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Direct approach to L^p estimates in homogenization theory

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Abstract We derive interior L^p-estimates for solutions of linear elliptic systems with oscillatory coefficients. The estimates are independent of ε, the small length scale of the rapid oscillations. So far, such results are based on potential theory and restricted to periodic coefficients. Our approach relies on BMO-estimates and an interpolation argument, gradients are treated with the help of finite differences. This allows to treat coefficients that depend on a fast and a slow variable. The estimates imply an L^p-corrector result for approximate solutions.

Keywords Heterogeneous media · Elliptic systems · Regularity theory · BMO · Two-scale convergence

Mathematics Subject Classification (2000) 35B27 · 49N60 · 35J15

1 Introduction

The classical (and most important) example of homogenization theory is the family of equations or systems for $u^\varepsilon : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$

$$-\nabla \cdot \big(A\,(x, x/\varepsilon)\,\nabla u^\varepsilon(x)\big) = f(x) \quad \text{in } \Omega, \tag{1.1}$$

for a bounded domain $\Omega \subset \mathbb{R}^n$ and $f \in H^{-1}(\Omega)$, accompanied with the boundary condition $u^\varepsilon|_{\partial\Omega} = 0$. On the coefficients A one assumes the Y-periodicity in the second variable, with $Y = (0, 1)^n$ the unit cube, see Fig. 1. Under appropriate assumptions on A, it is known that the family of solutions u^ε converges strongly in $L^2(\Omega)$ and weakly in

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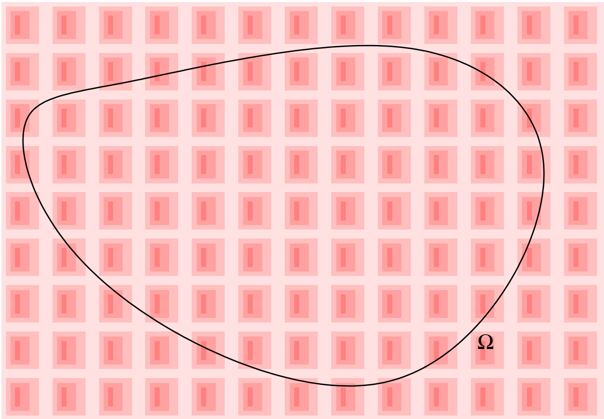

Fig. 1 The function $u = u^\varepsilon : \Omega \to \mathbb{R}^m$ solves $\nabla \cdot (A\nabla u) = f$, the coefficients $A = A(x, x/\varepsilon)$ are periodic in the fast variable and vary additionally on the macroscopic scale. The figure indicates the fast oscillations of the coefficients

$H^1(\Omega)$ to a limit function $u^0 \in H^1_0(\Omega)$, which is the solution of the homogenized problem $-\nabla \cdot (A^*(x)\nabla u^0(x)) = f(x)$. The most direct way to derive this result is the method of two-scale convergence introduced by Allaire [1], which provides additionally a corrector result: starting from the homogenized solution u^0 one can study the two-scale expansion of the solution in the form $\eta^\varepsilon(x) = u^0(x) + \varepsilon u^1(x, x/\varepsilon)$ and prove that $u^\varepsilon - \eta^\varepsilon \to 0$ strongly in $H^1(\Omega)$.

Current interest in homogenization analysis stems from questions in the failure of materials. Of particular interest are norms of the strain that are sensitive to peaks. In these applications, one is usually interested in L^q-norms or the L^∞-norm of the gradient rather than L^2-norms, see e.g. [16].

While homogenization theory is well developed in the Hilbert space setting, much less is known for L^q-norms. We emphasize that the interest here is to have estimates that are independent of the small parameter $\varepsilon > 0$. Obviously, regularity theory could be used to find estimates for $u^\varepsilon \in H^2(\Omega)$ and embedding results yield L^q-estimates for ∇u^ε—but due to the oscillations of u^ε on the ε scale, the H^2-norms will necessarily behave like $1/\varepsilon$. A more profound obstruction to regularity in L^q-spaces is given in [6]: Without the periodicity assumption on $A(x, .)$, no ε-independent estimate for $\|\nabla u^\varepsilon\|_{L^q(\Omega')}$ can hold (except for $q-2$ small, in which case Meyers estimate holds).

Concerning positive results, fundamental contributions are due to Avellaneda and Lin [2], where optimal (in terms of exponents) estimates in L^q-spaces were derived for the solutions of the above equation. As in the related articles [3–5,7], the assumption is that $A(x, y) = A(y)$ does not depend on the slow variable, the regularity is $A \in C^\alpha(Y)$. The authors derive estimates for the singular kernels of the corresponding Greens-functions in order to prove the regularity result for the solutions. Uniform regularity results for nonlinear elliptic systems $-\nabla \cdot A(x/\varepsilon, \nabla u(x)) = 0$ in small dimensions were obtained in [14]. The strength of these results is the global character, the estimates hold up to the boundary of the domain. Results under weaker regularity assumptions on the coefficients can be deduced from an approximation method of Caffarelli and Peral [7]. Based on local energy comparison and Calderón–Zygmund type decomposition, the authors provide ε-uniform local $W^{1,p}$ bounds for elliptic equations $-\nabla \cdot (A(x/\varepsilon)\nabla u(x)) = 0$ with continuous periodic coefficients. For an analysis of the behavior near macroscopic interfaces we refer to [15].

This work follows another approach. The improvement over existing work lies in the fact that we study coefficients that may additionally depend on the slow variable, $A = A(x, y)$. Under the asssumption of uniform continuity and uniform ellipticity, L^p estimates are shown to hold globally. Under refined regularity assumptions on the coefficients the result can be lifted to gradient estimates that hold in the interior.

Theorem 1 provides an estimate for $\|u^\varepsilon\|_{L^q(\Omega')}$ with an optimal exponent. The method uses no potential theory but rather follows the approach sketched, e.g., in [11]: the solution operator is bounded as a map $L^2 \to L^{2^*}$, and as a map $L^n \to$ BMO. The L^2-result is a direct consequence of the Sobolev embedding, the BMO-result is based on a decomposition of the solution on cubes. Inhomogeneous solutions with homogeneous boundary data are treated by a perturbation argument, homogeneous solutions by comparison with the solutions of the homogenized system. An interpolation argument between BMO and L^{2^*} yields the L^q-estimate. The proof of Theorem 1 is given in Sect. 2.

In Theorem 2 we prove an estimate for $\|\nabla u^\varepsilon\|_{L^q(\Omega')}$, again with the optimal exponent. The method is based on finite difference quotients of the solutions. Finite differences solve an equation of the same type if the x-differences are in accordance with the ε-periodicity of the equation. For such finite differences we can apply Theorem 1 to find L^q-estimates. We conclude with Lemma 2, the “local lemma”, which asserts that gradients can be estimated by the ε-size finite differences. This program is carried out in Sect. 3. We note that this idea was used in [19] for Lipschitz estimates in a perforated domain.

In Sect. 4 we prove the second part of Theorem 2, which transfers the estimates for compactly supported solutions to interior estimates. The localization procedure is intricate, since the product of a solution with a smooth cut-off function behaves badly under the application of the operator with oscillatory coefficients. We circumvent this effect by using a multiplication of the solution with two-scale approximations of cut-off functions.

The a priori bound on solutions provides an improvement of the previously mentioned corrector result. In Corollary 1 we show, for $f \in L^p(\Omega)$ and $\Omega' \subset \Omega$ a compact subset, that the two-scale expansions η^ε of solutions u^ε satisfy $u^\varepsilon - \eta^\varepsilon \to 0$ with the strong convergence of $W^{1,q}(\Omega')$.

Results

On a domain $\Omega \subset \mathbb{R}^n$ we study the family of operators $\mathcal{L}^\varepsilon$ acting on maps $v : \Omega \to \mathbb{R}^m$ as

$$\mathcal{L}^\varepsilon v(x) = -\nabla \cdot (A\,(x, x/\varepsilon)\, \nabla v(x)) \quad \text{in } \Omega. \tag{1.2}$$

Here, with a periodicity cell $Y = (0, 1)^n$, we consider coefficients $A = A(x, y)$ that satisfy

$$A : \Omega \times \mathbb{R}^n \to \mathbb{R}^{m^2 \times n^2} \text{ uniformly continuous and } Y\text{-periodic in } y, \tag{1.3}$$

$$\text{uniform ellipticity: for } \nu > 0 \text{ holds } A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \geq \nu |\xi|^2 \text{ for all } \xi \in \mathbb{R}^{n \times m}. \tag{1.4}$$

For the gradient estimates of Theorem 2 further regularity conditions on the coefficient are imposed:

$$A \in W^{1,\rho}(\Omega, C^0(Y)) \quad \text{for some } \rho > n, \tag{1.5}$$

$$A \in C^{0,1}(\Omega, W^{1,n}(Y)) \cap C^{1,1}(\Omega, L^n(Y)). \tag{1.6}$$

In our strongest result, we therefore assume the regularity $A(x, .) \in C^0(Y) \cap W^{1,n}(Y)$ for almost every x. We note that this condition is neither stronger nor weaker than Hölder continuity. Our main results are interior estimates for solutions of the boundary value problem.

Theorem 1 *Let the coefficients of the elliptic operator $\mathcal{L}^\varepsilon$ satisfy* (1.3) *and* (1.4)*, $\Omega \subset \mathbb{R}^n$ bounded, and $u \in H_0^1(\Omega)$ be a weak solution of*

$$\mathcal{L}^\varepsilon u = \operatorname{div} f \quad \text{in } \Omega.$$

Let $\Omega' \subset \Omega$ be compactly contained, $p \in [2, n)$ and $q = np/(n-p)$. Then there holds

$$\|u\|_{L^q(\Omega')} \le c\|f\|_{L^p(\Omega)}, \tag{1.7}$$

with a constant c depending on Ω, Ω', A, and p, but independent of f and ε. If $\Omega \subset \mathbb{R}^n$ is a domain with C^1-boundary, the above estimate holds globally, i.e., with $\Omega' = \Omega$. In the case $p = n$ and Ω' a cube, (1.7) *holds with L^q replaced by* BMO. *In case $p > n$ the estimate holds with $q = \infty$.*

Our second theorem lifts the orders of differentiability by one.

Theorem 2 *On $\Omega \subset \mathbb{R}^n$ we consider weak solutions $U \in H^1(\Omega)$ of*

$$\mathcal{L}^\varepsilon U = F \quad \text{in } \Omega.$$

Let $\Omega' \subset \Omega$ be compactly contained, $p \in [2, n)$ and $q = np/(n-p)$. Then, with a constant c depending on Ω, Ω', A, and p, but independent of F and ε, there holds:

1. *If the support of U is contained in Ω' and A satisfies* (1.3)–(1.5)*, then*

$$\|\nabla U\|_{L^q(\Omega)} \le c\, \|F\|_{L^p(\Omega)}. \tag{1.8}$$

2. *If A satisfies* (1.3)–(1.6)*, then*

$$\|\nabla U\|_{L^q(\Omega')} \le c \left(\|F\|_{L^p(\Omega)} + \|U\|_{H^1(\Omega)}\right). \tag{1.9}$$

In the case $p > n$ the above estimates hold with $q = \infty$.

2 BMO-estimates and interpolation

Proof of Theorem 1 We realize that the theorem holds trivially in the case $p = 2$, $q = 2^* = 2n/(n-2)$ by the continuous embedding $H^1 \to L^{2^*}$ for $n \ge 3$ and $H^1 \to \mathrm{BMO}$ for $n = 2$. Proposition 1 provides an BMO-estimate on compactly contained cubes $Q \subset \Omega$ for data in $L^n(\Omega)$. We consider homogeneous Dirichlet conditions, the operator $(\mathcal{L}^\varepsilon)^{-1}\mathrm{div} : L^2(\Omega) \to H_0^1(\Omega)$, and its restriction to Q, i.e., the operators $T^\varepsilon f := ((\mathcal{L}^\varepsilon)^{-1}\mathrm{div} f)\lfloor_Q$,

$$\begin{aligned} T^\varepsilon &: L^2(\Omega) \to L^{2^*}(Q), \\ T^\varepsilon &: L^n(\Omega) \to \mathrm{BMO}(Q). \end{aligned}$$

An interpolation yields the inner L^q-estimate (1.7). For the interpolation result we refer to Appendix A. For the global result we can take any cube $Q \supset \Omega$ so that Ω is compactly included and apply Proposition 2 with the same interpolation argument. □

We recall that for cubes $Q_0 \subset \mathbb{R}^n$ the homogeneous space of functions of bounded mean oscillation $\mathrm{BMO}(Q_0)$ is defined via the semi-norm

$$\|u\|_{\mathrm{BMO}(Q_0)} \equiv \sup_{Q \subset Q_0} \fint_Q |u - \fint_Q u|. \tag{2.1}$$

According to a well-known result of John and Nirenberg [13] an equivalent semi-norm is given by $\|u\|_{\mathrm{BMO}}^2 = \sup_{Q \subset Q_0} \fint_Q |u - \fint_Q u|^2$. Observe that for a bounded measurable set $B \subset \mathbb{R}^n$ and $u \in L^2(B)$, the function $\lambda \mapsto \int_B |u - \lambda|^2 \,\mathrm{d}x$ is minimal for $\lambda = \fint_B u$. Thus there is a universal constant $c = c(n)$ such that for a function $u \in L^1_{loc}(B_R(0))$ we have

$$\|u\|^2_{\mathrm{BMO}(Q_R(0))} \le c \sup \left\{ \fint_{B_r} |u - \fint_{B_r} u|^2 : B_r = B_r(x) \subset B_R(0) \right\}, \tag{2.2}$$

where $Q_R(0) := (-R, R)^n \subset B_R(0)$ denotes the cube of sidelength R centered at the origin.

The main steps in this section are the following. We have seen that the local part of Theorem 1 is a consequence of the BMO-estimate of Proposition 1. We derive this Proposition with Campanato's device of a local decomposition of the solution, $u = v + w$, where v solves a homogeneous problem. While the w-part can be handled directly, the v-part is treated seperately in Lemma 1. In that Proposition, we consider the regime of large and small radii separately. While small radii can be treated with standard Hölder estimates of Theorem 3, large radii are treated with homogenization and compactness arguments similar to the one in [2].

Proposition 1 (BMO inner estimate) *Suppose that the coefficients of the elliptic operator $\mathcal{L}^\varepsilon$ satisfy (1.3) and (1.4). Let u be a weak solution of*

$$\mathcal{L}^\varepsilon u = \operatorname{div} f \quad in\ B_R(0).$$

Then

$$\|u\|_{\mathrm{BMO}(Q_{R/2}(0))} \le c \left(\|f\|_{L^n(B_R(0))} + \|u\|_{H^1(B_R(0))} \right), \tag{2.3}$$

where the constant c depends on R and A, but is independent of f and ε.

Proof Let $x \in B_{R/2}(0)$ and $B_r = B_r(x)$ with $0 < r < \min\{R/4, 1\}$. We show that there is a constant $C > 0$, depending only on A, such that for any $0 < \rho < r/2$

$$\fint_{B_\rho} |u - \fint_{B_\rho} u|^2 \le C \left(\|f\|^2_{L^n(B_{2r})} + \|u\|^2_{H^1(B_{2r})} \right).$$

To this end we decompose $u = v + w$ where v is the weak solution of the homogeneous problem $\mathcal{L}^\varepsilon v = 0$ on B_r with $v|_{\partial B_r} = u|_{\partial B_r}$. With Poincaré's inequality

$$\int_{B_\rho} |u - \fint_{B_\rho} u|^2 \le 2 \int_{B_\rho} |v - \fint_{B_\rho} v|^2 + c\,\rho^2 \int_{B_r} |\nabla w|^2.$$

Using that $w \in H^1_0(\Omega)$ solves $\mathcal{L}^\varepsilon w = \operatorname{div} f$, Young's and Hölder's inequality imply

$$\int_{B_r} |\nabla w|^2 \le c \int_{B_r} |f|^2 \le c\, r^{n-2} \|f\|^2_{L^n(\mathrm{B}_r)}. \tag{2.4}$$

As a consequence of (2.7) of Lemma 1 below, we find that

$$\int_{B_\rho} |u - \fint_{B_\rho} u|^2 \le C \left(\frac{\rho}{r}\right)^n \int_{B_{r/2}} |v - \fint_{B_{r/2}} v|^2 + c\, r^n \, \|f\|^2_{L^n(B_r)}.$$

Taking into account that v is an $\mathcal{L}^\varepsilon$-minimal extension of u in B_r we find

$$\int_{B_{r/2}} |v - \fint_{B_{r/2}} v|^2 \le c\, r^2 \int_{B_r} |\nabla v|^2 \le c\, r^2 \int_{B_r} |\nabla u|^2.$$

But then Caccioppoli's inequality

$$r^2 \int_{B_r} |\nabla u|^2 \le c \int_{B_{2r}} |u - \fint_{B_{2r}} u|^2 + c\, r^2 \int_{B_{2r}} |f|^2 \tag{2.5}$$

and Hölder's inequality $\|f\|^2_{L^2(B_{2r})} \le c r^{n-2} \|f\|^2_{L^n(B_{2r})}$ imply

$$\int_{B_\rho} |u - \fint_{B_\rho} u|^2 \le C \left(\frac{\rho}{r}\right)^n \int_{B_{2r}} |u - \fint_{B_{2r}} u|^2 + c\, r^n\, \|f\|^2_{L^n(\mathrm{B}_{2r})}.$$

By a standard iteration method, cf. [10] Chap. III, Lemma 2.1 or [12], the factor r^n in the last term can be replaced by ρ^n. Adapting constants we find

$$\fint_{B_\rho} |u - \fint_{B_\rho} u|^2 \le C \left[\fint_{B_{2r}} |u - \fint_{B_{2r}} u|^2 + \|f\|^2_{L^n(B_{2r})} \right]$$

for any $0 < \rho < r/2$. This completes the proof. □

It remains to derive uniform bounds for the homogeneous problem. The main strategy will be to reduce everything to the following basic regularity result for elliptic systems with continuous coefficients in divergence form, that is originally due to Campanato [9] and Morrey [17]:

Theorem 3 (cf. [10] Chap. III Theorem 3.1) *Suppose that $A \in C^0(B_R(0))$ is uniformly elliptic. If $v \in H^1(B_R(0))$ is a weak solution of $\nabla \cdot (A(x)\nabla v) = 0$ in $B_R(0)$, then ∇v belongs to the Morrey space $L^{2,\lambda}(B_{R/2}(0))$ for any $0 < \lambda < n$. More specifically, for any $\gamma \in [0, 1)$, the estimate*

$$\rho^2 \fint_{B_\rho(0)} |\nabla v|^2 \le C \left(\frac{\rho}{R}\right)^{2\gamma} R^2 \fint_{B_{R/2}(0)} |\nabla v|^2 \tag{2.6}$$

holds true for any $0 < \rho < R/2$ with a constant C that only depends on A and γ.

More precisely, the bounds only depend on the ellipticity properties and the modulus of continuity of A. Thus, the proposition holds in a uniform fashion for equi-continuous and uniformly elliptic families $(A_\varepsilon)_{\varepsilon>0}$ of coefficient matrices.

In order to apply Theorem 3 we distinguish two regimes determined by the size of ρ relative to ε. For small radii $\rho \lesssim \varepsilon$ estimate (2.7) follows from a scaling argument that provides a standard situation. In the opposite regime of large radii when $\rho \gtrsim \varepsilon$, Theorem 3 will be applied to the homogenized problem and we conclude with a compactness argument similar to the one in [2].

Lemma 1 (C^γ inner estimate for the homogeneous problem) *Let the coefficients of the elliptic operator $\mathcal{L}^\varepsilon$ satisfy (1.3) and (1.4), and let $\gamma \in (0, 1)$. There exist constants $R_0, C_0 > 0$*

that depend only on γ and the coefficients A with the following property: If $r < R_0$ and v is a weak solution of

$$\mathcal{L}^{\varepsilon} v = 0 \text{ in } B_r(0),$$

then $v \in C^{\gamma}(B_{r/2}(0))$. More precisely, for any $B_\rho = B_\rho(x)$ with $0 < \rho < r/2$ and $x \in B_{r/2}(0)$ holds

$$\fint_{B_\rho} |v - \fint_{B_\rho} v|^2 \le C_0 \left(\frac{\rho}{r}\right)^{2\gamma} \fint_{B_{r/2}} |v - \fint_{B_{r/2}} v|^2. \tag{2.7}$$

Proof After translation it is enough to prove (2.7) for $x = 0$.

1. *Large radii.* In this step of the proof we consider only radii $\rho > \varepsilon/\varepsilon_0$, where the universal constant $\varepsilon_0 > 0$ is determined below. The estimate is based on an improvement estimate for the (squared) mean oscillation. More precisely, we show that there exist $\varepsilon_0 > 0$ and $\theta \in (0, 1)$ such that

$$\fint_{B_{\theta\rho}} |v - \fint_{B_{\theta\rho}} v|^2 \le \theta^{2\gamma} \fint_{B_\rho} |v - \fint_{B_\rho} v|^2. \tag{2.8}$$

From (2.8), the decay estimate (2.7) follows by a k-fold iteration, with k determined by $\theta^{k+1} r < 2\rho \le \theta^k r$. The constant in (2.7) depends only on θ. In order to prove (2.8) we first observe that the inequality is scaling invariant. Rescaling x and ε by ρ we obtain

$$\mathcal{L}^{\varepsilon}_{\rho} v := -\nabla \cdot (A(\rho x, x/\varepsilon)\nabla v) = 0 \quad \text{in } B_1 = B_1(0).$$

In view Caccioppoli's inequality (2.5) for $f = 0$, it is enough to show

$$\fint_{B_\theta} |v - \fint_{B_\theta} v|^2 \le \lambda\, \theta^{2\gamma} \fint_{B_{1/2}} |\nabla v|^2 \tag{2.9}$$

for a small constant $\lambda > 0$ that compensates the constant c of (2.5).

We first verify (2.9) with some suitable $\theta = \theta_0$ for solutions v of the corresponding homogenized equation at $x = 0$:

$$\mathcal{L}^{*}_{0} v = -\nabla \cdot (A^*(0)\nabla v) = 0 \quad \text{in } B_{1/2} = B_{1/2}(0).$$

Recall that the coefficients $A^*(0) = A(0, \cdot)^*$ are strictly elliptic. Thus by Theorem 3 for $\gamma^* = \frac{1}{2}(1 + \gamma)$ and Poincaré's inequality we obtain

$$\theta^{-2} \fint_{B_\theta} |v - \fint_{B_\theta} v|^2 \,\mathrm{d}x \le C\, \theta^{(1-\gamma)}\, \theta^{2\gamma} \int_{B_{1/2}} |Dv|^2 \,\mathrm{d}x \tag{2.10}$$

for any $\theta \in (0, 1/2]$ where $C = C(\gamma)$. We select θ_0 to be the maximal $\theta \in (0, 1/2]$ so that $C\,\theta^{(1-\gamma)} \le \lambda/2$ (which is sufficient for the assertion) and fix $\theta = \theta_0$.

Let us extend this result by compactness to small but finite $0 < \varepsilon < \varepsilon_0$ and $0 < \rho < \rho_0$. We argue by contradiction and suppose that there exist sequences $\varepsilon_k \to 0$ and $\rho_k \to 0$, and a corresponding sequence (v_k) of weak solutions of $\mathcal{L}^{\varepsilon_k}_{\rho_k} v = 0$ in B_1 so that

$$\fint_{B_\theta} |v_k - \fint_{B_\theta} v_k|^2 > \lambda\, \theta^{2\gamma} \fint_{B_{1/2}} |\nabla v_k|^2.$$

We define a sequence of blow-up functions

$$w_k = \left(\fint_{B_{1/2}} |\nabla v_k|^2 \right)^{-1/2} \left(v_k - \fint_{B_\theta} v_k \right)$$

that satisfy the equation $\mathcal{L}_{\rho_k}^{\varepsilon_k} w_k = 0$ in B_1 which we write as

$$\mathcal{L}_0^{\varepsilon_k} w_k = \left(\mathcal{L}_0^{\varepsilon_k} - \mathcal{L}_{\rho_k}^{\varepsilon_k} \right) w_k = \nabla \cdot f_k \quad \text{in } B_1. \tag{2.11}$$

Note that as $k \to \infty$

$$f_k = [A(\rho_k\, x, x/\varepsilon_k) - A(0, x/\varepsilon_k)]\, \nabla w_k \to 0 \quad \text{strongly in} \quad L^2(B_{1/2}) \tag{2.12}$$

by uniform L^2 boundedness of ∇w_k and uniform continuity of the coefficients $A(\cdot, y)$. By assumption we have

$$\fint_{B_\theta} |w_k|^2 > \lambda\, \theta^{2\gamma} \quad \text{for any } k \in \mathbb{N}. \tag{2.13}$$

Since $\fint_{B_\theta} w_k = 0$, Poincaré's inequality implies an L^2 estimate for (w_k). In particular, for a subsequence we find $w_k \rightharpoonup w$ weakly in $H^1(B_{1/2})$. By standard homogenization results, (2.11) and (2.12) imply that w is a weak solution of the homogenized equation $\mathcal{L}_0^* w = 0$ in $B_{1/2}$. By lower semicontinuity $\fint_{B_{1/2}} |\nabla w|^2 \le 1$. Thus (2.10) and Poincaré's inequality imply

$$\fint_{B_\theta} |w|^2 \le \frac{\lambda}{2}\, \theta^{2\gamma}. \tag{2.14}$$

But this is a contradiction to (2.13) since (w_k) is strongly pre-compact L^2.

2. *Small radii.* In order to treat radii $0 < \rho < \varepsilon/\varepsilon_0$ we rescale by $\varepsilon/\varepsilon_0$ (for the argument we can assume that $\varepsilon_0 = 1$). The rescaled coefficients $y \mapsto A\,(\varepsilon\, y, y)$ are equi-continuous and uniformly elliptic as $\varepsilon \to 0$. Accordingly, weak solutions $v_\varepsilon \in H^1(B_1)$ of the rescaled equation

$$\nabla \cdot (A\,(\varepsilon\, y, y)\, \nabla v_\varepsilon(y)) = 0 \quad \text{in } B_1$$

are (uniformly) locally Hölder continuous with exponent $\gamma \in (0, 1)$ and exhibit, in view of (2.6), Poincaré's and Caccioppoli's inequality, an estimate

$$\fint_{B_{\rho/\varepsilon}} |v_\varepsilon - \fint_{B_{\rho/\varepsilon}} v_\varepsilon|^2 \le C \left(\frac{\rho}{\varepsilon} \right)^{2\gamma} \fint_{B_1} |v_\varepsilon - \fint_{B_1} v_\varepsilon|^2$$

for any $0 < \rho < \varepsilon$ with a constant C that only depends on γ and A. Hence we get in the original scaling

$$\fint_{B_\rho} |v - \fint_{B_\rho} v|^2 \le C \left(\frac{\rho}{\varepsilon} \right)^{2\gamma} \fint_{B_\varepsilon} |v - \fint_{B_\varepsilon} v|^2.$$

This matches with the regime of large radii ($\varepsilon_0 = 1$) and completes the proof. □

Remark 1 (*Hölder estimates*) In the case $p > n$, the decay power in (2.4) can be improved. In combination with Lemma 1, this implies uniform Hölder estimates and, in particular, L^∞-estimates for u. The detailed arguments can be found e.g. in [10], Chapt. III.

Global BMO-estimates

This section is devoted to the statement of global estimates in Theorem 1. Let us therefore assume that Ω is a domain of class C^1 that is compactly contained within some cube Q. After trivial extension we have $u \in H_0^1(Q)$. The goal is to extend the local estimate (2.3) to the global estimate

$$\fint_B |u - \fint_B u|^2 \le c\left(\|f\|^2_{L^n(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right) \tag{2.15}$$

for any ball $B \subset Q$ and some universal constant c that only depends on A and Ω. Observe that $\|\nabla u\|_{L^2(\Omega)} \le C(\Omega)\|f\|_{L^n(\Omega)}$. Hence (2.15) implies:

Proposition 2 (BMO global estimate) *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain of class C^1. Let the coefficients of the elliptic operator $\mathcal{L}^\varepsilon$ satisfy (1.3) and (1.4). Let $u \in H_0^1(\Omega; \mathbb{R}^m)$ be a weak solution of*

$$\mathcal{L}^\varepsilon u = \operatorname{div} f \quad \textit{in } \Omega.$$

Then for any cube Q that compactly contains Ω

$$\|u\|_{\mathrm{BMO}(Q)} \le c\,\|f\|_{L^n(\Omega)}, \tag{2.16}$$

where the constant c only depends on A and Ω, but is independent of f and ε.

We only sketch the proof of the global estimate (2.15). The general device is well established, cf. e.g. [11,12] and the literature therein: In order to complement the local bounds we essentially have to show that they remain valid when balls are replaced by relative balls $\Omega(x_0, r) = \Omega \cap B_r(x_0)$ centered at a boundary point $x_0 \in \partial\Omega$. Indeed, the claim of Theorem 3 persists for $R < R_0$ and with a modified constant $C = C_0$ both depending on the C^1 structure of the boundary. Note that rescaling in space by a magnifying factor only flattens the boundary and will not increase these constants. In particular, the arguments from the proof of Lemma 1 carry over with slight modifications utilizing Dirichlet conditions: there are constants $R_0 > 0$ and $C_0 < \infty$ that only depend on A, the C^1 structure of $\partial\Omega$, and γ, with the following property: If $r < R_0$ and v is a weak solution of

$$\mathcal{L}^\varepsilon v = 0 \quad \text{in } \Omega(x_0, r) \quad \text{with} \quad v = 0 \quad \text{on } \partial\Omega \cap B_r(x_0).$$

then for any $0 < \rho < r/2$

$$\fint_{\Omega(x,\rho)} |v - \fint_{\Omega(x,\rho)} v|^2 \le C_0 \left(\frac{\rho}{r}\right)^{2\gamma} \fint_{\Omega(x,\frac{r}{2})} |v - \fint_{\Omega(x,\frac{r}{2})} v|^2. \tag{2.17}$$

In combination with a decomposition and an iteration argument as in the proof of Proposition 1, this yields for $0 < 4\rho < R < R_0$

$$\fint_{\Omega(x_0,\rho)} |u - \fint_{\Omega(x_0,\rho)} u|^2 \le c\left(\|f\|^2_{L^n(\Omega(x_0,R))} + \|\nabla u\|^2_{L^2(\Omega(x_0,R))}\right). \tag{2.18}$$

The constant only depends on A and Ω, but it is independent of ε and f. Then the result can be deduced by a standard covering argument, see e.g. [11,12].

3 Finite difference method

Theorem 2 provides uniform estimates for the gradient of solutions. Our approach is to consider difference quotients that are aligned with the periodicity of the problem, that is, with u evaluated at points x and $x+\varepsilon e_d$, where e_d is a coordinate vector in $\mathbb{R}^n$. Such difference quotients satisfy an equation of the same type and we can apply Theorem 1 to find estimates. The "local lemma", Lemma 2 below, allows to transfer the estimate for the difference quotients to an estimate for the gradient.

Proof of Theorem 2, item 1 We observe that Poincaré's inequality yields the estimate $\|U\|_{H^1} \le c\|F\|_{L^2}$. Our aim is to find better integrability properties of ∇U. We first study the case $q < \infty$. The main idea of our proof is to study the discrete difference quotients of the form

$$v_d := \nabla_d^\varepsilon U(x) := \frac{U(x+\varepsilon e_d) - U(x)}{\varepsilon},$$

where $e_d \in \mathbb{R}^n$ is the d'th unit vector, $d = 1, \ldots, n$, and $v = (v_1, \ldots, v_n)$. The functions v_d are compactly supported in Ω for ε sufficiently small. They satisfy the equation

$$\mathcal{L}^\varepsilon v_d(x) = (\nabla_d^\varepsilon F)(x) + \operatorname{div}\left(\nabla_d^\varepsilon A(x)\nabla U(x+\varepsilon e_d)\right). \tag{3.1}$$

For ease of notation and without loss of generality we assume $d = n$ and write $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. We can write $\nabla_n^\varepsilon F = \operatorname{div}(0, \ldots, 0, \tilde{F}_n)$ by setting

$$\tilde{F}_n(x', x_n) = \frac{1}{\varepsilon}\int_{x_n}^{x_n+\varepsilon} F(x', \xi)\,\mathrm{d}\xi.$$

Since $\tilde{F}_n$ is constructed as a local average of F, Jensens inequality allows to compare the L^p-norms, $\|\tilde{F}_n\|_{L^p} \le \|F\|_{L^p}$. We now apply Theorem 1 to v_d.

$$\begin{aligned}\|v_d\|_{L^q(\Omega)} &\le c\left(\|\tilde{F}_d\|_{L^p(\Omega)} + \|\nabla_d^\varepsilon A(.,./\varepsilon)\nabla U(.+\varepsilon e_d)\|_{L^p}\right)\\ &\le c\left(\|F\|_{L^p(\Omega)} + \|\nabla_d^\varepsilon A\|_{L^\rho(\Omega,C^0(Y))}\cdot\|\nabla U\|_{L^{q-\delta}}\right)\end{aligned}$$

for some $\delta > 0$, since, by assumption, we have the strict inequality $1/\rho < 1/n = 1/p - 1/q$.

We next use Lemma 2 below to transfer this estimate into a result on ∇U. We start by writing the L^q-norm as a sum over local L^q-norms. We use $Q_0 = (0,1)^n$ and take the sum over all $j \in \mathbb{Z}^n$ such that $\varepsilon j \in \Omega$.

$$\int_\Omega |\nabla U|^q = \sum_j \int_{\varepsilon(j+Q_0)} |\nabla U|^q.$$

In the single cell $\varepsilon(j + Q_0)$ we use the local lemma for the rescaled functions $u(y) = U(\varepsilon j + \varepsilon y)$, $f(y) = \varepsilon^2 F(\varepsilon j + \varepsilon y)$, and $g(y) = \varepsilon v(\varepsilon j + \varepsilon y)$. In their scaling laws, they are related to the original functions by $\nabla_y u = \varepsilon\nabla_x U$, $\mathcal{L}^1 u = f$, and $\nabla^1 u = \varepsilon\nabla^\varepsilon U = \varepsilon v = g$. Inequality (3.5) implies for every $s \ge 2$

$$\|\nabla_y u\|_{L^q(Q_0)} \le c\left(\|g\|_{L^q(Q_4)} + \|f\|_{L^p(Q_5)} + \varepsilon^\alpha\|\nabla_y u\|_{L^s(Q_5)}\right),$$

where α depends only on A and c is independent of g, f, ε, and s. The sets Q_l are the enlarged cubes $Q_l = (-l, l)^n$. Scaling back to the original variables and taking the q'th power yields for the single cell

$$\int_{\varepsilon(j+Q_0)} |\nabla_x U|^q = \varepsilon^n \varepsilon^{-q} \|\nabla_y u\|^q_{L^q(Q_0)}$$

$$\leq c\varepsilon^{n-q}\left(\varepsilon^q \varepsilon^{-n} \int_{\varepsilon(j+Q_4)} |v|^q + \varepsilon^{2q}\left(\varepsilon^{-n}\int_{\varepsilon(j+Q_5)} |F|^p\right)^{q/p}\right.$$

$$\left. + \varepsilon^{\alpha q}\varepsilon^q \left(\varepsilon^{-n}\int_{\varepsilon(j+Q_5)} |\nabla U|^s\right)^{q/s}\right)$$

$$= c\int_{\varepsilon(j+Q_4)} |v|^q + \left(\int_{\varepsilon(j+Q_5)} |F|^p\right)^{q/p} + \varepsilon^{n+\alpha q - nq/s}\left(\int_{\varepsilon(j+Q_5)} |\nabla U|^s\right)^{q/s},$$

where in the last equality we used $n + q - nq/p = 0$. Summing over all j we find for $q > s$

$$\|\nabla U\|^q_{L^q(\Omega)} \leq c\left(\|v\|^q_{L^q(\Omega)} + \max_j \left(\int_{\varepsilon(j+Q_5)} |F|^p\right)^{(q/p)-1} \sum_j \left(\int_{\varepsilon(j+Q_5)} |F|^p\right)\right.$$

$$\left. + \varepsilon^{n+\alpha q - nq/s} \max_j \left(\int_{\varepsilon(j+Q_5)} |\nabla U|^s\right)^{(q/s)-1} \sum_j \left(\int_{\varepsilon(j+Q_5)} |\nabla U|^s\right)\right)$$

$$\leq c\left(\|v\|^q_{L^q(\Omega)} + \|F\|^q_{L^p(\Omega)} + \varepsilon^{n+\alpha q - nq/s}\|\nabla U\|^q_{L^s(\Omega)}\right).$$

Inserting the v-estimate from above and exploiting $\|\nabla^\varepsilon_d A\|_{L^\rho(\Omega, C^0(Y))} \leq c$ from (1.5), we find

$$\|\nabla U\|_{L^q(\Omega)} \leq c\left(\|F\|_{L^p(\Omega)} + \|\nabla U\|_{L^{q-\delta}(\Omega)} + \varepsilon^{(n+\alpha q - nq/s)/q}\|\nabla U\|_{L^s(\Omega)}\right)$$

$$\leq c\left(\|F\|_{L^p(\Omega)} + \eta\|\nabla U\|_{L^q(\Omega)} + C_\eta \|U\|_{H^1(\Omega)} + \varepsilon^{(n+\alpha q - nq/s)/q}\|\nabla U\|_{L^s(\Omega)}\right),$$

where $\eta > 0$ can be chosen arbitrarily small such that we can absorb the second term of the right hand side into the left hand side. With the observation from the beginning of the proof we finally have

$$\|\nabla U\|_{L^q(\Omega)} \leq c\left(\|F\|_{L^p(\Omega)} + \varepsilon^{(n+\alpha q - nq/s)/q}\|\nabla U\|_{L^s(\Omega)}\right). \tag{3.2}$$

We note that the exponent of ε is positive for $(q/s) - 1 > 0$ small. We can therefore conclude the result by using (3.2) a finite number of times with indices s_k and $s_{k+1} = q_k = \Theta s_k$, $\Theta > 1$ fixed, starting with $s_0 = 2$. We note that we have to iterate only until $s_k > n/\alpha$, therefore the number of iterations is independent of q.

The case $q = \infty$. Only minor changes in the above arguments are necessary to treat the case $q = \infty$. Theorem 1 provided the estimate for the finite difference quotients v, which now reads

$$\|v\|_{L^\infty(\Omega)} \leq c\left(\|F\|_{L^p(\Omega)} + \|\nabla U\|_{L^s}\right),$$

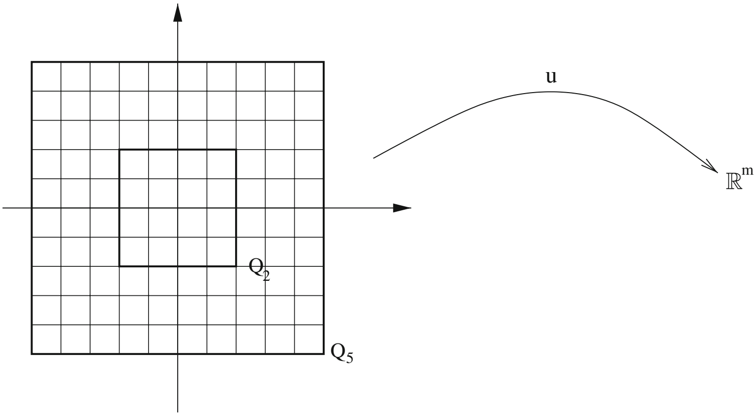

Fig. 2 Sketch of the geometry. Indicated are, for the two-dimensional case, the cubic domain Q_5 of the solution u, and the domain Q_2, for which L^2-estimates are derived

for some large $s < \infty$, the constant c then depends on s. The local lemma implies

$$\begin{aligned} &\sup_{\varepsilon(j+Q_0)} |\varepsilon \nabla_x U| \\ &\leq c\left(\sup_{\varepsilon(j+Q_4)} |\varepsilon v| + \varepsilon^{2-(n/p)} \|F\|_{L^p(\varepsilon(j+Q_5))} + \varepsilon^{\alpha-(n/s)} \|\varepsilon \nabla_x U\|_{L^s(\varepsilon(j+Q_5))} \right). \end{aligned}$$

Dividing by ε and taking the supremum over j we find, for $s = n/\alpha$,

$$\begin{aligned} \|\nabla U\|_{L^\infty(\Omega)} &\leq c\left(\|v\|_{L^\infty(\Omega)} + \|F\|_{L^p(\Omega)} + \varepsilon^{\alpha-\frac{n}{s}} \|\nabla U\|_{L^s(\Omega)} \right) \\ &\leq c\left(\|F\|_{L^p(\Omega)} + \|\nabla U\|_{L^s} \right). \end{aligned}$$

Together with the L^s-estimate of the first part of the proof ($q < \infty$), this provides the desired L^∞-estimate. □

The key in our finite difference approach was the local lemma, which we show next. Loosely speaking, the lemma asserts the following: locally, the gradient of a solution is as good as we can expect from the finite differences and from the right hand side.

The situation is as sketched in Fig. 2. We consider cubes $Q_l = (-l, l)^n$, $l = 1, \ldots, 5$, solutions $u : Q_5 \to \mathbb{R}^m$, and investigate the gradient ∇u on the smallest cube Q_1. We assume that for $\Omega \subset \mathbb{R}^n$ the coefficients are maps $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^{n^2 \times m^2}$ which are $[0, 1]^n$-periodic in y, continuous, and uniformly elliptic. For some exponent $\alpha \in (0, 1)$, which provides a small factor in the final estimate, we assume that for every y the map $A(., y)$ is Hölder-continuous with exponent 2α with a y-independent upper bound. The assumptions are met by A satisfying (1.3)–(1.5).

Lemma 2 (Local lemma) *Let $K \subset \Omega \subset \mathbb{R}^n$ be compact, $\xi \in K$ a parameter, $f \in L^p(Q_5)$ a right hand side, and $\varepsilon > 0$ sufficiently small. Let the pair (p, q) satisfy either $p \in [2, n)$, $q \leq np/(n-p)$, or $p > n$, $q = \infty$. We consider solutions $u : Q_5 \to \mathbb{R}^m$ of*

$$\nabla \cdot (A(\xi + \varepsilon y, y)\nabla u(y)) = f(y) \quad \forall y \in Q_5. \tag{3.3}$$

We assume to have a control on difference quotients of length 1,

$$u(y + e_d) - u(y) = g_d(y) \quad \forall y \in Q_4, \tag{3.4}$$

for $d = 1, \ldots, n$, $g : Q_4 \to \mathbb{R}^{m\times n} \in L^q$. *Then*

$$\|\nabla u\|_{L^q(Q_1)} \leq c\left(\|g\|_{L^q(Q_4)} + \|f\|_{L^p(Q_5)} + \varepsilon^\alpha \|\nabla u\|_{L^2(Q_5)}\right) \tag{3.5}$$

with c *depending on* q *and* A, *but independent of* f, g, u, ε, *and* ξ.

Proof We first show the estimate by a contradiction argument for $p = 2$, $q = 2$, and on the set Q_2 instead of Q_1. To this end, let us assume that the estimate fails for some A in dimension n. We then find sequences u^k, g^k, f^k, $\xi^k \to \xi$, and ε_k of solutions of (3.3) and (3.4) such that, after rescaling and subtraction of averages,

$$g^k \to 0 \text{ in } L^2(Q_4), \quad f^k \to 0 \text{ in } L^2(Q_5), \quad \varepsilon_k^\alpha \nabla u^k \to 0 \text{ in } L^2(Q_5), \tag{3.6}$$

$$\|\nabla u^k\|_{L^2(Q_2)} = 1, \quad \int_{Q_2} u^k = 0. \tag{3.7}$$

We see that necessarily $\varepsilon_k \to 0$. We observe that each function $g_d^k : Q_4 \to \mathbb{R}^m$ is an H^1-solution of

$$\begin{aligned}\nabla\cdot(A(\xi^k + \varepsilon_k y, y)\nabla g_d^k(y)) = f^k(y + e_d) - f^k(y)\\ -\nabla\cdot\left(\left[A(\xi^k + \varepsilon_k y + \varepsilon_k e_d, y) - A(\xi^k + \varepsilon_k y, y)\right]\right.\\ \left.\times\ \nabla u^k(y + e_d)\right).\end{aligned}$$

The difference of the coefficients in the squared brackets is pointwise bounded by $C\varepsilon_k^{2\alpha}$. Multiplication of this equation with $g_d^k \eta$ with a cut-off function $\eta \in C_0^\infty(Q_4)$ yields

$$\|g^k\|_{H^1(Q_3)} \leq c\left(\|g^k\|_{L^2(Q_4)} + \|f^k\|_{L^2(Q_5)} + \varepsilon_k^\alpha \|\nabla u^k\|_{L^2(Q_5)}\right) \to 0.$$

This, together with (3.4) and (3.7) implies

$$\|u^k\|_{H^1(Q_3)} \leq C.$$

Choosing a subsequence we may assume for some limit function $u \in H^1(Q_3)$

$$u^k \to u \text{ strongly in } L^2(Q_3) \text{ and weakly in } H^1(Q_3),$$

and u is a weak solution of

$$\nabla\cdot(A(\xi, y)\nabla u(y)) = 0.$$

The strong convergence of u^k implies that u satisfies relation (3.4) with $g \equiv 0$ on Q_2. Hence u is a periodic solution of the homogeneous problem and must therefore be constant, thus, by (3.7), $u \equiv 0$. Finally, exploiting that u^k is a solution of (3.3), we conclude

$$\|u^k\|_{H^1(Q_2)} \leq c\left(\|u^k\|_{L^2(Q_3)} + \|f^k\|_{L^2(Q_3)}\right) \to 0,$$

which contradicts (3.7).

The general estimate on Q_1 with exponent $p \geq 2$, for $q \leq np/(n-p)$, follows from interior regularity estimates for solutions with bounded H^1-norm on Q_2. □

4 Two-scale expansions

In this section we exploit the two-scale expansion of solutions and complete the proof of Theorem 2. We always assume the situation of Theorem 2, item 2, in particular the regularity assumption $A \in C^{0,1}(\Omega, W^{1,n}_{per}(Y)) \cap C^{1,1}(\Omega, L^n(Y))$ from (1.6).

We perform all calculations in the scalar case $m = 1$, the case $m > 1$ introduces only notational difficulties. Let $w_k = w_k(x, y)$ be the solutions of the cell-problems

$$\nabla_y \cdot \big(A(x, y)[\nabla_y w_k(x, y) + e_k]\big) = 0 \quad \text{in } Y,$$
$$w_k(x, .)\ Y\text{-periodic.}$$

We first check boundedness properties of w_k. The uniform continuity of A allows to conclude, for every compact subset $\Omega' \subset \Omega$ and every $s < \infty$, the uniform bound $\|w_k(x, \cdot)\|_{W^{1,s}} \le C(\Omega')$ for all $x \in \Omega'$, i.e., $w_k \in L^\infty(\Omega'; W^{1,s}(Y))$, see e.g. [11], p. 73. But even a much stronger estimate can be shown. An arbitrary x-derivative $W(x, y) = \partial_{x_l} w_k(x, y)$ satisfies the equation

$$\nabla_y \cdot \big(A(x, y)[\nabla_y W(x, y)]\big) = -\nabla_y \cdot \big(\partial_{x_l} A(x, y)[\nabla_y w_k(x, y) + e_k]\big)\,.$$

For every $x \in \Omega$, the right hand side is the divergence of a bounded function in $L^p(Y)$ for every $p < \infty$. We conclude the uniform boundedness of

$$w_k \in C^{0,1}(\Omega', W^{1,s}(Y)).$$

Second derivatives can be treated in the same way to find bounds for $w_k \in C^{1,1}(\Omega', L^s(Y))$.

With the help of the functions w_k we may, for an arbitrary smooth function η_0, construct the two-scale approximation function

$$\eta^\varepsilon(x) = \eta_0(x) + \varepsilon \sum_{k=1}^n \partial_k \eta_0(x)\, w_k(x, x/\varepsilon). \tag{4.1}$$

The function is constructed in such a way that the application of $\mathcal{L}^\varepsilon$ yields a bounded object.

$$\begin{aligned}
\nabla\eta^\varepsilon(x) &= \sum_{k=1}^n \partial_k\eta_0(x)[e_k + \nabla_y w_k(x, x/\varepsilon)] + \varepsilon \sum_{k=1}^n \nabla_x(\partial_k\eta_0(x)\, w_k(x, x/\varepsilon))\\
\mathcal{L}^\varepsilon\eta^\varepsilon(x) &= -\sum_{k=1}^n \nabla_x \cdot (A(x, x/\varepsilon)\, \partial_k\eta_0(x)\, [e_k + \nabla_y w_k(x, x/\varepsilon)])\\
&\quad -\varepsilon \sum_{k=1}^n \nabla \cdot (A(x, x/\varepsilon) \cdot \nabla_x(\partial_k\eta_0(x)\, w_k(x, x/\varepsilon))).
\end{aligned}$$

In the case $m > 1$ we use also scalar test-functions $\eta^\varepsilon : \Omega \to \mathbb{R}$, but they are interpreted as representing a variation in direction $\beta \in \{1, \dots, m\}$, the cell solutions are $w_k^\beta : Y \to \mathbb{R}^m$, and in the last line above we then calculate $\mathcal{L}^\varepsilon(\eta^\varepsilon e_\beta) : \Omega \to \mathbb{R}^m$.

Proof of Theorem 2, item 2 We consider an H^1-solution U of $\mathcal{L}^\varepsilon U = F$ on $B_R = B_R(0) \subset \mathbb{R}^n$ for $F \in L^p(B_R)$. Our aim is to derive, for some $\Theta \in (0, 1)$, an estimate

$$\|\nabla U\|_{L^q(B_{\Theta R})} \le c\left(\|F\|_{L^p(B_R)} + \|U\|_{H^1(B_R)}\right). \tag{4.2}$$

By a covering argument, this yields the claimed estimate on arbitrary compactly contained subsets $\Omega' \subset\subset \Omega$.

Let $\eta_0 \in C_0^\infty(B_{R/2})$ be a cut-off function with $\eta_0 \equiv 1$ on $B_{R/4}$. We use η^ε of (4.1) with support in $B_{R/2}$. The function $V^\varepsilon := U \cdot \eta^\varepsilon$ satisfies

$$\begin{aligned}\mathcal{L}^\varepsilon V^\varepsilon(x) &= -\nabla \cdot \big(A(x, x/\varepsilon)\nabla U(x)\eta^\varepsilon(x)\big) - \nabla \cdot \big(U(x)A(x, x/\varepsilon)\nabla\eta^\varepsilon(x)\big)\\ &= \eta^\varepsilon(x)\mathcal{L}^\varepsilon U(x) - 2\nabla U(x)A(x, x/\varepsilon)\nabla\eta^\varepsilon(x) + U(x)\mathcal{L}^\varepsilon\eta^\varepsilon(x).\end{aligned}$$

The regularity estimates for w_k imply uniform bounds for any $s < \infty$,

$$\nabla\eta^\varepsilon \in L^s(B_R), \qquad \mathcal{L}^\varepsilon\eta^\varepsilon \in L^n(B_R).$$

Inserting this above we find for $q = np/(n-p)$ the estimate

$$\begin{aligned}\|\mathcal{L}^\varepsilon V^\varepsilon\|_{L^p(B_R)} &\le c\ \big(\|F\|_{L^p(B_R)} + \|\nabla U\|_{L^{p+\delta}(B_R)} + \|U\|_{L^q(B_R)}\big)\\ &\le c\ \big(\|F\|_{L^p(B_R)} + \|U\|_{W^{1,p+\delta}(B_R)}\big)\end{aligned}$$

for some small $\delta > 0$. We can apply Theorem 2, item 1 to V^ε and find the $L^q(B_R)$-estimate for ∇V^ε. We note that in $B_{R/4}$ the gradients coincide, $\nabla V^\varepsilon = \nabla U^\varepsilon$, therefore

$$\|U\|_{W^{1,q}(B_{R/4})} \le c\left(\|F\|_{L^p(B_R)} + \|U\|_{W^{1,p+\delta}(B_R)}\right). \tag{4.3}$$

We can iterate this estimate, starting with $p = 2$. We arrive at an arbitrary q (including $q = \infty$) after a number of iterations that depends only on n and δ. This yields (4.2).

We note that, in order to start the iteration process with $p = 2$, we need a bound for $\nabla U \in L^{2+\delta}_{loc}$. This estimate is a consequence of Meyers estimate [18] in the scalar case. For systems, the estimate follows from reverse Hölder inequalities and Gehring's Lemma (cf. e.g. [11] p. 24 and p. 107, or [12]). In both cases, the estimate is independent of the modulus of continuity of the coefficients. □

Application to a corrector result

On a domain $\Omega \subset \mathbb{R}^n$ we study the homogenization problem

$$\mathcal{L}^\varepsilon u^\varepsilon = f \quad \text{in } \Omega,\ u^\varepsilon = 0 \quad \text{on } \partial\Omega$$

with coefficients $A(x, y)$ of the operator satisfying (1.3)–(1.6). We denote by $\eta_0 : \Omega \to \mathbb{R}$ the solution of the homogenized problem

$$\mathcal{L}^*\eta_0 = f \quad \text{in } \Omega,\ \eta_0 = 0 \quad \text{on } \partial\Omega,$$

and by η^ε from (4.1) the approximate solution to the ε-problem. For $f \in L^2$, the following corrector result holds ([1], Theorem 2.6): If

$$\eta_1(x, y) = \sum_{k=1}^n \partial_k\eta_0(x)\, w_k(x, y) \tag{4.4}$$

is such that η_1, $\nabla_x\eta_1$, and $\nabla_y\eta_1$ are admissible, then

$$u^\varepsilon - \eta^\varepsilon \to 0 \text{ strongly in } H^1(\Omega). \tag{4.5}$$

For the concept of admissibility we refer to [1], Definition 1.4, and the discussion thereafter.

Corollary 1 *Let coefficients A satisfy (1.3)–(1.6) and let $f \in L^p(\Omega)$. For $q' < q = np/(n-p)$ and Ω' a compactly included subdomain of Ω there holds*

$$u^\varepsilon - \eta^\varepsilon \to 0 \textit{ strongly in } W^{1,q'}(\Omega'). \tag{4.6}$$

Proof It suffices to verify the admissibility hypothesis for (4.5) and to provide uniform $L^q(\Omega')$-estimates for ∇u^ε and $\nabla \eta^\varepsilon$. Then the convergence of (4.5) implies the strong convergence in intermediate Lebesgue spaces as claimed. We note that Theorem 2, item 2, provides the uniform bound for $\nabla u^\varepsilon \in L^q(\Omega')$ with $q = np/(n-p) > p$. The boundedness of $\nabla \eta^\varepsilon \in L^s(\Omega')$ for every $s < \infty$ was already observed in the proof of Theorem 2, item 2.

It remains to analyze the regularity properties of η_0 and η_1. The homogenized operator $\mathcal{L}^*$ has Hölder-continuous coefficients $A^*(x)$, hence $\nabla_x \eta_0 \in L^q(\Omega)$. Furthermore, every x-derivative $\partial_k \eta_0$ of η_0 satisfies the equation

$$-\nabla \cdot (A^* \nabla \partial_k \eta_0) = \partial_k f + \nabla \cdot (\partial_k A^* \cdot \nabla \eta_0).$$

The right hand side is the divergence of a function in $L^p(\Omega)$ and we conclude $\eta_0 \in W^{2,p}(\Omega')$.

Regarding η_1 we have to study the cell problem. We find

$$\nabla_x \eta_1(x, y) = \sum_{k=1}^{n} \left(\nabla \partial_k \eta_0(x)\, w_k(x, y) + \partial_k \eta_0(x)\, \nabla_x w_k(x, y) \right),$$

hence $\nabla_x \eta_1 \in L^p(\Omega, C^0(Y))$, and

$$\nabla_y \eta_1(x, y) = \sum_{k=1}^{n} \partial_k \eta_0(x)\, \nabla_y w_k(x, y) \;\Rightarrow\; \nabla_y \eta_1 \in L^p(\Omega, C^0(Y)).$$

Therefore η_1, $\nabla_x \eta_1$, and $\nabla_y \eta_1$ are admissible and (4.5) holds. This concludes the proof. □

Appendix A: Remarks on the interpolation argument

The interpolation argument requires an off-diagonal version of the well-known interpolation theorem of Stampacchia, cf. [20], that only requires an (L^∞, BMO) bound at the upper endpoint. For the readers' convenience we briefly sketch the argument in our specific situation, mainly based on the classical Marcinkiewizc interpolation theorem along the lines of [8].

We let $Q \subset \Omega \subset \mathbb{R}^n$ and suppose that T is linear and bounded as a mapping $T : L^2(\Omega) \to L^{2^*}(Q)$ $(2^* = 2n/(n-2))$ and $T : L^n(\Omega) \to \mathrm{BMO}(Q)$, respectively. We take any subdivision $\{Q_i\}_{i \in I}$ of the cube Q. Accordingly, we define

$$\mathcal{T} f(x) = \fint_{Q_i} |Tf - \fint_{Q_i} Tf| \quad \text{if} \quad x \in Q_i.$$

Then $\mathcal{T}$ is subadditive and bounded as a mapping $\mathcal{T} : L^2(\Omega) \to L^{2^*}(Q)$ and $\mathcal{T} : L^n(\Omega) \to L^\infty(Q)$, respectively. We infer from the Marcinkiewicz interpolation theorem, cf. [21], Chapt. V, Theorem 2.4, that $\mathcal{T} : L^p(\Omega) \to L^q(Q)$ continuously for any admissible (p, q) pair, i.e.,

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{2^*} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{n} \quad \text{for some} \quad \theta \in (0, 1),$$

i.e., $q = np/(n-p)$ with $2 < p < n$, with bounds that are independent of the choice of subdivisions. Thus taking the supremum over the set Δ of subdivisions of Q (and associated operators $\mathcal{T}$) we find

$$\sup_{\{Q_i\}\in\Delta}\sum_{i\in I}|Q_i|\left(\fint_{Q_i}|Tf-\fint_{Q_i}Tf|\right)^q=\sup_{\mathcal{T}\in\Delta}\|\mathcal{T}f\|^q_{L^q(Q)}\leq C\,\|f\|^q_{L^p(\Omega)}$$

with a constant C that only depends on θ and the known bounds on T. By a result of John and Nirenberg, cf. [13], the latter quantity bounds the weak L^q norm of $\tilde{T}f=Tf-\fint_Q Tf$. Thus, for any admissible pair (p,q), the operator $\tilde{T}$ is weakly bounded. Further interpolation and application of the Marcinkiewicz interpolation theorem implies in turn

$$\|\tilde{T}f\|_{L^q(Q)}\leq C\,\|f\|_{L^p(\Omega)}\quad\text{thus}\quad\|Tf\|_{L^q(Q)}\leq C\left(\|f\|_{L^p(\Omega)}+\fint_Q|Tf|\right)$$

where C only depends on the previously known bounds on T and $p\in(2,n)$. Now, if we take as in our application $T=\mathcal{L}_\varepsilon^{-1}\mathrm{div}:f\mapsto u$ with end-point bounds that are independent of $\varepsilon>0$, we get with Hölder's inequality

$$\|\tilde{T}f\|_{L^q(Q)}\leq C\,\|f\|_{L^p(\Omega)}\quad\text{thus}\quad\|\mathcal{L}_\varepsilon^{-1}f\|_{L^q}\leq C(n,p,Q)\left(\|f\|_{L^p(\Omega)}+\|u\|_{L^2(\Omega)}\right)$$

for $n>2$, any $p\in(2,n)$ and $q=np/(n-p)$.

References

1. Allaire, G.: Homogenization and two-scale convergence. SIAM J. Math. Anal. **23**(6), 1482–1518 (1992)
2. Avellaneda, M., Lin, F.H.: Compactness methods in the theory of homogenization. Commun. Pure Appl. Math. **40**(6), 803–847 (1987)
3. Avellaneda, M., Lin, F.H.: Counterexamples related to high-frequency oscillation of Poisson's kernel. Appl. Math. Optim. **15**(2), 109–119 (1987)
4. Avellaneda, M., Lin, F.H.: Compactness methods in the theory of homogenization. II. Equations in non-divergence form. Commun. Pure Appl. Math. **42**(2), 139–172 (1989)
5. Avellaneda, M., Lin, F.H.: L^p bounds on singular integrals in homogenization. Commun. Pure Appl. Math. **44**(8–9), 897–910 (1991)
6. Astala, K., Faraco, D., Székelyhidi, L.: Convex integration and the L^p theory of elliptic equations. MPI-MIS (2004, preprint)
7. Caffarelli, L.A., Peral, I.: On $W^{1,p}$ estimates for elliptic equations in divergence form. Commun. Pure Appl. Math. **51**(1), 1–21 (1998)
8. Campanato, S.: Su un teorema di interpolatione di G Stampacchia. Ann. Sc. Norm. Sup. Pisa **20**, 649–652 (1966)
9. Campanato, S.: Equazioni ellittiche del secondo ordine e spazi $\mathcal{L}^{2,\lambda}$, Ann. Mat. Pura Appl. **69**, 321–380 (1965)
10. Giaquinta, M.: Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems. Princeton University Press, Princeton (1983)
11. Giaquinta, M.: Introduction to regularity theory for nonlinear elliptic systems. Lectures in Mathematics ETH Zürich. Birkhäuser, Basel (1993)
12. Giusti, E.: Direct Methods in the Calculus of Variations. World Scientific, Singapore (2003)
13. John, F., Nirenberg, L.: On functions of bounded mean oscillation. Commun. Pure Appl. Math. **14**, 415–426 (1961)
14. Kristensen, J., Melcher, C.: Regularity in oscillatory nonlinear elliptic systems. Math. Z. (2008, in press)
15. Li, Y., Nirenberg, L.: Estimates for elliptic systems from composite material. Commun. Pure Appl. Math. **56**(7), 892–925 (2003)
16. Lipton, R.: Homogenization theory and the assessment of extreme field values in composites with random microstructure. SIAM J. Appl. Math. **65**(2), 475–493, 2004/2005 (electronic)

17. Morrey, C.B. Jr. : Multiple Integrals in the Calculus of Variations. Springer Heidelberg, New York (1966)
18. Meyers, N.G.: An L^p-estimate for the gradient of solutions of second order elliptic divergence equations. Ann. Sc. Norm. Sup. Pisa **17**(3), 189–206 (1963)
19. Schweizer, B.: Uniform estimates in two periodic homogenization problems. Commun. Pure Appl. Math. **53**(9), 1153–1176 (2000)
20. Stampacchia, G.: The spaces $L^{p,\lambda}$, $N^{p,\lambda}$ and interpolation. Ann. Sc. Norm. Sup. Pisa **19**, 443–462 (1965)
21. Stein, E.M., Weiss, G.: Introduction to Fourier Analysis on Euclidean Space. Princeton University Press, Princeton (1975)