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Bochner's theorem and Stepanov almost periodic functions

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Abstract Motivated by a renewed interest in generalizations of classical almost periodicity (originally due to Harald Bohr), we develop a theorem of Bochner within the framework of almost periodic functions in the sense of Stepanov. As a result we establish some conditions that guarantee the existence of Stepanov almost periodic solutions to differential equations with Stepanov almost periodic coefficients. Finally, we extend a now classic theorem of Favard originally stated for classical almost periodic functions to the Stepanov almost periodic case.

Keywords Bochner's theorem · Almost periodic · Stepanov · Favard · Weyl · Besicovitch · Almost periodic functions · Existence · Minimizing norm · Differential equations · Systems

Mathematics Subject Classification (2000) Primary 34C27

1 Introduction

Although the concept of Stepanov almost periodic functions was introduced more than 60 years ago, some of their properties which play an important role in discussing the solutions of differential equations were not established until recently. For instance, we know that Bochner's Theorem (see [5]) can be used to derive many results on existence of (Bohr) almost periodic (a.p.) solutions of almost periodic differential equations. However, there seems to be no corresponding analogue known so far for Stepanov almost periodic functions. As a matter of notation, we emphasize that the compound words *almost periodic* will refer exclusively

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to Bohr almost periodic while other types of almost periodicity will usually have a prefix named after its originator before it (e.g, Stepanov a.p., Weyl a.p., or Besicovitch a.p., [4]).

Recall that a real-valued function f defined on the real line is almost periodic if it belongs to the completion of the space $\mathcal{T}$ of all finite trigonometric polynomials equipped with the norm of uniform convergence on the whole line, see [4,6] and Bochner [5] for an equivalent definition. In a sense (made precise in the cited reference [6]) such functions are equipped with a relatively dense set of *almost periods* as Bohr called them, extending the usual notion of the period of a periodic function. Thus, for instance, the well-known Weierstrass continuous yet nowhere differentiable function is an example of such an almost periodic function. They need not be differentiable anywhere but they do need to be continuous, in fact, uniformly continuous on the whole line. Other such basic results can be found in [4].

On the other hand, the main drawback for applications is precisely the requirement of *continuity* of such functions, an hypothesis which is enforced throughout the theory. In order to contend with such awkward matters, the originators of the more general theory of almost periodic functions (which now even have their development to almost periodic distributions [19]), considered the completion of the basic space of finite trigonometric polynomials under less restrictive metrics. In this way, Stepanov [25] produced the first generalization of Bohr's almost periodic functions and it is this theory that we will develop further in this paper (see also [3,20]). Fortunately, such Stepanov almost periodic functions need not be continuous by their very definition, and thus their applications may be of more interest. Also, as one would expect this larger class of generalized almost periodic functions includes Bohr's class. This generalization was expanded even further by both H. Weyl and A. Besicovitch, see e.g., [4] to produce what we now call Weyl a.p. and Besicovitch a.p. functions. We recall that the crucial difference between these various definitions lies in the nature of the completion of the space $\mathcal{T}$ relative to different metrics.

Recently, there has been somewhat of a renewal in the study and applications of generalized forms of almost periodicity. In [12], we show that there exists a second order real linear differential equation on the line with almost periodic coefficients for which every solution is bounded but no non-trivial solution is almost periodic. This surprising phenomenon shows that boundedness, by itself, is not sufficient to guarantee the existence of almost periodic solutions of even the simplest linear equations. Furthermore, in [11] we constructed an example of an almost periodic differential equation in which all solutions are bounded but there is no any non-trival solution which is (even) Stepanov almost periodic. This raises the following points: If these solutions are not almost periodic in the usual sense can they be almost periodic in a more general sense? Such a question even arose recently within the context of number theory, [15]: Here, under the assumption of a Generalized Riemann Hypothesis, the authors obtain the Stepanov almost periodicity of some remainder terms of functions in a Selberg class, see [15] for more details. Another question then deals with finding conditions under which there exists at least one Stepanov a.p. solution (maybe all?) to a Stepanov a.p. equation (or system of such equations). As we have seen this is a difficult problem even in the Bohr a.p. case. Therefore, our ultimate goal here is to find some conditions that guarantee the existence of at least one (non-trivial) Stepanov almost periodic solution to a given Stepanov a.p. differential equation or system.

It is known that some stability properties of the bounded solution such as total stability (Miller [18]), $\Sigma-$stability (Seifert [21–23]), stability under a disturbance from the hull (Sell [24]), can imply the existence of almost periodic solutions for almost periodic differential equations. Another such condition is the *separation property* proposed by Amerio [1] which, in a sense, is a generalization of Favard's condition, [8]. Later, Fink [9] considered a *semi-separation* property of a solution. In the paper [13], we obtain another condition that extends Favard's theorem for linear almost periodic differential equations in R^n. These results have been extended to evolution equations in Banach spaces (see [14,17]). However, there do not appear to be corresponding results for Stepanov almost periodic differential equations in the literature. Still, one needs to exercise caution here: In [10], Haraux shows that if a C^1-solution u of the linear first order differential inclusion

$$\frac{du}{dt} + Au(t) \ni f(t),$$

where A is maximal monotone and f is Stepanov a.p., which is bounded then it is in fact Bohr a.p. (see also [2]). In order to gain further generality in our applications we must, therefore, address the question of whether there exists a C^1-Stepanov a.p. function which is not Bohr a.p. and we do this in Sect. 5: Motivated by an article of Ursell [26], we show that there exists C^1-Stepanov a.p. functions (that must be unbounded) which are not Bohr a.p.; hence the results may not be reduced, in general, to the study of Bohr a.p. solutions in abstract spaces by means of the Bochner transform [20].

We first provide a formulation of Bochner's theorem within the framework of Stepanov almost periodic functions and then establish some conditions which imply the existence of Stepanov almost periodic solutions for Stepanov almost periodic differential equations. At the end we produce a version of Favard's Theorem for systems of Stepanov almost periodic differential equations.

2 Stepanov almost periodic functions

For completeness, we begin with some basic properties of Stepanov almost periodic functions and those results that will be used in the sequel.

Let X be a Banach space. We define the Stepanov norm $S_l^p(f)$ of a function $f \in L_p^{\text{loc}}(R, X)$. The quantity

$$S_l^p(f) = \sup_{t\in R} \left(\frac{1}{l} \int_t^{t+l} ||f(s)||^p \mathrm{d}s \right)^{\frac{1}{p}}$$

where $l > 0$ and $p \geq 1$ are some constants, is the Stepanov norm (or S_l^p-norm) of f.

Replacing the supremum norm by the S_l^p-norm in the definition of continuity (respectively, uniform continuity, boundedness) of f, we can introduce the concept of S_l^p-continuity (respectively, S_l^p -uniform continuity, S_l^p-boundedness) of f. For example, we call $f \in L_1^{loc}(R, X)$ S_l-bounded if there exists a constant $M > 0$ such that $S_l^p(f) \leq M$. It is easy to show that S_l^p-boundedness (S_l^p-continuity, S_l^p-uniform continuity) is not dependent on the constant l. So, we simply call such functions S^p-bounded, S^p -continuous, and S^p-uniformly continuous whenever these notions are applied.

We define the quantity $S_l^p(t, f)$ as follows: for $f \in L_p^{\text{loc}}(R, X)$,

$$S_l^p(t, f) = \left(\frac{1}{l} \int_t^{t+l} ||f(s)||^p \mathrm{d}s \right)^{\frac{1}{p}} \quad \text{for all } t \in R. \tag{2.1}$$

From (2.1) we have that for any $t, s \in R$,

$$S_l^p(t, f_s) = S_l^p(t + s, f).$$

where f_s is the translate of f. We use $S_l^p C(R, X)$ to denote the set of all S_l^p-continuous functions. Obviously, $C(R, X) \subset S_l^p C(R, X)$. As in the case of almost periodic functions, we can introduce another definition of a Stepanov almost periodic function.

Definition 1 Let $f \in S_l^p C(R, X)$. If for any sequence $\{\alpha_n\} \subset R$, there exist a subsequence $\{\alpha_n'\}$ of $\{\alpha_n\}$ and a function $g \in S_l^p C(R, X)$ such that

$$\lim_{n\to\infty} S_l^p(t, f_{\alpha_n'} - g) = 0, \quad \text{uniformly on} \quad R,$$

then f is called S_l^p-almost periodic on R.

This definition is equivalent to that of Stepanov's (see [4]). Yet another equivalent definition is found in [16]. We now state the basic properties of Stepanov almost periodic functions. The proofs can be found in [4].

It is easy to show that if there is a constant $l_0 > 0$ such that $f(t)$ is $S_{l_0}^p$-almost periodic, then f is S_l^p-almost periodic for any $l > 0$. So, we simply say that f is S^p-almost periodic and use $S^p\mathrm{AP}(R, X)$ to denote the set of all S^p-almost periodic functions.

Proposition 1 *For any $f \in S^p\mathrm{AP}(R, X)$, f is S^p-bounded on R.*

Proposition 2 *For any $f \in S^p\mathrm{AP}(R, X)$, f is S^p-uniformly continuous on R.*

Proposition 3 *$S^p\mathrm{AP}(R, X)$ is closed in the sense of the S^p-norm.*

Proposition 4 *Let $f \in S^p\mathrm{AP}(R, X)$. If for a given l, $S_l^p(t, f) \to 0$ as $t \to \infty$, then $S_l^p(t, f) \equiv 0$ for all $t \in R$.*

We use the notations in [9]: that is, whenever $\alpha = \{\alpha_n\}$ is a sequence of real numbers, $\alpha' \subset \alpha$ means that $\alpha' = \{\alpha_n'\} \subset \{\alpha_n\}$ is a subsequence of α. Let $f, g \in S^p C(R, X)$. If there exists a sequence α and a real number $l > 0$ such that

$$\lim_{n\to\infty} \left(\frac{1}{l} \int_t^{t+l} |f(s+\alpha_n) - g(s)|^p \mathrm{d}s \right)^{\frac{1}{p}} = 0, \tag{2.2}$$

pointwise for $t \in R$, we write $S^p T_\alpha f = g$. If (2.2) holds uniformly on $t \in R$, we say $US^p T_\alpha f = g$.

Now we give the definition of the *uniform Stepanov hull* of a S^p -almost periodic function, and then establish some properties of the hull.

Definition 2 Let $f \in S^p C(R, X)$. We call the set

$$\{g | g \in \mathrm{SC}(R, X), \text{ there exists a sequence } \{\alpha_n\} \subset R \text{ such that } US^p T_\alpha f = g\}$$

the **uniform Stepanov hull**, or simply **uniform $\mathbf{S}^p$ -hull**, denoted by $S^p H(f)$.

Obviously, for any $f \in \mathrm{SC}(R, X)$, $S^p H(f)$ is not empty since $f \in S^p H(f)$.

Now we discuss the properties of the uniform S^p-hull of a function .

Proposition 5 *If f is Bohr almost periodic on R, then*

$$H(f) \subseteq S^p H(f)$$

where $H(f)$ is the uniform Bohr hull of f.

Proposition 6 *$S^pH(f)$ is compact in the sense of the S^p-norm if and only if f is S^p-almost periodic on R.*

Proof Necessity. Suppose that $S^pH(f)$ is compact. For any sequence $\{\alpha_n\} \subset R$, let $f_n(t) = f_{\alpha_n}(t)$. Then $f_n \in S^pH(f)$ for each $n = 1, 2, \ldots$. So there exists a subsequence $\{n_k\} \subseteq \{n\}$ and a function $g \in S^pH(f)$ such that $S_l^p(t, f_{n_k} - g) \to 0$ uniformly on R as $k \to \infty$. So, $S_l^p(t, f_{\alpha_n} - g) \to 0$ as $n \to \infty$ uniformly on R. This implies that f is S-almost periodic on R.

Sufficiency. Let $f \in S^p\mathrm{AP}(R, X)$, $g_n \in S^pH(R, X)$, $n = 1, 2, \ldots$. For any n, choose $\left\{\alpha_k^{(n)}\right\} \subset R$ such that

$$S_l^p(f_{\alpha_k^{(n)}} - g_n) = \sup_{t\in R} S_l^p(t, f_{\alpha_k^{(n)}} - g_n) \to 0, \quad \text{as } n \to \infty.$$

So, we can choose a sequence $\{\alpha_n\} \subset R$ such that

$$S_l^p(f_{\alpha_n} - g_n) < \frac{1}{n}, \quad \text{for each } n = 1, 2, \ldots.$$

Since $f \in S^p\mathrm{AP}(R, X)$, there exists $\left\{\alpha_{n_k}\right\} \subseteq \{\alpha_n\}$ and a function $g \in S^pC(R, X)$ such that

$$\lim_{n\to\infty} S_l^p(t, f_{\alpha_{n_k}} - g) = 0, \quad \text{uniformly on } R.$$

So, $g \in S^pH(R, X)$ and

$$\begin{aligned} S_l^p(t, g_{n_k} - g) &\le S_l^p(t, g_{n_k} - f_{\alpha_{n_k}}) + S_l^p(t, f_{\alpha_{n_k}} - g) \\ &\le S_l^p(t, g_{n_k} - f_{\alpha_{n_k}}) + \sup_{t\in R} S_l^p(t, f_{\alpha_{n_k}} - g) \\ &\le \frac{1}{n_k} + S_l^p(f_{\alpha_{n_k}} - g). \end{aligned}$$

Therefore,

$$\lim_{k\to\infty} S_l^p(t,\ g_{n_k} - g) = 0.$$

This means that g_{n_k} converges to g in the S^p-norm sense and so, $S^pH(f)$ is compact. □

Proposition 7 *If $f \in S^p\mathrm{AP}(R, X)$, $g \in S^pH(f)$, then $g \in S^p\mathrm{AP}(R, X)$ and $f \in S^pH(g)$.*

Proof Since $g \in S^pH(f)$, there exists a sequence $\{\alpha_n\} \subset R$ such that

$$\lim_{n\to\infty} S_l^p(t, f_{\alpha_n} - g) = 0 \quad \text{uniformly on } R.$$

For any sequence $\{\beta_n\} \subset R$, let $\gamma_n = \beta_n + \alpha_n$, $n = 1, 2, \ldots$. Since $f \in S^p\mathrm{AP}(R, X)$, there exists a subsequence $\left\{\gamma_n'\right\} \subseteq \{\gamma_n\}$ and a function $g_1 \in S^pH(f)$ such that

$$\lim_{n\to\infty} S_l^p(t, f_{\gamma_n'} - g_1) = 0 \quad \text{uniformly on } R,$$

so,

$$\begin{aligned} S_l^p(t, g_{\beta_n'} - g_1) &\le S_l^p(t, g_{\beta_n'} - f_{\gamma_n'}) + S_l^p(t, f_{\gamma_n'} - g_1) \\ &= S_l^p(t + \beta_n', g - f_{\alpha_n'}) + S_l^p(t, f_{\gamma_n'} - g_1) \end{aligned}$$

and thus

$$\lim_{n\to\infty} S_l^p(t, g_{\beta_n'} - g_1) = 0 \quad \text{uniformly on } R.$$

This implies that $g \in S^p\mathrm{AP}(R, X)$. Choosing $\delta_n = -\alpha_n$, we get $f \in S^pH(g)$. □

Proposition 8 *If $f \in S^p\mathrm{AP}(R, X)$, then for any $g \in S^pH(f)$, $S^pH(g) = S^pH(f)$.*

Proof We first show that for any $f \in S^pH(f)$, $S^pH(g) \subset S^pH(f)$.

Let $h \in S^pH(g)$. Then there are two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ such that

$$\lim_{n\to\infty} S_l^p(f_{\alpha_n} - g) = 0 \quad \text{uniformly on } R,$$

and

$$\lim_{n\to\infty} S_l^p(g_{\beta_n} - h) = 0 \quad \text{uniformly on } R.$$

So,

$$\lim_{n\to\infty} S_l^p(f_{\alpha_n+\beta_n} - h) \le \lim_{n\to\infty} S_l^p(f_{\alpha_n+\beta_n} - g_{\beta_n}) + \lim_{n\to\infty} S_l^p(g_{\beta_n} - h) = 0 \text{ uniformly on } R.$$

Therefore, $h \in S^pH(f)$. This implies that $S^pH(g) \subset S^pH(f)$. From Proposition 7, $f \in S^pH(g)$. Using what we have proved above, we have that $S^pH(f) \subset S^pH(g)$. Thus, $S^pH(g) = S^pH(f)$ and the proof is complete. □

3 Formulation of a Bochner-type theorem

In this section, we present a version of Bochner's theorem for almost periodic functions that applies to the case of Stepanov almost periodic functions. This theorem plays an important role in discussing the existence of Stepanov almost periodic solutions for Stepanov almost periodic differential equations. Now we state a Bochner-type theorem (see [5]) for Stepanov almost periodic functions.

Theorem 1 *Let $f \in S^pC(R, X)$. Then f is Stepanov almost periodic on R if and only if for any pair of sequences $\alpha, \beta \subset R$, one can extract common subsequences $\alpha' \subset \alpha$, $\beta' \subset \beta$ such that*

$$S^pT_{\alpha'+\beta'}f = S^pT_{\alpha'}(S^pT_{\beta'}f),$$

i.e., there exist two functions $g, h \in S^pC(R, X)$ such that

$$\lim_{n\to\infty} S_l^p(t, f_{\alpha_n'+\beta_n'} - g) = 0 \quad \text{for all } t \in R$$
$$\lim_{n\to\infty} S_l^p(t, f_{\beta_n'} - h) = 0 \quad \text{for all} t \in R$$
$$\lim_{n\to\infty} S_l^p(t, h_{\alpha_n'} - g) = 0 \quad \text{for all } t \in R.$$

Proof Necessity. Let $f \in S^p\mathrm{AP}(R, X)$. For any sequences $\alpha = \{\alpha_n\} \subset R$, $\beta = \{\beta_n\} \subset R$, we can extract a subsequence $\beta'' \subset \beta$ and a function $h \in S^pC(R, X)$ such that $US^pT_{\beta''}f = h$ and $h \in S^p\mathrm{AP}(R, X)$ by Proposition 7. So, we can extract a subsequence $\alpha' = \left\{\alpha_n'\right\} \subset \alpha''$, which is a subsequence of α with the same subscripts as β'', and a function $g \in S^pC(R, X)$ such that $US^pT_{\alpha'}h = g$. Now let $\beta' \subset \beta''$ have the same subscripts as α' and let $\gamma' = \alpha' + \beta'$. Then we can choose a subsequence $\gamma \subset \gamma'$ and a function $k \in S^pC(R, X)$ such that $US^pT_\gamma f = k$. Since

$$S_l^p(t, k - g) \le S_l^p(t, k - f_{\gamma_n}) + S_l^p(t + \alpha_n, f_{\beta_n} - h) + S_l^p(t, h_{\alpha_n} - g),$$

we can let $n \to \infty$. Then we see that

$$S_l^p(t, k - g) = 0 \quad \text{for any } t \in R.$$

But

$$S_l^p(t, f_\gamma - g) \le S_l^p(t, f_\gamma - k) + S_l^p(t, k - g) \to 0 \quad \text{as } n \to \infty.$$

This implies that $S^pT_\gamma f = g$. Since $\gamma \subset \gamma' = \alpha' + \beta'$, we can write $\gamma = \alpha + \beta$ where $\alpha \subset \alpha'$ and $\beta \subset \beta'$ have the same subscripts and $S^pT_\alpha h = g$, $S^pT_\beta f = h$. This completes the proof of the necessity.

Sufficiency. For any sequence $\gamma = \{\gamma_n\}$, we will show that there exist a subsequence $\gamma' \subset \gamma$ and a function $g \in S^pC(R, X)$ such that $US^pT_{\gamma'} f = g$.

For any sequence $\gamma = \{\gamma_n\}$, let $\alpha = 0$, $\beta = \gamma$. From the assumptions, we can choose common sequences $\alpha' \subset \alpha$, $\beta' \subset \beta$ and the functions $g, h \in S^pC(R, X)$ such that $S^pT_{\alpha'+\beta'} f = g$, $S^pT_{\beta'} f = h$, $S^pT_{\alpha'}h = g$. So, $S^pT_{\beta'} f = g$, and $S^pT_{\beta'} f = h$. Therefore,

$$S_l^p(t, g - h) \le S_l^p(t, g - f_{\beta_n'}) + S_l^p(t, f_{\beta_n'} - h) \to 0 \quad \text{as } n \to \infty \quad \text{for any } t \in R.$$

So,

$$S_l^p(t, g - h) \equiv 0 \quad \text{for any } t \in R.$$

Now we will show that $US^pT_{\beta'} f = g$., i.e.,

$$\lim_{n\to\infty} S_l^p(t, f_{\beta_n'} - g) = 0 \quad \text{uniformly on } R.$$

Assuming to the contrary, we suppose that the convergence of the above is not uniform on R. Then there exists a real number $\varepsilon_0 > 0$, a subsequence $\nu = \{\nu_n\}$ of β and a sequence $t = \{t_n\} \subset R$ such that

$$S_l^p(t_n, \quad f_{\nu_n} - g) \ge \varepsilon_0 \quad \text{for any } n = 1, 2, \dots.$$

From the given conditions, we can choose a pair of common subsequences $\nu' \subset \nu$, $t' = \{t_n'\} \subset t$ and functions $g_1, h_1 \in S^pC(R, X)$ such that $S^pT_{\nu'+t'} f = g_1$, $\quad S^pT_{\nu'} f = h_1$ and $S^pT_{t'}h_1 = g_1$. Since $\nu' \subset \nu \subset \beta$, we have that

$$S_l^p(t, h - h_1) \le S_l^p(t, f_{\nu_n'} - h) + S_l^p(t, f_{\nu_n'} - h_1).$$

Letting $n \to \infty$, we have that $S_l^p(t, h - h_1) \equiv 0$ for all $t \in R$. But,

$$S_l^p(t, f_{t_n'+\nu_n'} - g_{t_n'}) \le S_l^p(t, f_{t_n'+\nu_n'} - g_1) + S_l^p(t, g_1 - h_{1t_n'}) + S_l^p(t, h_{1t_n'} - g_{t_n'}).$$

So, if we let $n \to \infty$, we have that

$$S_l^p(t, f_{t_n'+\nu_n'} - g_{t_n'}) \to 0, \quad \text{for any } t \in R.$$

We set $t = 0$, and use the fact that $S_l^p(0, f_{t+S^p}) = S_l^p(t, f_s)$. We have that

$$S_l^p(t_n', f_{\nu_n'} - g) \to 0 \quad \text{as } n \to \infty.$$

Since $t' \subset t$, we obtain a contradiction because

$$S_l^p(t_n, \quad f_{\nu_n} - g) \ge \quad \varepsilon_0 \quad \text{for any } n = 1, 2, \dots$$

This completes the proof of this theorem. □

4 Stepanov almost periodic differential equations

In this section, we consider S^p-almost periodic differential equations. First, we introduce some concepts related to S^p-almost periodic functions with parameters, i.e., $f(t,x) : R \times R^n \to R^n$. We assume that for any $x \in R^n$, $f(t,x) \in S^pC(R,R^n)$ and for any $t \in R$, $f(t,x)$ is continuous on R^n. We write the set of all such functions as $S^pC_tC_x(R \times R^n, R^n)$. Let K be a compact subset of R^n. For any $f \in S^pC_tC_x(R \times R^n, R^n)$, we write

$$S_l^p(t, f, K) = \sup_{x\in K} S_l^p(t, f(\cdot, x))$$

and

$$S_l^p(f, K) = \sup_{t\in R} S_l^p(t, f, K)$$

Let $f, g \in S^pC_tC_x(R \times R^n, R^n)$, $\alpha = \{\alpha_n\} \subset R$ be a sequence, and K a compact subset of R^n. If

$$S_\ell^p(t, f_{\alpha_n} - g, K) \to 0 \quad \text{as } n \to +\infty \quad \text{pointwise for } t \in R \tag{4.1}$$

we write $S_K^pT_\alpha f = g$. If (4.1) also holds uniformly for $t \in R$, we write it as $US_K^pT_\alpha f = g$. Similarly, we denote the uniform S^p-hull of f by $S_K^pH(f)$, i.e.,

$$S_K^pH(f) = \left\{g \mid \text{there is a sequence } \alpha \subset R \text{ such that } US_K^pT_\alpha f = g\right\}.$$

Definition 3 Let $f \in L_p^{\text{loc}}C_x(R \times R^n, R^n)$. We say that f S^p-almost periodic in t uniformly in x if for any compact subset K of R^n, and any sequence $\alpha = \{\alpha_n\} \subset R$, one can extract a subsequence $\alpha' \subset \alpha$ and a function $g \in L_p^{loc}C_x(R \times R^n, R^n)$ such that $US_K^pT_\alpha f = g$.

Definition 4 Let f be S^p-almost periodic in t uniformly in x and K be a compact subset of R^n. If there exist a contant $q > 0$ and a S^p -almost periodic function $l(t) \in L_q(R)$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and for any $x, y \in K$, we have

$$|f(t,x) - f(t,y)| \le l(t)\,|x-y|, \quad \text{for almost all } t \in R,$$

we say that f is S^p-Lipschitzian in K.

Proposition 9 *Let K be a compact subset of R^n and $f \in S^pC_tC_x(R \times R^n, R^n)$. If f is S^p-Lipschitzian in K, then for any $g \in S_K^pH(f)$, g is also S^p-Lipschitzian in K.*

Proof Since f is S^p-Lipschitizan in K, there exist a constant $q > 0$ and a S^p-almost periodic function $l(t) \in L_q(R)$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and for any $x, y \in K$, we have

$$|f(t,x) - f(t,y)| \le l(t)\,|x-y|, \quad \text{for almost all } t \in R.$$

For any $g \in S_K^pH(f)$, there is a sequence $\alpha \subset R$ such that $US_K^pT_\alpha f = g$, that is,

$$\lim_{n\to\infty}\left(\frac{1}{l}\int_t^{t+l} |f(s+\alpha_n, x) - g(s,x)|^p \mathrm{d}s\right)^{\frac{1}{p}} = 0$$

uniformly in $R \times K$. Since $l(t)$ is S^p-almost periodic, for the sequence α, there exist a subsequence $\alpha' \subset \alpha$ and a function $k \in S^pH(l)$ such that

$$\lim_{n\to\infty}\left(\frac{1}{l}\int_t^{t+l} |l(s+\alpha_n') - k(s)|^p \mathrm{d}s\right)^{\frac{1}{p}} = 0.$$

So, by Minkowski's inequality,

$$\left(\frac{1}{l}\int_t^{t+l}|g(s,x)-g(s,y)|^p\mathrm{d}s\right)^{\frac{1}{p}} \le \left(\frac{1}{l}\int_t^{t+l}|g(s,x)-f_{\alpha_n'}(s,x)|^p\mathrm{d}s\right)^{\frac{1}{p}}$$

$$+\left(\frac{1}{l}\int_t^{t+l}|f_{\alpha_n'}(s,x)-f_{\alpha_n'}(s,y)|^p\mathrm{d}s\right)^{\frac{1}{p}}+\left(\frac{1}{l}\int_t^{t+l}|f_{\alpha_n'}(s,y)-g(s,y)|^p\mathrm{d}s\right)^{\frac{1}{p}}$$

$$\le\left(\frac{1}{l}\int_t^{t+l}|g(s,x)-f_{\alpha_n'}(s,x)|^p\mathrm{d}s\right)^{\frac{1}{p}}+\left(\frac{1}{l}\int_t^{t+l}|l_{\alpha_n'}(s)-k(s)|^p\mathrm{d}s\right)^{\frac{1}{p}}|x-y|$$

$$+\left(\frac{1}{l}\int_t^{t+l}k(s)^p\mathrm{d}s\right)^{\frac{1}{p}}|x-y|+\left(\frac{1}{l}\int_t^{t+l}|f_{\alpha_n'}(s,y)-g(s,y)|^p\mathrm{d}s\right)^{\frac{1}{p}}$$

Passing to the limit as $n\to+\infty$, we have that

$$\left(\frac{1}{l}\int_t^{t+l}|g(s,x)-g(s,y)|^p\mathrm{d}s\right)^{\frac{1}{p}}\le\left(\frac{1}{l}\int_t^{t+l}k(s)^p\mathrm{d}s\right)^{\frac{1}{p}}|x-y|.$$

So,

$$\int_t^{t+l}|g(s,x)-g(s,y)|^p ds\le|x-y|^p\int_t^{t+l}k(s)^p\mathrm{d}s$$

and this completes the proof of the proposition. □

Now consider the system of S^p-almost periodic differential equations

$$x'=f(t,x) \tag{4.2}$$

where $f\in L_p^{loc}C_x(R\times R^n,R^n)$ is S^p-almost periodic in t uniformly in $x\in R^n$. We always assume that for any $(t_0,x_0)\in R\times R^n$, the equation (4.2) has a solution through (t_0,x_0), defined in $[t_0,+\infty)$ and we denote such a solution by $x(t,t_0,x_0,f)$. Our main goal is to discuss the existence of S^p-almost periodic solutions of (4.2).

The next few results deal with (Lebesgue) measure theory and are included for completeness. Let I be a subset of R. We use $m(I)$ to denote the measure of I. If I' is another subset of R, $I\backslash I'$ is the set

$$\{t|t\in I,\text{but},t\notin I'\}$$

Let $I=[a,b]$ be a closed interval of R and $F=\{f\}$ a subset of $S^pC(R,R^n)$. We say F is *almost everywhere equi-continuous* on I if there exists at most a subset I' of I with $m(I')=0$ such that F is equi-continuous on $I\backslash I'$. F is said *almost everywhere uniformly bounded* on I if there exists at most a subset I' of I with $m(I')=0$ such that F is uniformly bounded on $I\backslash I'$. Let $\{f_n\}\subset F$ be a sequence of functions. $\{f_n\}$ is *almost everywhere uniformly convergent* on I if there exists at most a subset I' of I with $m(I')=0$ such that $\{f_n\}$ is uniformly convergent on $I\backslash I'$.

Lemma 1 *Let $I = [a, b]$ be a closed interval of R. For any subset I_0 of I with $m(I_0) = 0$, there exists a subset Q_0 of $I \backslash I_0$, which is countable and dense in $I \backslash I_0$.*

Proof Let $Q_I = \{r_1, r_2, \ldots, r_n, \ldots\}$ be the set of all rationals in I. Then Q_I is dense in I. For any subset I_0 of I with $m(I_0) = 0$, we show that for each $r_n \in Q_I$, there exists a sequence $\{t_k^n\} \subset I \backslash I_0$ such that $t_k^n \to r_n$ as $k \to \infty$. In fact, if there exists an r_n, and there exists no such sequence, then there must exist a real number $\delta > 0$ such that $(r_n - \delta, r_n + \delta) \cap (I \backslash I_0) = \emptyset$. We can take $\delta > 0$ sufficiently small so that $(r_n - \delta, r_n + \delta) \subset I$. So, $(r_n - \delta, r_n + \delta) \subset I_0$. This contradicts the fact that $m(I_0) = 0$. Now, let $T = \cup_{n=1}^{\infty} \{t_k^n\}$. Obviously, $T \subset I \backslash I_0$ and it is countable and dense in $I \backslash I_0$. This completes the proof of this lemma. □

Theorem 2 *Let $F = \{f\}$ be a set of functions defined on a bounded interval I. If F is almost everywhere uniformly bounded and almost everywhere equi-continuous on I, then F contains a sequence which is almost everywhere uniformly convergent on I.*

Proof Let I_0 is subset of I with $m(I_0) = 0$ such that F is uniformly bounded and equi-continuous on $I \backslash I_0$. From Lemma 1, there exists a subset T of $I \backslash I_0$, which is countable and dense in $I \backslash I_0$.

The rest of the proof is the same as that of the Ascoli Theorem (see [7], by replacing the set of all rationals in I by the set T. We omit it. □

Corollary 1 *Let $\{f_n\}$ be a sequence of functions defined on R. If $\{f_n\}$ is almost everywhere bounded and almost everywhere equi-continuous on R, then there exists a subsequence of $\{f_n\}$, which is almost everywhere convergent on every compact subset of R.*

Proof Using Theorem 2 and a diagonalization argument, we can find a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which is almost everywhere uniformly convergent on every compact subset of R. For details, see [9]].

Now, we discuss the differential equation (4.2) and the equation

$$x' = g(t, x) \tag{4.3}$$

where $g(t, x) \in S_K^p H(f)$. We will establish some properties of the bounded solutions of (4.3). Firstly, we have the following lemma. □

Lemma 2 *Let K be a compact subset of R^n, f be S^p-Lipschitzian in K, and ϕ a solution of (4.2) with $\{\phi(t)|t \in R\} \subset K$. If there exist a sequence $\alpha = \{\alpha_n\} \subset R$, a function $g \in S^p H_K(f)$ and a function $\varphi \in S^p H(\phi)$ such that $\{\varphi(t)|t \in R\} \subset K$, $S_K^p T_\alpha f = g$ and $S^p T_\alpha \phi = \varphi$ holds uniformly in any compact subset of R, then there exists a solution of (4.3), say $\widetilde{\varphi}$, such that $S^p T_\alpha \phi = \widetilde{\varphi}$ holds uniformly in any compact subset of R, and $\widetilde{\varphi} \in K$ for all $t \in R$.*

Proof By assumption, there is a constant $l > 0$ such that

$$\begin{aligned}
&\lim_{n\to\infty} \left(\frac{1}{l} \int_t^{t+l} |f(s+\alpha_n, x) - g(s, x)|^p \mathrm{d}s \right)^{\frac{1}{p}} = 0, \\
&\lim_{n\to\infty} \left(\frac{1}{l} \int_t^{t+l} |\phi(s+\alpha_n) - \varphi(s)|^p \mathrm{d}s \right)^{\frac{1}{p}} = 0,
\end{aligned} \tag{4.4}$$

uniformly for $x \in K$ and pointwise for $t \in R$. In particular, for each $t \in R$, it follows that $\lim_{n\to\infty}\left(\int_0^l |\phi(s+t+\alpha_n)-\varphi(t+s)|^p \mathrm{d}s\right)^{\frac{1}{p}} = 0$; in other words, the sequence $\{\phi(t+s+\alpha_n)\}$ in $L^p[0,l]$ converges to $\varphi(t+s)$, and hence the sequence converges to $\varphi(t+s)$ in the sense of measure on $[0,l]$. Since $t \in R$ is arbitrary, the sequence of measurable functions $\{\phi(\tau+\alpha_n)\}$ converges to $\varphi(\tau)$ in the sense of measure on R; hence from a well known result in the theory of Lebesgue measure we know that there is a subsequence α' of α such that $\lim_{n\to\infty}\phi(\tau+\alpha'_n)=\varphi(\tau)$ a.e. on R. Take a point t_0 in R such that $\lim_{n\to\infty}\phi(t_0+\alpha'_n)=\varphi(t_0)$. Since ϕ is a solution of (4.2), we get the following relations:

$$\phi(t+\alpha'_n)=\phi(t_0+\alpha'_n)+\int_{t_0}^{t} f(s+\alpha'_n,\phi(s+\alpha'_n))\mathrm{d}s, \tag{4.5}$$

for $t \in R$, $n=1,2,\ldots$. Note that

$$\begin{aligned}\lim_{n\to\infty}\left(\int_{t_0}^{t}|f(s+\alpha'_n,\varphi(s))-g(s,\varphi(s))|^p\mathrm{d}s\right)^{\frac{1}{p}}&=0,\\ \lim_{n\to\infty}\left(\int_{t_0}^{t}|\phi(s+\alpha'_n)-\varphi(s)|^p\mathrm{d}s\right)^{\frac{1}{p}}&=0,\end{aligned}\tag{4.6}$$

locally uniformly for $t \in R$ (which means the uniformity of convergence on any finite interval in R). We now show that

$$\lim_{n\to\infty}\int_{t_0}^{t} f(s+\alpha'_n,\phi(s+\alpha'_n))\mathrm{d}s=\int_{t_0}^{t} g(s,\varphi(s))\mathrm{d}s, \tag{4.7}$$

locally uniformly for $t \in R$. From the assumption, Holder's inequality and Cauchy's inequality, we have that for $t \geq t_0$,

$$\begin{aligned}&\left|\int_{t_0}^{t} f_{\alpha'_n}(s,\phi_{\alpha'_n}(s))\mathrm{d}s-\int_{t_0}^{t} g(s,\varphi(s))\mathrm{d}s\right|=\left|\int_{t_0}^{t}\left(f_{\alpha'_n}(s,\phi_{\alpha'_n}(s))-g(s,\varphi(s))\right)\mathrm{d}s\right|\\
&\leq\int_{t_0}^{t}\left|\left(f_{\alpha'_n}(s,\phi_{\alpha'_n}(s))-f_{\alpha'_n}(s,\varphi(s))\right)\right|\mathrm{d}s+\int_{t_0}^{t}\left|\left(f_{\alpha'_n}(s,\varphi(s))-g(s,\varphi(s))\right)\right|\mathrm{d}s\\
&\leq\int_{t_0}^{t} l_{\alpha'_n}(s)\left|\phi_{\alpha'_n}(s)-\varphi(s)\right|\mathrm{d}s+\int_{t_0}^{t}\left|\left(f_{\alpha'_n}(s,\varphi(s))-g(s,\varphi(s))\right)\right|\mathrm{d}s\\
&\leq\left(\int_{t_0}^{t} l(s+\alpha'_n)^q\mathrm{d}s\right)^{\frac{1}{q}}\left(\int_{t_0}^{t}\left|\phi(s+\alpha'_n)-\varphi(s)\right|^p\mathrm{d}s\right)^{\frac{1}{p}}\\
&\quad+\left(\int_{t_0}^{t}|f_{\alpha'_n}(s,\varphi(s))-g(s,\varphi(s))|^p\mathrm{d}s\right)^{\frac{1}{p}}\end{aligned}$$

Using (4.6), we see that (4.7) holds.

Define a (continuous) function $\tilde{\varphi}(t)$ by

$$\tilde{\varphi}(t) = \varphi(t_0) + \int_{t_0}^{t} g(s, \varphi(s))\mathrm{d}s, \quad t \in R.$$

From (4.5) and (4.7), we see that $\lim_{n\to\infty} \phi(t + \alpha'_n) = \tilde{\varphi}(t)$, locally uniformly for $t \in R$. Consequently, it follows that $\varphi(t) \equiv \tilde{\varphi}(t)$ a.e. on R. Hence

$$\lim_{n\to\infty} \left(\frac{1}{l} \int_{t}^{t+l} |\phi(s + \alpha'_n) - \tilde{\varphi}(s)|^p \mathrm{d}s \right)^{\frac{1}{p}}$$
$$= \lim_{n\to\infty} \left(\frac{1}{l} \int_{t}^{t+l} |\phi(s + \alpha'_n) - \varphi(s)|^p \mathrm{d}s \right)^{\frac{1}{p}} = 0,$$

locally uniformly for $t \in R$, which shows $S^p T_{\alpha'}\phi = \tilde{\varphi}$. Furthermore, we get

$$\tilde{\varphi}(t) = \varphi(t_0) + \int_{t_0}^{t} g(s, \varphi(s))\mathrm{d}s$$
$$= \tilde{\varphi}(t_0) + \int_{t_0}^{t} g(s, \tilde{\varphi}(s))\mathrm{d}s, \quad t \in R,$$

which shows that $\tilde{\varphi}$ is a solution of (4.3). The last conclusion is obvious. This completes the proof of Lemma. □

Remark According to this lemma, if ϕ is a solution of (4.2), and there exists a sequence $\alpha \subset R$ such that (4.4) hold, we can simply write $S^p T_\alpha \phi = \widetilde{\varphi}$ as a solution of (4.3).

Theorem 3 *Suppose that $f(t, x)$ is S^p-almost periodic in t uniformly in $x \in R^n$ and bounded almost everywhere in t on $R \times R^n$. Let K be a compact subset of R^n, $\phi(t)$ a solution of (4.2) with $\phi(t) \in K$ for all $t \in R$. If f is S^p-Lipschitzian in K, then for any given sequence $\alpha \subset R$, one can extract a subsequence α' of α and functions $g \in S^p_K H(f)$, $\varphi, \phi_1 \in S^p C(R, R^n)$ such that $S^p T_{\alpha'}\phi = \varphi$, $S^p T_{-\alpha'}\varphi = \phi_1$ holds uniformly on any compact subset of R, φ and ϕ_1 are solutions of (4.3) and (4.3), respectively, and $\varphi(t) \in K$, $\phi_1(t) \in K$ for all $t \in R$.*

Proof From the assumptions, for any given sequence $\alpha \subset R$, we can choose a subsequence $\beta \subset \alpha$ and a function $g \in S^p_K H(f)$ such that $g = U S^p_K T_\beta f$ and $f = U S^p_K T_{-\beta} g$. Let $I_N = [-N, N]$. From the assumptions on f, we have that $f(t, x)$, $f(t+\beta_n, x)$ are uniformly bounded almost everywhere on I_N for all $x \in K$. So, $\phi(t + \beta_n)$ is uniformly bounded and almost everywhere equi-continuous on I_N. Using Corollary 1, we can extract a subsequence $\gamma \subset \beta$ and a function $\widetilde{\varphi}$ such that $\{\phi(t + \gamma_n)\}$ is almost everywhere uniformly convergent to $\widetilde{\varphi}$ on I_N. So, we have that $S^p_K T_\gamma f = g$ and $S^p T_\gamma \phi = \widetilde{\varphi}$ holds uniformly on each I_N, $N = 1, 2, \ldots$. From Lemma 2 , there exists a solution φ of (4.3) such that $S^p T_\gamma \phi = \varphi$ holds in each I_N and $\varphi(t) \in K$ for all $t \in R$.

Using the same argument, we can find a subsequence $\alpha' \subset \gamma$ and $\widetilde{\phi}$ such that $U S^p_K T_{-\alpha} g = f$ and $U S^p_K T_{-\alpha}\varphi = \widetilde{\phi}$ hold on each I_N. Again using Lemma 2, there exists a solution ϕ_1

of (4.2) such that $US^pT_{-\alpha}\varphi = \phi_1$ holds on each I_N and $\phi_1(t) \in K$ for all $t \in R$. This completes the proof of the theorem. □

Theorem 4 *Suppose that $f(t, x)$ satisfies all conditions in Theorem 3. Let K be compact subset of R^n, $\phi(t)$ a solution of (4.2) defined on $[t_0, +\infty)$ for some $t_0 \in R$, valued in K. If f is S^p-Lipschitzian in K, then for any $g \in S_K^p H(f)$, the equation (4.3) has a solution φ defined on R with values in K.*

Proof Let $\alpha_n = n$. Then $\phi(t+n)$ is a solution of

$$x' = f(t+n, x)$$

defined on $[t_0 - n, +\infty)$. Using arguments similar to Theorem 3, there exist a function $g \in S_K^p H(f)$, φ and $\alpha' \subset \alpha$ such that $S_K^p T_\alpha f = g$, $S^p T_\alpha \phi = \varphi$ and φ is a solution of (4.3). Obviously, φ is defined on R. At the same time, there exists ϕ_1 with $\phi_1 = S^p T_{-\alpha}\varphi$ and ϕ_1 is a solution of (4.2) defined on R. Now, using Theorem 3, for any $g \in S_K^p H(f)$, we get a solution of (4.3) which is defined on R with values in K. The proof is complete. □

We can prove the following lemma easily using the definitions of Stepanov hull and of S^p-norm.

Lemma 3 *Let $\phi \in S^pC(R, R^n)$, $l > 0$ a real number. Then for any $\varphi \in S^pH(\phi)$, we have $S_l^p(\varphi) \le S_l^p(\phi)$.*

We also have

Lemma 4 *Let $\{x_n(t)\}$ be a sequence of solutions of (4.2), defined on R. If $\{x_n(t)\}$ is almost everywhere uniformly convergent on any compact subset of R, then there exists a solution $\phi(t)$ of (4.2) such that $\{x_n(t)\}$ almost everywhere uniformly converges to $\phi(t)$ on any compact subset of R.*

Proof From the conditions, we can take a function $x^*(t)$, defined almost everywhere on R and $\{x_n(t)\}$ almost everywhere uniformly converges to $x^*(t)$ on any compact subset of R. Let $I_0 \subset R$, with $m(I_0) = 0$ and let $x^*(t)$ be defined on $R \backslash I_0$. Now choose $t_0 \in R \backslash I_0$ and define

$$\phi(t) = \begin{cases} x^*(t_0) + \int_{t_0}^t f(s, x^*(s))\mathrm{d}s & t \in I_0 \\ x^*(t) & t \in R \backslash I_0. \end{cases}$$

It is easy to show that $\phi(t)$ is a solution of (4.2) and $\{x_n(t)\}$ almost everywhere uniformly converges to $\phi(t)$ on any compact subset of R. This ends the proof. □

Theorem 5 *Suppose that $f(t, x)$ is almost everywhere bounded in t uniformly in $x \in R$. Let K be a compact subset of R^n, ϕ a solution of (4.2) with $\phi(t) \in K$ for all $t \in R$. Then there is a solution ϕ_0 of (4.2) such that $\phi_0(t) \in K$ for all $t \in R$ and $\phi_0(t)$ minimizes the S^p-norm in K, i.e., for any solution φ of (4.2) with $\varphi(t) \in K$ for all $t \in R$, $S_l^p(\phi_0) \le S_l^p(\varphi)$.*

Proof Let

$$\mathcal{F} = \{x(t) | x(t) \text{ is a solution of (4.2)}, x(t) \in K \quad \text{for all } t \in R\}$$

and define

$$\lambda = \inf_{x \in \mathcal{F}} \{S_l^p(x)\}.$$

Then λ exists and $\lambda \leq S_l^p(\phi)$. Now we define

$$\lambda_n = \inf_{x \in \mathcal{F}} \left\{ \sup_{|t| \leq n} \left(\frac{1}{l} \int_t^{t+l} |x(s)|^p \mathrm{d}s \right)^{\frac{1}{p}} \right\}.$$

Obviously, $\lambda_n \leq \lambda_{n+1}$ and $\lim_{n\to\infty} \lambda_n = \lambda$. So, we can find a sequence $\{x_n(t)\}$ such that $x_n(t) \in \mathcal{F}$ and for each $n \in Z^+$

$$\sup_{|t| \leq n} \left(\frac{1}{l} \int_t^{t+l} |x(s)|^p \mathrm{d}s \right)^{\frac{1}{p}} \leq \lambda_n + \frac{1}{n}. \tag{4.8}$$

According to the assumptions, $\{x_n(t)\}$ are uniformly bounded and almost everywhere equicontinuous. So, from Theorem 2, there exists a subsequence $\{x_{n_k}(t)\}$ of $\{x_n(t)\}$ which is almost everywhere uniformly convergent on any compact subset of R. From Lemma 4, there exists a function $\phi_0 \in \mathcal{F}$ such that $\{x_{n_k}(t)\}$ converges to ϕ_0 almost everywhere uniformly. Now, using Minkovskii's inequality and (4.8), we have that

$$\begin{aligned}
\sup_{|t| \leq n} \left(\frac{1}{l} \int_t^{t+l} |\phi_0(s)|^p \mathrm{d}s \right)^{\frac{1}{p}} &\leq \sup_{|t| \leq n} \left(\frac{1}{l} \int_t^{t+l} |\phi_0(s) - x_{n_k}(s)|^p \mathrm{d}s \right)^{\frac{1}{p}} \\
&\quad + \sup_{|t| \leq n} \left(\frac{1}{l} \int_t^{t+l} |x_{n_k}(s)|^p \mathrm{d}s \right)^{\frac{1}{p}} \\
&\leq \sup_{|t| \leq n} \left(\frac{1}{l} \int_t^{t+l} |\phi_0(s) - x_{n_k}(s)|^p \mathrm{d}s \right)^{\frac{1}{p}} + \lambda_n + \frac{1}{n}
\end{aligned}$$

for each $n \in Z^+$. Letting $n \to \infty$, we have that $S_l^p(\phi_0) \leq \lambda$. Since $\phi_0 \in \mathcal{F}$, from the definition of λ, we obtain $S_l^p(\phi_0) = \lambda$. This ends the proof of the theorem. □

Lemma 5 *Suppose that $f(t, x)$ is S^p-almost periodic in t uniformly in $x \in R^n$ and almost everywhere bounded in t uniformly in $x \in R$. Let K be a compact subset of R^n, ϕ a solution of (4.2) defined on R with minimizing S^p-norm in K. If $f(t, x)$ is S^p-Lipschitzian in K and there are $g \in S_K^p H(f)$ and a sequence $\alpha \subset R$ such that $U S^p T_\alpha f = g$ and $S^p T_\alpha \phi$ exists uniformly on any compact subset of R, then, $S^p T_\alpha \phi$ is a solution of the equation*

$$x' = g(t, x) \tag{4.9}$$

with minimizing S^p-norm in K.

Proof From the Remark of Lemma 2, we know that $S^p T_\alpha \phi$ is a solution of (4.9) valued in K. Firstly, we show that $S_l^p(S^p T_\alpha \phi) = S_l^p(\phi)$. In fact, by Lemma 3, we have $S_l^p(S^p T_\alpha \phi) \leq S_l^p(\phi)$. On the other hand, from Theorem 3 we can take a subsequence $\alpha' \subset \alpha$ such that $S^p T_{-\alpha'}(S^p T_\alpha \phi)$ is a solution of (4.2) valued in K. Using Lemma 3 again, we have that

$S_l^p(S^pT_{-\alpha'}(S^pT_{\alpha'}\phi)) \le S_l^p(S^pT_{\alpha'}\phi) \le S_l^p(\phi)$. But, according to the assumption, ϕ minimizes the S^p-norm in K, and so $S_l^p(\phi) \le S_l^p(S^pT_{-\alpha'}(S^pT_{\alpha'}\phi))$. Therefore, we have proved that $S_l^p(S^pT_\alpha\phi) = S_l^p(\phi)$.

Now, we shall show that $S^pT_\alpha\phi$ minimizes S^p-norm in K . Assuming the contrary, the equation (4.9) has another solution φ such that $S_l^p(\varphi) < S_l^p(S^pT_\alpha\phi)$. Then, there exists a subsequence $\alpha' \subset \alpha$ such that $US_K^pT_{-\alpha'}g = f$ and from Theorem 3, there exists a solution ϕ_1 of (4.2) such that $S^pT_{-\alpha'}\varphi = \phi_1$ and $\phi_1(t) \in K$ for all $t \in R$. From Lemma 3, we have that $S_l^p(\phi_1) = S_l^p(S^pT_{-\alpha'}\varphi) \le S_l^p(\varphi) < S_l^p(S^pT_\alpha\phi) = S_l^p(\phi)$. This contradicts the fact that ϕ minimizes the S^p-norm in K. This ends the proof of this lemma. □

Now we state and prove the main result of this section, i.e., an existence theorem for an S^p-almost periodic solution of (4.2).

Theorem 6 *Suppose that $f(t,x)$ is S^p-almost periodic in t uniformly in $x \in R$ and almost everywhere bounded in t uniformly in $x \in R^n$. Let K be a compact subset of R^n. If $f(t,x)$ is S^p-Lipschitzian in K and* (4.2) *has a solution $\phi(t)$ defined on $[t_0,\infty)$ for some $t_0 \in R$ with $\phi(t) \in K$ for all $t \in [t_0,\infty)$ and for any $g \in S_K^pH(f)$, the equation* (4.9) *has at most one solution minimizing the S^p-norm in K, then, for every $g \in S_K^pH(f)$, the equation* (4.9) *has an S^p-almost periodic solution on R.*

Proof By Theorem 4 and Theorem 5, there exists a solution ϕ_1 of (4.2) defined on R with minimizing S^p-norm in K. We show that ϕ_1 is an S^p-almost periodic solution of (4.2).

To this end, we use the formulation of Bochner's theorem above and show that for any pair of sequences $\alpha, \beta \subset R$, there are two common sequences $\alpha' \subset \alpha, \beta' \subset \beta$ such that $S^pT_{\alpha'}(S^pT_{\beta'}\phi_1) = S^pT_{\alpha'+\beta'}\phi_1$.

Now, let $\alpha, \beta \subset R$ be any pair of sequences. Since $f(t,x)$ is S^p-almost periodic in t uniformly in $x \in R^n$, we can find two common sequences $\alpha' \subset \alpha, \beta' \subset \beta$ and functions $g, g_1 \in S_K^pH(f)$ such that $g_1 = US^pT_{\beta'}f$, $g = US^pT_{\alpha'}g_1 = US^pT_{\alpha'+\beta'}f$. From Theorem 3 and the Remark of Lemma 2, we can take the sequences α', β' such that $S^pT_{\beta'}\phi_1$, $S^pT_{\alpha'}(S^pT_{\beta'}\phi_1)$, $S^pT_{\alpha'+\beta'}\phi_1$ exist uniformly on any compact subset of R and $S^pT_{\beta'}\phi_1$ is a solution of the equation $x' = g_1(t,x)$, while $S^pT_{\alpha'}(S^pT_{\beta'}\phi_1)$, $S^pT_{\alpha'+\beta'}\phi_1$ are all solutions of (4.9). Furthermore, from Lemma 3, $S^pT_{\alpha'}(S^pT_{\beta'}\phi_1)$, $S^pT_{\alpha'+\beta'}\phi_1$ are all solutions of (4.9) with minimizing S^p-norm in K. By the assumption of the theorem, we have that $S^pT_{\alpha'}(S^pT_{\beta'}\phi_1) = S^pT_{\alpha'+\beta'}\phi_1$ for all $t \in R$. This is what we desire.

Obviously, for any $g \in S_K^pH(f)$, there exists a sequence $\alpha \subset R$ such that $g = US_K^pT_\alpha f$ and therefore $S^pT_\alpha\phi$ is a S^p-almost periodic solution of (4.9). This completes the proof of the theorem. □

5 A Favard-type theorem for Stepanov almost periodic systems

We show here that there exists a C^1-Stepanov a.p. function that is not Bohr a.p. Indeed, the idea, originally due to Ursell [26], can be improved upon to show the existence of C^∞-Stepanov a.p. functions that are not Bohr a.p.

Let ε_n $(n = 1, 2, \ldots)$ satisfy

$$0 < \varepsilon_n < 1, \quad \sum_{n=1}^{n=\infty} \varepsilon_n < \infty$$

For each n, define a function $f_n(x)$ as follows:

$$f_n(x) = \begin{cases} \frac{2(x-y_{n,k}+\varepsilon_n)^2}{\varepsilon_n^2}, & \left(y_{n,k}-\varepsilon_n,\ y_{n,k}-\frac{\varepsilon_n}{2}\right] \\ 1-\frac{2(x-y_{n,k})^2}{\varepsilon_n^2}, & \left(y_{n,k}-\frac{\varepsilon_n}{2},\ y_{n,k}+\frac{\varepsilon_n}{2}\right] \\ \frac{2(x-y_{n.k}-\varepsilon_n)^2}{\varepsilon_n^2}, & \left(y_{n,k}+\frac{\varepsilon_n}{2},\ y_{n.k}+\varepsilon_n\right] \\ 0, & \text{elsewhere} \end{cases}$$

where

$$y_{n,k} = (2k+1)n, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

Then, for each n, $f_n(x)$ is a periodic function with period $2n$.

Let

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

Then, $f(x)$ is continous on $(-\infty,\ \infty)$. For any positive number A and any $x \in [-A,\ A]$

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x)$$

where

$$f'(x) = \begin{cases} \frac{4(x-y_{n,k}+\varepsilon_n)}{\varepsilon_n^2}, & \left(y_{n,k}-\varepsilon_n,\ y_{n,k}-\frac{\varepsilon_n}{2}\right] \cap [-A,\ A] \\ -\frac{4(x-y_{n,k})}{\varepsilon_n^2}, & \left(y_{n,k}-\frac{\varepsilon_n}{2},\ y_{n,k}+\frac{\varepsilon_n}{2}\right] \cap [-A,\ A] \\ \frac{4(x-y_{n.k}-\varepsilon_n)}{\varepsilon_n^2}, & \left(y_{n,k}+\frac{\varepsilon_n}{2},\ y_{n.k}+\varepsilon_n\right] \cap [-A,\ A] \\ 0, & \text{elsewhere on } [-A,\ A]. \end{cases}$$

So, $f'(x)$ is continuous on $[-A,\ A]$ for any $A > 0$. Since $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, $f(x)$ is Stepanov a.p. but since $f'(x)$ is unbounded, $f(x)$ is not uniformly continuous on the real line and so it is not Bohr a.p. By smoothing out (mollifying) the peaks in the example we can find an appropriate infinitely differentiable Stepanov a.p. function that is not a.p.

Now consider a linear S^p-almost periodic system

$$x' = A(t)x + f(t) \tag{5.1}$$

where $A(t)$ is an $n \times n$ matrix function and $f(t)$ is vector function. We always assume that $A(t)$ and $f(t)$ are S^2-almost periodic on R, but do not indicate this again in this section.

Lemma 6 *Let $A(t)$ and $f(t)$ be almost everywhere bounded on any compact subset of R. Let $B \in S^2H(A)$ and $g \in S^2H(f)$ be such that there exists a sequence $\alpha \subset R$ such that $US^2T_\alpha A = B$, $US^2T_\alpha f = g$. If every non-trivial bounded solution $y(t)$ of the equation*

$$x' = B(t)x \tag{5.2}$$

satisfies

$$\inf_{t \in R} S_l^2(t, y) > 0, \tag{5.3}$$

then the equation

$$x' = B(t)x + f(t) \tag{5.4}$$

has at most one solution in K with minimizing S_l^2-norm.

Proof Assuming the contrary. Suppose that there are two distinct solutions $x_1(t), x_2(t)$ of (5.4) with minimizing S^2-norm in K, i.e., $S_l^2(x_1) = S_l^2(x_2)$. Let $y_1(t) = \frac{1}{2}(x_1(t) + x_2(t))$, $y_2(t) = \frac{1}{2}(x_1(t) - x_2(t))$. Then, $y_1(t)$ is a solution of (5.4), but $y_2(t)$ is a solution of (5.2). From the condition (5.3), we have

$$\delta = \inf_{t\in R} S_l^2(t, y_2) > 0.$$

Now we have

$$\begin{aligned}\left[S_l^2(t, y_1)\right]^2 + \left[S_l^2(t, y_2)\right]^2 &= \frac{1}{2}\left[S_l^2(t, x_1)\right]^2 + \frac{1}{2}\left[S_l^2(t, x_2)\right]^2 \\ &\le \left[S_l^2(x_1)\right]^2.\end{aligned}$$

Thus,

$$\begin{aligned}\left[S_l^2(y_1)\right]^2 &\le \left[S_l^2(x_1)\right]^2 - \left[S_l^2(t, y_2)\right]^2 \\ &\le \left[S_l^2(x_1)\right]^2 - \delta^2 \\ &< \left[S_l^2(x_1)\right]^2.\end{aligned}$$

So, we get $S_l^2(y_1) < S_l^2(x_1)$. This contradicts the fact that x_1 minimizes the S_l^2-norm. This ends the proof. □

Now, we can state a generalization of Favard's Theorem for S^p-almost periodic differential equations.

Theorem 7 *Suppose that $A(t)$ and $f(t)$ are almost everywhere bounded on any compact subset of R and that for any $B \in S^2H(A)$, any nontrivial solution of* (5.2) *satisfies* (5.3)*. Let K be a compact subset of R^n, $\phi(t)$ a solution of* (5.1) *defined on $[\tau, +\infty)$, for $\tau \in R$, with values in K. Then for any $B \in S^2H(A)$, $g \in S^2H(f)$, the equation* (5.1) *has an S^2-almost periodic solution on R.*

This theorem can easily follow from Theorem 6 and Lemma 6. We leave out the details.

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