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Uniform Schauder estimates for regularized hypoelliptic equations

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Abstract In this paper we are concerned with a family of elliptic operators represented as sum of square vector fields: $L_\epsilon = \sum_{i=1}^m X_i^2 + \epsilon\Delta$ in $\mathbb{R}^n$, where Δ is the Laplace operator, $m < n$, and the limit operator $L = \sum_{i=1}^m X_i^2$ is hypoelliptic. Here we establish Schauder's estimates, uniform with respect to the parameter ϵ, of solution of the approximated equation $L_\epsilon u = f$, using a modification of the lifting technique of Rothschild and Stein. These estimates can be used in particular while studying regularity of viscosity solutions of nonlinear equations represented in terms of vector fields.

Keywords Hypoelliptic operators · Carnot groups · Fundamental solution · Schauder estimates

Mathematics Subject Classification (2000) 35H10 · 35A08 · 43A80 · 35B45

1 Introduction

Let $X_1, \ldots, X_m$ be smooth real vector fields on an open set $\Omega \subset \mathbb{R}^n$ satisfying the Hörmander condition of hypoellipticity

$$rank\ Lie(X_1, \ldots, X_m)(x) = n, \quad \forall x \in \Omega. \tag{1}$$

It is well known that the operator

$$L = \sum_{i=1}^m X_i^2 \tag{2}$$

is hypoelliptic, see [15]. However, in many applications it is necessary to study elliptic regularization of this type of operators: For every fixed point x_0 there exist ν and vector fields

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$X_{m+1}, \ldots, X_\nu$ (for example the complete list of commutators up to a fixed step s) such that

$$X_1, \ldots, X_m, X_{m+1}, \ldots, X_\nu \tag{3}$$

span the tangent space at x for every $x \in \Omega$. Then the operator

$$L_\epsilon = \sum_{i=1}^{m} X_i^2 + \epsilon^2 \sum_{i=m+1}^{\nu} X_i^2 \tag{4}$$

is uniformly elliptic in Ω. This approximation can be used, for example, to study interior regularity of viscosity solutions of nonlinear problems, when the vector fields $(X_i)_{i=1,\ldots,m}$ depend on the solution: $X_i = X_i(u, \nabla u)$. We refer to [7,9] for nonlinear differential equation of this type, arising in complex analysis or mathematics finance.

To illustrate our results in a simple case we consider in $\mathbb{R}^3$:

$$Lu = \partial_1^2 u + (\partial_2 + u\partial_3)^2 u = f, \quad u = u_0 \text{ on } \partial\Omega. \tag{5}$$

This problem cannot be studied directly but, under very general assumptions on the open set Ω and the boundary datum u_0, the approximating problem:

$$L_\epsilon u = \partial_1^2 u + (\partial_2 + u\partial_3)^2 u + \epsilon^2 \partial_3^2 u = f, \quad u = u_0 \text{ on } \partial\Omega, \tag{6}$$

has a smooth solution u_ϵ. In order to prove the existence of a classical solution of (5) it is natural to establish interior estimates uniform in ϵ for u_ϵ and then let ϵ goes to 0. Regularized technique is a classical approach, indeed it allows us to work with smooth solutions of an elliptic problem $L_\epsilon u = f$ in order to obtain similar results for the limit equation.

Local Schauder estimates for solutions of $L_\epsilon u = f$ hold with a suitable constant C_ϵ and the dependence of C_ϵ on the variable $\epsilon > 0$ is completely unknown, to the best of our knowledge. However it is known that the control distance d_ϵ associated to L_ϵ tends to the control distance d of L as ϵ tends to 0. This has been proved in [17]. We also refer to [1,14], where the relation between the metric d_ϵ and d is investigated.

We will prove the following a priori Schauder estimates for the approximated equation $L_\epsilon u = f$ that are uniform with respect to the regularization parameter ϵ:

Theorem 1.1 *Let $\alpha \in]0, 1[$. Assume that $u \in C^{2,\alpha}_{d_\epsilon}(\Omega)$ such that $L_\epsilon u \in C^{\alpha}_{d_\epsilon}(\Omega)$. Let $0 < t < s$ be such that the ball $B_{d_\epsilon}(s)$ of radius s, with respect to the distance d_ϵ, is contained in Ω. Then there exists a constant C, independent of ϵ, such that*

$$||u||_{C^{2,\alpha}_{d_\epsilon}(B_{d_\epsilon}(t))} \le C\,(||L_\epsilon u||_{C^{\alpha}_{d_\epsilon}(B_{d_\epsilon}(s))} + ||u||_{L^\infty(B_{d_\epsilon}(s))}). \tag{7}$$

We refer to Sect. 2 for a precise definition of Hölder spaces and distance d_ϵ.

In [2] the authors proved local Schauder estimates for operator of kind $L = \sum_{i,j=1}^m a_{ij} X_i X_j$ and for their parabolic analogue, where $(a_{ij}(x))_{ij}$ is a symmetric uniformly positive definite matrix.

Let us briefly remember some other results in literature. In [26] Schauder estimates are proved imposing an additional structure condition on the vector fields. Pointwise Schauder estimates are given in [6]. Local Schauder estimates are established in [21] for a particular class of nonsmooth vector fields. We also refer to [20] where Schauder estimates are proved for Kolmogorov–Fokker–Planck operator. See also [18] for Hölder estimates. These works contain more comprehensive lists of references.

In this paper we essentially follow the same approach introduced in [2] but we also study the dependence on the variable ϵ. The main idea of the proof is to make use of a new lifting method introduced in [8] which is a variant of the lifting method of Rothschild and Stein and a Campanato-type generalization of Hölder continues functions introduced in [4,5,12,19] and recently in [2], where an integral equivalent formulation of Hölder continuous function, proved in [5] in Euclidean setting, has been extended in this setting. The lifting method has been first introduced by Rothschild and Stein [23], and subsequently improved by [11,13,16]. In [8], adding suitable variables and vector fields, the operator L_ϵ is lifted to a new operator $\tilde{L}_\epsilon$ (which is sum of squares of a family of stratified and nilpotent vector fields), to obtain a lifting independent of the variable ϵ. Besides, by choosing the added vector in an accurate way, the metric d_ϵ induced by the old vectors is the projection on the initial space of the lifted metric $\tilde{d}_\epsilon$, according to estimate (22). Campanato's definition and the generalization of a Sánchez-Calle result, (Lemma 3.1), allows us to prove that a function u is Hölder continuous with respect to d_ϵ if and only if its lifting $\tilde{u}$ is a Hölder continuous function with respect to the lifted distance $\tilde{d}_\epsilon$ and

$$||\tilde{u}||_{C^\alpha_{\tilde{d}_\epsilon}(\tilde{\Omega})} \leq ||u||_{C^\alpha_{d_\epsilon}(\Omega)} \leq C\,||\tilde{u}||_{C^\alpha_{\tilde{d}_\epsilon}(\tilde{\Omega})},$$

with a suitable positive constant C independent of ϵ.

Additional basic roles are played by the representation formula in Theorem 4.2 and by the Hölder continuity of integral operators, on metric-measure spaces which satisfy the doubling property, recalled in Theorem 4.3.

The plain of the paper is the following. In Sect. 2 we introduce some notations, in Sect. 3 we recall the lifting procedure and in Sect. 4 we prove a representation formula and we give the proof of Theorem 1.1.

2 Preliminaries and notations

In this section we recall the properties of a Hörmander type operator already proved by [10,23,24]. Indeed, we lift the operator in (4) to a new operator of this type. Consider now an arbitrary Hörmander type operator

$$L = \sum_{i=1}^{\nu} X_i^2 \quad \text{in} \quad \Omega \subset \mathbb{R}^n \tag{8}$$

where $X_1, \ldots, X_\nu$ satisfy the rank condition (1) at every point. We say that a commutator has degree s, and denote $\deg(X) = s$, if $X = [X_{i_1}, [X_{i_2}, \ldots [X_{i_{s-1}}, X_{i_s}]] \ldots]$ with $i_1, \ldots, i_s \in \{1, \ldots, \nu\}$. For a fixed point x_0 there is a number s such that the set of all commutators of degree smaller than s span the whole tangent space at every point in a neighborhood of x_0. Then we complete $X_1, \ldots, X_\nu$ with the collection

$$X_{\nu+1}, \ldots, X_N \tag{9}$$

of all the commutators of degree less or equal to s.

Different equivalent definitions of the control distance have been provided in [22].

If $X_1, \ldots, X_\nu$ are free up to step s, (see [25]) then $n = \nu$ and the distance is defined in terms of the exponential map. Indeed, for every fixed point x_0 in $\mathbb{R}^n$ there exist a neighborhood V of x_0 and for every $x \in V$ a neighborhood U_x of x in the Lie algebra, such that for every $x \in V$ the exponential mapping

$$u \longmapsto y = \exp\left(\sum_{i=1}^{n} u_i X_i\right)(x) \tag{10}$$

is defined in U_x. The definition of distance d simply reduces to

$$d(x, y) = \sum_{i=1}^{n} |u_i|^{\frac{1}{\deg(X_i)}}, \quad x, y \in V. \tag{11}$$

Suitable restricting V and choosing $W \subset\subset V$ we can assume that for every $x \in W$ the map in (10) is defined on the same $U \subset U_x$ and it is a diffeomorphism from U onto the image. Its inverse map denoted Θ_x satisfies $U \subseteq \Theta_x(V)$ for every $x \in W$. Finally

$$\Theta : W \times W \to \mathbb{R}^n, \quad \Theta(x, y) = \Theta_x(y) \quad \text{on } W \times W \tag{12}$$

is smooth. For a fixed x, the function Θ_x introduces a change of variable called canonical.

If g is a nilpotent graded Lie algebra, i.e. g is a direct sum decomposition $g = \oplus_{j=1}^{p} V_j$ such that $[V_1, V_{j-1}] = V_j$ for $2 \le j \le p$ and $[V_1, V_p] = \{0\}$, then g has a natural family of automorphisms, called dilations, defined on each V_j as $\delta_\lambda(X) = \lambda^j X$, $X \in V_j$ and $\lambda > 0$. By the exponential map, the family $\{\delta_\lambda\}_{\lambda>0}$ can be lifted to a family of automorphisms of the simply connected Lie group G corresponding to g. The group G, equipped with these dilations, is called a homogeneous group and the natural number $\sum_{j=1}^{p} j(dim V_j)$ is called the homogeneous dimension of G, see [23].

If the vector fields $X_1, \ldots, X_\nu$ defined on $\mathbb{R}^n$ are free up to step s, then, in general, the Lie algebra generated by the vector fields is not a nilpotent graded Lie algebra and in particular there is not an underlying structure of homogeneous group G on $\mathbb{R}^n$ such that the vector fields are left translation invariant on G with respect to composition law and such that they are homogeneous with respect to the dilations. But, due to the fact that the vector fields $X_1, \ldots, X_\nu$ are free up to step s and together with their commutators of order s span the tangent space at every point, the following number is constant and it has the same rule of the homogeneous dimension in the homogeneous setting

$$Q = \sum_{i=1}^{n} \deg(X_i). \tag{13}$$

In particular, for any compact set $K \subset \mathbb{R}^n$ there exists $R > 0$ such that for any $x \in K$ and $0 < r < R$ the Lebesgue measure of the ball is

$$C_0 r^Q \le |B_d(x, r)| \le C_1 r^Q, \tag{14}$$

with suitable positive constants C_0, C_1 depending only on K.

Let us go back to the properties of a general operator L. Note that, if the family of Lie algebra generated by $(X_i)_{i=1,\ldots,\nu}$ is not free, it is always possible to lift it to a Lie algebra free up to a step s, via the lifting procedure introduced by Rothschild and Stein. Precisely

Theorem 2.1 (Theorems 4 and 5 in [23]) *Let $X_1, \ldots, X_\nu$ be C^∞ vector fields, which, together with their commutators up to step s, span the tangent space at a point x_0. Then we can find new variables $\hat{x}$ and vector fields defined in a neighborhood of x_0*

$$\tilde{X}_i(x, \hat{x}) = X_i(x) + Z_i, \quad Z_i = \sum_{j=1}^{l} a_i^j(x, \hat{x}) \frac{\partial}{\partial \hat{x}_j} \quad i = 1, \ldots, \nu$$

such that the system $(\tilde{X}_i)_{i=1,\ldots,\nu}$ is free up to order s and span $R^{\tilde{N}}$, where $\tilde{N} = n + l$. In the coordinate given by Θ in (12)

$$\tilde{X}_i(f(\Theta_\xi(\cdot)))(\eta) = (Y_i f + R_i^\xi f)(\Theta_\xi(\eta)) \quad \text{on } U \quad i = 1, \ldots, \nu$$

where Y_i are homogeneous left invariant vector fields and R_i^ξ are differential operator of local degree ≤ 0 (*according to definition p. 272 in* [23]), *depending smoothly on* ξ.

In order to simplify notations we give the following definitions

Definition 2.1 We say that K is a regular kernel of type $\lambda > 0$ with respect to the vectors $X_1, \ldots, X_\nu$, the distance d, in an open set W, and we denote $K \in F_\lambda(X, d, W)$, if there exists a positive constant C such that, one has

$$|K(x, y)| \leq C \frac{d^\lambda(x, y)}{|B_d(x, d(x, y))|}, \quad \text{for every } x, y \in W \text{ with } x \neq y \tag{15}$$

and

$$|K(x, y) - K(z, y)| \leq C \frac{d(z, x)}{d(z, y)^{Q-\lambda+1}} \tag{16}$$

for every $x, y, z \in W$ with $d(z, y) \geq Md(z, x)$, $M \geq 1$.

If $\lambda = 0$ we also require that there exists positive constant C such that for any $0 < a < b$

$$\int\limits_{a \leq d(x,y) \leq b} K(x, y) dy \leq C\,(b - a) \tag{17}$$

and (see condition (2.9) in [2])

$$\lim_{\delta \to 0} \left| \int\limits_{d(x,y) \geq \delta} K(x, y) dy - \int\limits_{d(x_1,y) \geq \delta} K(x_1, y) dy \right| \leq C\, d(x, x_1)^\beta \tag{18}$$

for every $\beta \in]0, 1[$.

Definition 2.2 We say that T is a operator of type $\lambda > 0$ if $K(x, y)$ is a kernel of type λ and

$$Tf(x) = \int K(x, y)\, f(y)\, dy.$$

T is a operator of type 0 if $K(x, y)$ is a kernel of type 0 and there exists a smooth function a such that

$$Tf(x) = p.v. \int K(x, y)\, f(y)\, dy + a(x) f(x)$$

where the integral is a principal value integral.

Let us introduce the function spaces used to establish Schauder estimates:

Definition 2.3 (Hölder space) Let $\alpha \in]0, 1[$ and let d be a distance associated to $X_1, \ldots, X_\nu$. We denote by $C_d^\alpha(\Omega)$ the Hölder space induced by d. Let $k \in \mathbb{N}$, we say that $u \in C_d^{k,\alpha}(\Omega)$ if

$$X_{i_1} \cdots X_{i_h} u \in C_d^\alpha(\Omega)$$

for $i_j \in \{1, \ldots, \nu\}$, $\sum_{j=1}^h i_j \leq k$. We define

$$||u||_{C_d^\alpha(\Omega)} = \sup \left\{ \frac{|u(x) - u(y)|}{d(x, y)^\alpha} \; : \; x, y \in \Omega, x \neq y \right\} + ||u||_{L^\infty(\Omega)},$$

and

$$||u||_{C_d^{k,\alpha}(\Omega)} = \sum_{\sum_{j=1}^h i_j \leq k} ||X_{i_1} \cdots X_{i_h} u||_{C_d^\alpha(\Omega)}.$$

3 The lifting procedure

We first note that the notion of degree of each vector field is not preserved by the approximating operator (4). Indeed at every point the vector fields $(X_i)_{i=m+1,\dots,\nu}$, are the complete list of the commutators, so that it would be natural to associate to them a degree grater that 1, but multiplying them by ϵ, we get the vector fields defining L_ϵ, which could be considered of degree 1. In order to overcome this difficulty in [8] the authors introduced a new lifting method: The vector fields $(\epsilon X_i)_{i=m+1,\dots,\nu}$ are lifted to new vector fields linearly independent of the commutators. In order to do so, we define $\nu - m$ new vector fields free and nilpotent of step s, in term of completely new variables. These vector fields will be denoted

$$\tilde{X}_{m+1}, \dots, \tilde{X}_\nu \quad \text{in} \quad \mathbb{R}^{\tilde{N}} = \mathbb{R}^n \times \mathbb{R}^\mu. \tag{19}$$

In this way the new vector fields have the same step of the starting ones and

$$X_1, \dots, X_m, \dots, \tilde{X}_{m+1} + \epsilon X_{m+1}, \dots, \tilde{X}_\nu + \epsilon X_\nu$$

are linearly independent from their commutators. We call this list of vectors $(\tilde{X}_{\epsilon,i})_{i=1,\dots,\nu}$ and define a lifted regularized operator as

$$\tilde{L}_\epsilon = \sum_{i=1}^{\nu} \tilde{X}_{\epsilon,i}^2 \quad \text{in } \mathbb{R}^{\tilde{N}}.$$

We will need a third operator $\tilde{L}$, with the same structure of $\tilde{L}_\epsilon$, but independent of ϵ. We simply eliminate the dependence on ϵ in the vector defining $\tilde{L}_\epsilon$ choosing the new family as

$$\tilde{X}_1 = X_1, \dots, \tilde{X}_m = X_m, \dots, \tilde{X}_{m+1}, \dots, \tilde{X}_\nu,$$

and defining

$$\tilde{L} = \sum_{i=1}^{\nu} \tilde{X}_i^2 \ \text{ in } \mathbb{R}^{\tilde{N}}.$$

See [8] for more details.

The family $(\tilde{X}_i)_{i=1,\dots,\nu}$ will be completed to a basis of the space

$$\tilde{X}_1, \dots, \tilde{X}_n, \tilde{X}_{n+1}, \dots, \tilde{X}_{\tilde{N}} \tag{20}$$

with the list of all commutators ordered as the list $(\tilde{X}_{\epsilon,i})_{i=n+1,\dots,\tilde{N}}$.

It is possible to define a Lie algebra isomorphism ψ_ϵ between the vector fields defining $\tilde{L}_\epsilon$ and $\tilde{L}$ in the follow way:

$$\psi_\epsilon(\tilde{X}_i) = \tilde{X}_{\epsilon,i}, \quad i = 1, \dots, \nu$$

and ψ_ϵ can be extended on the whole algebra via the bracket. Since ψ_ϵ is linear and lower diagonal, with 1 on the diagonal, it has Jacobian determinant 1. The function ψ_ϵ induces, via the exponential map, a change of variables on $\mathbb{R}^{\tilde{N}}$

$$\Phi_\epsilon : \mathbb{R}^{\tilde{N}} \to \mathbb{R}^{\tilde{N}}, \quad \Phi_\epsilon = Exp \circ \psi_\epsilon \circ \Theta_0. \tag{21}$$

Since the function Exp and Θ_0 are local diffeomorphisms, with determinant 1 in 0, and independent of ϵ, the Jacobian determinant of Φ_ϵ is locally bounded by constants independent of ϵ, see [8].

In what follows, the generic point of the lifted space $\mathbb{R}^{\tilde{N}}$ is denoted by $\tilde{x} = (x, \hat{x})$, where $x \in \mathbb{R}^n$ is the initial variable and $\hat{x} \in \mathbb{R}^\mu$ the added ones.

If $\tilde{d}$ is the distance associated to $(\tilde{X}_i)_{i=1,\dots,\tilde{N}}$ and $\tilde{d}_\epsilon$ the distance associated to $(\tilde{X}_{\epsilon,i})_{i=1,\dots,\tilde{N}}$ then the following relation holds:

$$\tilde{d}_\epsilon(\tilde{x}, \tilde{y}) = \tilde{d}(\Phi_\epsilon(\tilde{x}), \Phi_\epsilon(\tilde{y})).$$

In particular, this provides a estimate of the measure of the balls of the metric $\tilde{d}_\epsilon$: there exist positive constants C_0 and C_1 independent of ϵ such that

$$C_0 r^{\tilde{Q}} \leq |B_{\tilde{d}_\epsilon}(\tilde{x}, r)| \leq C_1 r^{\tilde{Q}},$$

where $\tilde{Q}$ is defined in (13) with respect to the basis $\tilde{X}_1, \dots, \tilde{X}_{\tilde{N}}$.

Let us study the relation between the metric induced by the family of vector fields $(X_{\epsilon,i})_{i=1,\dots,\nu}$, and their lifted counterpart.

The following crucial lemma is a generalization of the projection result of Sánchez-Calle in [24]:

Lemma 3.1 *For every compact set $K \subset \mathbb{R}^n$ there exist positive constants C_1, C_2, independent of ϵ, such that if $\chi_{B_{\tilde{d}_\epsilon}((x,0),r)}$ is the characteristic function of the ball $B_{\tilde{d}_\epsilon}((x,0),r)$, then for every $x \in K$ and $r > 0$,*

$$C_1 \frac{r^{\tilde{Q}}}{|B_{d_\epsilon}(x,r)|} \leq \int \chi_{B_{\tilde{d}_\epsilon}((x,0),r)}(y, \hat{y}) d\hat{y} \leq C_2 \frac{r^{\tilde{Q}}}{|B_{d_\epsilon}(x,r)|}. \tag{22}$$

The second inequality in (22) has been proved Lemma 4.3 in [8], the proof of the first one is very similar and we omit it.

The local inclusions between the balls with respect the distances d and d_ϵ proved Lemma 4.4 in [8], ensure that, in our setting, local estimates uniform in ϵ with respect to d_ϵ are local estimates uniform in ϵ with respect to the distance d.

Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. Consider its lifting $\tilde{u}$ defined in the follow way: $\tilde{u} : \tilde{\Omega} \subset \mathbb{R}^{\tilde{N}} \to \mathbb{R}$ where $\tilde{\Omega} = \Omega \times I$, $I \subset\subset \mathbb{R}^\mu$ and $\tilde{u}(x, \hat{x}) = u(x)$.

Lemma 3.2 *Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and let $\tilde{u} : \tilde{\Omega} \subset \mathbb{R}^{\tilde{N}} \to \mathbb{R}$ be its lifting. Then $u \in C^\alpha_{d_\epsilon}(\Omega)$ if and only if $\tilde{u} \in C^\alpha_{\tilde{d}_\epsilon}(\tilde{\Omega})$. Besides, there exists a positive constant C, independent of ϵ, such that*

$$||\tilde{u}||_{C^\alpha_{\tilde{d}_\epsilon}(B_{\tilde{d}_\epsilon}(r))} \leq ||u||_{C^\alpha_{d_\epsilon}(B_{d_\epsilon}(r))} \leq C||\tilde{u}||_{C^\alpha_{\tilde{d}_\epsilon}(B_{\tilde{d}_\epsilon}(r))}, \tag{23}$$

for r small enough such that $B_{\tilde{d}_\epsilon}(r) \subset \tilde{\Omega}$ and $B_{d_\epsilon}(r) \subset \Omega$.

By Lemma 3.1 and reasoning as in the proof of Proposition 8.3 in [2] (using the Campanato definition of Hölder continuous functions), we obtain the assertion of Lemma 3.2. The constant C in (23) depends only on the constant C_1 in (22) and on some other constants independent of ϵ. Hence, C is independent of ϵ.

There are suitable cut-off functions associated to the distance $\tilde{d}_\epsilon$, independent of ϵ in the following sense:

Lemma 3.3 (Cut-off functions) *For any $0 < s < r$, $\tilde{x}_0 \in \mathbb{R}^{\tilde{N}}$ there exists $\phi_\epsilon \in C^\infty_0(\mathbb{R}^{\tilde{N}})$ such that $0 \leq \phi_\epsilon \leq 1$, $\phi_\epsilon \equiv 1$ on $B_{\tilde{d}_\epsilon}(\tilde{x}_0, s)$, sprt $\phi_\epsilon \subset B_{\tilde{d}_\epsilon}(\tilde{x}_0, r)$ and for every $k \in \mathbb{N}$ there exists C_k, independent of ϵ, such that*

$$|\tilde{X}_{\epsilon,i_1} \cdots \tilde{X}_{\epsilon,i_h} \phi_\epsilon| \leq \frac{C_k}{(r-s)^k} \quad \text{on sprt } \phi_\epsilon$$

and

$$||\tilde{X}_{\epsilon,i_1}\cdots\tilde{X}_{\epsilon,i_h}\phi_\epsilon||_{C^\alpha_{\tilde{d}_\epsilon}} \le \frac{C_k}{(r-s)^{k+1}},$$

for $r-s$ small enough and for $i_j \in \{1,\dots,\tilde{N}\}$, $\sum_{j=1}^h i_j \le k$.

Proof Consider the balls $B_{\tilde{d}}(\Phi_\epsilon(\tilde{x}_0), s)$ and $B_{\tilde{d}}(\Phi_\epsilon(\tilde{x}_0), r)$ and choose ϕ as in Lemma 6.2 in [2]: $\phi \in C_0^\infty(\mathbb{R}^{\tilde{N}})$, $0 \le \phi \le 1$, $\phi \equiv 1$ on $B_{\tilde{d}}(\Phi_\epsilon(\tilde{x}_0), s)$, sprt $\phi \subset B_{\tilde{d}}(\Phi_\epsilon(\tilde{x}_0), r)$ which satisfies the conditions of lemma.

We set

$$\phi_\epsilon(\tilde{x}) = \phi(\Phi_\epsilon(\tilde{x})),$$

where the function Φ_ϵ is the change of variables defined in (21).

Note that

$$\tilde{X}_{\epsilon,i}\phi_\epsilon(\tilde{x}) = (\tilde{X}_i\phi)(\Phi_\epsilon(\tilde{x})),$$

and remember that the Jacobian determinant of Φ_ϵ is locally bounded by constants independent of ϵ. This implies that ϕ_ϵ satisfies the condition of lemma. □

4 Representation formulas

Remark 4.1 Let us consider a kernel $K \in F_\lambda(\tilde{X}, \tilde{d}, W)$ and the function Φ_ϵ defined in (21). Then the kernel

$$K_\epsilon(\tilde{x}, \tilde{y}) = K(\Phi_\epsilon(\tilde{x}), \Phi_\epsilon(\tilde{y}))$$

belongs to $F_\lambda(\tilde{X}_\epsilon, \tilde{d}_\epsilon, U)$, for some U, and satisfies the estimates in Definition 2.1 with the same constants as K, then with constants independent of ϵ.

For example, if K is a kernel of type $\lambda > 0$ then

$$|K_\epsilon(\tilde{x}, \tilde{y})| \le C(\tilde{d}(\Phi_\epsilon(\tilde{x}), \Phi_\epsilon(\tilde{x}))^{\lambda-\tilde{Q}} = C(\tilde{d}_\epsilon(\tilde{x}, \tilde{y}))^{\lambda-\tilde{Q}} \le C\frac{\tilde{d}_\epsilon(\tilde{x}, \tilde{y})^\lambda}{B_{\tilde{d}_\epsilon}(\tilde{x}, d_\epsilon(\tilde{x}, \tilde{y}))}, \tag{24}$$

with C independent of ϵ.

Besides, if $K \in F_\lambda(\tilde{X}, \tilde{d}, W)$ with $\lambda \ge 1$ then $\tilde{X}_{\epsilon,i}K_\epsilon \in F_{\lambda-1}(\tilde{X}_\epsilon, \tilde{d}_\epsilon, U)$. The proof is a slight variant of the proof of Theorem 8 in [23], in fact

$$(\tilde{X}_{\epsilon,i}K_\epsilon(\tilde{x}, \cdot))(\tilde{y}) = (\tilde{X}_i K)(\Phi_\epsilon(\tilde{x}), \cdot))(\Phi_\epsilon(\tilde{y})).$$

The next theorem provides a representation formula for u in terms of integral operators. The idea of this theorem is based on the parametrix method in [24].

Theorem 4.1 *Let $\tilde{a} \in C_0^\infty(U)$ and let $p \in \mathbb{N}$. There exist kernels $\tilde{K}_\epsilon \in F_2(\tilde{X}_\epsilon, \tilde{d}_\epsilon, U)$ and $\tilde{H}_{\epsilon,p} \in F_p(\tilde{X}_\epsilon, \tilde{d}_\epsilon, U)$, satisfying estimates in Definition 2.1 with constants independent of ϵ, such that for every $\tilde{x}$, $\tilde{y} \in U$ we have*

$$\tilde{L}_\epsilon^{\tilde{y}}(\tilde{K}_\epsilon(\tilde{x}, \tilde{y})) = \tilde{a}(\tilde{x})\delta_{\tilde{y}}(\tilde{x}) + \tilde{H}_{\epsilon,p}(\tilde{x}, \tilde{y}) \tag{25}$$

where $\delta_{\tilde{y}}$ is the Dirac distribution at $\tilde{y}$ and $\tilde{L}_\epsilon^{\tilde{y}}$ means that the differentiation is in the $\tilde{y}$-variable.

Besides, if $\tilde{T}_\epsilon$ and $\tilde{T}_{\epsilon,p}$ are the operators of kernels $\tilde{K}_\epsilon$ and $\tilde{H}_{\epsilon,p}$ respectively, then for every compactly supported function $\tilde{u} \in C^{2,\alpha}_{\tilde{d}_\epsilon}(U)$ we have

$$\tilde{L}_\epsilon\,(\tilde{T}_\epsilon \tilde{u}) = \tilde{a}\,\tilde{u} + \tilde{T}_{\epsilon,p}\,\tilde{u}. \tag{26}$$

Analogously, it is possible to find an operator $\tilde{T}_\epsilon$ of type 2 and an operator $\tilde{T}_{\epsilon,p}$ of type p such that

$$\tilde{T}_\epsilon\,(\tilde{L}_\epsilon \tilde{u}) = \tilde{a}\,\tilde{u} + \tilde{T}_{\epsilon,p}\,\tilde{u}. \tag{27}$$

Proof We approximate the vector fields $\tilde{X}_1, \ldots, \tilde{X}_\nu$ as in Theorem 2.1. We consider the approximating operator $L_{\tilde{Y}} = \sum_{i=1}^{\nu} \tilde{Y}_i^2$ and its fundamental solution $\Gamma_{\tilde{Y}}$, see [10].

We now choose

$$\tilde{K}_0(\xi,\eta) = \Gamma_{\tilde{Y}}(\Theta(\eta,\xi)),$$

so that $\tilde{K}_0 \in F_2(\tilde{Y}, \tilde{d}, W)$. Since, $\tilde{Y}_i$ approximates $\tilde{X}_i$ up to a differential operator of degree less or equal to zero then $\tilde{K}_0 \in F_2(\tilde{X}, \tilde{d}, W)$.

Further, we define

$$\tilde{K}_{\epsilon,0}(\tilde{x},\tilde{y}) = \tilde{K}_0(\Phi_\epsilon(\tilde{x}), \Phi_\epsilon(\tilde{y})).$$

By Remark 4.1, the kernel $\tilde{K}_{\epsilon,0}$ belongs to $F_2(\tilde{X}_\epsilon, \tilde{d}_\epsilon, U)$ and satisfies (15), (16) with constants independent of ϵ.

Note that

$$\begin{aligned}(\tilde{X}_{\epsilon,i} K_{\epsilon,0}(\tilde{x},\cdot))(\tilde{y}) &= \tilde{X}_i(K_0(\Phi_\epsilon(\tilde{x}),\cdot))(\Phi_\epsilon(\tilde{y}))\\ &= \tilde{X}_i(\Gamma_{\tilde{Y}}(\Theta(\cdot,\Phi_\epsilon(\tilde{x}))))(\Phi_\epsilon(\tilde{y})) = (\tilde{Y}_i\Gamma_{\tilde{Y}}(\cdot) + R_i\Gamma_{\tilde{Y}}(\cdot))(\Theta(\Phi_\epsilon(\tilde{y}), \Phi_\epsilon(\tilde{x}))),\end{aligned}$$

where R_i is a differential operator of degree ≤ 0 according to [23]. Then, we check that

$$\tilde{L}_\epsilon^{\tilde{y}}(\tilde{a}(\tilde{x})\,\tilde{K}_{\epsilon,0})(\tilde{x},\tilde{y}) = \tilde{a}(\tilde{x}) \sum_i (\tilde{Y}_i + R_i)^2(\Gamma_{\tilde{Y}}(\cdot))(\Theta(\Phi_\epsilon(\tilde{y}), \Phi_\epsilon(\tilde{x}))) =$$

(for a suitable kernel $\tilde{H}_0$ of type 1)

$$= \tilde{a}(\tilde{x})\delta_{\Phi_\epsilon(\tilde{y})}(\Phi_\epsilon(\tilde{x})) + \tilde{H}_0(\Phi_\epsilon(\tilde{x}), \Phi_\epsilon(\tilde{y})) = a(\tilde{x})\delta_{\tilde{y}}(\tilde{x}) + \tilde{H}_{\epsilon,0}(\tilde{x},\tilde{y}),$$

where $\tilde{H}_{\epsilon,0}(\tilde{x},\tilde{y})$ belongs to $F_1(\tilde{X}_\epsilon, \tilde{d}_\epsilon, U)$ with constants in (15), (16) independent of ϵ.

We now define

$$\tilde{K}_{\epsilon,1}(\tilde{x},\tilde{y}) = \tilde{a}(\tilde{x})\,\tilde{K}_{\epsilon,0}(\tilde{x},\tilde{y}) - (\tilde{H}_0 * \tilde{K}_0)(\Phi_\epsilon(\tilde{x}), \Phi_\epsilon(\tilde{y})).$$

Then $\tilde{K}_{\epsilon,1} \in F_2(\tilde{X}_\epsilon, \tilde{d}_\epsilon, U)$ and satisfies (15), (16) with constants independent of ϵ. Besides, we infer

$$\begin{aligned}\tilde{L}_\epsilon^{\tilde{y}}(\tilde{K}_{\epsilon,1})(\tilde{x},\tilde{y}) &= \tilde{L}_\epsilon^{\tilde{y}}(\tilde{a}\,\tilde{K}_{\epsilon,0})(\tilde{x},\tilde{y}) - \tilde{L}_\epsilon^{\tilde{y}}(\tilde{H}_0 * \tilde{K}_0)(\Phi_\epsilon(\tilde{x}), \Phi_\epsilon(\tilde{y}))\\ &= a(\tilde{x})\delta_{\tilde{y}}(\tilde{x}) + \tilde{H}_0(\Phi_\epsilon(\tilde{x}), \Phi_\epsilon(\tilde{y})) - \tilde{L}_\epsilon^{\tilde{y}}(\tilde{H}_0 * \tilde{K}_0)(\Phi_\epsilon(\tilde{x}), \Phi_\epsilon(\tilde{y})) =\end{aligned}$$

(for a suitable differential operator R of degree ≤ 1)

$$= \tilde{a}(x)\delta_{\tilde{y}}(\tilde{x}) + R(\tilde{H}_0 * \tilde{K}_0)(\Phi_\epsilon(\tilde{x}), \Phi_\epsilon(\tilde{y})) = \tilde{a}(x)\delta_{\tilde{y}}(\tilde{x}) + \tilde{H}_1(\Phi_\epsilon(\tilde{x}), \Phi_\epsilon(\tilde{y})).$$

We put

$$\tilde{H}_{\epsilon,1}(\tilde{x}, \tilde{y}) = \tilde{H}_1(\Phi_\epsilon(\tilde{x}), \Phi_\epsilon(\tilde{y})).$$

Hence, $\tilde{H}_{\epsilon,1} \in F_2(\tilde{X}_\epsilon, \tilde{d}_\epsilon, U)$ and it satisfies (15), (16) with constants independent of ϵ.
Iterating this procedure we define

$$\tilde{K}_{\epsilon,p}(\tilde{x}, \tilde{y}) = \tilde{a}\,\tilde{K}_{\epsilon,p-1}(\tilde{x}, \tilde{y}) - (\tilde{H}_{p-1} * \tilde{K}_0)(\Phi_\epsilon(\tilde{x}), \Phi_\epsilon(\tilde{y})).$$

By repeating the above arguments we get the assertion.
If $\tilde{T}_\epsilon$ is the operator of kernel $\tilde{K}_\epsilon$ and $\tilde{T}_{\epsilon,p}$ the operator of kernel $\tilde{H}_{\epsilon,p}$ then it follow easily that they verify (26). □

Taking two derivatives along the vector fields $\tilde{X}_{\epsilon,i}$ of both side the representation formula in Theorem 4.1 we obtain:

Theorem 4.2 *Let $\tilde{a} \in C_0^\infty(U)$. For every $p \in \mathbb{N}$ and $i, j \in \{1, \ldots, \nu\}$ there exist operators $\tilde{T}_\epsilon$ of type zero and $\tilde{T}_{\epsilon,p}$ of type p such that*

$$\tilde{X}_{\epsilon,i}\tilde{X}_{\epsilon,j}(\tilde{a}\,\tilde{u}) = \tilde{T}_\epsilon\,(\tilde{L}_\epsilon\tilde{u}) + \tilde{T}_{\epsilon,p}\,\tilde{u}. \tag{28}$$

The kernels defining the operators $\tilde{T}_\epsilon$ and $\tilde{T}_{\epsilon,p}$ satisfy conditions in Definition 2.1 with constants independent of ϵ.

The next theorem concerning the Hölder continuity of singular and fractional integrals is proved in [2]:

Theorem 4.3 *Let T be an operator of type $\lambda \geq 0$ with kernel $K \in F_\lambda(\tilde{X}_\epsilon, \tilde{d}_\epsilon, U)$. If r is sufficiently small then T is continuous on $C^\alpha_{\tilde{d}_\epsilon}(B_{\tilde{d}_\epsilon}(r))$ and there exists a positive constant C such that*

$$||Tu||_{C^\alpha_{\tilde{d}_\epsilon}(B_{\tilde{d}_\epsilon}(r))} \leq C\,||u||_{C^\alpha_{\tilde{d}_\epsilon}(B_{\tilde{d}_\epsilon}(r))}$$

for every $u \in C^\alpha_{\tilde{d}_\epsilon}(B_{\tilde{d}_\epsilon}(r))$ with compact support.

The constant C in the above theorem depends on the constants in Definition 2.1 but not explicitly on ϵ.

4.1 Proof of Theorem 1.1

We briefly explain how to prove the theorem, (see the computations in [2] for more details). The proof is based on three steps:

1. Schauder estimates for $\tilde{u}$ compactly supported in some small ball $B_{\tilde{d}_\epsilon}(r)$ with constant independent of ϵ. Using representation formula in Theorem 4.2 for the derivatives $\tilde{X}_{\epsilon,i}\tilde{X}_{\epsilon,j}$, applying Theorem 4.3 and arguing as in [2], we obtain, for r sufficiently small,
$$||\tilde{u}||_{C^{2,\alpha}_{\tilde{d}_\epsilon}(B_{\tilde{d}_\epsilon}(r))} \leq C\,(||\tilde{L}_\epsilon\tilde{u}||_{C^\alpha_{\tilde{d}_\epsilon}(B_{\tilde{d}_\epsilon}(r))} + ||\tilde{u}||_{L^\infty(B_{\tilde{d}_\epsilon}(r))}), \tag{29}$$
where C, r are independent of ϵ. Indeed C depends on the constants in Definition 2.1 which are independent of ϵ.

2. Schauder estimates for $\tilde{u}$ not necessarily with compact support. Arguing as in [2], as a consequence of the existence of suitable cut-off functions (Lemma 3.3) and an interpolation inequality (see Theorem 7.4 in [2]) we get: for r sufficiently small and $0 < t < s < r$

$$||\tilde{u}||_{C^{2,\alpha}_{\tilde{d}_\epsilon}(B_{\tilde{d}_\epsilon}(t))} \le \frac{C}{(s-t)^\beta} (||\tilde{L}_\epsilon \tilde{u}||_{C^{\alpha}_{\tilde{d}_\epsilon}(B_{\tilde{d}_\epsilon}(s))} + ||\tilde{u}||_{L^\infty(B_{\tilde{d}_\epsilon}(s))}), \tag{30}$$

where C, r, β are independent of ϵ.

3. Schauder estimates for u. Combining step 2 and Lemma 3.2 we obtain the thesis. □

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