Annali di Matematica (2009) 188:235–245
DOI 10.1007/s10231-008-0073-6

Characterization of balls by Riesz-potentials

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Received: 25 October 2007 / Revised: 22 January 2008 / Published online: 26 March 2008

Abstract For a bounded convex domain $G \subset \mathbb{R}^N$ and $2 < \alpha \neq N$ consider the unit-density Riesz-potential $u(x) = \int_G |x-y|^{\alpha-N}\,dy$. We show in this paper that $u =$ const. on ∂G if and only if G is a ball. This result corresponds to a theorem of L.E. Fraenkel, where the ball is characterized by the Newtonian-potential ($\alpha = 2$) of unit density being constant on ∂G. In the case $\alpha = N$ the kernel $|x-y|^{\alpha-N}$ is replaced by $-\log|x-y|$ and a similar characterization of balls is given. The proof relies on a recent variant of the moving plane method which is suitable for Green-function representations of solutions of (pseudo-)differential equations of higher-order.

Keywords Riesz-potential · Pseudo-differential operator · Moving plane method · Radial symmetry

Mathematics Subject Classification (2000) Primary: 31B30, 35J30 · Secondary: 31B35, 35S99

1 Introduction

In Newton's theory of gravitation the potential of a ball $B_R(0) \subset \mathbb{R}^3$ of constant mass density $\rho > 0$ is given by

$$u(x) = \frac{1}{4\pi} \int\limits_{B_R(0)} \frac{\rho}{|x-y|}\,\mathrm{d}y = \begin{cases} \rho\left(\dfrac{R^2}{2} - \dfrac{|x|^2}{6}\right), & |x| \leq R, \\ \dfrac{\rho R^3}{3|x|}, & |x| \geq R. \end{cases}$$

Outside the ball the gravitational potential coincides with that of a single point centered at the origin whose mass equals the mass of the entire ball. This observation (and its generalization

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to radially symmetric mass densities) allows to reduce celestial mechanics of stars and planets to the interaction of point masses. Similar properties hold for the Newtonian potential of an N-dimensional ball $N \geq 4$ and for the two-dimensional logarithmic potential of a disk in $\mathbb{R}^2$. Note that the gravitational potential of a ball of constant mass density is constant on the surface of the ball. This property in fact uniquely characterizes the balls, as it was shown by Fraenkel [7] through the following theorem.

Theorem 1 (Fraenkel 2000) *Let $G \subset \mathbb{R}^N$ be a bounded open set and let ω_N be the surface measure of the unit-sphere in $\mathbb{R}^N$. Consider*

$$u(x) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_G \log \frac{1}{|x-y|}\, \mathrm{d}y, & N = 2, \\ \dfrac{1}{(N-2)\omega_N} \displaystyle\int_G \frac{1}{|x-y|^{N-2}}\, \mathrm{d}y, & N \geq 3. \end{cases}$$

If u is constant on ∂G then G is a ball.

One of the striking aspects of Fraenkel's theorem is that no regularity of G is assumed a priori. The goal of this paper is to prove for Riesz-potentials the following analogue of the above result. Unlike in Theorem 1 we need to a priori restrict the class of open sets.

Theorem 2 *Let $G \subset \mathbb{R}^N$ be a bounded convex domain. For $\alpha > 2$ consider*

$$u(x) = \begin{cases} \displaystyle\int_G \log \frac{1}{|x-y|}\, \mathrm{d}y, & N = \alpha, \\ \displaystyle\int_G \frac{1}{|x-y|^{N-\alpha}}\, \mathrm{d}y, & N \neq \alpha. \end{cases} \tag{1}$$

If u is constant on ∂G then G is a ball.

It is easy to see that the converse of both Theorems 1 and 2 hold. Suppose $G = B_R(0)$ is a ball centered at the origin. Then u is radially symmetric and hence u is constant on ∂G.

Let us give some heuristic arguments for Fraenkel's theorem. The Newtonian potential in Theorem 1 satisfies

$$-\Delta u = 1 \quad \text{in } G, \qquad -\Delta u = 0 \quad \text{in } \mathbb{R}^N \backslash \overline{G}$$

and by assumption $u = \beta$ on ∂G. If one considers the two boundary value problems (here we assume $N \geq 3$)

$$(*) \begin{cases} -\Delta u_i = 1 & \text{in } G, \\ u_i = \beta & \text{on } \partial G \end{cases} \qquad (**) \begin{cases} -\Delta u_e = 0 & \text{in } \mathbb{R}^N \backslash \overline{G}, \\ u_e = \beta & \text{on } \partial G,\ u_e \to 0 \quad \text{at } \infty \end{cases}$$

then there exist unique solutions u_i, u_e, and they must coincide with u. The fact that u is a $C^1(\mathbb{R}^N)$ function means that next to the boundary values $u_i = u_e = \beta$ on ∂G also the normal derivatives of u_i, u_e have to coincide on ∂G. For an arbitrary domain G this would not be the case. Thus, $(*)$, $(**)$ together with matched normal derivatives is an overdetermined problem, which explains why the shape of G cannot be arbitrary. In fact, the only way to resolve $(*)$, $(**)$ and simultaneously match the normal derivatives is by G being a ball. Note that in Fraenkel's theorem no regularity of ∂G is assumed, so that in general normal derivatives of u_i, u_e cannot be understood in the classical sense.

Let us discuss similarly the Riesz-potentials of Theorem 2. First we recall fundamental solutions $G(x,y)$ of the pseudo-differential operators $(-\Delta)^{\alpha/2}$ in $\mathbb{R}^N$, $\alpha > 0$. In case $\frac{1}{2}(\alpha - N) \notin \mathbb{N}_0$ (i.e., either $0 < \alpha < N$ or $\alpha \geq N$ but $\alpha - N$ is not an even natural number) then

$$G(x,y) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{2^\alpha \pi^{N/2}\Gamma\left(\frac{\alpha}{2}\right)}|x-y|^{\alpha-N}$$

whereas if $\alpha - N = 2k$, $k \in \mathbb{N}_0$ then

$$G(x,y) = \frac{(-1)^k}{2^{\alpha-1}\pi^{N/2}\Gamma\left(\frac{\alpha}{2}\right)}|x-y|^{\alpha-N}\log\frac{1}{|x-y|}.$$

It follows that for $(\alpha - N)/2 \notin \mathbb{N}_0$ the potential u of Theorem 2 satisfies in the distributional sense (χ_G is the characteristic function of the set G)

$$(-\Delta)^{\alpha/2}u = \text{const.}\,\chi_G \quad \text{in } \mathbb{R}^N$$

together with $u = \beta$ on ∂G, $u \in C^l(\mathbb{R}^N)$ for $1 \leq l < \alpha$. Note that for $\alpha = 2m$ the potential u satisfies a polyharmonic equation in $\mathbb{R}^N$. For general $\alpha > 2$ there is no analogue of the two boundary value problems $(*)$, $(**)$ as in the second-order case. It is therefore remarkable that the mere information of ∂G being a level set of u completely determines u and G. Even in the case $\alpha = 2m$ the boundary value problems analogous to $(*)$, $(**)$ are underdetermined individually since only one boundary datum is prescribed. But if they are viewed as a system coupled by the fact that $u \in C^{2m-1}(\mathbb{R}^N)$ coincides with u_i in G and u_e outside G then they become overdetermined.

We finish this discussion with the following two open problems:

(i) Is Theorem 2 true if the assumption of convexity of G is dropped?
(ii) Is there an analogous result as in Theorem 2 for potentials

$$u(x) = \int_G |x-y|^{\alpha-N}\log\frac{1}{|x-y|}\,\mathrm{d}y?$$

The most interesting case would be the case where $\alpha - N = 2k$ with $k \in \mathbb{N}_0$ since then the kernel function is (up to a normalization constant) the fundamental solution of $(-\Delta)^{\alpha/2}$.

The main reason why both questions remain open is the fact that the validity of Lemma 4 is not clear under these assumptions, cf. the remark following Lemma 4.

In the Newtonian case a number of potential-theoretic characterizations of balls are known in the literature. If instead of a volume potential one considers a single-layer potential u concentrated on ∂G with constant density, then G is a ball if and only if u is constant on ∂G. This conjecture of P. Gruber (cf. Heil and Martini [10]) has been verified for different smoothness classes of domains. The two-dimensional case was considered by Martensen [14], Gardiner [8] and Ebenfelt et al. [6] and the higher-dimensional case by Reichel [18], Mendez and Reichel [15] and Sirakov [21]. We mention that in [15] only convexity of the underlying domain was assumed. Similar characterizations of annuli were given by Payne, Philippin [16] and Philippin [17] and different single-layer characterizations of balls were achieved by Shahgholian [20] and Mikyoung Lim [13].

Our approach is based on a new variant of the moving plane method. The classical moving plane method is based on the pointwise maximum principle for second order elliptic equations. It was developed by Alexandrov [1], Serrin [19] and Gidas et al. [9]. Very recently some important improvements of the moving plane method were achieved by Chang and Yang [4], Berchio et al. [2], Li [12], Chen et al. [5], Jin and Li [11] and Birkner et al. [3]. These new variants of the moving plane method are applied to the integral equation resulting from the Green-function representation, cf. Lemma 10 below. In this way symmetry results for higher-order elliptic problems as well as pseudo-differential equations can be achieved although pointwise maximum principles are not available.

The paper is organized as follows. In Sect. 2 we provide some basic estimates for the far-field of the potential. In Sect. 3 the moving-plane procedure is carried out.

2 Estimates for the Riesz-potentials

Throughout the paper let $\alpha > 2$ and let u denote the function defined in (1).

Lemma 3 *Let $l \in \mathbb{N}$ with $1 \le l < \alpha$. Then $u \in C^l(\mathbb{R}^N)$ and differentiation of order l can be taken under the integral.*

Proof The result is standard. We give a proof for the reader's convenience. We consider the case $\alpha \neq N$; the proof for $\alpha = N$ is just a slight variant. Let $\eta : [0,\infty) \to [0,1]$ be a C^∞-function with $\eta \equiv 0$ on $[0,1]$ and $\eta \equiv 1$ on $[2,\infty)$. Let $\mu = (\mu_1,\dots,\mu_N)$ be a multi-index of order $|\mu| = l$ and let $c_1(l), c_2(l), \dots$ denote constants which only depend on l. For $\epsilon > 0$ let $\eta_\epsilon(t) := \eta(t/\epsilon)$ and define

$$u_\epsilon(x) := \int_G \frac{\eta_\epsilon(|x-y|)}{|x-y|^{N-\alpha}}\,\mathrm{d}y, \qquad v_\mu(x) := \int_G D_x^\mu \frac{1}{|x-y|^{N-\alpha}}\,\mathrm{d}y.$$

Note that $\left|D_x^\mu |x-y|^{\alpha-N}\right| \le \text{const.}\,|x-y|^{\alpha-N-l}$ with $\alpha - N - l > -N$. Therefore $v_\mu(x)$ exists for all $x \in \mathbb{R}^N$. Furthermore

$$\begin{aligned}
|D^\mu u_\epsilon(x) - v_\mu(x)| &\le \int_G D_x^\mu \Big(\big(1-\eta_\epsilon(|x-y|)\big)|x-y|^{\alpha-N}\Big)\,\mathrm{d}y \\
&\le c_1(l) \sum_{|\nu|+|\nu'|=l} \int_G D_x^\nu\big(1-\eta_\epsilon(|x-y|)\big) D_x^{\nu'}|x-y|^{\alpha-N}\,\mathrm{d}y \\
&\le c_2(l) \sum_{|\nu|+|\nu'|=l} \int_G \epsilon^{-|\nu|}|x-y|^{\alpha-N-|\nu'|}\,\mathrm{d}y \\
&\le c_3(l)\epsilon^{\alpha-l} \to 0 \quad \text{as} \quad \epsilon \to 0.
\end{aligned}$$

Thus $D^\mu u_\epsilon$ converges uniformly on $\mathbb{R}^N$ to v_μ for all multi-indices μ with $|\mu| < \alpha$. This establishes the proof. □

In the following we assume that G is convex and that $u = \text{const.} = \beta$ on ∂G.

Lemma 4 *If $N \ge \alpha$ then $u(x) < \beta$ for $x \in \mathbb{R}^N \setminus \overline{G}$ and $u(x) > \beta$ for $x \in G$. If $N < \alpha$ then $u(x) > \beta$ for $x \in \mathbb{R}^N \setminus \overline{G}$ and $u(x) < \beta$ for $x \in G$.*

Remark In the computations below we use that the kernel function $|x-y|^{\alpha-N}$ has monotonicity and sub-/superharmonicity properties. In general this is not the case for kernels of the form $|x-y|^{\alpha-N}\log 1/|x-y|$. Moreover, it is an open problem how to overcome the convexity assumption of G in the proof below.

Proof Lemma 3 shows that u is a $C^2(\mathbb{R}^N)$-function since $\alpha > 2$. Note that $\Delta|x|^{\alpha-N} = (\alpha-N)(\alpha-2)|x|^{\alpha-N-2}$ and $\Delta \log\frac{1}{|x|} = (2-N)|x|^{-2}$. Let us first consider the case $N \geq \alpha > 2$. In this case u is superharmonic and hence inside G the function u is larger than the value β of u on ∂G. In the case $2 \leq N < \alpha$ the function u is subharmonic and hence inside G the function u is smaller than its value β on the boundary. It remains to consider u outside G. We show that the convexity of G implies that u has no local extremum outside G. Since either $u(x) \to 0, \infty$ or $-\infty$ as $|x| \to \infty$ this implies that u is smaller (larger) than β outside G. So let $x \in \mathbb{R}^N \backslash \overline{G}$. By the convexity of G we can separate x from G through a hyperplane, i.e., there exists a unit vector $e \in \mathbb{R}^N$ and a point $z_0 \in \mathbb{R}^N \backslash \overline{G}$ such that

$$(y-z_0)\cdot e < 0 < (x-z_0)\cdot e \quad \text{for all } y \in G.$$

In particular $(x-y)\cdot e > 0$ for all $y \in G$. Since

$$\nabla u(x)\cdot e = c_{\alpha,N}\int_G \frac{(x-y)\cdot e}{|x-y|^{N-\alpha+1}}\,\mathrm{d}y$$

and the integrand is strictly positive we see that u has no local extremum outside G. □

By Lemma 4 we see that G is a sub- or super-level set of u. This observation led Fraenkel [7] to rewrite u as the Newtonian potential of the nonlinear density function $f_H(u(x)-\beta)$ over all of $\mathbb{R}^N$, where f_H is the Heaviside-function. Hence u fulfilled a nonlinear integral equation in $\mathbb{R}^N$ with no explicit appearance of the set G. The same is clearly true in the context of Riesz-potentials as expressed by the following corollary.

Corollary 5 *Let $f_H(t) = 1$ for $t > 0$ and $f_H(t) = 0$ for $t \leq 0$ be the Heaviside-function and χ_G be the characteristic function of G. Then $\chi_G = f_H(u-\beta)$ if $N \geq \alpha$ and $\chi_G = f_H(\beta-u)$ if $N < \alpha$. Hence*

$$u(x) = \begin{cases} \int\limits_{\mathbb{R}^N} \log\frac{1}{|x-y|} f_H(u(y)-\beta)\,dy, & N=\alpha,\\ \int\limits_{\mathbb{R}^N} \frac{f_H(u(y)-\beta)}{|x-y|^{N-\alpha}}\,dy, & N>\alpha,\\ \int\limits_{\mathbb{R}^N} \frac{f_H(\beta-u(y))}{|x-y|^{N-\alpha}}\,dy, & N<\alpha. \end{cases}$$

Lemma 6 *Let $q = \frac{1}{\operatorname{vol} G}\int_G y\,dy$ be the barycentre of G and let $v(x) = u(x+q)$. Then*

$$v(x) = \begin{cases} \operatorname{vol} G \log\frac{1}{|x|} + h(x) & \text{if } N=\alpha,\\ \operatorname{vol} G|x|^{\alpha-N} + h(x) & \text{if } N\neq\alpha \end{cases}$$

where h satisfies $|h(x)| \leq C|x|^{\alpha-N-2}$, $|\nabla h(x)| \leq C|x|^{\alpha-N-3}$ for some constant $C > 0$.

Proof Let $N \neq \alpha$. A direct application of Taylor's theorem to the function $g(t) := |x-t\eta|^{\alpha-N}$ yields

$$|x-\eta|^{\alpha-N} = |x|^{\alpha-N} - (\alpha-N)|x|^{\alpha-N-2}x\cdot\eta + k(x,\eta) \tag{2}$$

where there exists a constant $C > 0$ and a radius $R_0 > 0$ such that

$$|k(x,\eta)| \le C|x|^{\alpha-N-2}, \quad |\nabla_x k(x,\eta)| \le C|x|^{\alpha-N-3} \quad \text{for all } |x| \ge R_0, \eta \in G-q. \tag{3}$$

Here $R_0 > 0$ is chosen such that $\overline{G} - q \subset B_{R_0}(0)$. Note that

$$v(x) = \int_G \frac{1}{|x+q-y|^{N-\alpha}}\,\mathrm{d}y = \int_{G-q} \frac{1}{|x-\eta|^{N-\alpha}}\,\mathrm{d}\eta.$$

Since the barycentre of $G-q$ is zero the claim of the lemma follows from integrating (2). The estimate for $h(x) := \int_{G-q} k(x,\eta)\,d\eta$ follows from (3). The proof for $N = \alpha$ is similar. $\square$

3 Proof of Theorem 2 by the method of moving planes

For a point $x \in \mathbb{R}^N$ let $x^\lambda = (2\lambda - x_1, x')$ be the reflection of x at the hyperplane $T_\lambda := \{x \in \mathbb{R}^N : x_1 = \lambda\}$. Hence $|x^\lambda|^2 - |x|^2 = 4\lambda(\lambda - x_1)$. Also define the half-space $H_\lambda := \{x \in \mathbb{R}^N : x_1 < \lambda\}$ and note that $\partial H_\lambda = T_\lambda$. On $\overline{H}_\lambda$ define the function $w_\lambda(x) := v(x) - v(x^\lambda)$. We will show that for $\alpha \le N (\alpha > N)$ the function w_λ satisfies

$$w_\lambda(x) > 0 (< 0) \quad \text{in } H_\lambda, \qquad \frac{\partial w_\lambda}{\partial x_1}(x) = 2\frac{\partial v}{\partial x_1}(x) < 0 (> 0) \quad \text{on } T_\lambda \tag{4}$$

for all $\lambda > 0$. By continuity this implies for $\alpha \le N$ that $v(x_1, x') \ge v(-x_1, x')$ for all $x \in \mathbb{R}^N$, $x_1 \ge 0$ while for $\alpha > N$ the reverse inequality holds. In both cases the corresponding reverse inequalities also hold by repeating the moving plane argument with the $-x_1$-direction. Hence $v(-x_1, x') = v(x_1, x')$ for all $x \in \mathbb{R}^N$ and moreover v is strictly monotone in the positive x_1-direction. Repeating the moving-plane argument with an arbitrary unit-direction instead of the x_1-direction one obtains that the function v is radially symmetric with respect to the origin and moreover radially strictly monotone. Together with the fact that $\partial(G-q)$ is a level-surface of the function v this implies that $G-q$ must be a ball centered at the origin. Thus, Theorem 2 is proved if we show (4) for all values of $\lambda > 0$. This will be done next. Theorem 2 follows from the preceeding explanation and Lemma 10 and Lemma 12.

Lemma 7 *For every $\lambda > 0$ there exists a value $R(\lambda) > 0$ such that for all $x \in H_\lambda$ with $|x| \ge R(\lambda)$ we have*

$$w_\lambda(x) \begin{cases} > 0 & \text{if } 2 < \alpha \le N, \\ < 0 & \text{if } \alpha > N. \end{cases}$$

The function $R(\lambda)$ and a value $\lambda_0 > 0$ can be chosen such that $R(\lambda)$ is non-increasing in λ and constant for $\lambda \ge \lambda_0 > 0$.

Proof According to the value of α we divide the proof into several cases. If h is the function of Lemma 6 then

$$v(x) - v(x^\lambda) = \begin{cases} \operatorname{vol} G(|x|^{\alpha-N} - |x^\lambda|^{\alpha-N}) + h(x) - h(x^\lambda), & \alpha \ne N, \\ \operatorname{vol} G(-\log|x| + \log|x^\lambda|) + h(x) - h(x^\lambda), & \alpha = N. \end{cases}$$

Case 1 $2 < \alpha < N$. Assume first that $|x^\lambda|^2 \le 2|x|^2$. By convexity of the function $s \mapsto s^{\frac{\alpha-N}{2}}$ for $s > 0$ we have

$$|x|^{\alpha-N} - |x^\lambda|^{\alpha-N} > \frac{N-\alpha}{2}|x^\lambda|^{\alpha-N-2} 4\lambda(\lambda - x_1) \ge C_1|x|^{\alpha-N-2}\lambda(\lambda - x_1)$$

where $C_1 := (N-\alpha)2^{\frac{\alpha-N}{2}}$. By Lemma 6 $|h(x)-h(x^\lambda)| \le 2C|x|^{\alpha-N-3}(\lambda - x_1)$. Hence

$$v(x) - v(x^\lambda) > |x|^{\alpha-N-3}(\lambda - x_1)\big(\operatorname{vol} G C_1|x|\lambda - 2C\big) > 0$$

provided $|x| > \frac{2C}{\operatorname{vol} G C_1 \lambda}$. Next assume that $|x^\lambda|^2 \ge 2|x|^2$. Then

$$|x|^{\alpha-N} - |x^\lambda|^{\alpha-N} \ge |x|^{\alpha-N}(1 - 2^{\frac{\alpha-N}{2}}) =: C_2|x|^{\alpha-N},$$

where $C_2 > 0$. Again by Lemma 6 $|h(x)-h(x^\lambda)| \le 2C|x|^{\alpha-N-2}$. Thus

$$v(x) - v(x^\lambda) \ge |x|^{\alpha-N}\left(\operatorname{vol} G C_2 - \frac{2C}{|x|^2}\right) > 0$$

provided $|x| > \sqrt{\frac{2C}{\operatorname{vol} G C_2}}$. Hence the statement of the lemma follows if we set

$$R(\lambda) := \max\left\{\frac{2C}{\operatorname{vol} G C_1 \lambda}, \sqrt{\frac{2C}{\operatorname{vol} G C_2}}\right\}.$$

Case 2 $\alpha = N$. The structure of proof is the same as in Case 1. Assume first that $|x^\lambda|^2 \le 2|x|^2$. The convexity of the function $s \mapsto -\log s$ for $s > 0$ implies

$$-\log|x| + \log|x^\lambda| > |x^\lambda|^{-2}2\lambda(\lambda - x_1) \ge \frac{1}{2}|x|^{-2}\lambda(\lambda - x_1).$$

With the estimate for h as above we find $v(x) - v(x^\lambda) > 0$ provided $|x| > \frac{4C}{\operatorname{vol} G\lambda}$. Likewise, if $|x^\lambda|^2 \ge 2|x|^2$ then

$$-\log|x| + \log|x^\lambda| \ge \frac{1}{2}\log 2$$

and with the estimate for h as above we find $v(x) - v(x^\lambda) > 0$ provided $|x| > \sqrt{\frac{4C}{\operatorname{vol} G \log 2}}$. Hence we may set

$$R(\lambda) := \max\left\{\frac{4C}{\operatorname{vol} G\lambda}, \sqrt{\frac{4C}{\operatorname{vol} G \log 2}}\right\}.$$

Case 3 $N < \alpha < N+2$. Again we assume first that $|x^\lambda|^2 \le 2|x|^2$. The concavity of the function $s \mapsto s^{\frac{\alpha-N}{2}}$ for $s > 0$ implies

$$|x|^{\alpha-N} - |x^\lambda|^{\alpha-N} < \frac{N-\alpha}{2}|x^\lambda|^{\alpha-N-2}4\lambda(\lambda - x_1) \le -C_1|x|^{\alpha-N-2}\lambda(\lambda - x_1)$$

with $C_1 := (\alpha - N)2^{\frac{\alpha-N}{2}}$. Using the estimate for h as in Case 1 we find

$$v(x) - v(x^\lambda) < |x|^{\alpha-N-3}(\lambda - x_1)\big(-\operatorname{vol} G C_1|x|\lambda + 2C\big) < 0$$

provided $|x| > \frac{2C}{\operatorname{vol} G C_1 \lambda}$. For $|x^\lambda|^2 \ge 2|x|^2$ we get

$$|x|^{\alpha-N} - |x^\lambda|^{\alpha-N} \le |x|^{\alpha-N}(1 - 2^{\frac{\alpha-N}{2}}) =: -C_2|x|^{\alpha-N}$$

where $C_2 > 0$. Together with the estimate $|h(x) - h(x^\lambda)| \le 2C|x|^{\alpha-N-2}$ we obtain

$$v(x) - v(x^\lambda) < |x|^{\alpha-N}\left(-\operatorname{vol} GC_2 + \frac{2C}{|x|^2}\right) < 0$$

provided $|x| > \sqrt{\frac{2C}{\operatorname{vol} GC_2}}$. Therefore it suffices to set

$$R(\lambda) := \max\left\{\frac{2C}{\operatorname{vol} GC_1\lambda}, \sqrt{\frac{2C}{\operatorname{vol} GC_2}}\right\}.$$

Case 4 $\alpha \ge N + 2$. For $|x^\lambda|^2 \le 2|x|^2$ the convexity of $s \mapsto s^{\frac{\alpha-N}{2}}$ for $s > 0$ implies

$$|x|^{\alpha-N} - |x^\lambda|^{\alpha-N} < \frac{N-\alpha}{2}|x|^{\alpha-N-2}4\lambda(\lambda - x_1) =: -C_1|x|^{\alpha-N-2}\lambda(\lambda - x_1)$$

where $C_1 = 2(\alpha - N) > 0$. For h we obtain this time a different estimate:

$$\begin{aligned}|h(x) - h(x^\lambda)| &\le \begin{cases} 2C|x^\lambda|^{\alpha-N-3}(\lambda - x_1) & \text{if } \alpha - N - 3 \ge 0,\\ 2C|x|^{\alpha-N-3}(\lambda - x_1) & \text{if } \alpha - N - 3 < 0\end{cases}\\ &\le D|x|^{\alpha-N-3}(\lambda - x_1),\end{aligned}$$

where either $D = 2^{\frac{\alpha-N-1}{2}}C$ or $D = 2C$. Thus

$$v(x) - v(x^\lambda) < |x|^{\alpha-N-3}(\lambda - x_1)\big(-\operatorname{vol} GC_1|x|\lambda + D\big) < 0$$

provided $|x| > \frac{D}{\operatorname{vol} GC_1\lambda}$. Finally, if $|x^\lambda|^2 \ge 2|x|^2$ then

$$|x|^{\alpha-N} - |x^\lambda|^{\alpha-N} \le |x^\lambda|^{\alpha-N}(2^{\frac{N-\alpha}{2}} - 1) =: -C_2|x^\lambda|^{\alpha-N}$$

where $C_2 > 0$. Together with the estimate $|h(x) - h(x^\lambda)| \le 2C|x^\lambda|^{\alpha-N-2}$ we conclude

$$v(x) - v(x^\lambda) < |x^\lambda|^{\alpha-N}\left(-\operatorname{vol} GC_2 + \frac{2C}{|x^\lambda|^2}\right) < 0$$

provided $|x| > \sqrt{\frac{2C}{\operatorname{vol} GC_2}}$ (recall that $|x^\lambda| \ge |x|$ in H_λ). Therefore let us set in this case

$$R(\lambda) := \max\left\{\frac{D}{\operatorname{vol} GC_1\lambda}, \sqrt{\frac{2C}{\operatorname{vol} GC_2}}\right\}.$$

□

Lemma 8 *There exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*$ we have*

$$w_\lambda(x)\begin{cases} > 0 & \textit{if } 2 < \alpha \le N,\\ < 0 & \textit{if } \alpha > N.\end{cases}$$

in H_λ.

Proof The proof is again divided according to the value of α. Let $R(\lambda)$ be the function defined in Lemma 7.

Case 1 $2 < \alpha < N$. Let $c_1 := \min_{|x| \le R(1)} v(x)$. Hence $c_1 > 0$, and since $v(x)$ decays to 0 as $|x| \to \infty$ there exists a value $\lambda^* \ge 1$ such that $|x| \ge \lambda^*$ implies $v(x) \le c_1/2$. Let now $\lambda > \lambda^*$. Consider $x \in H_\lambda$ with $|x| > R(1)$. For such x we have $|x| > R(\lambda)$ and hence

$v(x) > v(x^\lambda)$ by Lemma 7. Now consider $x \in H_\lambda$ with $|x| \le R(1)$. Since $|x^\lambda| \ge \lambda > \lambda^*$ we find $v(x) \ge c_1 > v(x^\lambda)$, and the claim is proved.

Case 2 $\alpha = N$. The proof is as above, but now c_1 is not necessarily positive. But now $v(x)$ decays to $-\infty$ as $|x| \to \infty$ so that we can choose the value $\lambda^* \ge 1$ such that $|x| \ge \lambda^*$ implies $v(x) \le c_1 - 1$. The rest of the proof is the same.

Case 3 $\alpha > N$. Choose $c_1 := \max_{|x|\le R(1)} v(x)$ so that $c_1 > 0$. This time $v(x)$ tends to ∞ as $|x| \to \infty$ so that we can choose $\lambda^* \ge 1$ such that $|x| \ge \lambda^*$ implies $v(x) \ge 2c_1$. Similar consideration as before imply the claim. □

Lemma 9 *Let* $\lambda > 0$.

(a) *For all* $x, y \in H_\lambda$:

$$\begin{aligned} 2<\alpha<N: &\quad \frac{1}{|x-y|^{N-\alpha}} > \frac{1}{|x^\lambda-y|^{N-\alpha}},\\ \alpha=N: &\quad \log\frac{1}{|x-y|} > \log\frac{1}{|x^\lambda-y|},\\ \alpha>N: &\quad \frac{1}{|x-y|^{N-\alpha}} < \frac{1}{|x^\lambda-y|^{N-\alpha}}. \end{aligned}$$

(b) *For all* $x \in T_\lambda$, $y \in H_\lambda$:

$$\begin{aligned} 2<\alpha<N: &\quad \frac{\partial}{\partial x_1}\frac{1}{|x-y|^{N-\alpha}} < 0, \quad \frac{\partial}{\partial x_1}\left(\frac{1}{|x-y|^{N-\alpha}} + \frac{1}{|x-y^\lambda|^{N-\alpha}}\right) = 0,\\ \alpha=N: &\quad \frac{\partial}{\partial x_1}\log\frac{1}{|x-y|} < 0, \quad \frac{\partial}{\partial x_1}\left(\log\frac{1}{|x-y|} + \log\frac{1}{|x-y^\lambda|}\right) = 0,\\ \alpha>N: &\quad \frac{\partial}{\partial x_1}\frac{1}{|x-y|^{N-\alpha}} > 0, \quad \frac{\partial}{\partial x_1}\left(\frac{1}{|x-y|^{N-\alpha}} + \frac{1}{|x-y^\lambda|^{N-\alpha}}\right) = 0. \end{aligned}$$

Proof The proof of (a) follows from

$$|x^\lambda - y|^2 = 4\underbrace{(\lambda - x_1)}_{>0}\underbrace{(\lambda - y_1)}_{>0} + |x-y|^2.$$

The proof of the first part of (b) follows from

$$\frac{\partial}{\partial x_1}|x-y| = \frac{x_1 - y_1}{|x-y|} = \frac{\lambda - y_1}{|x-y|} > 0$$

and the chain rule. For the second part of (b) note that if $x \in T_\lambda$ and $y \in H_\lambda$ then $|x-y| = |x^\lambda - y^\lambda| = |x - y^\lambda|$. Hence for every C^1-function g we have that

$$\frac{\partial}{\partial x_1}\big(g(|x-y|) + g(|x-y^\lambda|)\big) = \frac{g'(|x-y|)}{|x-y|}\big((x_1-y_1) + x_1 - (2\lambda - y_1)\big) = 0$$

since $x \in T_\lambda$. □

Lemma 10 *Let* $\lambda > 0$.

(a) *Suppose* $2 < \alpha \le N$. *If* $w_\lambda \ge 0$ *in* H_λ *then* $w_\lambda > 0$ *in* H_λ *and* $\frac{\partial w_\lambda}{\partial x_1}(x) < 0$ *on* T_λ.

(b) *Suppose* $\alpha > N$. *If* $w_\lambda \le 0$ *in* H_λ *then* $w_\lambda < 0$ *in* H_λ *and* $\frac{\partial w_\lambda}{\partial x_1}(x) > 0$ *on* T_λ.

Proof We give the proof in the case $2 < \alpha < N$; the proof in the case $\alpha \geq N$ is very similar. Note first that by Corollary 5 we have

$$v(x) = \int\limits_{\mathbb{R}^N} \frac{f_H(v(y)-\beta)}{|x-y|^{N-\alpha}}\,\mathrm{d}y = \int\limits_{H_\lambda} \ldots\,\mathrm{d}y + \int\limits_{\mathbb{R}^N\setminus H_\lambda} \ldots\,\mathrm{d}y$$
$$= \int\limits_{H_\lambda} \frac{f_H(v(y)-\beta)}{|x-y|^{N-\alpha}} + \frac{f_H(v(y^\lambda)-\beta)}{|x-y^\lambda|^{N-\alpha}}\,\mathrm{d}y.$$

Therefore

$$\begin{aligned} v(x)-v(x^\lambda) &= \int\limits_{H_\lambda} f_H(v(y)-\beta)\left(\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x^\lambda-y|^{N-\alpha}}\right)\mathrm{d}y \\ &\quad + \int\limits_{H_\lambda} f_H(v(y^\lambda)-\beta)\left(\frac{1}{|x-y^\lambda|^{N-\alpha}} - \frac{1}{|x^\lambda-y^\lambda|^{N-\alpha}}\right)\mathrm{d}y \\ &= \int\limits_{H_\lambda} \big(f_H(v(y)-\beta) - f_H(v(y^\lambda)-\beta)\big)\underbrace{\left(\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x^\lambda-y|^{N-\alpha}}\right)}_{>0 \text{ by Lemma 9(a)}}\mathrm{d}y. \end{aligned} \tag{5}$$

Moreover, $f_H(v(y)-\beta) - f_H(v(y^\lambda)-\beta)\big) \geq 0$ since f_H is non-decreasing and $w_\lambda \geq 0$ by assumption. If we assume for contradiction that $f_H(v(y)-\beta) \equiv f_H(v(y^\lambda)-\beta)$ for almost all $y \in H_\lambda$ then we would find $v(x)-v(x^\lambda) \equiv 0$ in H_λ, which contradicts Lemma 7 and the assumption $\lambda > 0$. Therefore there exists a subset $M_\lambda \subset H_\lambda$ of positive measure such that $f_H(v(y)-\beta) > f_H(v(y^\lambda)-\beta)$ for all $y \in M_\lambda$. As a consequence we see from (5) that $w_\lambda(x) > 0$ for all $x \in H_\lambda$.

To see the second part of the claim, note that for $x \in T_\lambda$ we have $\frac{\partial w_\lambda}{\partial x_1}(x) = 2\frac{\partial v}{\partial x_1}(x)$ so that

$$\frac{1}{2}\frac{\partial w_\lambda}{\partial x_1}(x) = \int\limits_{H_\lambda} f_H(v(y)-\beta)\underbrace{\frac{\partial}{\partial x_1}\left(\frac{1}{|x-y|^{N-\alpha}}\right)}_{<0 \text{ by Lemma 9(b)}} + f_H(v(y^\lambda)-\beta)\frac{\partial}{\partial x_1}\left(\frac{1}{|x-y^\lambda|^{N-\alpha}}\right)\mathrm{d}y$$

Moreover, we have seen that $f_H(v(y)-\beta) > f_H(v(y^\lambda)-\beta)$ on a subset $M_\lambda \subset H_\lambda$ of positive measure. Therefore, for all $x \in T_\lambda$ we find

$$\frac{1}{2}\frac{\partial w_\lambda}{\partial x_1}(x) < \int\limits_{H_\lambda} f_H(v(y^\lambda)-\beta)\frac{\partial}{\partial x_1}\left(\frac{1}{|x-y|^{N-\alpha}} + \frac{1}{|x-y^\lambda|^{N-\alpha}}\right)\mathrm{d}y = 0$$

due to Lemma 9(b). This establishes the claim. □

For the final part of this section let us define the set

$$J := \begin{cases} \{\lambda > 0 : w_\lambda > 0 \text{ in } H_\lambda\} & \text{if } 2 < \alpha \leq N, \\ \{\lambda > 0 : w_\lambda < 0 \text{ in } H_\lambda\} & \text{if } \alpha > N. \end{cases}$$

Lemma 11 *The set $J \subset (0,\infty)$ is open.*

Proof We give the proof only in the case $2 < \alpha \leq N$. Assume that J is not open. Then for some $\lambda \in J$ there exists a sequence $\lambda_n \to \lambda$ as $n \to \infty$ and $x_n \in H_{\lambda_n}$ such that $w_{\lambda_n}(x_n) \leq 0$. Let $R(\lambda)$ be the function from Lemma 7. Clearly $|x_n| \leq R(\lambda/2)$, because $|x_n| > R(\lambda/2)$ would imply $|x_n| > R(\lambda_n)$ for large n and hence $w_{\lambda_n}(x_n) > 0$ for large n, which cannot hold. Hence, by extracting a subsequence if necessary, we may assume that $x_n \to x_0 \in \overline{B_{R(\lambda/2)}(0)}$, $x_0 \in \overline{H(\lambda)}$. Since $w_\lambda > 0$ in H_λ we must have $x_0 \in T_\lambda$. Thus, by Lemma 10(a) we find $\frac{\partial v}{\partial x_1}(x_0) < 0$, which contradicts $v(x_n) \leq v(x_n^{\lambda_n})$ for large n. □

The proof of Theorem 2 will be completed through the following, final lemma.

Lemma 12 *The set* $J = (0, \infty)$.

Proof Again let us stay with the case $2 < \alpha \leq N$. Let (μ, ∞) be the largest open interval contained in J. By Lemma 8, μ is a finite value in $[0, \infty)$. Assume for contradiction that $\mu > 0$. Then $w_\mu \geq 0$ in H_μ and by Lemma 10(a) we see that $w_\mu > 0$ in H_μ so that $\mu \in J$. A contradiction is reached since by Lemma 11 we know that J is open. □

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