction of $y(x)$ on $I_i \subseteq \bar{I}$ and I_i are defined by $I_1 = [0, a_{k(\varepsilon)}]$, $I_2 = [a_{k(\varepsilon)}, a_{k_0}]$, and $I_3 = [a_{k_0}, 1]$. Hence,

$$|G_\varepsilon(y)| \leq |G_\varepsilon(y_{I_1})| + |G_\varepsilon(y_{I_2})| + \varepsilon L_G(y_{I_3}) \leq |G_\varepsilon(y_{I_1})| + |G_\varepsilon(y_{I_2})| + \varepsilon M_0, \tag{75}$$

where $L_G(y_{I_3})$ denote the arclength of $y(x)$ over the interval I_3, and M_0 is the arclength of $y(x)$ over $[a_{k_0}, 1]$ which is finite. Therefore, in order to estimate $|G_\varepsilon(y)|$, we need to estimate $|G_\varepsilon(y_{I_1})|$ and $|G_\varepsilon(y_{I_2})|$. Here $k(\varepsilon)$ is determined in (66), and k_0 is chosen so that (61) and (62) hold, and also, since $f(x) \sim \lambda x^{-\alpha}$ near $x = 0$, where $\alpha > p$, for a $\lambda_2 > 0$ and for this k_0, we have that $f(x) \leq \lambda_2 x^{-\alpha}$ for all $x \in (0, a_{k_0})$. It together with (3), (5), and Proposition 2 gives that

$$|y(s_k)| \leq c_1 s_k^{\frac{\alpha}{pq}} \leq c_1 a_k^{\frac{\alpha}{pq}} \quad \text{for all } k > k_0. \tag{76}$$

Also, from (62) and (66) follows that for all $\varepsilon \in (0, \varepsilon_0)$,

$$a_{k(\varepsilon)} \leq c_4\left(\frac{1}{k(\varepsilon) - k_0}\right)^{\frac{p}{\alpha-p}} \leq c_5(\varepsilon^{\frac{\alpha-p}{\alpha}})^{\frac{p}{\alpha-p}} = c_5\varepsilon^{\frac{p}{\alpha}}. \tag{77}$$

Now, according to (76) and (77) we derive that $|y(s_{k(\varepsilon)})| \leq c\varepsilon^{\frac{1}{q}}$ and so,

$$|G_\varepsilon(y_{I_1})| \leq 2(a_{k(\varepsilon)} + 2\varepsilon)(|y(s_{k(\varepsilon)})| + \varepsilon) \leq c_6(\varepsilon^{\frac{p}{\alpha}+\frac{1}{q}} + \varepsilon^{1+\frac{p}{\alpha}} + \varepsilon^{1+\frac{1}{q}} + \varepsilon^2).$$

Since $\alpha > p^2$, it is clear that $\frac{p}{\alpha} + \frac{1}{q} < 1$, and hence,

$$|G_\varepsilon(y_{I_1})| \leq c_6\varepsilon^{\frac{p}{\alpha}+\frac{1}{q}}. \tag{78}$$

Furthermore, from (61), (62), and (76), we derive that

$$|G_\varepsilon(y_{I_2})| \leq \sum_{k=k_0+1}^{k(\varepsilon)} \varepsilon\,(2|y(s_k)| + a_k - a_{k+1}) \leq c_7\varepsilon \sum_{k=k_0+1}^{k(\varepsilon)} (a_k^{\frac{\alpha}{pq}} + a_k^{\frac{\alpha}{p}})$$

$$\leq c_7\varepsilon \sum_{k=k_0+1}^{k(\varepsilon)} a_k^{\frac{\alpha}{pq}} \leq c_8\varepsilon \sum_{k=k_0+1}^{k(\varepsilon)} \left(\frac{1}{k-k_0}\right)^{\frac{p}{\alpha-p}\frac{\alpha}{pq}} \leq c_9\varepsilon\left(\frac{1}{k(\varepsilon)+1-k_0}\right)^{\frac{\alpha}{q(\alpha-p)}-1}$$

because $\frac{\alpha}{q(\alpha-p)} < 1$ since $\alpha > p^2$, which together with (66) gives that

$$|G_\varepsilon(y_{I_2})| \leq c_9\varepsilon\,(k(\varepsilon) + 1 - k_0)^{1-\frac{\alpha}{q(\alpha-p)}} \leq c_{10}\varepsilon^{\frac{p}{\alpha}+\frac{1}{q}}. \tag{79}$$

Since $\frac{p}{\alpha} + \frac{1}{q} < 1$, from (75), (78), and (79), we observe that

$$|G_\varepsilon(y)| \le c\varepsilon^{\frac{p}{\alpha}+\frac{1}{q}}. \tag{80}$$

Multiplying (80) by $(2\varepsilon)^{s-2}$, where $s = 2 - \frac{1}{q} - \frac{p}{\alpha}$ and passing to the limit superior when ε tends to 0, we obtain that

$$M^s(G(y)) = \limsup_{\varepsilon\to 0}(2\varepsilon)^{s-2}|G_\varepsilon(y)| \le c_1 \limsup_{\varepsilon\to 0} \varepsilon^{s-2+\frac{1}{q}+\frac{p}{\alpha}} = c_1 < \infty, \tag{81}$$

where $c_1 = c2^{s-2}$. Also, from (80) we derive that

$$\begin{aligned} \dim_M G(y) &= \limsup_{\varepsilon\to 0}\left(2 - \frac{\log|G_\varepsilon(y)|}{\log\varepsilon}\right), \\ &\le \limsup_{\varepsilon\to 0}\left(2 - \frac{\log c}{\log\varepsilon} - \frac{\log\varepsilon^{\frac{1}{q}+\frac{p}{\alpha}}}{\log\varepsilon}\right) = 2 - \frac{1}{q} - \frac{p}{\alpha} = s. \end{aligned} \tag{82}$$

Note that (69) follows from (81) and (82). Now by (68) and (69), the proof of this theorem is complete. □

5 Singular behaviour of L^p norm of $y'(x)$

In this section, we pay attention to L^p nonintegrability of the derivative $y'(x)$ of all solutions $y(x)$ of Eq. (1) on the interval I, where $f(x) \sim \lambda x^{-\alpha}$ near $x = 0$ and $\alpha > p$.

We know that the regularity of the function F in the p-Laplacian equation (E_p): $(\Phi(y'))' = F$ at any interior point x_0 is closely related to L^p integrability of all solutions of (E_p) in an open neighbourhood of x_0; in most general setting, see in [19,20]. It is comparable with our attention here to prove that the order of growth for singularity of $f(x)$ at boundary point $x = 0$ implies the order of growth for L^p nonintegrability of all solutions of Eq. (1) near $x = 0$.

At the first, we show that the L^1 integrability of $y'(x)$ depends on a relation between α and p^2 in this way.

Corollary 2 *Let $f(x)$ satisfy (2) and (8), and let $f(x) \sim \lambda x^{-\alpha}$ near $x = 0$, where $\alpha > p$. Then for all solutions $y(x)$ of Eq. (1) we have:*

(i) *$y' \in L^1(I)$ provided $p < \alpha < p^2$;*
(ii) *$y' \notin L^1(I)$ provided $\alpha \ge p^2$.*

Proof Since the rectifiability of the graph $G(y)$ of a smooth function $y(x)$ on I is equivalent to the integrability of $y'(x)$, see the proof of Theorem 2, the conclusions (i) and (ii) of this corollary easy follow from Theorem 3. □

Because of Corollary 2, it is worth to consider the L^p nonintegrability of the derivative $y'(x)$ on I. We will show that $y'(x)$ does not L^p integrable on I for any case of $\alpha > p$. Also, we determine a lower bound for the singular behaviour of L^p norm of $y'(x)$ on the interval $(\varepsilon, 1)$ when $\varepsilon \to 0$.

Theorem 7 *Let $f(x)$ satisfy (2) and (8), and let $f(x) \sim \lambda x^{-\alpha}$ near $x = 0$, where $\alpha > p$. Then $y' \notin L^p(I)$ for all solutions $y(x)$ of Eq. (1) and*

$$\liminf_{\varepsilon\to 0} \frac{\log||y'||_{L^p(\varepsilon,1)}}{\log 1/\varepsilon} \ge \frac{\alpha}{p^2} - \frac{1}{p} > 0, \tag{83}$$

where $||y'||_{L^p(\varepsilon,1)}$ *denotes the* L^p *norm of* $y'(x)$ *on* $(\varepsilon, 1)$. *Moreover, if* $\alpha \geq p^2$ *then*

$$\liminf_{\varepsilon\to 0} \frac{\log ||y'||_{L^p(\varepsilon,1)}}{\log 1/\varepsilon} \geq \frac{1}{q}. \tag{84}$$

Proof Since $p > 1$ and $y(a_{k+1}) = y(a_k) = 0$ for all $k \in \mathbf{N}$, and $y \in C^1(I)$, we have that $y \in W_0^{1,p}(a_{k+1}, a_k)$ and by using [3, Theorem 9.12] we obtain a constant $c > 0$ depending only on p such that

$$\sup_{(a_{k+1},a_k)} |y(x)| \leq c(a_k - a_{k+1})^{1-\frac{1}{p}} ||y'||_{L^p(a_{k+1},a_k)}, \quad k \in \mathbf{N}. \tag{85}$$

Let $k_0 \in \mathbf{N}$ be from Lemma 3 and let $k(\varepsilon) \in \mathbf{N}$ be from (66). We claim that

$$k(\varepsilon) \geq k_0 + 1 \quad \text{for all } \varepsilon \in (0, \varepsilon_0), \tag{86}$$

where ε_0 is from (67). Also, there is a constant $c_0 > 0$ which does not depend on ε such that

$$a_{k(\varepsilon)+1} \geq c_0 \varepsilon^{\frac{p}{\alpha}} \quad \text{for all } \varepsilon \in (0, \varepsilon_0). \tag{87}$$

Indeed, from (66) and (67), for all $\varepsilon \in (0, \varepsilon_0)$, we derive that

$$k(\varepsilon) \geq k_0 + d_0 \varepsilon^{-\frac{\alpha-p}{\alpha}} \geq k_0 + d_0 \varepsilon_0^{-\frac{\alpha-p}{\alpha}} = k_0 + d_0 \left(\frac{d_0}{2k_0+2}\right)^{-\frac{\alpha-p}{\alpha}\frac{\alpha}{\alpha-p}} \geq k_0 + 1.$$

Also, by (62) and (66), for all $\varepsilon \in (0, \varepsilon_0)$, we obtain that

$$a_{k(\varepsilon)+1} \geq c_3 \left(\frac{1}{k(\varepsilon) + k_0 + 1}\right)^{\frac{p}{\alpha-p}} \geq c_4 \varepsilon^{\frac{\alpha-p}{\alpha}\frac{p}{\alpha-p}} = c_4 \varepsilon^{\frac{p}{\alpha}}.$$

Now, because of (85), (86), and (87), for all $\varepsilon \in (0, \varepsilon_0)$, we can write that

$$\begin{aligned} ||y'||^p_{L^p(c_0\varepsilon^{p/\alpha},1)} &\geq ||y'||^p_{L^p(a_{k(\varepsilon)+1},a_{k_0+1})} = \sum_{k=k_0+1}^{k(\varepsilon)} ||y'||^p_{L^p(a_{k+1},a_k)} \\ &\geq c^{-p} \sum_{k=k_0+1}^{k(\varepsilon)} \frac{\left[\sup_{(a_{k+1},a_k)} |y(x)|\right]^p}{(a_k - a_{k+1})^{p-1}}. \end{aligned} \tag{88}$$

Let us remark that by combining (61), (62) and (63), for all $k > k_0$, we have:

$$\sup_{(a_{k+1},a_k)} |y(x)| = |y(s_k)| \geq c_1 \left(\frac{1}{k+1+k_0}\right)^{\frac{\alpha}{q(\alpha-p)}}, \tag{89}$$

$$a_k - a_{k+1} \leq c_2 \left(\frac{1}{k-k_0}\right)^{\frac{\alpha}{\alpha-p}}. \tag{90}$$

Since $p - 1 = p/q$ and

$$\frac{k-k_0}{k+1+k_0} \geq \frac{1}{2k_0+2} \quad \text{for all } k \geq k_0 + 1,$$

by putting (89) and (90) into (88) and by (66), we obtain

$$||y'||^p_{L^p(c_0\varepsilon^{p/\alpha},1)} \geq c_3 \sum_{k=k_0+1}^{k(\varepsilon)} \left(\frac{k-k_0}{k+1+k_0}\right)^{\frac{\alpha}{\alpha-p}(p-1)}$$
$$\geq c_3 \left(\frac{1}{2k_0+2}\right)^{\frac{\alpha}{\alpha-p}(p-1)} \sum_{k=k_0+1}^{k(\varepsilon)} 1 = c_4(k(\varepsilon)-k_0) \geq c_4 d_0 \varepsilon^{-\frac{\alpha-p}{\alpha}},$$

that is, $||y'||_{L^p(c_0\varepsilon^{p/\alpha},1)} \geq c_5\varepsilon^{-\frac{\alpha-p}{p\alpha}}$ for all $\varepsilon \in (0, \varepsilon_0)$. Hence,

$$||y'||_{L^p(\varepsilon,1)} \geq c_6\varepsilon^{-\frac{\alpha-p}{p\alpha}\frac{\alpha}{p}} = c_6\varepsilon^{-\frac{\alpha-p}{p^2}} \quad \text{for all } \varepsilon \in (0, c_0\varepsilon_0^{\frac{p}{\alpha}}).$$

Tacking the logarithm and the limit inferior in the previous inequality, we get

$$\liminf_{\varepsilon\to 0} \frac{\log ||y'||_{L^p(\varepsilon,1)}}{\log 1/\varepsilon} \geq \frac{\alpha-p}{p^2} \geq \frac{\alpha}{p^2} - \frac{1}{p} > 0,$$

since $\alpha > p$. Thus, the desired inequality (83) is proved. Finally, if $\alpha \geq p^2$ then $\frac{\alpha}{p^2} - \frac{1}{p} \geq 1 - \frac{1}{p} = \frac{1}{q}$ and so, (84) immediately follows from (83). Thus, this theorem is proved. □

Appendix

At the end of this paper, we give the proofs of some technical results which have been used in the previous sections.

Proof of (15) From (16) we derive:

$$\begin{cases} |y(x)|^p = a^p f^{-Ap}(x)V(x)|w(\varphi(x))|^p, & x > 0, \\ |y'(x)|^p = b^q f^{Bq}(x)V(x)|w'(\varphi(x))|^p, & x > 0. \end{cases} \tag{91}$$

It is clear that $(\Phi(u))' = (p-1)|u|^{p-2}u'$.

At the first, in all points $x \in (0, 1)$ where $y(x) \neq 0$, from Eq. (1) follows $(\Phi(y'))'/\Phi(y) = -f(x)$. According to (16) and (17), it implies

$$\begin{aligned}\left(\frac{\Phi(y')}{\Phi(y)}\right)' &= \frac{(\Phi(y'))'}{\Phi(y)} - \frac{\Phi(y')}{(\Phi(y))^2}(p-1)|y|^{p-2}y'(x) \\ &= -f(x) - (p-1)\frac{|y'|^p}{|y|^p} = -f(x) - (p-1)\frac{b^q f^{Bq}(x)V(x)|w'(\varphi(x))|^p}{a^p f^{-Ap}(x)V(x)|w(\varphi(x))|^p} \\ &= -f(x) - f(x)\frac{|w'(\varphi(x))|^p}{|w(\varphi(x))|^p} = -\frac{f(x)}{|w(\varphi(x))|^p},\end{aligned} \tag{92}$$

where in the last equality, the identity (5) is used. In the same way, from Eq. (4) follows $(\Phi(w'))'/\Phi(w) = -(p-1)$ and

$$\begin{aligned}\left(\frac{\Phi(w'(\varphi(x)))}{\Phi(w(\varphi(x)))}\right)' &= \frac{\left(\Phi(w'(\varphi(x)))\right)'}{\Phi(w(\varphi(x)))} - \Phi(w'(\varphi(x)))\left(\frac{1}{\Phi(w(\varphi(x)))}\right)' \\ &= -(p-1)\varphi'(x) - (p-1)\frac{|w'|^p}{|w|^p}\varphi'(x) = -\frac{p-1}{|w(\varphi(x))|^p}\varphi'(x),\end{aligned}$$

where in the last equality, the identity (5) is used.

On the other hand, it is clear that

$$\Phi(y) = a^{p-1} f^{-A(p-1)}(x) V^{\frac{p-1}{p}}(x) \Phi(w(\varphi(x))),$$

and so,

$$\begin{aligned}\left(\frac{\Phi(y')}{\Phi(y)}\right)' &= \left[\frac{-bf^{B}(x)V^{\frac{1}{q}}(x)\Phi(w'(\varphi(x)))}{a^{p-1}f^{-A(p-1)}(x)V^{\frac{p-1}{p}}(x)\Phi(w(\varphi(x)))}\right]' \\ &= \frac{-b}{a^{p-1}}\left[f^{B+A(p-1)}(x)\frac{\Phi(w'(\varphi(x)))}{\Phi(w(\varphi(x)))}\right]' \\ &= \frac{-bf^{B+A(p-1)}(x)}{a^{p-1}}\left[(B+A(p-1))\frac{f'(x)}{f(x)}\frac{\Phi(w'(\varphi(x)))}{\Phi(w(\varphi(x)))} - \frac{p-1}{|w(\varphi(x))|^{p}}\varphi'(x)\right].\end{aligned} \tag{93}$$

From (92) and (93) follows that

$$\begin{aligned}&(B+A(p-1))\frac{f'(x)}{f(x)}\frac{\Phi(w'(\varphi(x)))}{\Phi(w(\varphi(x)))} - \frac{p-1}{|w(\varphi(x))|^{p}}\varphi'(x) \\ &\quad = \frac{a^{p-1}}{b}\frac{f^{1-B-A(p-1)}(x)}{|w(\varphi(x))|^{p}}.\end{aligned} \tag{94}$$

Let us remark that from (17) follows

$$\frac{B+A(p-1)}{p-1} = \frac{1}{p}, \quad \frac{a^{p-1}}{b(p-1)} = (p-1)^{-\frac{1}{p}}, \quad \text{and} \quad 1-B-A(p-1) = 1/p. \tag{95}$$

Hence, from (94) and (95) we derive the desired differential equation (15). □

Proof of (25) *and* (26) From (16) we obtain that:

$$\begin{cases} |w(\varphi(x))|^{p} = a^{-p}f^{Ap}(x)V^{-1}(x)|y(x)|^{p}, & x > 0, \\ |w'(\varphi(x))|^{p} = b^{-q}f^{-Bq}(x)V^{-1}(x)|y'(x)|^{p}, & x > 0. \end{cases} \tag{96}$$

Now, by putting (96) into (5) in particular for $a = (p-1)^{\frac{1}{p^2}}$, $b = 1/a$, and $A = B = 1/(pq)$, the desired equality (25) for the function $V(x)$ immediately follows.

Next, it is clear that the equality (26) is a particular case of the following fact which says that for any real numbers e and E,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\left[(p-1)ef^{E}(x)|y'|^{p} + ef^{E+1}(x)|y|^{p}\right] \\ &\quad = \frac{\mathrm{d}}{\mathrm{d}x}\left[(p-1)ef^{E}(x)\right]|y'|^{p} + \frac{\mathrm{d}}{\mathrm{d}x}\left[ef^{E+1}(x)\right]|y|^{p},\end{aligned}$$

where y is any solution of Eq. (1). Indeed, from Eq. (1) we have that

$$\left(|y'|^{p}\right)' = -\frac{1}{p-1}f(x)\left(|y|^{p}\right)',$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(p-1)ef^{E}(x)|y'|^{p}+ef^{E+1}(x)|y|^{p}\right]$$
$$=\Big((p-1)ef^{E}(x)\Big)'|y'|^{p}+(p-1)ef^{E}(x)\left(|y'|^{p}\right)'+\Big(ef^{E+1}(x)\Big)'|y|^{p}$$
$$+ef^{E+1}(x)\left(|y|^{p}\right)'$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}\Big[(p-1)ef^{E}(x)\Big]|y'|^{p}+\frac{\mathrm{d}}{\mathrm{d}x}\Big[ef^{E+1}(x)\Big]|y|^{p}.$$

□

Proof of Proposition 3 Since $\psi^{A-1}\psi''\in L^{1}(I)$, there is a constant $K>0$ such that

$$K=\int_{0}^{1}\psi^{A-1}(x)|\psi''(x)|\mathrm{d}x<\infty.$$

Hence,

$$\psi^{A-1}(1)\psi'(1)-\psi^{A-1}(s)\psi'(s)-(A-1)\int_{s}^{1}\psi^{A-2}(x)(\psi'(x))^{2}\mathrm{d}x$$
$$=\int_{s}^{1}\psi^{A-1}(x)\psi''(x)\mathrm{d}x\geq -K. \tag{97}$$

On the first hand, from (97) follows,

$$\psi^{A-1}(s)\psi'(s)\leq K+\psi^{A-1}(1)|\psi'(1)|-(A-1)\int_{s}^{1}\psi^{A-2}(x)(\psi'(x))^{2}\mathrm{d}x$$
$$\leq K+\psi^{A-1}(1)|\psi'(1)|,$$

which gives $(\psi^{A}(x))'\leq c$ for all $x\in I$. Since $\psi(0)=0$, we observe that $\psi^{A}(x)\leq cx$, $x\in I$, which shows that $1/\psi^{A}(x)\geq c/x$ and hence, $1/\psi^{A}\notin L^{1}(I)$. Thus, the conclusion (i) of this proposition is proved.

On the other hand, from (97) we obtain that,

$$(A-1)\int_{s}^{1}\psi^{A-2}(x)(\psi'(x))^{2}\mathrm{d}x\leq K+\psi^{A-1}(1)|\psi'(1)|-\psi^{A-1}(s)\psi'(s).$$

Since $\psi(0)=0$ and $\psi(x)>0$ for all $x\in I$, from the mean value theorem we get a sequence $s_n\in I$ such that $s_n\to 0$ and $\psi'(s_n)>0$. Putting for $s=s_n$ in the previous inequality and passing to the limit, we obtain that

$$\int_{0}^{1}\psi^{A-2}(x)(\psi'(x))^{2}\mathrm{d}x=\lim_{s_n\to 0}\int_{s_n}^{1}\psi^{A-2}(x)(\psi'(x))^{2}\mathrm{d}x\leq\frac{K+\psi^{A-1}(1)|\psi'(1)|}{A-1},$$

which proves that $\psi^{A-2}(\psi')^2 \in L^1(I)$ which together with (97) implies that there exists constant c such that $\lim_{s\to 0}\psi^{A-1}(s)\psi'(s) = c$. If $c \neq 0$, then

$$\int_0^1 \frac{1}{\psi^A(x)}\mathrm{d}x = \int_0^1 \frac{\psi^{A-2}(x)(\psi'(x))^2}{(\psi^{A-1}(x)\psi'(x))^2}\mathrm{d}x < \infty,$$

which is not possible since $1/\psi^A \notin L^1(I)$. Hence, $c = 0$ and the conclusion (ii) is proved.

Finally, the conclusion (iii) easily follows from the facts that $\psi^{A-2}(\psi')^2 \in L^1(I)$, $\psi^{A-1}\psi'' \in L^1(I)$, and $(\psi^{A-1}(x)\psi'(x))' = (A-1)\psi^{A-2}(x)(\psi'(x))^2 + \psi^{A-1}(x)\psi''(x)$. Hence, all conclusions of this proposition are proved. □

Proof of (61) We firstly show the left inequality in (61). In this direction, according to the assumption that $f(x) \sim \lambda x^{-\alpha}$ near $x = 0$, where $\alpha > p$, there is a constant $\lambda > 0$ such that $\lim_{x\to 0}\frac{f(x)}{x^\alpha} = \lambda$. It gives two constants λ_1 and λ_2 such that $0 < \lambda_1 < \lambda < \lambda_2$ and for which there is a $\delta > 0$ such that

$$\frac{\lambda_1}{x^\alpha} \leq f(x) \leq \frac{\lambda_2}{x^\alpha} \quad \text{for all } x \in (0, \delta). \tag{98}$$

Next, let $y(x)$ be a non-trivial solution of Eq. (1), and let $a_k \in I$ (resp., $s_k \in I$) be a decreasing sequence of consecutive zeros of $y(x)$ (resp., of $y'(x)$), see Remark 2. Since $a_k \searrow 0$ as $k \to \infty$, there is a $k_0 \in \mathbf{N}$ such that $a_k \in (0, \delta)$ for all $k > k_0$. Also, since Eq. (1) can be rewritten in the form

$$(p-1)|y'(x)|^{p-2}y''(x) = -f(x)|y(x)|^{p-2}y(x), \tag{99}$$

and since $f(x) > 0$, it is clear that the inequality $y(x) > 0$ on (a_{k+1}, a_k) (resp., $y(x) < 0$) implies that $y(x)$ is a concave (resp. a convex) function on (a_{k+1}, a_k) and in this case $y'(x) > 0$ (resp., $y'(x) < 0$) on (a_{k+1}, s_k). For this moment, let

$$y(x) > 0 \quad \text{and} \quad y'(x) > 0 \text{ on } (a_{k+1}, s_k). \tag{100}$$

The opposite case of (100): $y(x) < 0$ and $y'(x) < 0$ on (a_{k+1}, s_k), can be analogously considered. Now, by integrating Eq. (1) over (x, s_k) where $x \in (a_{k+1}, s_k)$ and $k > k_0$, we obtain

$$y'(x) = \left[\int_x^{s_k} f(t)y^{p-1}(t)\mathrm{d}t\right]^{\frac{1}{p-1}}.$$

Integrating previous equality over the interval (a_{k+1}, s_k), we obtain

$$y(s_k) = \int_{a_{k+1}}^{s_k}\left[\int_x^{s_k} f(t)y^{p-1}(t)\mathrm{d}t\right]^{\frac{1}{p-1}}\mathrm{d}x. \tag{101}$$

Now, from (98), (100), and (101) easily follows

$$y(s_k) \leq \left(\frac{\lambda_2}{a_{k+1}^\alpha}\right)^{\frac{1}{p-1}} y(s_k)(a_k - a_{k+1})^{\frac{p}{p-1}}, \quad k > k_0.$$

Hence, we observe that

$$\lambda_2^{-\frac{1}{p}} a_{k+1}^{\frac{\alpha}{p}} \leq a_k - a_{k+1} \quad \text{for all } k > k_0,$$

which proves the left inequality in (61).

To prove the right inequality of (61), we compare Eq. (1) with

$$(\Phi(u'))' + \frac{\lambda_1}{a_k^\alpha}\Phi(u) = 0, \quad x \in I. \tag{102}$$

Apply Sturm Comparison Theorem to Eqs. (1) and (102) with $f(x)$ satisfying (98), we know that between two consecutive zeros of $u(x)$, there must lie one zero of $y(x)$, see [5, p. 177, Theorem 2.4]. Let $b_n \in I$ be the sequence of consecutive zeros of $u(x)$. So $a_k \in (b_n, b_{n-1})$ where $y(a_k) = u(b_n) = u(b_{n-1}) = 0$, $0 < b_n < a_k < b_{n-1} < 1$ and n depends on k. The next zero a_{k+1} of $y(x)$ must lie between (b_{n+1}, b_{n-1}). This shows that

$$a_k - a_{k+1} < b_{n-1} - b_{n+1} \le 2(b_n - b_{n+1}), \tag{103}$$

because solutions of (102) are periodic and of equal length between zeros.

To estimate $b_{n-1} - b_{n+1}$, we make a scale change through a transformation of variables: $v(\xi) = u(x)$, $\xi = \mu_k x$ of Eq. (102):

$$(\Phi(\dot{v}))^{\cdot} + (p-1)\Phi(v) = 0, \tag{104}$$

where $\mu_k = [\lambda_1/(p-1)]^{\frac{1}{p}} a_k^{-\alpha/p}$ and "dot" denotes differentiation with respect to ξ. Since $a_k \to 0$ as $k \to \infty$, so $\xi = \mu_k x \to \infty$ for every fixed $x \in I$ as $k \to \infty$ and zeros of $u(x)$ correspond to that of $v(\xi)$. Now consider (104) on the semi-infinite interval and note that the distance between any two consecutive zeros of $v(\xi)$ is constant $d_p = \frac{2\pi}{p}(\sin\frac{\pi}{p})^{-1}$. Thus the length of the interval $[b_n, b_{n-1}]$ is given by d_p/μ_k. Hence it follows from (103) that

$$a_k - a_{k+1} \le 2d_p\left(\frac{p-1}{\lambda_1}\right)^{\frac{1}{p}} a_k^{\alpha/p} \quad \text{for all } k \ge k_0.$$

This proves the right inequality of (61). □

Proof of (62) Let $y(x)$ be a non-trivial solution of Eq. (1). Let $\varphi(x)$ be from (3) and let T_k be from (6). As in Remark 2, let $a_k = \varphi^{-1}(T_k) \in I$ such that $y(a_k) = 0$ (the inverse function $\varphi^{-1}(x)$ of function $\varphi(x)$ exists because of (9)). Now by means of Proposition 1, there exists $x_0 \in (0, 1]$ such that

$$\frac{c_0}{2}\int_x^{x_0} f^{\frac{1}{p}}(s)\mathrm{d}s + \varphi(x_0) \le \varphi(x) \le 2c_0\int_x^{x_0} f^{\frac{1}{p}}(s)\mathrm{d}s + \varphi(x_0), \tag{105}$$

where $c_0 = (p-1)^{-1/p}$ and $\varphi(x_0)$ depends on $y(x_0)$ and $y'(x_0)$ by (3). Note that $\varphi(a_k) = T_k = d_p k$, where $d_p = \frac{2\pi}{p}(\sin\frac{\pi}{p})^{-1}$ by (6), so (105) implies

$$\frac{c_0}{2}\int_{a_k}^{x_0} f^{\frac{1}{p}}(s)\mathrm{d}s + \varphi(x_0) \le \varphi(a_k) = d_p k \le 2c_0\int_{a_k}^{x_0} f^{\frac{1}{p}}(s)\mathrm{d}s + \varphi(x_0). \tag{106}$$

Using (98), we obtain from (106),

$$\frac{p\lambda_1 c_0}{2(\alpha-p)}\left[a_k^{\frac{p-\alpha}{p}} - 1\right] \le d_p k - \varphi(x_0) \le \frac{2p\lambda_2 c_0}{\alpha-p}\left[a_k^{\frac{p-\alpha}{p}} - 1\right]. \tag{107}$$

Choose a natural number $k_0 > \max\left[\varphi(x_0), \frac{p\lambda_1 c_0}{2(\alpha-p)}\right]$, we then deduce from (107) the desired inequalities for a_k in (62). □

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