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Nontrivial large-time behaviour in bistable reaction–diffusion equations

Jean-Michel Roquejoffre · Violaine Roussier-Michon

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Abstract Bistable reaction–diffusion equations are known to admit one-dimensional travelling waves which are globally stable to one-dimensional perturbations—Fife and McLeod [7]. These planar waves are also stable to two-dimensional perturbations—Xin [30], Levermore-Xin [19], Kapitula [16]—provided that these perturbations decay, in the direction transverse to the wave, in an integrable fashion. In this paper, we first prove that this result breaks down when the integrability condition is removed, and we exhibit a large-time dynamics similar to that of the heat equation. We then apply this result to the study of the large-time behaviour of conical-shaped fronts in the plane, and exhibit cases where the dynamics is given by that of two advection–diffusion equations.

Keywords Reaction–diffusion equations · Travelling fronts · Nontrivial dynamics

Mathematics Subject Classification (2000) 35K57 · 35B40 · 35B35

1 Introduction

Consider the following scalar parabolic equation:

$$\begin{aligned} u_t - \Delta u = f(u)\,, &\quad (x, y) \in \mathbb{R}^2,\ t > 0 \\ u(0) = u_0\,, &\quad (x, y) \in \mathbb{R}^2 \end{aligned} \tag{1.1}$$

J.-M. Roquejoffre (✉)
Institut de Mathématiques (UMR CNRS 5219) and Institut Universitaire de France,
Université Paul Sabatier, 118 Route de Narbonne, 31062 Toulouse Cedex 4, France
e-mail: roque@mip.ups-tlse.fr

V. Roussier-Michon
Institut de Mathématiques (UMR CNRS 5219), INSA de Toulouse, 135 Avenue de Rangueil,
31077 Toulouse Cedex 4, France
e-mail: roussier@insatoulouse.fr

where $u : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$. The function f is of class $C^2(\mathbb{R})$ and it is assumed to be of the 'bistable' type. Namely, there exists $\theta \in (0, 1)$ such that

$$\begin{cases} f(0) = f(\theta) = f(1) = 0, \\ f < 0 \text{ on } (0, \theta) \cup (1, +\infty), \quad f > 0 \text{ on } (-\infty, 0) \cup (\theta, 1), \\ f'(0) < 0, \quad f'(1) < 1, \quad f'(\theta) > 0. \end{cases}$$

Moreover, we shall assume that $\int_0^1 f(u)\mathrm{d}u > 0$.

This reaction–diffusion equation is a classical model for spreading and interacting particules—see [1,8,17]—and the transport of information is often represented by some particular solutions to (1.1) characterized by their time independent profile, uniformly translating at some constant speed c. Plugging the ansatz $u(t, x, y) = \phi(x, y + ct)$ yields the elliptic equation

$$-\Delta\phi + c\partial_y\phi = f(\phi) \ \text{ in } \mathbb{R}^2, \tag{1.2}$$

completed by the following conditions at infinity, understood in the *pointwise* sense in x:

$$\phi(x, -\infty) = 0, \ \phi(x, +\infty) = 1. \tag{1.3}$$

Looking for planar travelling waves (i.e solutions of (1.2)–(1.3) independent of x), it is well known, see [7], that there is a unique speed $c_0 > 0$ and a unique profile ϕ_0 (up to translations) such that the ordinary differential equation

$$-\phi_0'' + c_0\phi_0' = f(\phi_0) \ \text{ in } \mathbb{R}, \quad \phi_0(-\infty) = 0, \ \phi_0(+\infty) = 1 \tag{1.4}$$

has a solution. The function $\phi_0(y + c_0t)$ is a planar solution of (1.1).

It is also known that (1.1) has genuinely nonplanar, conical-shaped, travelling wave solutions. Taking a uniform limit in x in (1.3) automatically yields that ϕ is a planar wave $\phi(x, y) = \phi_0(y+y_0)$ for some translate $y_0 \in \mathbb{R}$; see [3]. Taking the limit in (1.3) pointwise—as opposed to uniformly—in x, the papers [10–12]—see also [9,22]—prove the existence of solutions $(c, \phi) = (c_0/\sin\alpha, \phi)$ of (1.2)–(1.3) for some angle $\alpha \in (0, \pi/2)$ satisfying the following properties:

(P1) $0 < \phi < 1$ *in* $\mathbb{R}^2$,

(P2) $\phi(x, y) = \tilde{\phi}(|x|, y)$, $\partial_{|x|}\tilde{\phi} \geq 0$, $\partial_y\phi > 0$,

(P3) *the function* ϕ *satisfies*

$$\begin{cases} \displaystyle\limsup_{A\to+\infty,\ y\geq A-|x|\cot\alpha} (1 - \phi(x, y)) = 0, \\ \displaystyle\limsup_{A\to-\infty,\ y\leq A-|x|\cot\alpha} \phi(x, y) = 0. \end{cases} \tag{1.5}$$

(P4) *the function* ϕ *is decreasing in any unit direction* $\tau = (\tau_x, \tau_y) \in \mathbb{R}^2$ *such that* $\tau_y < -\cos\alpha$,

(P5) *there is exponential convergence of* $\phi(x, y)$ *to the planar fronts* $\phi_0(\pm x\cos\alpha + y\sin\alpha)$ *in the directions* $(\pm\sin\alpha, -\cos\alpha)$*; moreover the slopes of the level lines of* ϕ *converge exponentially, in the same directions, to* $\mp\cot\alpha$*. More precisely, if we set*

$$X = x\sin\alpha - y\cos\alpha, \quad Y = x\cos\alpha + y\sin\alpha \tag{1.6}$$

and still denote $\phi(x, y)$ *by* $\phi(X, Y)$ *with an obvious abuse of notations,then the level line* $\{\phi(X, Y) = a\}$ *is described in the half-plane* $\{x \geq 0\}$ *by an equation* $\{Y = \psi_a(X)\}$*, and* *there is* $\omega = \omega(\alpha, f) > 0$ *such that, for all* $a \in (0, 1)$ *and* $X > 0$,

$$|\psi_a'(X)| \le C_a e^{-2\omega|X|} \tag{1.7}$$

for some constant $C_a = C_a(a, \alpha, f, \phi)$. *Also, for all* Y *such that the point* $(X, Y + \psi_a(X))$ *is in the half-plane* $\{x > 0\}$, *we have*

$$|\phi(X, Y + \psi_a(X)) - \phi_0(Y + \phi_0^{-1}(a))| \le C_a e^{-2\omega(|X|+|Y|)}.$$

The constant C_a *degrades as* a *converges to* 0 *or* 1.

As far as the Cauchy problem for (1.1) is concerned, if u_0 is a continuous function from $\mathbb{R}^2$ to (0, 1) trapped between two (planar or conical) waves, then there exists a unique solution $u(t, x, y)$ of equation (1.1) emanating from u_0 with the same properties as u_0 for any time $t > 0$.

One question of interest for this reaction diffusion equation (1.1) is the behaviour as t goes to infinity of $u(t, x, y)$. A prominent role is played by the family of the travelling waves, and much is understood about their stability. What is already known is summarised in the following set of properties:

(P6) *Let* $u_0(y)$ *be a—one-dimensional—Cauchy datum to* (1.1), *satisfying*

$$\limsup_{y\to-\infty} u_0(y) < \theta, \quad \liminf_{y\to+\infty} u_0(y) > \theta.$$

Then there is $y_0 \in \mathbb{R}$ *and* $\gamma > 0$ *such that, if* $u(t, y)$ *is the solution of* (1.1) *emanating from* u_0, *we have—Fife–McLeod* [7] - $u(t, y) - \phi_0(y + y_0 + c_0 t) = O(e^{-\gamma t})$, *uniformly in* $y \in \mathbb{R}$.

(P7) *Let* $u_0(x, y)$ *be a—possibly two-dimensional—Cauchy datum to* (1.1), *satisfying*

$$\varepsilon := \|u_0 - \phi_0\|_{H^1(\mathbb{R}^2)} << 1. \tag{1.8}$$

Then—see Xin [30], *Levermore-Xin* [19], *Kapitula* [16]—*we have, for some* $\omega > 0$: $u(t, x, y) - \phi_0(y + c_0 t) = O(t^{-\omega})$, *uniformly in* $(x, y) \in \mathbb{R}^2$.

(P8) *Let* $u_0(x, y)$ *be a—two-dimensional—Cauchy datum to* (1.1), *satisfying*

$$|u_0(x, y) - \phi(x, y)| = O(e^{-2\omega(|x|+|y|)}), \tag{1.9}$$

where ω *is some positive number, and* $\phi(x, y)$ *a solution of* (1.2)–(1.3)–(1.5) – *hence a conical-shaped solution. Then—see Hamel–Monneau–Roquejoffre* [10]—*we have, for some* $\gamma > 0$ *uniformly in* $(x, y) \in \mathbb{R}^2$:

$$u(t, x, y) - \phi(x, y + ct) = O(e^{-\gamma t})$$

(P9) *Let* $u(t, x, y)$ *be a time-global—i.e. defined on* $\{(t, x, y) \in \mathbb{R}^3\}$—*solution of* (1.1), *such that there is* $(X_1, X_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ *for which we have uniformly in* $(t, x, y) \in \mathbb{R}^3$

$$\phi((x, y + ct) + X_1) \le u(t, x, y) \le \phi((x, y + ct) + X_2).$$

where phi is still a conical-shaped wave. Then—see Hamel–Monneau–Roquejoffre [10]—*we have, for some* $X_0 \in \mathbb{R}^2$: $u(t, x, y) = \phi((x, y + ct) + X_0)$.

Let us examine the differences between these four properties. Let u_0 be a Cauchy datum for (1.1), lying between two conical waves:

$$\phi((x, y) + X_1) \le u_0 \le \phi((x, y) + X_2)$$

Define its ω-limit set as

$$\omega(u_0) = \{\psi(x, y) \in C(\mathbb{R}^2) \,|\, \exists (t_n)_n \to +\infty \text{ s.t. } \lim_{n\to+\infty} u(t_n, x, y + ct_n) = \psi(x, y)\}.$$

It is important to note that the convergence in the above definition of the ω-limit set should *a priori* be understood uniformly on every compact subset of $\mathbb{R}^2$: at this stage, we only have at our disposition the derivative estimates, which are not strong enough to imply uniform convergence properties. In fact, $\omega(u_0)$ might well be empty if we insist in talking about uniform convergence on $\mathbb{R}^2$.

There is a gap between the behaviour described in (P8) and that described in (P9). Applying (P9) yields that $\omega(u_0)$ is made up of solutions of (1.2)–(1.3). However, due to the translational invariance of (1.2)–(1.3), $\omega(u_0)$ may well be homeomorphic to a nontrivial compact subset of $\mathbb{R}^2$. On the contrary, applying (P8) yields that $\omega(u_0)$ is reduced to a single conical wave and is homeomorphic to a single point of $\mathbb{R}^2$. It is therefore natural to ask whether a conclusion similar to that of (P8) is kept, even if its assumptions are relaxed. See [20,21] for a result in this direction: the difference $u_0 - \phi$ is only supposed to vanish at infinity instead of doing it in an exponential fashion; in return no particular rate of convergence holds. However, assuming only that the initial datum lies between two waves is still weaker than this last assumption. Finally, let us just remark that a similar gap exists between data which converge to a planar wave at infinity—property (P7)—and data which simply sit between two planar waves—one can prove, in a similar fashion as in (P9), that their ω-limit sets are made up of planar waves.

The contribution of this paper is to prove that the ω-limit set of a Cauchy datum to (1.1) is nontrivial in general. We will, in particular, construct Cauchy data u_0, trapped between two waves, such that $\omega(u_0)$ is homeomorphic to a compact of $\mathbb{R}^2$ with nonempty interior. To this end, we will first have to understand what happens with planar fronts and extend those results to conical fronts. In other words, this paper shows that the asymptotic stability of planar (resp. conical) traveling waves proved in (P7) (resp. (P8)) breaks down as soon as the assumptions are relaxed as low as "the initial datum u_0 to (1.1) lies between two planar (resp. conical) waves". Comparing these results to (P6) highlights the gap between the dynamics in dimension $n = 1$ and dimensions $n \geq 2$.

Such nontrivial behaviour has already been observed in reaction-diffusion equations: see, for instance work of the second author [27] or [31]: it is proved there that an expanding, initially compactly supported solution of (1.1) does not necessarily attain eventual spherical symmetry. See also [25] for different aspects of the problem in bounded domains. Concerning the nonlinear supercritical heat equation $u_t - \Delta u = u^p$ in $\mathbb{R}^N$ for large N and p (Fujita equation), there is an interesting parallel between our results and a series of works by Poláčik–Yanagida [23,24]: for instance [23] presents the construction of a solution of the the Fujita equation that oscillates indefinitely between two spatially localised steady solutions; in [24], a Liouville property for entire solutions, close to Property (P9), is proved. Coming back to travelling waves, one may guess that things can become much worse for the KPP equation—same model as (1.1), but with this time $f(u) = u(1-u)$—because of the existence of a full range of planar wave velocities. This is indeed the case, as was proved by Hamel–Nadirashvili [13,14]: in [13], it is proved that planar waves of different velocities may mix in order to form generalised waves—i.e. entire solutions, whose front is localised, and which are not travelling waves; in [14] the equation is proved to have an extermely complex dynamics: the global attractor contains a set homeomorphic to the set of Borel measures! For a different point of view on the KPP equation in spherical geometry, see the recent paper [32]. Still remaining with the KPP equation, one may wonder what will happen to solutions initially trapped between two waves of the same speed: once again nontrivial behaviour will occur; it will be studied in the forthcoming paper [2].

The above considerations draw the plan of the paper: after presenting our results in Sect. 2 and deriving some consequences, we will prove in Sect. 3 that the large-time dynamics of (1.1), complemented by a datum lying between two planar waves, is that of a one-dimensional heat equation. Such an equation is, counter-intuitively enough, known to exhibit nontrivial dynamics, see Collet–Eckmann [5] and later papers such as, for instance [29]. Section 4 will be devoted to conical-shaped—with the same angle α—data; we will prove that the resulting dynamics is that of the product of two advection–diffusion equations. The last section is really an appendix in which we shall recall, for the reader's convenience, some classical interpolation inequalities deduced from the scaling properties of the heat equation.

2 Results and their consequences

The large-time behaviour of (1.1) will be described by two asymptotic estimates—one for the planar case, one for the conical case—in which we will show that the solution of (1.1) evolves to a shifted travelling wave, with the property that the shift will be varying in space and time. What will allow us to say something is that the shift will be slowly varying in time.

2.1 Main results

Let us start with almost planar initial data.

Theorem 2.1 *Given $u_0 \in C(\mathbb{R}^2)$, assume the existence of two reals $y_1 \le y_2$ such that*

$$\forall (x, y) \in \mathbb{R}^2 : \quad \phi_0(y + y_1) \le u_0(x, y) \le \phi_0(y + y_2),$$

where $\phi_0(y)$ is a solution of (1.4).

(i) *There is $t_0 > 0$ and a function $s(t, x) \in C^2([t_0, +\infty) \times \mathbb{R})$ such that the solution $u(t, x, y)$ of (1.1), emanating from u_0, satisfies, for all $\delta \in (0, 1)$:*

$$\sup_{t \ge t_0, (x,y) \in \mathbb{R}^2} |u(t, x, y) - \phi_0(y + c_0 t + s(t, x))| = O(t^{\delta - 1}). \tag{2.1}$$

Moreover, for all $\delta \in (0, 1)$, there is $C_\delta(u_0) > 0$ such that the function $\sigma(t, x) := e^{c_0 s(t,x)/2}$ satisfies, for $t \ge t_0$:

$$|\sigma_t - \sigma_{xx}| \le \frac{C_\delta(u_0)}{(1 + t)^{2 - 2\delta}}. \tag{2.2}$$

(ii) *Assume the existence of $\varepsilon > 0$ and of a smooth function $s_0(x)$ such that*

$$\sup_{(x,y) \in \mathbb{R}^2} |u_0(x, y) - \phi_0(y + s_0(x))| + \|\partial_{xx} \sigma_0\|_{L^\infty(\mathbb{R})} \le \varepsilon, \tag{2.3}$$

where we have set $\sigma_0 = e^{c_0 s_0/2}$. Then, if ε is small enough, we may choose

$$t_0 = 0, \quad \text{and } C_\delta(u_0) = O(\varepsilon^\delta). \tag{2.4}$$

We note that a result similar to [ii] was already proved by Brauner-Hulshof-Lunardi [4], in the case of the following free boundary problem:

$$\begin{aligned} u_t - \Delta u = 0 \quad &\text{in } \{u < 1\} \\ [u] = 0, \quad [u_\nu] = -1 \quad &\text{on } \partial(\{u < 1\}) \end{aligned} \tag{2.5}$$

Problem (2.5) is very much related to our equation (see [6]). It is indeed, at least in a formal fashion (passing to the limit in a mathematically rigorous way is a difficult question) the limit, as $\varepsilon \to 0$, of the reaction-diffusion equation

$$u_t - \Delta u = \frac{1}{\varepsilon^2}(1 - u)\varphi\left(\frac{u - 1}{\varepsilon}\right).$$

The function φ is, for instance, the characteristic function of the interval $[-1, +\infty)$.

Turn now to the conical case. Define the tilted coordinates $(X_\pm, Y_\pm)$:

$$\begin{cases} X_+ = x\sin\alpha - y\cos\alpha, & Y_+ = x\cos\alpha + y\sin\alpha \\ X_- = -x\sin\alpha - y\cos\alpha, & Y_- = -x\cos\alpha + y\sin\alpha \end{cases} \tag{2.6}$$

Theorem 2.2 *Let $\phi(x, y)$ be the only solution of (1.2)–(1.3) that is even in x and satisfies $\phi(0,0) = \theta$. Consider a Cauchy datum $u_0(x, y) \in C^2(\mathbb{R}^2)$ satisfying the following requirements.*

- *there exist a small $\varepsilon > 0$ and a couple $(X_1, X_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that:*

$$\phi((x, y) + X_1) \le u_0(x, y) \le \phi((x, y) + X_2), \quad |X_1 - X_2| \le \varepsilon, \tag{2.7}$$

- *there holds $\partial_y u_0 > 0$. Moreover there is $\rho_\varepsilon \in (0, \varepsilon^5]$, such that*

$$\limsup_{X_\pm \to +\infty} \|\partial_{X_\pm X_\pm} u_0(X_\pm, .)\|_{L^\infty(\mathbb{R})} \le \rho_\varepsilon^4. \tag{2.8}$$

Choose $\lambda \in (0, 1)$, let the set $\{u_0(x, y) = \lambda\}$ be written as $\{Y_+ = s_0^+(X_+)\}$ - resp. $\{Y_- = s_0^-(X_-)\}$ in the right half-plane $\{x > 0\}$ - resp. in the left half-plane $\{x < 0\}$ (the dependence in λ is deleted for commodity). Define the functions $\sigma_0^\pm(X_\pm)$ as

$$\sigma_0^\pm(X_\pm) = \begin{cases} e^{c_0 s_0^\pm(X_\pm)/2} & \text{if } X_\pm \ge 1 \\ e^{c_0 s_0^\pm(1)/2} & \text{if } X_\pm \le 1 \end{cases} \tag{2.9}$$

Let $\sigma^\pm(t, X_\pm)$ be the solutions of the advection-diffusion equations

$$\begin{aligned} (\partial_t - \partial_{X_\pm X_\pm} - c\cos\alpha\,\partial_{X_\pm})\sigma^\pm &= 0 \\ \sigma^\pm(0, X_\pm) &= \sigma_0^\pm(X_\pm) \end{aligned} \tag{2.10}$$

Let $u(t, x, y)$ be the solution of (1.1) emanating from u_0. For a given $\lambda \in (0, 1)$, there exists $A > 0$ such that the set $\{u(t, x, y) = \lambda\}$ can be described as of the form $\{Y_+ = \chi^+(t, X_+)\}$ in the half-plane $\{x \ge A\}$ - resp. $\{Y_- = \chi^-(t, X_-)\}$ in the half-plane $\{x \le -A\}$. Moreover there is a constant $C_\varepsilon > 0$ - possibly going to $+\infty$ as $\varepsilon \to 0$ - and another constant $C > 0$ independent of ε, such that there holds, for all $\delta \in \left(0, \frac{1}{2}\right)$, and uniformly in $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^2$:

$$|\chi^\pm(t, X_\pm) - \text{Log}\,\sigma^\pm(t, X_\pm)| \le C_\varepsilon \left(\frac{1}{(1+t)^{1-2\delta}} + e^{-\omega(|x|+|y|)}\right) + C\rho_\varepsilon^{\delta/2}. \tag{2.11}$$

This theorem calls the following

Remark 2.3 (i) The assumption $\partial_y u_0 > 0$ is a commodity assumption that can certainly be removed. See [10], Theorem 1.7, how it is possible to take into account fluctuations at infinity. Notice, however, that the strong maximum principle and (2.7) imply that $\partial_y u(1, ., .) > 0$ on a very large subset of $\mathbb{R}^2$.

(ii) If we set $u_0 = \phi$, then we may take $\rho_\varepsilon = 0$ by Property (P5). We wish to express here that the level sets of u_0 deviate from those of ϕ in a non-integrable fashion, but that the oscillation is very mild - and in any case, smaller than the distance between u_0 and the travelling wave closest to it in the L^∞ norm.

(iii) The assumption that the initial datum is L^∞-close to a front can also certainly be removed. However, it is quite sufficient to display explicit examples of nontrivial behaviour.

2.2 Interpretation and consequences of Theorems 2.1 and 2.2

2.2.1 Interpretation of Theorem 2.1

The presence of the term $\frac{C_\delta(u_0)}{(1+t)^{2-2\delta}}$ in Eq. (2.2) does allow us to conclude—because of the time-integrability of this term—that the eventual dynamics of $\sigma(t,x) = e^{c_0 s(t,x)/2}$ is the one of the heat equation, but does not allow us to conclude that this dynamics is nontrivial. In order to exhibit a nontrivial dynamics, we resort to Part [ii] of Theorem 2.1.

Let us consider an initial datum u_0 satisfying (2.3). We note that the smallness assumption concerns the derivatives of s_0, but not the function s_0 itself: hence this function has a lot of room to oscillate. In particular, we may take

$$\sup_{\mathbb{R}} s_0 = 1, \quad \inf_{\mathbb{R}} s_0 = 0, \tag{2.12}$$

while keeping s_0' and s_0'' small. If $\sigma^0(t,x)$ is the solution of the heat equation

$$\sigma^0_t = \sigma^0_{xx}, \quad \sigma^0(0,.) = e^{c_0 s_0/2} := \sigma_0,$$

we denote by $\omega(\sigma_0)$ the ω-limit set of σ_0 with respect to the above dynamical system. Let us construct s_0 in such a way that we have $\omega(\sigma_0) = [1, e^{c_0/2}]$. Let $(a_n)_n$ be an increasing sequence such that

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = +\infty \tag{2.13}$$

and $s_0(x)$ defined by

$$s_0(x) = \begin{cases} 1 & \text{if } a_{2n} \le |x| < a_{2n+1} \\ 0 & \text{if } a_{2n+2} \le |x| < a_{2n+3} \end{cases} \tag{2.14}$$

with smooth matching in the intervals $[a_{2n+1}, a_{2n+2}]$ and $[a_{2n+3}, a_{2n+4}]$ - this is to keep the derivatives of s_0 small. We have

$$\sigma^0(t,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-y^2} \sigma_0(\sqrt{t}y)\,\mathrm{d}y\,. \tag{2.15}$$

Let $(t_n)_n$ be an increasing sequence such that

$$\lim_{n\to+\infty} \frac{a_n}{\sqrt{t_n}} = 0, \quad \lim_{n\to+\infty} \frac{a_{n+1}}{\sqrt{t_n}} = +\infty; \tag{2.16}$$

this is possible by (2.13). A possible choice is $a_n = (n+n_0)!$ and $t_n = n a_n^2$; the integer n_0 is chosen large enough so that s_0' and s_0'' are suitably small. In any case, Eq. (2.16) and the dominated convergence theorem permit us to infer from (2.15):

$$\lim_{n\to+\infty} \sigma^0(t_{2n},0) = e^{c_0/2}, \quad \lim_{n\to+\infty} \sigma^0(t_{2n+3},0) = 1.$$

This is exactly the behaviour that we were looking for.

The just constructed example is, of course, by no means new. It was first identified in [5], where the reader may find a much more exhaustive study.

Apply Theorem 2.1 to u_0: estimate (2.2) implies

$$\|\sigma(t,.) - \sigma^0(t,.)\|_{L^\infty(\mathbb{R})} = O(\varepsilon^\delta), \quad \text{uniformly in } t. \tag{2.17}$$

From (2.12), for all $x \in \mathbb{R}$, the function $t \mapsto s(t,x)$ has an interval of asymptotic values of length at least $1 - O(\varepsilon^\delta)$. This implies the nontrivality of $\omega(u_0)$, and this also implies that the dynamics of the function $\sigma(t,x) = e^{c_0 s(t,x)}$ is ε^δ-close to a nontrivial dynamics of the pure heat equation.

2.2.2 *Interpretation of Theorem 2.2*

This time, the difference between the two translates of the conical wave bounding the initial datum u_0 is small; however, we still have the freedom to choose how slowly the level lines of u_0 will oscillate at infinity. In particular, we may decide that their oscillation rate will be much smaller than their amplitude, and this is the meaning of Condition (2.8). In particular, we may take

$$\sup_{\mathbb{R}} s_0^\pm = \varepsilon, \quad \inf_{\mathbb{R}} s_0^\pm = 0. \tag{2.18}$$

while keeping the derivatives of both functions $s_0^\pm$ of order ρ_ε^2 . Let us construct $s_0^\pm$ in such a way that $\omega(\sigma_0^\pm)$ is non-trivial, where $\sigma_0^\pm = e^{c_0 s_0^\pm/2}$ and where the ω-limit set is taken with respect to the advection–diffusion equations (2.10), with solutions $\sigma^\pm(t, X_\pm)$.

If $(a_n)_n$ is a sequence satisfying (2.13), and if $s_0(X_\pm)$ is defined by (2.14), we have

$$\sigma^\pm(t,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-y^2} \sigma_0^\pm(\sqrt{t}\,y + ct\cos\alpha)\,\mathrm{d}y\,. \tag{2.19}$$

If $(t_n)_n$ satisfies (2.16), then we have

$$\lim_{n\to+\infty} \sigma^\pm(t_{2n},0) = e^{c_0\varepsilon/2}, \quad \lim_{n\to+\infty} \sigma^\pm(t_{2n+3},0) = 1.$$

This, and much more, is explained in Vázquez–Zuazua [29].

Apply Theorem 2.2 to u_0: if $\delta \in \left(\frac{1}{2}, 1\right)$, estimate (2.11) implies:

$$\|\chi^\pm(t,.) - \mathrm{Log}\sigma^\pm(t,.)\|_{L^\infty(\{|(x,y)|\geq\omega^{-1}|\mathrm{Log}\rho_\varepsilon|\})} = O(\rho_\varepsilon^{\delta/2}), \tag{2.20}$$

as soon as $t > 0$ is large enough. Now, choose any $\delta > \frac{1}{2}$. From (2.20), for all $x \in \mathbb{R}$, the function $t \mapsto \chi^\pm(t, X_\pm)$ has an interval of asymptotic values of length at least $\varepsilon(1+O(\rho_\varepsilon^{\delta/2}\varepsilon^{-1})) = \varepsilon(1+O(\varepsilon^{2\delta-1})) = \varepsilon(1+o_{\varepsilon\to 0}(1))$. As a consequence, we once again recover the nontrivality of $\omega(u_0)$.

2.3 Notations

Let us close the section by setting up some notations that will be used all along the paper. We will extensively work with Hölder's spaces defined as follows: If I is an open, not necessarily bounded interval of $\mathbb{R}_+$, let us denote - as is classical - by $C^{\frac{\alpha}{2},\alpha}(I\times\mathbb{R}^n)$ the space of all functions $u(t,X) \in L^\infty(I\times\mathbb{R}^n)$ such that

$$\|u\|_{\dot{C}^{\frac{\alpha}{2},\alpha}(I\times\mathbb{R}^n)} := \sup \frac{|u(t,X) - u(t',X')|}{|t-t'|^{\frac{\alpha}{2}} + |X-X'|^\alpha} < +\infty, \tag{2.21}$$

where the supremum is taken over all quadruples $(t,t',X,X') \in I^2 \times \mathbb{R}^{2n}$ such that $t \neq t'$ and $X \neq X'$. The set $C^{1+\frac{\alpha}{2},2+\alpha}(I\times\mathbb{R}^n)$ is the space of functions $u(t,X) \in L^\infty(I\times\mathbb{R}^n)$

such that $\partial_t u$ and $\partial_X^2 u$ exist and belong to $C^{\frac{\alpha}{2},\alpha}(I \times \mathbb{R}^2)$. See [18] for an extensive study of the properties of these spaces. The spaces $C^\alpha(\mathbb{R}^n)$ and $C^{2+\alpha}(\mathbb{R}^n)$—the functions of these spaces do not depend of t—are defined similarly.

Let now $\phi_0(y)$ be a solution of (1.4). If $BUC(\mathbb{R})$ is the set of all bounded, uniformly continuous functions of $\mathbb{R}$, and if $BUC^k(\mathbb{R})$ is the set of all bounded, C^k functions of $\mathbb{R}$ whose k^{th} derivative is in $BUC(\mathbb{R})$, define L_0 by

$$D(L_0) = BUC^2(\mathbb{R}), \quad L_0 = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + c_0\frac{\mathrm{d}}{\mathrm{d}y} - f'(\phi_0).$$

L_0 stands for the linearised operator of equation (1.4) around the wave ϕ_0. Recall that 0 is a simple isolated eigenvalue of L_0 with eigenvector ϕ_0'. Therefore, see [15,16,28], the space $BUC(\mathbb{R})$ may be broken as

$$BUC(\mathbb{R}) = <\phi_0'> \oplus R(L_0) = N(L_0) \oplus R(L_0),$$

and the projector P onto $N(L_0)$ parallel to $R(L_0)$ is given by

$$(Pu)(y) = \left(\alpha \int_{\mathbb{R}} e^{-c_0 z}\phi_0'(z)u(z)\,\mathrm{d}z\right)\phi_0'(y) = \left(\int_{\mathbb{R}} \psi_0(z)u(z)\mathrm{d}z\right)\phi_0'(y). \quad (2.22)$$

where $\psi_0(y) = \alpha e^{-c_0 y}\phi_0'(y)$ and α is chosen so that $\int_{\mathbb{R}} \psi_0\phi_0' = 1$. We set

$$Q = I - P.$$

The spectral subspace corresponding to the eigenvalue 0 is defined by $N(L_0) = \{u \in BUC^2(\mathbb{R}) \,|\, u = Pu\}$ and its supplementary by $R(L_0) = \{u \in BUC^2(\mathbb{R}) \,|\, Pu = 0\}$. Then, $R(L_0)$ equipped with the $L^\infty(\mathbb{R})$ norm is a Banach space and $L_0|_{R(L_0)}$ generates an analytic semigroup which satisfies $\|e^{tL_0}\|_{\mathcal{L}(R(L_0))} \leq Ce^{-\gamma t}$ for all $t \geq 0$ and some given positive constants C and γ.

Finally, we denote by C a generic positive constant, which may differ from place to place even in the same chain of inequalities.

3 Almost planar fronts

The proof of Theorem 2.1, presented in this section, is broken into two parts. In the first part, we assume that the initial datum is L^∞-close to a wave, and more precisely that (2.3) holds. In the second part, we prove that the problem may be reduced to the model situation of the first part, provided a sufficiently large time has elapsed.

3.1 Local study

Here is the exact statement that we are going to prove here.

Theorem 3.1 *Fix $\alpha \in (0, 1)$. Consider $u_0(x, y) \in C^{2+\alpha}(\mathbb{R}^2)$ for which we may find a couple $(s_0(x), v_0(x, y))$, and two positive numbers C and ε such that*

(i) $s_0 \in C^{2+\alpha}(\mathbb{R})$, $v_0 \in C^{2+\alpha}(\mathbb{R}^2)$*; moreover, if* $\sigma_0 = e^{c_0 s_0/2}$ *we have*

$$\|v_0\|_{C^{2+\alpha}(\mathbb{R}^2)} \leq \varepsilon, \quad \|\sigma_0\|_{L^\infty(\mathbb{R})} \leq C, \quad \|\partial_{xx}\sigma_0\|_{\dot{C}^\alpha(\mathbb{R})} \leq \varepsilon. \quad (3.1)$$

(ii) *For all* $x \in \mathbb{R}$ *we have* $Pv_0(x,.) = 0$*; moreover we have the equality*

$$u_0(x,y) = \phi_0(y + s_0(x)) + v_0(x, y + s_0(x)). \tag{3.2}$$

Then, there exists a unique global in time solution u *of Eq.* (*1.1*) *emanating from* u_0 *and there is a unique decomposition for any* $(t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^2$

$$u(t,x,y) = \phi_0(y + c_0 t + s(t,x)) + v(t,x,y + c_0 t + s(t,x)), \quad Pv(t,x,.) = 0 \tag{3.3}$$

such that, for all $\delta \in (0,1)$ *we have*

$$\|v(t)\|_{L^\infty(\mathbb{R}^2)} = O\left(\frac{\varepsilon^\delta}{(1+t)^{1-\delta}}\right)$$

and the function $\sigma(t,x) := e^{c_0 s(t,x)/2}$ *satisfies, for some* $C_\delta > 0$*:*

$$|\sigma_t - \sigma_{xx}| \le \frac{C_\delta \varepsilon^\delta}{(1+t)^{2-2\delta}}. \tag{3.4}$$

Proof of Theorem 3.1. Since $u_0 \in C^{2+\alpha}(\mathbb{R}^2)$, there exists a unique solution $u \in C^{1+\alpha/2, 2+\alpha}$ $(\mathbb{R}^+ \times \mathbb{R}^2)$ of Eq. (1.1) emanating from u_0. Let $u(t,x,y)$ undergo the three successive transformations.

- Set $u(t,x,y) = U(t,x,y + c_0 t + s(t,x))$—the function $s(t,x)$ is, at this stage, an unknown that satisfies $s(0,x) = s_0(x)$—the function U satisfies

$$U_t - \Delta U - 2 s_x U_{xy} - s_x^2 U_{yy} + (s_t + c_0 - s_{xx}) U_y = f(U) \tag{3.5}$$

  where U_y denotes the derivative of U with respect to its third variable.
- Denoting by (t,x,y) the new system of coordinates and setting $u(t,x,y) := U(t,x,y)$—the old reference frame will not be referred to anymore—we look for a decomposition of $u(t,x,y)$ as

$$u(t,x,y) = \phi_0(y) + v(t,x,y), \quad Pv(t,x,.) = 0, \; v(0,x,y) = v_0(x,y).$$

  Such a decomposition is certainly valid at time $t = 0$. To be valid for all later time, it must go with an equation for s. To derive it, we look for $s(t,x)$ as—Hopf–Cole transform—$\sigma(t,x) = e^{c_0 s(t,x)/2}$. Expand Eq. (3.5) about ϕ_0; then project it, pointwise in x, onto $N(L_0)$ and $R(L_0)$, this yield the system

$$\begin{cases} v_t + (-\partial_{xx} + L_0)v = f_1(\sigma, v) \\ \sigma_t - \partial_{xx}\sigma = f_2(\sigma, v) \end{cases} \tag{3.6}$$

  where the f_i's are functionals whose expressions can be explicitely computed from (3.5) and the Taylor's formula with integral remainder.
- Finally, let (σ_*, v_*) be the unique solution of the (linear) system

$$\begin{cases} \partial_t v_* + (-\partial_{xx} + L_0)v_* = \dfrac{4}{c_0^2}\left(\dfrac{\partial_x \sigma_*}{\sigma_*}\right)^2 Q(\phi_0'') \\ \partial_t \sigma_* - \partial_{xx}\sigma_* = 0 \\ \sigma_*(0,x) = \sigma_0(x), \; v_*(0,x,y) = v_0(x,y) \end{cases} \tag{3.7}$$

  The unknown (σ, v) is sought for under the form $(\sigma_* + \sigma_1, v_* + v_1)$, and the new unknown satisfy

$$\begin{cases} \partial_t v_1 + (-\partial_{xx} + L_0)v_1 = F_1(\sigma_1, v_1) \\ \partial_t \sigma_1 - \partial_{xx}\sigma_1 = F_2(\sigma_1, v_1) \\ \sigma_1(0,x) = 0, \; v_1(0,x,y) = 0 \end{cases} \tag{3.8}$$

where the expressions of the functionals F_i are given by

$$F_1(\sigma_1, v_1) = Q(K_{\phi_0}[v]v^2) + \frac{4}{c_0}\frac{\sigma_x}{\sigma}Q(v_{xy}) + \frac{4}{c_0^2}\left(\frac{\sigma_x}{\sigma}\right)^2 Q(v_{yy})$$
$$+\frac{4}{c_0^2}\left(\left(\left(\frac{\sigma_x}{\sigma}\right)^2 - \left(\frac{\partial_x\sigma_*}{\sigma_*}\right)^2\right)Q(\phi_0'') - \frac{2}{c_0}\left(\frac{\sigma_t}{\sigma} - \frac{\sigma_{xx}}{\sigma} - \left(\frac{\sigma_x}{\sigma}\right)^2\right)Q(v_y)\right)$$
$$F_2(\sigma_1, v_1) = \frac{c_0}{2}\sigma\int_R \psi_0(y)K_{\phi_0}[v]v^2\,\mathrm{d}y + 2\sigma_x\int_R \psi_0(y)v_{xy}\,\mathrm{d}y$$
$$+\frac{2}{c_0}\frac{\sigma_x^2}{\sigma}\int_R \psi_0(y)v_{yy}\,dy - \left(\sigma_t - \sigma_{xx} + \frac{\sigma_x^2}{\sigma}\right)\int_R \psi_0(y)v_y\,\mathrm{d}y$$

where we have noted, for commodity: $(\sigma, v) = (\sigma_* + \sigma_1, v_* + v_1)$ and

$$K_{\phi_0}[v]v^2 = f(\phi_0 + v) - f(\phi_0) - f'(\phi_0)v = \frac{v^2}{2}\int_0^1 (1-\zeta)f''(\phi_0 + \zeta v)\,\mathrm{d}\zeta\,.$$

The expressions of F_1 and F_2 look formidable, but they are only standard quadratic terms in the unknowns that we wish to keep small i.e. v_1 and σ_1. From now on, *we fix* $\delta \in (0, 1)$. *All the constants in the rest of the section will depend on* δ.

Lemma 3.2 (Estimates on (σ_*, v_*)) *Under the assumptions of Theorem 3.1, we have, for some* $C > 0$ *independent of* ε*:*

$$\|\sigma_*(t)\|_{L^\infty(\mathbb{R})} \le C,$$
$$\|\partial_x\sigma_*(t)\|_\infty \le \frac{C\varepsilon^{\frac{\delta}{2+\alpha}}}{(1+t)^{1/2-\delta/2}}, \qquad \|\partial_{xx}\sigma_*(t)\|_\infty \le \frac{C\varepsilon^{\frac{\delta}{2+\alpha}}}{(1+t)^{1-\delta}}$$
$$\|\sigma_*\|_{\dot C^{\alpha,\frac{\alpha}{2}}} \le \frac{C\varepsilon^{\frac{\delta}{2+\alpha}}}{(1+t)^{1/2-\delta/2}}, \qquad \|\partial_x\sigma_*\|_{\dot C^{\alpha,\frac{\alpha}{2}}} \le \frac{C\varepsilon^{\frac{\delta}{2+\alpha}}}{(1+t)^{1-\delta}}$$
$$\|v_*(t)\|_{L^\infty(\mathbb{R}^2)} \le \frac{C\varepsilon^{\frac{2\delta}{2+\alpha}}}{(1+t)^{1-\delta}},$$
$$\|\partial_x v_*(t)\|_\infty \le \frac{C\varepsilon^{\frac{3\delta}{2+\alpha}}}{(1+t)^{3/2-3\delta/2}}, \qquad \|\partial_y v_*(t)\|_\infty \le \frac{C\varepsilon^{\frac{2\delta}{2+\alpha}}}{(1+t)^{1-\delta}}$$
$$\|\partial_{xy} v_*(t)\|_\infty \le \frac{C\varepsilon^{\frac{3\delta}{2+\alpha}}}{(1+t)^{3/2-3\delta/2}}, \qquad \|\partial_{yy} v_*(t)\|_\infty \le \frac{C\varepsilon^{\frac{2\delta}{2+\alpha}}}{(1+t)^{1-\delta}}$$
$$\|v_*\|_{\dot C^{\alpha,\alpha/2}} \le \frac{C\varepsilon^{\frac{3\delta}{2+\alpha}}}{(1+t)^{3/2-3\delta/2}}, \qquad \|\partial_y v_*\|_{\dot C^{\alpha,\alpha/2}} \le \frac{C\varepsilon^{\frac{3\delta}{2+\alpha}}}{(1+t)^{3/2-3\delta/2}}$$
$$\|\partial_{xy} v_*\|_{\dot C^{\alpha,\alpha/2}} \le \frac{C\varepsilon^{\frac{4\delta}{2+\alpha}}}{(1+t)^{2-2\delta}}, \qquad \|\partial_{yy} v_*\|_{\dot C^{\alpha,\alpha/2}} \le \frac{C\varepsilon^{\frac{3\delta}{2+\alpha}}}{(1+t)^{3/2-3\delta/2}}$$

These estimates will be proved in Appendix.

Proof of Theorem 3.1 (continued). By a standard analytic semigroup argument—see [15], Chap. 3—system (3.8), endowed with the initial datum $(\sigma_1, v_1)(t=0) = (0, 0)$, has a unique local in time solution $(\sigma_1, v_1) \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T^*[\times\mathbb{R}) \times C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T^*[\times\mathbb{R}^2)$ for some $T^* > 0$. Let $T > 0$ be the largest time T' such that, forall $t \in [0, T']$, we have

$$\begin{cases} \|\sigma_1(t)\|_{L^\infty(\mathbb{R})} \le \varepsilon^{\frac{2\delta}{2+\alpha}} \\ \|\partial_x\sigma_1(t)\|_{L^\infty(\mathbb{R})} \le \dfrac{\varepsilon^{\frac{2\delta}{2+\alpha}}}{\sqrt{1+t}} \\ \|\partial_{xx}\sigma_1\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((t,2t)\times\mathbb{R})} + \|\partial_t\sigma_1\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((t,2t)\times\mathbb{R})} \le \dfrac{\varepsilon^{\frac{2\delta}{2+\alpha}}}{(1+t)^{1+\frac{\alpha}{2}}} \\ \|v_1(t)\|_{L^\infty(\mathbb{R}^2)} \le \dfrac{\varepsilon^{\frac{2\delta}{2+\alpha}}}{1+t} \\ \|\partial_{yy}v_1\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((t,2t)\times\mathbb{R}^2)} \le \dfrac{\varepsilon^{\frac{2\delta}{2+\alpha}}}{1+t} \\ \|\partial_{xx}v_1\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((t,2t)\times\mathbb{R}^2)} + \|\partial_{xy}v_1\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((t,2t)\times\mathbb{R}^2)} \le \dfrac{\varepsilon^{\frac{2\delta}{2+\alpha}}}{(1+t)^{1+\frac{\alpha}{2}}} \end{cases} \tag{3.9}$$

Since at time $t = 0$, $\sigma_1 = 0$ and $v_1 = 0$, the definition of T makes sense and by continuity, $T > 0$. We claim that $T = T^*$ which also implies $T = T^* = +\infty$. Indeed, if $T < T^*$, for any $t \in [0, T]$, inequalities (3.9) hold and by appendix 5.4,

$$\begin{aligned} \|F_2(t)\|_{L^\infty} \le& C\left(\|v(t)\|_\infty^2 + \|\sigma_x(t)\|_\infty\|v_{xy}(t)\|_\infty + \|\sigma_x(t)\|_\infty^2\|v_{yy}(t)\|_\infty\right) \\ &+ C(\|\partial_t\sigma_1(t)\|_\infty + \|\partial_{xx}\sigma_1(t)\|_\infty + \|\partial_x\sigma_*(t)\|_\infty^2)\|v_y(t)\|_\infty \\ \le& \frac{\varepsilon^{\frac{4\delta}{2+\alpha}}}{(1+t)^{2(1-\delta)}} \\ \|F_2\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((t,2t)\times\mathbb{R})} \le& \frac{\varepsilon^{\frac{5\delta}{2+\alpha}}}{(1+t)^{\frac{5}{2}(1-\delta)}} \end{aligned}$$

Let us deal with σ_1. We have

$$\sigma_1(t,x) = \int_0^t e^{(t-s)\partial_{xx}} F_2(s)\,\mathrm{d}s;$$

from the above estimates on F_2 norms and Proposition 5.4, we have the following more precise estimates

$$\begin{cases} \|\sigma_1(t)\|_{L^\infty(\mathbb{R})} \le \varepsilon^{\frac{4\delta}{2+\alpha}} & \|\sigma_1\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((t,2t)\times\mathbb{R})} \le \dfrac{\varepsilon^{\frac{4\delta}{2+\alpha}}}{\sqrt{1+t}} \\ \|\partial_x\sigma_1(t)\|_{L^\infty(\mathbb{R})} \le \dfrac{\varepsilon^{\frac{4\delta}{2+\alpha}}}{\sqrt{1+t}} & \|\partial_x\sigma_1\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((t,2t)\times\mathbb{R})} \le \dfrac{\varepsilon^{\frac{4\delta}{2+\alpha}}}{1+t} \\ \|\partial_t\sigma_1(t)\|_\infty + \|\partial_{xx}\sigma_1(t)\|_\infty \le \dfrac{\varepsilon^{\frac{4\delta}{2+\alpha}}}{1+t} & \|\partial_t\sigma_1\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} + \|\partial_{xx}\sigma_1\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} \le \dfrac{c\varepsilon^{\frac{4\delta}{2+\alpha}}}{(1+t)^{1+\frac{\alpha}{2}}} \end{cases}$$

We now plug these last inequalities into the equation for v_1. By Appendix 5.4 there holds

$$\begin{aligned} \|F_1(t)\|_{L^\infty(\mathbb{R}^2)} \le& C\left(\|v(t)\|_\infty^2 + \|\sigma_x(t)\|_\infty\|v_{xy}(t)\|_\infty + \|\sigma_x(t)\|_\infty^2\|v_{yy}(t)\|_\infty\right) \\ &+ C(\|\partial_x\sigma_1(t)\|_\infty\|\sigma_*(t)\|_\infty + \|\sigma_1(t)\|_\infty\|\partial_x\sigma_*(t)\|_\infty)\|\partial_x\sigma_*(t)\|_\infty \\ &+ C(\|\partial_t\sigma_1(t)\|_\infty + \|\partial_{xx}\sigma_1(t)\|_\infty + \|\partial_x\sigma_*(t)\|_\infty^2)\|v_y(t)\|_\infty \\ \le& \frac{C\varepsilon^{\frac{4\delta}{2+\alpha}}}{1+t} \end{aligned}$$

In a similar way, we get the same decay, rate for $\|F_1\|_{\dot C^{\frac{\alpha}{2},\alpha}((t,2t)\times\mathbb{R}^2)}$ and thus,

$$\|v_1(t)\|_{L^\infty(\mathbb{R}^2)} + \|\partial_{yy}v_1(t)\|_{L^\infty(\mathbb{R}^2)} \le \int_0^t e^{-\gamma(t-\tau)}\frac{\varepsilon^{\frac{4\delta}{2+\alpha}}}{1+\tau}\mathrm{d}\tau \le \frac{\varepsilon^{\frac{4\delta}{2+\alpha}}}{1+t}.$$

Finally, using Proposition 5.4 once again, we get

$$\|\partial_{xx}v_1\|_{\dot C^{\frac{\alpha}{2},\alpha}} + \|\partial_{xy}v_1\|_{\dot C^{\frac{\alpha}{2},\alpha}} \le \frac{C\varepsilon^{\frac{4\delta}{2+\alpha}}}{(1+t)^{1+\frac{\alpha}{2}}}.$$

Thus, at time $t = T$, the inequalities satisfied by the norms of σ_1 and v_1 are better than expected and we contradict the maximal nature of T. Thus $T = T^* = +\infty$ and estimates (3.9) are satisfied for all times. This concludes the proof of Theorem 3.1. □

3.2 Global study

The aim of this section is to bridge the gap between Theorems 2.1 and 3.1. We assume here that the initial datum lies between two waves and we show that, provided a large time has elapsed, the solution satisfies the assumptions (3.1) and (3.2) of Theorem 3.1, that is to say the model situation (2.3) in which the solution u can be split, in each point $x \in \mathbb{R}$, into a translate of the wave ϕ_0 and a small perturbation v_0.

Theorem 3.3 *Given $u_0 \in C(\mathbb{R}^2)$, assume the existence of $y_1 \le y_2$ such that*

$$\forall (x,y) \in \mathbb{R}^2: \quad \phi_0(y+y_1) \le u_0(x,y) \le \phi_0(y+y_2),$$

where $\phi_0(y)$ is a solution of (1.4). We denote by $u(t,x,y)$ the solution of equation (1.1) emanating from u_0. Fix $\alpha \in (0,1)$. Then, for any $\varepsilon > 0$, there exist some time $t_\varepsilon > 0$ and some function $s_\varepsilon \in C^{2+\alpha}(\mathbb{R})$ such that

$$\|u(t_\varepsilon,x,y) - \phi_0(y+s_\varepsilon(x))\|_{C^{2+\alpha}(\mathbb{R}^2)} \le \varepsilon$$
$$\|\partial_{xx}s_\varepsilon\|_{\dot C^\alpha(\mathbb{R})} \le \varepsilon$$

Let us postpone the proof of this theorem to the end of this section and use it for the

Proof of Theorem 2.1. Let $u_0 \in C(\mathbb{R}^2)$ be as in the assumptions of Theorem 2.1. Let y_1 and y_2 be two real numbers such that

$$\forall (x,y) \in \mathbb{R}^2, \quad \phi_0(y+y_1) \le u_0(x,y) \le \phi_0(y+y_2).$$

Let $u(t,x,y)$ be the unique solution to the Cauchy problem

$$\begin{aligned} &\partial_t u - \Delta u = f(u) \; t > 0, \quad (x,y) \in \mathbb{R}^2 \\ &u(0,x,y) = 0 \qquad (x,y) \in \mathbb{R}^2 \end{aligned}$$

Fix $\alpha \in (0,1)$ and $\varepsilon > 0$. By Theorem 3.3, there exist $t_\varepsilon > 0$ and a function $s_\varepsilon \in C^{2+\alpha}(\mathbb{R})$ such that

$$\|u(t_\varepsilon,x,y) - \phi_0(y+s_\varepsilon(x))\|_{C^{2+\alpha}(\mathbb{R}^2)} \le \varepsilon$$
$$\|\partial_{xx}s_\varepsilon\|_{\dot C^\alpha(\mathbb{R})} \le \varepsilon.$$

Let us define the following functions

$$v_0(x, y + s_\varepsilon(x)) = u(t_\varepsilon, x, y) - \phi_0(y + s_\varepsilon(x))$$
$$s_0(x) = s_\varepsilon(x)$$
$$\sigma_0(x) = e^{c_0 s_0(x)/2}$$

Then, $s_0 \in C^{2+\alpha}(\mathbb{R})$, $v_0 \in C^{2+\alpha}(\mathbb{R}^2)$, $\|v_0\|_{C^{2+\alpha}(\mathbb{R}^2)} \leq \varepsilon$ and there exists a constant $C > 0$ such that $\|\sigma_0\|_\infty \leq C$. In ordrer to use Theorem 3.1, we just need to estimate the norm $\dot{C}^\alpha(\mathbb{R})$ of $\partial_{xx}\sigma_0$, which is easily computed from the previous estimates, interpolation inequalities developed in Appendix 5.1 and Taylor's formula for the exponential function. Thus,

$$\|\partial_{xx}\sigma_0\|_{\dot{C}^\alpha} \leq C\varepsilon^{\frac{4}{(2+\alpha)^2(1+\alpha)}}$$

where $C > 0$ is some positive constant.

Letting $\tilde{\varepsilon} = \varepsilon^{\frac{4}{(2+\alpha)^2(1+\alpha)}}$, we finally have $(v_0, s_0) \in C^{2+\alpha}(\mathbb{R}) \times C^{2+\alpha}(\mathbb{R}^2)$, the estimates $\|v_0\|_{C^{2+\alpha}(\mathbb{R}^2)} \leq \tilde{\varepsilon}$, $\|\partial_{xx}\sigma_0\|_{\dot{C}^\alpha(\mathbb{R})} \leq \tilde{\varepsilon}$. By modifying v_0 and σ_0 in an $\tilde{\varepsilon}$-fashion—this only requires the Implicit functions Theorem—we may also assume that the decomposition $u_0(x, y) = \phi_0(y + s_0(x)) + v_0(x, y + s_0(x))$, with $Pv_0(x, .) = 0$, holds. Applying Theorem 3.1, there exists a unique decomposition for $t > t_\varepsilon$

$$u(t, x, y) = \phi_0(y + c_0 t + s(t, x)) + v(x, y + c_0 t + s(t, x)) \text{ and } Pv(x, .) = 0$$

where s and v satisfy the expected estimates. □

Turn to the proof of Theorem 3.3, which will be divided in a few lemmas for clarity. The idea is to show that the distance in y between the fonction $u(t, x, .)$ and the family of the travelling waves $\{\phi_0(. + y_0)\}_{y_0 \in \mathbb{R}}$ goes to zero as t goes to infinity.

Lemma 3.4 *Under the assumptions of Theorem 3.3,* $\lim_{t\to+\infty} \partial_t u = 0$ *uniformly in* $(x, y) \in \mathbb{R}^2$.

Proof The—by now classical (see [26])—idea is to use a sliding method both in time and space. Pick $h > 0$, $t > 0$ and $s \geq t$. Define $u^k(s, x, y) = u(s + h, x, y + k)$. Then, $\partial_t u^k = \Delta u^k - c_0 \partial_y u^k + f(u^k)$. By the maximum principle, u stays between two travelling waves; therefore there holds $\lim_{y\to-\infty} u(s, x, y) = 0$ and $\lim_{y\to+\infty} u(s, x, y) = 1$ uniformly in $(s, x) \in [t, +\infty) \times \mathbb{R}$. Thus, because ϕ_0 is increasing there is $A > 0$ such that

$$\forall k \geq A\,, \quad \forall s \geq t\,, \quad \forall (x, y) \in \mathbb{R}^2\,, \quad u(s, x, y) \leq u^k(s, x, y)$$

Setting

$$k^*(t) = \inf\{k > 0 \,|\, \forall s \geq t\,,\ \forall (x, y) \in \mathbb{R}^2\,,\ u(s, x, y) \leq u^k(s, x, y)\}$$

we shall prove that $\lim_{t\to+\infty} k^*(t) = 0$. Denote by l the limit of this positive non-increasing function k^* and let us prove by contradiction that $l = 0$.

Indeed, if $l > 0$, we are able to build a sequence $(t_n)_{n\in\mathbb{N}}$ going to infinity, such that $(k^*(t_n))_n$ converges to l as $n \to +\infty$, and for any $n \in \mathbb{N}$, there is $(s_n, x_n, y_n) \in [t_n, +\infty) \times \mathbb{R}^2$ with

$$\lim_{n\to+\infty} \big(u(s_n, x_n, y_n) - u(s_n + h, x_n, y_n + k^*(t_n))\big) = 0\,. \tag{3.10}$$

Denote by $v_n(s, x, y) = u(s + s_n, x + x_n, y)$ for $s > -s_n$. Then, v_n satisfies $\partial_t v_n = \Delta v_n - c\partial_y v_n + f(v_n)$ and by standard parabolic estimates, Ascoli's Theorem and up to a sub-sequence, $(v_n)_{n\in\mathbb{N}}$ converges locally uniformly in $(s, x, y) \in \mathbb{R}^3$ towards a function v_∞ which is a global solution to

$$\partial_t v_\infty = \Delta v_\infty - c\partial_y v_\infty + f(v_\infty)\,.$$

Because u is between two fixed translates of ϕ_0, we may assume that $(y_n)_n$ converges to some $y_\infty \in \mathbb{R}$. From Property (P9) we have $v(t, x, y) = \phi_0(y + y_\infty)$. However, passing to the limit in (3.10) when n goes to infinity, we get $v_\infty(h, 0, k^*) = v_\infty(0, 0, 0)$. This is impossible; ϕ_0 cannot be periodic. Then, $l = 0$ and $\lim_{t\to+\infty} k^*(t) = 0$.

Now, notice that our argument is valid irrespective of the sign of h. Indeed, we only have to assume that $|h| \le 1$ and start the argument from $t > 1$. This implies:

$$\lim_{t\to+\infty} (u(t + h, x, y) - u(t, x, y) = 0) \quad \text{uniformly in } (x, y) \in \mathbb{R}^2. \tag{3.11}$$

To prove that $\lim_{t\to+\infty} \|\partial_t u(t, ., .)\|_\infty = 0$, we argue as follows: pick any $\varepsilon > 0$; from (3.11) with $h = \varepsilon$ there is $t_\varepsilon > 0$ such that

$$\forall t \ge t_\varepsilon,\ \forall (x, y) \in \mathbb{R}^2, \quad |u(t + \varepsilon, x, y) - u(t, x, y)| \le \varepsilon^2. \tag{3.12}$$

For $t \ge t_\varepsilon$ and $(x, y) \in \mathbb{R}^2$; (3.12) and the mean value theorem yield the existence of $t_{\varepsilon,x,y} \in [t, t + \varepsilon]$ such that

$$u_t(t_{\varepsilon,x,y}, x, y) = \frac{u(t + \varepsilon, x, y) - u(t, x, y)}{\varepsilon}, \quad \text{hence} \ \ |u_t(t_{\varepsilon,x,y}, x, y)| \le \varepsilon.$$

On the other hand, u_{tt} is uniformly bounded due to the parabolic estimates; therefore we have $|u_t(t, x, y)| = O(\varepsilon)$. □

Lemma 3.5 *Under the assumptions of Theorem 3.3,* $\lim_{t\to+\infty} \partial_x u = 0$ *and* $\lim_{t\to+\infty} \partial_{xx} u = 0$ *uniformly in* $(x, y) \in \mathbb{R}^2$.

Proof Proof of lemma 3.4 can be followed along the same lines since the time invariance and the space invariance in the x variable are the same in (1.1). Finally, parabolic regularisation gives the result for the second derivative in x. □

Lemma 3.6 *Under the assumptions of Theorem 3.3,*

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} dist(u(t, x, .), \{\phi_0\}) = 0$$

where $\{\phi_0\}$ *denotes the set of all translates of the one dimensional profile* ϕ_0.

Proof We prove lemma 3.6 by reducing it to the absurd. If the conclusions of lemma 3.6 were false, there would exist $\delta > 0$ and some sequences $(t_n, x_n) \in \mathbb{R}^+ \times \mathbb{R}$ such that t_n goes to infinity and $d(u(t_n, x_n, .), \{\phi_0\}) > \delta$. Define, for all $(t, x, y) \in \mathbb{R}^+ \times \mathbb{R}^2$, $v_n(t, x, y) = u(t + t_n, x + x_n, y)$. The idea is to show that v_n converges to a function v_∞ which satisfies Eq. (1.4) and by uniqueness is a travelling wave, which contradicts the above assumptions.

The function v_n verifies $\partial_t v_n = \Delta v_n - c\partial_y v_n + f(v_n)$ for $t > -t_n$. Once again, using parabolic estimates, Ascoli's Theorem and up to a subsequence, v_n converges to a function v_∞ global solution to $\partial_t v_\infty = \Delta v_\infty - c\partial_y v_\infty + f(v_\infty)$. Using Lemmas 3.4 and 3.5, we get $\lim_{n\to+\infty} \partial_t v_n = \lim_{n\to+\infty} \partial_{xx} v_n = 0$. Then, v_∞ verifies $\partial_{yy} v_\infty - c\partial_y v_\infty + f(v_\infty) = 0$.

Let us have a look at the limiting conditions satified by v_∞. Since u satisfies uniformly in $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, $\lim_{y\to\pm\infty} u(t, x, y) = 1$ or 0, v_∞ satifies the same limit conditions and by uniqueness, there exists a real b such that $v_\infty(t, x, y) = \phi_0(y - b)$ and $c = c_0$. This contradicts the initial assumption on u. □

Let us notice that, since u_0 is between two travelling waves, we have $\omega(u_0) \subset \{\phi_0(y-b), b \in [y_1, y_2]\}$. The inclusion may be strict.

Proof of Theorem 3.3. Let u_0 be a function trapped between two travelling waves as in Theorem 3.3. Define $u(t)$ the solution of (1.1) with u_0 as initial condition. By the above lemmas, we know that

$$\lim_{t\to+\infty} \partial_t u(t,x,y) = 0 \text{ uniformly in } (x,y) \in \mathbb{R}^2$$
$$\lim_{t\to+\infty} \partial_x u(t,x,y) = 0 \text{ uniformly in } (x,y) \in \mathbb{R}^2$$
$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} dist(u(t,x,.),\{\phi_0\}) = 0$$

We have therefore a function $s(t,x)$ such that

$$\forall\varepsilon > 0\,,\ \exists t_\varepsilon > 0 :\ \ \forall(t,x,y) \in [t_\varepsilon; +\infty) \times \mathbb{R}^2\ |u(t,x,y) - \phi_0(y - s(t,x))| \le \varepsilon$$

Let us denote by s_1 the piecewise constant function defined by $s_1(t_0,x) = s(t_0,k)$ when $x \in [k,k+1)$, $k \in \mathbb{Z}$. Thus

$$\begin{aligned}|u(t_0,x,y) - \phi(y - s_1(t_0,x))| &\le |u(t_0,x,y) - u(t_0,k,y)| + |u(t_0,k,y) - \phi(y - s(t_0,k))|\\ &\le \|\partial_x u(t_0,x,y)\|_{L^\infty(k,k+1)}|x-k| + \varepsilon \le 2\varepsilon\,.\end{aligned}$$

We thus construct the function s_1 such that

$$\forall\varepsilon > 0\,,\ \exists t_0 > 0\,|\,\forall(x,y) \in \mathbb{R}^2\,,\ |u(t_0,x,y) - \phi(y - s_1(t_0,x))| \le 2\varepsilon\,.$$

Let us show that the jumps of s_1 are not much larger than a few ε's. Let $(p,q) \in \mathbb{Z}^2$.

$$\begin{aligned}|s_1(t_0,p) - s_1(t_0,q)| &\le \frac{|\phi_0(y - s_1(t_0,p)) - \phi_0(y - s_1(t_0,q))|}{\inf_{[y_1,y_2]}\phi_0'}\\ &\le C(4\varepsilon + \|\partial_x u(t_0)\|_{L^\infty(p,q)}|p-q|)\\ &\le C(4 + |p-q|)\varepsilon\end{aligned}$$

where C^{-1} is the infimum of ϕ_0' on the compact set $[y_1,y_2]$. Then, for all integer k, $|s_1(t_0, k+1) - s_1(t_0,k)| \le 5\varepsilon$ and s_1 is bounded in the compact set $[y_1,y_2]$.

Finally, let us define some mollifier $\rho \in C_0^\infty(\mathbb{R})$ such that $s_0 = \rho * s_1$ on each interval $[k - \frac{1}{2}, k + \frac{1}{2}]$ satisfies $s_0(t_0) \in C^{2+\alpha}(\mathbb{R})$ and $\|\partial_{xx}s_0(t_0)\|_{\dot{C}^\alpha([k-\frac{1}{2},k+\frac{1}{2}])} \le 5\varepsilon$.

Let us now prove that $u - \phi_0(y - s_0)$ satifies the conclusions of Theorem 3.3. We set $S_0(x) = s_0(t_0,x)$ and $v(t,x,y) = u(t,x,y) - \phi_0(y - S_0(x))$ on (t_0, t_0+1). Thus, v satifies the parabolic equation

$$\partial_t v = \Delta v - c\partial_y v + f(\phi_0 + v) + \phi_0'' - S_0''\phi_0 - S_0'\phi_0'$$

and by [18] on $(t_0 + \frac{1}{2}, t_0 + \frac{3}{2}) \times \mathbb{R}^2$, there exists some time T_ε in this interval satifying $\|v(T_\varepsilon)\|_{C^{2+\alpha}(\mathbb{R}^2)} \le C\varepsilon$. This ends the proof of Theorem 3.3. □

4 Conical fronts

To prove Theorem 2.2, the idea is to combine the results developped by [12] on the asymptotic behaviour of the conical wave and the previous Sect. 3 on almost planar fronts. What allows us to conclude is an exponential stability result in [10].

First, recall the expression of the tilted coordinates $(X_\pm, Y_\pm)$:

$$\begin{cases} X_+ = x\sin\alpha - y\cos\alpha, & Y_+ = x\cos\alpha + y\sin\alpha \\ X_- = -x\sin\alpha - y\cos\alpha, & Y_- = -x\cos\alpha + y\sin\alpha \end{cases} \tag{4.1}$$

The system (X_+, Y_+) will be used in the right half-plane, the system (X_-, Y_-) in the left half-plane. From now on, we will only work in the right half plane and all following calculations can be done in a symmetric way in the left half plane. We will therefore delete all $\pm$. For a given function $V \in C(\mathbb{R}^2)$, we will indifferently use the notation $V(x, y)$ or $V(X, Y)\ldots$ according to the system of coordinates we consider.

Let now u_0 satisfy the assumptions of Theorem 2.2, namely u_0 is sandwiched between two conical waves, distant from each other by a small translation. By Property (P5)—recall that it says that a conical wave is exponentially close to a planar wave in the directions X and up to some translation that we may, without loss of generality, assume to be zero—there is some large $X_\varepsilon > 0$, and a function $w_0(X, Y)$, defined when X is larger than X_ε, such that

$$u_0(X, Y) = \phi_0(Y) + w_0(X, Y), \quad |w_0(X, Y)| \le \varepsilon. \tag{4.2}$$

Using Property (P5) and the inequality $\inf(a, b) \le \sqrt{ab}$, valid for any set of positive numbers a and b, we derive the following estimate, as soon as $X \ge X_\varepsilon$:

$$|w_0(X, Y)| \le \sqrt{\varepsilon}e^{-\omega|Y|}, \quad |Dw_0(X, Y)| + |D^2 w_0(X, Y)| \le \rho_\varepsilon e^{-\omega|Y|}. \tag{4.3}$$

Extend the functions w_0 as

$$w_0(X, Y) = w_0(X_\varepsilon, Y) \quad \text{if } X \le X_\varepsilon - 1, \quad \|\partial_{XX} w_0\|_\infty \le C\rho_\varepsilon.$$

Finally, consider the solutions of the Cauchy Problem

$$\begin{aligned} &(\partial_t - \Delta - c\cos\alpha\,\partial_X + c_0\partial_Y)p = f(p), \quad t > 0, (X, Y) \in \mathbb{R}^2 \\ &p(t = 0, X, Y) = \phi_0(Y) + w_0(X, Y), \quad (X, Y) \in \mathbb{R}^2 \end{aligned} \tag{4.4}$$

Notice that the functions $p(t, X + ct\cos\alpha, Y)$ satisfy the assumptions of Theorem 2.1. Let us apply it: setting $\xi = X + ct\cos\alpha$ we may decompose p as

$$p(t, \xi, Y) = \phi_0(Y + S(t, \xi)) + w(t, \xi, Y)$$

where the functions $\Sigma(t, X) = e^{c_0 S(t,X)/2}$ and the function w satisfy, for every $\delta \in (0, 1)$:

$$\begin{aligned} &(\partial_t - \partial_{XX} - c\cos\alpha\,\partial_X)\Sigma = O\left(\frac{\varepsilon^\delta}{(1+t)^{2-2\delta}}\right) \\ &\|e^{\omega|Y|}w(t)\|_{C^2(\mathbb{R}^2)} \le \frac{C_\delta\rho_\varepsilon}{(1+t)^{1-\delta}} \end{aligned} \tag{4.5}$$

The constant C_δ may vary from one line of (4.5) to another, but will never depend on ε.

For a positive number X_0, let us denote by $\mathcal{C}(X_0, \alpha, \mu)$ the cone with vertex the point $(X = X_0, Y = 0)$, with axis the line $\{X \ge X_0,\ Y = 0\}$, and with angle $\mu > 0$. Let us once and for all fix

- a number $\mu \in (0, \min(\alpha, \frac{\pi}{2} - \alpha))$,
- a smooth, nonnegative, even function $\rho(x, y)$ with unit mass, suppported in the unit ball whose derivatives are small.

If $\mathbf{1}_A$ denotes the characteristic function of the set A, let us set

$$\gamma = \rho * \mathbf{1}_{\mathcal{C}(2X_\varepsilon,\alpha,\mu)}, \quad \gamma_0 = 1 - \gamma. \tag{4.6}$$

The following properties are clear, if $\varepsilon > 0$ is small enough:

$$\mathrm{supp}\gamma_0 \cap \mathrm{supp}\gamma \subset \mathcal{C}(X_\varepsilon, \alpha, \mu) \backslash \mathcal{C}(4X_\varepsilon, \alpha, \mu) \tag{4.7}$$

Finally, let $u(t, x, y)$ be the solution of (1.1) emanating from u_0. In the reference frame of the wave ϕ, (1.1) becomes

$$u_t - \Delta u + c\partial_y u = f(u), \quad (t > 0,\ (x, y) \in \mathbb{R}^2); \tag{4.8}$$

this new system of coordinates, still denoted by (x, y) will be used without further mention. The system (X, Y) will also be deduced from this new system by (4.1).

Let us finally set

$$u(t, x, y) = \gamma(X, Y)p(t, X, Y) + \gamma_0(x, y)\phi(x, y) + v(t, x, y) \tag{4.9}$$

Theorem 2.2 will be proved through the following intermediate result.

Proposition 4.1 *Under the assumptions of Theorem 2.2, for all* $\delta \in (0, 1)$*, there is a constant* $C_\delta > 0$*, independent of* ε *such that*

$$\|v(t)\|_\infty \le C_\delta(\rho_\varepsilon + \frac{\sqrt{\varepsilon}}{(1+t)^{1-\delta}}). \tag{4.10}$$

Proof Let us set, for a function $U(t, x, y) \in C^{2,1}(\mathbb{R}_+ \times \mathbb{R}^2)$:

$$\mathrm{NL}[U] = U_t - \Delta U + cU_y - f(U). \tag{4.11}$$

Also, introduce the space

$$X_\omega = \{u(x, y) \in \mathrm{BUC}(\mathbb{R}^2)|\ e^{\omega(|x|+|y|)}u(x, y) \ \in \mathrm{BUC}(\mathbb{R}^2)\}.$$

The operator L is defined as

$$D(L) = \{u \in X_\omega \mid \Delta u \in X_\omega\}; \quad \forall u \in D(L),\ Lu = -\Delta u + c\partial_y u - f'(\phi)u. \tag{4.12}$$

Let us compute $\mathrm{NL}[u]$, using the expression (4.9). For two given functions $\psi(x, y)$ and $v(x, y)$ let us set, for commodity

$$K_\psi[v] = \frac{1}{2}\int_0^1 (1-\zeta) f''(\psi + \zeta v)\, \mathrm{d}\zeta \tag{4.13}$$

1. The region $\{\gamma_0 = 1\}$. In this area we have $u = \phi + v$, therefore

$$\mathrm{NL}[u] = v_t + Lv + K_\phi[v]v^2 := v_t + Lv + H_1(t, x, y, v)v.$$

2. The region $\{\gamma = 1\}$**.** In this area, we have $\gamma_0 = 0$. Set—still for notational commodity:

$$\phi_0(x, y) = \phi_0(Y). \tag{4.14}$$

We obtain:

$$\mathrm{NL}[u] = v_t + Lv + (f'(\phi) - f'(p))v + K_p[v]v^2 := v_t + Lv + H_1(t, x, v)v.$$

The important feature to notice is that, in the area $\{\gamma = 1\}$ we have, from Property (P5), Assumption (2.7) and Property (4.5),

$$\begin{aligned} |H_1(t, x, y, 0)| &\le |f'(p) - f'(\phi_0)| + |f'(\phi_0) - f'(\phi)| \\ &\le C(\rho_\varepsilon + e^{-\omega X_\varepsilon}). \end{aligned} \tag{4.15}$$

The last quantity goes to 0 as ε goes to 0.

3. The region $\{\gamma_0 \neq 0\} \cap \{\gamma \neq 0\}$. Here we have $\gamma \neq 1$. Notice that, once this area is examined, we will have computed $NL[u]$ in the whole plane. Let us set

$$\psi(x, y) = \phi(x, y) - \phi_0(x, y);$$

we have

$$\begin{aligned}\mathrm{NL}[u] =& u_t - \Delta u + cu_y - f(u)\\ =& v_t - \Delta v + cv_y + \gamma f(p) + \gamma_0 f(\phi) - f(\gamma p + \gamma_0\phi + v) + r\end{aligned}$$

where

$$r = -p\Delta\gamma - \phi\Delta\gamma_0 - 2\nabla p \cdot \nabla\gamma - 2\nabla\phi \cdot \nabla\gamma_0 + cp\gamma_y + c\phi\partial_y\gamma_0\,.$$

Expand the nonlinear terms:

$$\begin{aligned}\gamma f(p)+\gamma_0 f(\phi)-f(\gamma p+\gamma_0\phi + v) &= \gamma(f(p) - f(\phi)) + f(\phi) - f(\phi + \gamma(p - \phi) + v)\\ &= \gamma f'(\phi)(p - \phi) + \gamma(p - \phi)^2 K_\phi[p - \phi]\\ &\quad + f(\phi) - f(\phi + \gamma(p - \phi))\\ &\quad - vf'(\gamma_0\phi + \gamma p) - v^2 K_{\gamma_0\phi+\gamma p}[v]\\ &= \gamma f'(\phi)(p - \phi) + \gamma(p - \phi)^2 K_\phi[p - \phi]\\ &\quad - \gamma f'(\phi)(p - \phi) - \gamma^2(p - \phi)^2 K_\phi[\gamma(p - \phi)]\\ &\quad - vf'(\gamma_0\phi + \gamma p) - v^2 K_{\gamma_0\phi+\gamma p}[v]\\ &= \gamma(p - \phi)^2 K_\phi[p - \phi] - \gamma^2(p-\phi)^2 K_\phi[\gamma(p-\phi)]\\ &\quad - vf'(\gamma_0\phi + \gamma p) - v^2 K_{\gamma_0\phi+\gamma p}[v]\end{aligned}$$

The final expression for $\mathrm{NL}[u]$ is therefore

$$\mathrm{NL}[u] = v_t + Lv + H_1(t, x, y, v)v + H_2(t, x, y),$$

where we have set

$$\begin{aligned}H_1(t, x, y, v) =& (f'(\phi) - f'(\gamma_0\phi + \gamma p))v - K_{\gamma_0\phi+\gamma p}[v]v^2\\ H_2(t, x, y) =& r + (p - \phi)^2(\gamma K_\phi[p - \phi] - \gamma^2 K_\phi[\gamma(p - \phi)])\end{aligned}$$

We have, from property (4.5):

$$\|\gamma(w - \psi)\|_{D(L)} \le C\rho_\varepsilon, \quad \|\partial_t(\gamma(w - \psi))\|_{X_\omega} \le \frac{C_\delta\rho_\varepsilon}{(1 + t)^{1-\delta}}.$$

This implies

$$\begin{aligned}\|(H_1(t, x, y, 0), e^{\omega(|x|+|y|)} H_2(t, x, y))\|_{C^2(\{\gamma_0\neq0,\gamma\neq0\})} &\le C\rho_\varepsilon\\ \|\partial_t(H_1(t, x, y, 0), e^{\omega(|x|+|y|)} H_2(t, x, y))\|_{L^\infty(\{\gamma_0\neq0,\gamma\neq0\})} &\le \frac{C_\delta\rho_\varepsilon}{(1 + t)^{1-\delta}}\end{aligned} \tag{4.16}$$

Therefore the part that is nonlinear in v can be decomposed into a quadratic part in v plus a small, exponentially decaying, part.

4. Decomposition of the function v and conclusion. Recall the following result—[10], Theorem 4.1: L is a sectorial operator of X_ω, whose spectrum lies in a cone of the complex plane with positive vertex. Hence there is $\lambda_0 > 0$ such that

$$\|e^{-tL}\|_{\mathcal{L}(X_\omega)} \le Ce^{-\lambda_0 t}. \tag{4.17}$$

The equation to solve for v is therefore

$$v_t + Lv + H_1(t,x,y,v)v + H_2(t,x,y) = 0 \tag{4.18}$$

with the estimates (4.16) extending to the whole real plane - indeed, $H_2 = 0$ outside $\{\gamma_0 \neq 0, \gamma \neq 0\}$. To get estimate (4.10) for v, we proceed as follows.

- Let $v_1^0(t,x,y)$ be the unique solution of

  $$Lv_1 + H_2(t,x,y) = 0\,,$$

  we have $\|v_1\|_{D(L)} \leq C\rho_\varepsilon$ and $\|\partial_t v_1\|_{D(L)} \leq \dfrac{C\rho_\varepsilon}{(1+t)^{1-\delta}}$. By the implicit functions Theorem, there is a unique solution to

  $$Lv_1 + H_1(t,x,y,v_1)v_1 + H_2(t,x,y) = 0, \quad \|v_1 - v_1^0\|_{D(L)} \leq C\rho_\varepsilon^2. \tag{4.19}$$

  We have, in addition:

  $$\|\partial_t(v_1 - v_1^0)\|_{X_\omega} \leq \frac{C\rho_\varepsilon^2}{(1+t)^{1-\delta}}. \tag{4.20}$$

- Set, finally: $v_2 = v - v_1$. We argue as in the proof of Theorem (2.22): suppose that $t_1 > 0$ is the maximal time such that we have

  $$\|v_2(t,.,.)\|_{X_\omega} \leq \frac{C\sqrt{\varepsilon}}{(1+t)^{1-\delta}}.$$

  Note that this is the only place where we use the poorer order of magnitude for $v(0)$, which is of order ε. We have

  $$v_2(t,x) = e^{-tL}v_2(0) - \int_0^t e^{(t-s)L}(H_1(s,x,y,v)v - H_1(s,x,y,v_1)v_1 + \partial_t v_1)\,\mathrm{d}s$$

  which implies, for $t \leq t_1$:

  $$\begin{aligned}\|v_2(t)\|_{X_\omega} &\leq C\sqrt{\varepsilon}e^{-\lambda_0 t} + C\int_0^t e^{-\lambda_0(t-s)}\left(\rho_\varepsilon|v(s)| + \frac{\rho_\varepsilon^2 + \rho_\varepsilon}{(1+s)^{1-\delta}}\right)\mathrm{d}s \\ &\leq \frac{C\varepsilon}{(1+t)^{1-\delta}}\end{aligned}$$

  implying in turn that $t_1 = +\infty$, provided ε is small enough.

This ends the proof of Proposition 4.1. □

Proof of Theorem 2.2 We have $\partial_Y u > 0$; therefore the level set $\{u(t,X,Y) = \lambda\}$ is a union of curves $\{Y = \chi(t,X)\}$. Also, we may assume, without loss of generality, that $\phi_0(0) = \lambda$. For any $t > 0$ and (x,y) in the right half plane, we have

$$\begin{aligned}Y = \chi(X) &\Leftrightarrow \gamma p + \gamma_0\phi + v = \lambda \\ &\Leftrightarrow \gamma\phi_0(Y + S(t,X)) + \gamma_0\phi_0(Y - \psi_\lambda(X)) \\ &\quad = \lambda + O\left(e^{-2\omega(|X|+|Y|)} + \rho_\varepsilon + \frac{\rho_\varepsilon}{(1+t)^{1-\delta}}\right)\end{aligned}$$

thanks to Theorem (2.1), Proposition (4.1) and Property (P5). Since $\lambda = \phi_0(0) = \phi(Y - \chi(X))$, we get

$$\gamma_0|\chi_\lambda - \psi_\lambda| + \gamma|S + \chi_\lambda| = O\left(e^{-2\omega(|X|+|Y|)} + \rho_\varepsilon + \frac{\rho_\varepsilon}{(1+t)^{1-\delta}}\right)$$

Finally, all we have to do is to compare Σ and σ. We recall that $\Sigma(t, X) = e^{c_0 S(t,X)/2}$ and σ is defined in Theorem 2.2 by (2.10) as the solution of the advection–diffusion equation

$$\begin{aligned}(\partial_t - \partial_{XX} - c\cos\alpha\,\partial_X)\sigma = 0\\ \sigma(0, X) = \sigma_0(X)\end{aligned} \tag{4.21}$$

where σ_0 is definied by (2.9) as

$$\sigma_0(X) = \begin{cases} e^{c_0 s_0(X)/2} & \text{if } X \geq 1\\ e^{c_0 s_0(1)/2} & \text{if } X \leq 1\end{cases} \tag{4.22}$$

Thus, by (4.5) and (4.21)

$$\begin{aligned}\Sigma(t, X) - \sigma(t, X) =& e^{t(\partial_{XX} + c\cos\alpha\partial_X)}(\sigma_0(X) - \sigma_0(X))\\ &+ \int_0^t e^{(t-s)(\partial_{XX}+c\cos\alpha\partial_X)} O\left(\frac{\varepsilon^\delta}{(1+s)^{2-2\delta}}\right)\\ =& O\left(\varepsilon^\delta + \frac{1}{(1+t)^{1-2\delta}}\right)\end{aligned}$$

This implies (2.11). □

5 Appendix: some interpolation inequalites

5.1 Basic C^α inequalities

We state here three standard propositions, whose proofs will be omitted. Take $\alpha \in (0, 1)$ and $f \in L^\infty(\mathbb{R})$.

Proposition 5.1 *If $f \in L^\infty(\mathbb{R})$ and $f' \in \dot{C}^\alpha(\mathbb{R})$ then $f \in C^{1+\alpha}(\mathbb{R})$ and there exists $C > 0$ such that*

$$\|f'\|_{L^\infty(\mathbb{R})} \leq C\|f\|_{L^\infty(\mathbb{R})}^{\frac{\alpha}{1+\alpha}}\|f'\|_{\dot{C}^\alpha(\mathbb{R})}^{\frac{1}{1+\alpha}}$$

Proposition 5.2 *If $f \in L^\infty(\mathbb{R})$ and $f'' \in \dot{C}^\alpha(\mathbb{R})$ then $f \in C^{2+\alpha}(\mathbb{R})$ and there exists $C > 0$ such that*

$$\|f''\|_{L^\infty(\mathbb{R})} \leq C\|f\|_{L^\infty(\mathbb{R})}^{\frac{\alpha}{2+\alpha}}\|f''\|_{\dot{C}^\alpha(\mathbb{R})}^{\frac{2}{2+\alpha}}$$

Proposition 5.3 *If $f \in C^{2+\alpha}(\mathbb{R})$, there exists $C > 0$ such that*

1. $\|f'\|_\infty \leq C\|f\|_\infty^{\frac{1}{2}}\|f''\|_\infty^{\frac{1}{2}} \leq C\|f\|_\infty^{\frac{1+\alpha}{2+\alpha}}\|f''\|_{\dot{C}^\alpha(\mathbb{R})}^{\frac{1}{2+\alpha}}$
2. $\|f'\|_{\dot{C}^\alpha(\mathbb{R})} \leq C\|f'\|_\infty^{\frac{\alpha}{1+\alpha}}\|f''\|_{\dot{C}^\alpha(\mathbb{R})}^{\frac{1}{1+\alpha}} \leq C\|f\|_\infty^{\frac{\alpha}{2+\alpha}}\|f''\|_{\dot{C}^\alpha(\mathbb{R})}^{\frac{2}{2+\alpha}}$
3. $\|f\|_{\dot{C}^\alpha(\mathbb{R})} \leq C\|f\|_\infty^{\frac{\alpha}{1+\alpha}}\|f'\|_{\dot{C}^\alpha(\mathbb{R})}^{\frac{1}{1+\alpha}} \leq C\|f\|_\infty^{\frac{\alpha(3+\alpha)}{(2+\alpha)(1+\alpha)}}\|f''\|_{\dot{C}^\alpha(\mathbb{R})}^{\frac{2}{(2+\alpha)(1+\alpha)}}$

5.2 Estimates on σ_*

The aim of the subsection is to prove one part of Lemma 3.2. We recall that σ_0 and σ_* are defined in Theorem 3.1 by the following inequalities and equations:

$$\begin{aligned} &\sigma_0 \in C^{2+\alpha}(\mathbb{R})\,, \quad \|\sigma_0\|_\infty \le C\,, \quad \|\partial_{xx}\sigma_0\|_{\dot{C}^\alpha(\mathbb{R})} \le \varepsilon \\ &\partial_t\sigma_* - \partial_{xx}\sigma_* = 0 \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R} \\ &\sigma_*(0,x) = \sigma_0(x) \quad x \in \mathbb{R} \end{aligned}$$

Let us prove the estimates of Lemma 3.2.

Estimates on the integral kernal of the heat equation lead to the existence of a constant $C > 0$ such that, for all $t \in \mathbb{R}^+$,

$$\|\sigma_*(t)\|_\infty \le C\,, \quad \|\partial_{xx}\sigma_*(t)\|_\infty \le C\|\partial_{xx}\sigma_0\|_\infty \le C\varepsilon^{\frac{2}{2+\alpha}}$$

and

$$\|\partial_{xx}\sigma_*(t)\|_{\dot{C}^\alpha(\mathbb{R})} \le C\|\partial_{xx}\sigma_0\|_{\dot{C}^\alpha(\mathbb{R})} \le C\varepsilon$$

Interpolating those estimates, we get bounds on all the derivatives up to the second order of σ_* in both norms L^∞ and $\dot{C}^\alpha$.

In the same way, we know time dependent estimates on the heat kernel:

$$\|\partial_x\sigma_*\|_\infty \le \frac{C}{\sqrt{t}}\|\sigma_0\|_\infty\,.$$

We can deduce from this inequality and from Proposition 5.3 similar time dependent estimates on the L^∞ and $\dot{C}^\alpha$ norm of the derivatives of σ_*.

Finally interpolating the first ones with the second ones, we get for any $\delta \in (0,1)$,

$$\begin{aligned} &\|\sigma_*\|_\infty \le C,\ \ \|\partial_x\sigma_*\|_\infty \le \frac{C\varepsilon^{\frac{\delta}{2+\alpha}}}{(1+t)^{1/2-\delta/2}},\ \ \|\partial_{xx}\sigma_*\|_\infty \le \frac{C\varepsilon^{\frac{2\delta}{2+\alpha}}}{(1+t)^{1-\delta}} \\ &\qquad \|\sigma_*\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((t,2t)\times\mathbb{R})} \le \frac{C\varepsilon^{\frac{\delta}{2+\alpha}}}{(1+t)^{1/2-\delta/2}},\ \ \|\partial_x\sigma_*\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((t,2t)\times\mathbb{R})} \le \frac{C\varepsilon^{\frac{2\delta}{2+\alpha}}}{(1+t)^{1-\delta}} \end{aligned}$$

5.3 Estimates on v_*

The aim of the subsection is to prove the second part of Lemma 3.2. We recall that v_0 and v_* are defined in Theorem 3.1 by

$$\begin{aligned} &v_0 \in C^{2+\alpha}(\mathbb{R}^2)\,, \quad \|v_0\|_{C^{2+\alpha}(\mathbb{R}^2)} \le \varepsilon\,, \quad Pv_0(x,.) = 0 \\ &\partial_t v_* + (-\partial_{xx} + L_0)v_* = \frac{4}{c_0^2}\left(\frac{\partial_x\sigma_*}{\sigma_*}\right)^2 Q(\phi_0") \\ &v_*(0,x,y) = v_0(x,y)\,, \quad (x,y) \in \mathbb{R}^2 \end{aligned}$$

Let us prove the estimates of Lemma 3.2. Written in its integral form, equation (3.7) satisfied by v_* reads

$$v_*(t) = e^{t(-\partial_{xx}+L_0)}v_0 + \int_0^t e^{(t-\tau)(-\partial_{xx}+L_0)}\frac{4}{c_0^2}\left(\frac{\partial_x\sigma_*}{\sigma_*}\right)^2 Q(\phi_0'')\mathrm{d}\tau\,.$$

Keeping in mind that L_0 generates an analytic semigroup which is exponentially decreasing in time in the supplementary $R(L_0)$ of its kernel and using the above section on σ_*, we can bound v_* and its derivative in the L^∞ norm. Let us just notice that the desired power of ε is obtained for $\partial_x v_*$ by inverting the derivative and the semi-group. Finally, $\dot{C}^\alpha$ estimates are obtained by the inequality $\|f\|_{\dot{C}^\alpha} \leq \|f'\|_\infty$.

5.4 Estimates on F_1 and F_2

We recall the expressions of the non-linear terms F_1 and F_2 that appear in the equations for v_1 and σ_1 in the local study of planar fronts (see Sect. 3.1):

$$\begin{aligned} F_1(\sigma_1, v_1) &= Q(K_{\phi_0}[v]v^2) + \frac{4}{c_0}\frac{\sigma_x}{\sigma} Q(v_{xy}) + \frac{4}{c_0^2}\left(\frac{\sigma_x}{\sigma}\right)^2 Q(v_{yy}) \\ &\quad + \frac{4}{c_0^2}\left(\left(\left(\frac{\sigma_x}{\sigma}\right)^2 - \left(\frac{\partial_x \sigma_*}{\sigma_*}\right)^2\right) Q(\phi_0'') - \frac{2}{c_0}\left(\frac{\sigma_t}{\sigma} - \frac{\sigma_{xx}}{\sigma} - \left(\frac{\sigma_x}{\sigma}\right)^2\right) Q(v_y)\right) \\ F_2(\sigma_1, v_1) &= \frac{c_0}{2}\sigma \int_R \psi_0(y) K_{\phi_0}[v]v^2 \,\mathrm{d}y + 2\sigma_x \int_R \psi_0(y) v_{xy}\,\mathrm{d}y \\ &\quad + \frac{2}{c_0}\frac{\sigma_x^2}{\sigma}\int_R \psi_0(y) v_{yy}\,\mathrm{d}y - \left(\sigma_t - \sigma_{xx} + \frac{\sigma_x^2}{\sigma}\right)\int_R \psi_0(y) v_y\,\mathrm{d}y \end{aligned}$$

where we have noted, for commodity: $(\sigma, v) = (\sigma_* + \sigma_1, v_* + v_1)$ and

$$K_{\phi_0}[v]v^2 = f(\phi_0 + v) - f(\phi_0) - f'(\phi_0)v = \frac{1}{2}\int_0^1 (1-\zeta) f''(\phi_0 + \zeta v)\,\mathrm{d}\zeta\, v^2 .$$

For any $t > 0$, we need some bounds on the norms $\|F_1(t)\|_{L^\infty(\mathbb{R}^2)}$, $\|F_2(t)\|_{L^\infty(\mathbb{R})}$, $\|F_1\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((t,2t)\times(\mathbb{R}^2))}$ and $\|F_2\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((t,2t)\times(\mathbb{R}))}$. To get the bounds of the L^∞ norms, all you have to know are the following ideas:

- Since $\sigma_0 > 0$ on the real line and $\partial_{xx}\sigma_0$ is small, due to the maximum principle, there exists $a > 0$ such that $\sigma(t, x) > a$ for any time and any real x.
- The operator Q is a projector.
- $\left|\int_{\mathbb{R}} \psi_0(y) v(t, x, y) dy\right| \leq \|v(t)\|_{L^\infty(\mathbb{R}^2)} \|\psi_0\|_{L^1(\mathbb{R})}$

Then,

$$\begin{aligned} \|F_1(t)\|_{L^\infty} \leq & C\left(\|v(t)\|_\infty^2 + \|\sigma_x(t)\|_\infty \|v_{xy}(t)\|_\infty + \|\sigma_x(t)\|_\infty^2 \|v_{yy}(t)\|_\infty\right) \\ & + C(\|\partial_x\sigma_1(t)\|_\infty \|\sigma_*(t)\|_\infty + \|\sigma_1(t)\|_\infty \|\partial_x\sigma_*(t)\|_\infty)\|\partial_x\sigma_*(t)\|_\infty \\ & + C(\|\partial_t\sigma_1(t)\|_\infty + \|\partial_{xx}\sigma_1(t)\|_\infty + \|\partial_x\sigma_*(t)\|_\infty^2)\|v_y(t)\|_\infty \end{aligned}$$

and

$$\begin{aligned} \|F_2(t)\|_{L^\infty} \leq & C\left(\|v(t)\|_\infty^2 + \|\sigma_x(t)\|_\infty \|v_{xy}(t)\|_\infty + \|\sigma_x(t)\|_\infty^2 \|v_{yy}(t)\|_\infty\right) \\ & + C(\|\partial_t\sigma_1(t)\|_\infty + \|\partial_{xx}\sigma_1(t)\|_\infty + \|\partial_x\sigma_*(t)\|_\infty^2)\|v_y(t)\|_\infty \end{aligned}$$

Going through the $\dot{C}^{\frac{\alpha}{2},\alpha}$ norms, the only new idea is that for any $(f,g) \in C^\alpha(\mathbb{R})$, $\|fg\|_{\dot{C}^\alpha} \le \|f\|_\infty \|g\|_{\dot{C}^\alpha} + \|g\|_\infty \|f\|_{\dot{C}^\alpha}$.

$$\begin{aligned}
&\|F_1\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((t,2t)\times\mathbb{R}^2)}\\
&\quad\le C(\|v(t)\|_\infty \|v\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} + \|\sigma_x(t)\|_\infty \|v_{xy}\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} + \|\sigma_x\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} \|v_{xy}(t)\|_\infty)\\
&\qquad + C(\|\sigma_x(t)\|_\infty^2 \|v_{yy}\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} + \|\sigma_x(t)\|_\infty \|\sigma_x\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} \|v_{yy}(t)\|_\infty)\\
&\qquad + C(\|\sigma_*(t)\|_\infty \|\partial_x\sigma_1(t)\|_\infty + \|\sigma_1(t)\|_\infty \|\partial_x\sigma_*(t)\|_\infty)(\|\partial_x\sigma_*(t)\|_\infty + \|\partial_x\sigma_*\|_{\dot{C}^{\frac{\alpha}{2},\alpha}})\\
&\qquad + C(\|\sigma_*(t)\|_\infty \|\partial_x\sigma_1\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} + \|\sigma_*\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} \|\partial_x\sigma_1(t)\|_\infty)\|\partial_x\sigma_*(t)\|_\infty\\
&\qquad + C(\|\sigma_1(t)\|_\infty \|\partial_x\sigma_*\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} + \|\sigma_1\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} \|\partial_x\sigma_*(t)\|_\infty)\|\partial_x\sigma_*(t)\|_\infty\\
&\qquad + C(\|\partial_t\sigma_1(t)\|_\infty + \|\partial_{xx}\sigma_1(t)\|_\infty + \|\sigma_x(t)\|_\infty^2)\|v_y\|_{\dot{C}^{\frac{\alpha}{2},\alpha}}\\
&\qquad + C(\|\partial_t\sigma_1\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} + \|\partial_{xx}\sigma_1\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} + \|\sigma_x\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} \|\sigma_x(t)\|_\infty)\|v_y(t)\|_\infty
\end{aligned}$$

and

$$\begin{aligned}
\|F_2\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((t,2t)\times\mathbb{R})} \le\, & C(\|\sigma(t)\|_\infty \|v(t)\|_\infty \|v\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} + \|\sigma\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} \|v(t)\|_\infty^2)\\
& + C(\|\sigma_x(t)\|_\infty \|v_{xy}\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} + \|\sigma_x\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} \|v_{xy}(t)\|_\infty)\\
& + C(\|\sigma_x(t)\|_\infty^2 \|v_{yy}\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} + \|\sigma_x(t)\|_\infty \|\sigma_x\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} \|v_{yy}(t)\|_\infty)\\
& + C(\|\partial_t\sigma_1(t)\|_\infty + \|\partial_{xx}\sigma_1(t)\|_\infty + \|\sigma_x(t)\|_\infty^2)\|v_y\|_{\dot{C}^{\frac{\alpha}{2},\alpha}}\\
& + C(\|\partial_t\sigma_1\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} + \|\partial_{xx}\sigma_1\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} + \|\sigma_x\|_{\dot{C}^{\frac{\alpha}{2},\alpha}} \|\sigma_x(t)\|_\infty)\|v_y(t)\|_\infty .
\end{aligned}$$

5.5 The inhomogeneous one-dimensional heat equation

Some estimates in Section 3 rely on the following simple equation:

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = f(t,x) & t>0 \quad x\in\mathbb{R}\\ u(0,x) = 0 & x\in\mathbb{R}\,. \end{cases} \tag{5.1}$$

where $f \in C^{\frac{\alpha}{2},\alpha}(\mathbb{R}^+\times\mathbb{R})$ is an external force which satisfies for any $t_0 > 0$

$$\|f(t_0)\|_{L^\infty(\mathbb{R})} \le \frac{\varepsilon^{\frac{4\delta}{2+\alpha}}}{(1+t_0)^{2-2\delta}} \quad \|f\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((t_0,2t_0)\times\mathbb{R})} \le \frac{\varepsilon^{\frac{5\delta}{2+\alpha}}}{(1+t_0)^{\frac{5}{2}(1-\delta)}} \tag{5.2}$$

The aim of this appendix is to estimate the $C^{1+\frac{\alpha}{2},2+\alpha}((t_0,2t_0)\times\mathbb{R})$ norm of the solution u of Eq. (5.1) and more precisely to prove the following

Proposition 5.4 *Let $u \in C^{1+\frac{\alpha}{2},2+\alpha}(\mathbb{R}^+\times\mathbb{R})$ be the solution of Eq. (5.1) where f satifies bounds (5.2), then, for any $t_0>0$,*

$$\begin{cases} \|u(t_0)\|_{L^\infty(\mathbb{R})} \le \varepsilon^{\frac{4\delta}{2+\alpha}}\\ \|u_t\|_{\dot{C}^{1+\frac{\alpha}{2},2+\alpha}((t_0,2t_0)\times\mathbb{R})} + \|u_{xx}\|_{\dot{C}^{1+\frac{\alpha}{2},2+\alpha}((t_0,2t_0)\times\mathbb{R})} \le \frac{\varepsilon^{\frac{4\delta}{2+\alpha}}}{(1+t_0)^{1+\frac{\alpha}{2}}}\,. \end{cases}$$

Proof Thanks to [18], we know that for any $t>0$

$$\|u\|_{C^{1+\frac{\alpha}{2},2+\alpha}((0,t)\times\mathbb{R})} \le C\left(\|u_0\|_\infty + \|f\|_{C^{\alpha/2,\alpha}(\mathbb{R}^+\times\mathbb{R})}\right)$$

This theorem is enough to bound the $C^{1+\frac{\alpha}{2},2+\alpha}((t_0,2t_0)\times\mathbb{R})$ norm of the solution u for $t_0\in(0,2)$ but we have to find another way to estimate this norm for $t_0>2$. Let us remind that

$$u(t,x)=\int_0^t\int_{\mathbb{R}} G(t-s,x-y)f(s,y)\mathrm{d}y\mathrm{d}s$$

where G is the heat kernel $G(t,x)=\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$. We immediately get $\|u(t)\|_\infty\le\varepsilon^{\frac{4\delta}{2+\alpha}}$. As far as the partial derivatives of u are concerned, we will only deal with $\partial_t u$ since they both play the same role and it is important to keep in mind that by interpolation, the three norms described in 5.4 are sufficient to bound the $C^{1+\frac{\alpha}{2},2+\alpha}$ norm of u.

Let us bound $\|u_t\|_{\dot{C}^{1+\frac{\alpha}{2},2+\alpha}((t_0,2t_0)\times\mathbb{R})}$. We devide the integral definition of u into two pieces: for any $0<t_0<t<2t_0$

$$\begin{aligned}u(t,x)&=\int_0^{\frac{t_0}{2}}\int_{\mathbb{R}} G(t-s,x-y)f(s,y)\mathrm{d}y\mathrm{d}s+\int_{\frac{t_0}{2}}^{t}\int_{\mathbb{R}} G(t-s,x-y)f(s,y)\mathrm{d}y\mathrm{d}s\\&=\qquad I(t,x)\qquad+\qquad J(t,x)\end{aligned}$$

Since $\partial_\tau G(\tau,\eta)=\dfrac{1}{\sqrt{4\pi}\tau^{3/2}}\left(-\dfrac{1}{2}+\dfrac{\eta^2}{4\tau}\right)e^{-\frac{\eta^2}{4\tau}}$ and by the classical change of variables $z=\dfrac{x-y}{2\sqrt{t-s}}$,

$$I_t(t,x)=\int_0^{\frac{t_0}{2}}\int_{\mathbb{R}}\frac{C}{t-s}\left(-\frac{1}{2}+z^2\right)e^{-z^2}f(s,x-2\sqrt{t-s}z)\mathrm{d}z\mathrm{d}s\,.$$

Denoting $X=x-2\sqrt{t-s}z$ for simplicity, for any $t\neq t'$ and $x\neq x'$,

$$|I_t(t,x)-I_t(t',x')|\le C\int_0^{\frac{t_0}{2}}\int_{\mathbb{R}}\frac{z^2+1}{t_0^2}e^{-z^2}\Big(|t-t'|\|f(s)\|_\infty+|t-s||X-X'|^\alpha\|f\|_{\dot{C}^{1+\frac{\alpha}{2},2+\alpha}}\Big)$$

Thus,

$$\begin{aligned}|I_t(t,x)-I_t(t',x')|&\le C\int_0^{\frac{t_0}{2}}\int_{\mathbb{R}}\frac{z^2+1}{t_0^2}e^{-z^2}t_0^{1-\frac{\alpha}{2}}\left(\|f(s)\|_\infty+\|f\|_{\dot{C}^{1+\frac{\alpha}{2},2+\alpha}}\right)\mathrm{d}z\mathrm{d}s\\&\le\frac{C}{(1+t_0)^{1+\frac{\alpha}{2}}}\int_0^{\frac{t_0}{2}}\left(\|f(s)\|_\infty+\|f\|_{\dot{C}^{1+\frac{\alpha}{2},2+\alpha}}\right)\mathrm{d}s\int_{\mathbb{R}}(z^2+1)e^{-z^2}\mathrm{d}z\,.\end{aligned}$$

The conclusion of this calculation is important: the $\dot{C}^{\frac{\alpha}{2},\alpha}$ norm of I_t does not depend on the decay rate in time of the external force f provided it is integrable in time. The assumptions (5.2) on f could have been

$$\|f(t_0)\|_{L^\infty(\mathbb{R})}\le\frac{\varepsilon}{(1+t_0)^{1+\lambda}}\qquad\|f\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((t_0,2t_0)\times\mathbb{R})}\le\frac{\varepsilon}{(1+t_0)^{1+\lambda'}}$$

with λ and λ' two strictly positive numbers.

Let us now turn to J. It satisfies

$$\begin{cases} J_t - J_{xx} = f\,, \quad t > \frac{t_0}{2}\,, \quad x \in \mathbb{R} \\ J(\frac{t_0}{2}, x) = 0\,, \quad x \in \mathbb{R} \end{cases}$$

We make the usual change of variables $\tau = \frac{t}{t_0}$, $\eta = \frac{x}{\sqrt{t_0}}$ and denote $v(\tau, \eta) = J(t, x)$, $F(\tau, \eta) = f(t, x)$. Then,

$$\begin{cases} v_\tau - v_{\eta\eta} = F\,, \quad \tau > \frac{1}{2}\,, \quad \eta \in \mathbb{R} \\ v(\frac{1}{2}, x) = 0\,, \quad \eta \in \mathbb{R} \end{cases}$$

By [18], $\|v\|_{C^{1+\frac{\alpha}{2},2+\alpha}((1,2)\times\mathbb{R})} \le C\|F\|_{C^{\frac{\alpha}{2},\alpha}((1,2)\times\mathbb{R})}$ and

$$\begin{aligned} \|J_t\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((t_0,2t_0)\times\mathbb{R})} &\le \frac{C}{(1+t_0)^{1+\frac{\alpha}{2}}}\|v_t\|_{\dot{C}^{\frac{\alpha}{2},\alpha}((1,2)\times\mathbb{R})} \le \frac{C}{(1+t_0)^{1+\frac{\alpha}{2}}}\|F\|_{C^{\frac{\alpha}{2},\alpha}((1,2)\times\mathbb{R})} \\ &\le \frac{C}{(1+t_0)^{1+\frac{\alpha}{2}}}\left(\|f(t_0)\|_\infty + (1+t_0)^{\frac{1}{2}}\|f\|_{\dot{C}^{\frac{\alpha}{2},\alpha}}\right) \le \frac{C\varepsilon^{\frac{4\delta}{2+\alpha}}}{(1+t_0)^{3-2\delta+\frac{\alpha}{2}}} \end{aligned}$$

The proof is terminated by putting together the estimates on I_t and J_t. □

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