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Invariant surfaces with constant mean curvature in $\mathbb{H}^2 \times \mathbb{R}$

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Abstract We classify the profile curves of all surfaces with constant mean curvature in the product space $\mathbb{H}^2 \times \mathbb{R}$, which are invariant under the action of a 1-parameter subgroup of isometries.

Keywords Minimal surfaces · Constant mean curvature surfaces · Invariant surfaces

Mathematics Subject Classification (2000) 53C42 · 53A10

1 Introduction

The theory of constant mean curvature (CMC) surfaces in $\mathbb{R}^3$ or, more generally, into a space form, has been an extensive field of research in the twentieth century, partially motivated by the possible applications and by the intrinsic beauty of the subject. In the last decade, the study of surfaces in three-dimensional manifolds with non constant sectional curvature has grown very rapidly and particular attention has been given to the study of surfaces in homogeneous three-manifolds with compact quotients (Thurston's geometries). Geometrically, the most interesting surfaces in these spaces are those with constant mean or Gauss curvature, and among these there are those which present many symmetries, i.e. invariant under the action of a subgroup of the isometry group of the ambient manifold. For surfaces in a three-dimensional manifold the unique interesting (not trivial) subgroups of isometries are the one-parameter subgroups and, in this case, any invariant surface can be rendered as the orbit of a curve (the *profile curve*) by the action of the subgroup.

I dedicate this work to my supervisor Francesco Mercuri, on the occasion of his 60th anniversary.

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The study of invariant surfaces in the three-dimensional Thurston's geometries has been initiated by Caddeo, Piu and Ratto in [2], where they characterized the SO(2)-invariant CMC surfaces in a three-dimensional homogenous space, and by Tomter, in [13], for the Heisenberg group $\mathbb{H}_3$. Also, in [3], the authors described the profile curves of the SO(2)-invariant surfaces with constant Gauss curvature in the Heisenberg group $\mathbb{H}_3$. Later, Montaldo and Onnis, in [7], classified all the invariant surfaces with constant Gauss curvature in $\mathbb{H}_3$ and in the space $\mathbb{H}^2 \times \mathbb{R}$, which is the product of the hyperbolic plane $\mathbb{H}^2$ (of constant curvature -1) and the real line. In [4], Figueroa, Mercuri and Pedrosa presented the final classification of all the invariant CMC surfaces in $\mathbb{H}_3$.

This paper is devoted to the study of CMC surfaces in the product space $\mathbb{H}^2 \times \mathbb{R}$ which are invariant under the action of a one-parameter subgroup of isometries of the ambient space. The aim is to complete the classification of such surfaces, that has been initiated by Montaldo and Onnis in [5,6] and by Toubiana and Sá Earp in [11].

We shall use standard techniques of equivariant geometry, in particular the Reduction Theorem of Back, do Carmo and Hsiang ([1], see also [4] for a detailed proof). The Reduction Theorem allows us to give a local description of the invariant surfaces by the study of a system of three ordinary differential equations that characterizes the profile curve of an invariant surface. Solutions of the above system can be studied qualitatively in virtue of the existence of a first integral, namely $J(s) = k$, with $k \in \mathbb{R}$. Thus the classification of the profile curves will be given in terms of the mean curvature H and of the parameter k (see Theorems 2 and 3).

One major problem in the classification of all invariant surfaces is to determine the minimum number of one-parameter subgroups of isometries that generate (up to isometries of the ambient space) all possible invariant surfaces. In this paper we solve this problem for the invariant surfaces in the space $\mathbb{H}^2 \times \mathbb{R}$ (see Proposition 2). We prove that there are four types of "independent" one-parameter subgroups: the group of rotations, the group of helicoidal (screw) motions and two other groups which are now called *parabolic* and *hyperbolic* screw motions.

The paper is organized as follows: in Sects. 2 and 3 we recall some basic results of equivariant geometry, in particular the Reduction Theorem, and some facts about the space $\mathbb{H}^2 \times \mathbb{R}$ and its 1-parameter subgroups of isometries. Then, in the subsequent sections, we classify the profile curves of the invariant CMC surfaces in the cases not considered in [5,6]. We end the paper with the tables showing the plots of the profile curves for all invariant surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

The results are part of the author's doctoral dissertation at the University of Campinas, Brazil, (see [9]) and were announced in [10]. Shortly after, in the pre-print [12], the author obtained essentially the same results by using a different approach.

2 Basic facts

Let (N, g) be a Riemannian manifold and let G be a closed subgroup of the isometry group $\mathrm{Isom}(N)$. If $x \in N$, we will denote by hx the action of an element $h \in G$ on x and by:

- $G(x) := \{hx : h \in G\}$, the *orbit* of x,
- $G_x := \{h \in G : hx = x\}$, the *isotropy subgroup of* x,
- $\mathscr{B} = N/G$, the *orbit space*.

From the theory of Riemannian actions we know that:

- There exists a unique minimal conjugacy class of isotropy subgroups. Orbits with isotropy in this class are called *principal* and the union of the principal orbits, denoted by N_{r}, is called the *regular part* of N. The set N_{r} is open and dense in N.
- The isotropy group G_x is compact and $G(x) = G/G_x$. All orbits with isotropy in the same conjugacy class are pairwise diffeomorphic. In particular all principal orbits are pairwise diffeomorphic.
- The regular part of the orbit space $\mathscr{B}_{\mathrm{r}} := N_{\mathrm{r}}/G$ is a connected differentiable manifold (if N is connected) and the quotient map $\pi : N_{\mathrm{r}} \to \mathscr{B}_r$ is a submersion.

Remark 1 The full orbit space may contain singularities due to the presence of non-principal orbits. However, for the case of principal orbits of codimension ≤ 2, which is the case of our study, the orbit space is always a manifold, with or without boundary. In this case, the analysis at the boundary (image of the non-principal orbits) may be carried out, even if conditioned by the differential equations involved.

In the sequel we assume that N is three-dimensional manifold. Let X be a complete Killing vector field on N. Then X generates a one-parameter subgroup G_X of the isometry group of (N, g). Let $f : M^2 \to N^3$ be an immersion from a surface M into N. We say that f is a G_X-*equivariant* immersion, and $f(M)$ a G_X-*invariant* surface of N, if there exists an action of G_X on M such that for any $x \in M$ and $h \in G_X$ we have $f(hx) = hf(x)$. We will endow M with the metric induced by f and we will assume that $f(M) \subset N_{\mathrm{r}}$ and that N/G_X is connected. Then f induces an immersion $\tilde{f} : M/G_X \to N_{\mathrm{r}}/G_X$ between the orbit spaces; moreover, the space N_{r}/G_X can be equipped with a Riemannian metric, the *quotient metric*, so that the quotient map $\pi : N_{\mathrm{r}} \to N_{\mathrm{r}}/G_X$ is a Riemannian submersion. Note that N_{r}/G_X is a surface and that $\tilde{f}$ defines a curve in N_{r}/G_X called the *profile curve*.

It is well known (see, for example, [8]) that N_{r}/G_X can be locally parametrized by the invariant functions of the Killing vector field X. If $\{f_1, f_2\}$ is a complete set of invariant functions on a G_X-invariant subset of N_{r}, then the quotient metric is given by $\tilde{g} = \sum_{i,j=1}^{2} h^{ij}\,\mathrm{d}f_i \otimes \mathrm{d}f_j$, where (h^{ij}) is the inverse of the matrix (h_{ij}) with entries $h_{ij} = g(\nabla f_i, \nabla f_j)$. Here ∇ is the gradient operator of (N, g).

The mean curvature of an invariant immersion is well related to the geodesic curvature of the profile curve as shown by the celebrated

Theorem 1 (Reduction Theorem [1]) *Let H be the mean curvature of $M_{\mathrm{r}} \subset N_{\mathrm{r}}$ and k_g the geodesic curvature of the profile curve $M_{\mathrm{r}}/G_X \subset \mathscr{B}_{\mathrm{r}}$ with respect to the orbital metric $\tilde{g}$. Then*

$$H(x) = k_g(\pi(x)) - D_{\boldsymbol{n}} \ln \omega(\pi(x)), \qquad x \in M_{\mathrm{r}},$$

where $\boldsymbol{n}$ is the unit normal to the profile curve and $\omega = \sqrt{g(X, X)}$ is the volume function of the principal orbit.

Remark 2 Theorem 1 was well known for compact groups. This version is due to Back, do Carmo and Hsiang and appeared in the unpublished manuscript [1]. A published proof may be found in [4].

3 The space $\mathbb{H}^2 \times \mathbb{R}$

In this section we will recall some basic geometric properties of the space $\mathbb{H}^2 \times \mathbb{R}$, in particular we will describe the conjugacy classes of 1-parameter subgroups of isometries.

Let $\mathbb{H}^2$ be represented by the upper half-plane model $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the metric $g_{\mathbb{H}} = (dx^2 + dy^2)/y^2$. The space $\mathbb{H}^2$, with the group structure derived by the composition of proper affine maps, is a Lie group and the metric $g_{\mathbb{H}}$ is left invariant. Therefore, the product $\mathbb{H}^2 \times \mathbb{R}$ is a Lie group with respect to the product

$$L_{(x,y,z)}(x', y', z') = (x, y, z) * (x', y', z') = (x'y + x, yy', z + z') \tag{1}$$

and the product metric $g = g_{\mathbb{H}} + dz^2$ is left invariant. With respect to the metric g an orthonormal basis of left invariant vector fields is given by:

$$E_1 = y\frac{\partial}{\partial x}, \quad E_2 = y\frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}.$$

It is well known that the isometry group of $\mathbb{H}^2 \times \mathbb{R}$ has dimension four, the maximal possible for a non constant sectional curvature three-dimensional space. In particular, we can choose the following basis of Killing vector fields:

$$X_1 = \frac{(x^2 - y^2)}{2}\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}; \quad X_2 = \frac{\partial}{\partial x}; \quad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}; \quad X_4 = \frac{\partial}{\partial z}.$$

Lets denote by G_i the one-parameter subgroup of isometries generated by X_i, by G_{ij} that generated by linear combinations of X_i and X_j and so on. Explicitly, indicating by L the left translation defined by (1), we have that:

$$\begin{aligned}
&G_1 = \{\mathscr{L}_{(t,0,0,0)} \mid t \in \mathbb{R}\} \quad \text{with} \\
&\quad \mathscr{L}_{(t,0,0,0)}(x, y, z) = \left(\frac{-2[t(x^2 + y^2) - 2x]}{(tx - 2)^2 + t^2y^2}, \frac{4y}{(tx - 2)^2 + t^2y^2}, z\right); \\
&G_2 = \{\mathscr{L}_{(0,t,0,0)} \mid t \in \mathbb{R}\} \quad \text{with} \quad \mathscr{L}_{(0,t,0,0)} \equiv L_{(t,1,0)}; \\
&G_3 = \{\mathscr{L}_{(0,0,t,0)} \mid t \in \mathbb{R}\} \quad \text{with} \quad \mathscr{L}_{(0,0,t,0)} \equiv L_{(0,e^t,0)}; \\
&G_4 = \{\mathscr{L}_{(0,0,0,t)} \mid t \in \mathbb{R}\} \quad \text{with} \quad \mathscr{L}_{(0,0,0,t)} \equiv L_{(0,1,t)}.
\end{aligned}$$

Remark 3 In the sequel we shall use the Killing vector field $X_{12}^* = X_1 + X_2/2$. The orbits of X_{12}^* are rotations in $\mathbb{H}^2 \times \mathbb{R}$ about the vertical straight line $(0, 1, z)$. In fact, the integral curve of this vector field through the point $(x_0, y_0, z_0) \in \mathbb{H}^2 \times \mathbb{R}$ is given by $\mathscr{L}_{(t,t/2,0,0)}(x_0, y_0, z_0) = (x(t), y(t), z_0)$, where

$$x(t)^2 + y(t)^2 - \beta\, y(t) + 1 = 0, \qquad \beta = (1 + x_0^2 + y_0^2)/y_0.$$

An easy computation shows that the hyperbolic distance from a point of the integral curve to the point $(0, 1, z_0)$ is constant and equal to $\ln\left(\frac{\beta+\sqrt{\beta^2-4}}{2}\right)$. Therefore, the integral curves of X_{12}^* are geodesic circles centred at $(0, 1, z_0)$ (see Fig. 1).

3.1 Congruent invariant surfaces

In the study of the surfaces which are invariant under the action of a one-parameter subgroup of isometries appears naturally the problem of reducing the number of cases. In fact, there are 15 essential possibilities to combine the basis of the Killing vector fields to produce a one-parameter subgroup of isometries. The key ingredient in order to reduce the number of cases is the following general criterium:

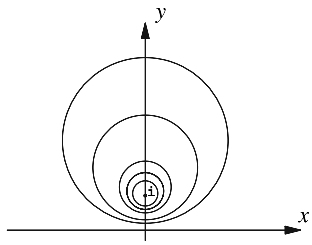


Fig. 1 Integral curves of X_{12}^*

Table 1 Conjugated groups with the respective isometry φ

$\mathbf{G_X}$	$\mathbf{G_Y}$	$\varphi(w, z)$
G_1	G_2	$(-2/w, z)$
$G_3 = \{\mathscr{L}_{(0,0,bt,0)} : t \in \mathbb{R}\}$	$G_{13} = \{\mathscr{L}_{(at,0,bt,0)} : t \in \mathbb{R}\}$	$\left(\frac{\frac{a}{b}w}{\frac{1}{2}w+\frac{b}{a}}, z\right)$
G_3	$G_{23} = \{\mathscr{L}_{(0,at,-bt,0)} : t \in \mathbb{R}\}$	$\left(\frac{aw-b}{bw}, z\right)$
G_3	$G_{12}^- = \{\mathscr{L}_{(t,at,0,0)} : t \in \mathbb{R}\}, \quad a < 0$	$\left(\frac{\frac{1}{2b}w-\frac{1}{2}}{w+b}, z\right)$
$G_{123} = \{\mathscr{L}_{(t,at,bt,0)} : t \in \mathbb{R}\}$	$G_{12} = \{\mathscr{L}_{(t,ct,0,0)} : t \in \mathbb{R}\}, \quad c = (2a - b^2)/2$	$(w + b, z)$
$G_{12}^+ = \{\mathscr{L}_{(t,at,0,0)} : t \in \mathbb{R}\}$	$G_{12}^* = \{\mathscr{L}_{(bt,bt/2,0,0)} : t \in \mathbb{R}\}, \quad b^2 = 2a$	$\left(\frac{w}{\sqrt{2a}}, z\right)$

Proposition 1 *If two groups G_X and G_Y are conjugated by an isometry φ of N (i.e. $G_Y = \varphi G_X \varphi^{-1}$), then the respective invariant surfaces are congruent by φ.*

Proof Suppose that there exists $\varphi \in \mathrm{Isom}(N)$ such that $\varphi\, G_X\, \varphi^{-1} = G_Y$. Let M_1 be a G_X-invariant surface in N and set $M_2 = \varphi(M_1)$. The surface M_2 is G_Y-invariant:

$$G_Y M_2 = G_Y\,(\varphi(M_1)) = (\varphi\, G_X\, \varphi^{-1})(\varphi(M_1)) = \varphi(G_X M_1) = \varphi(M_1) = M_2.$$

Conversely, if M_2 is a G_Y-invariant surface, then it is congruent to the G_X-invariant surface given by $\varphi^{-1}(M_2)$. $\square$

Therefore, using Proposition 1, we can reduce the study of the invariant surfaces by analyzing the conjugacy classes of one-parameter subgroups of isometries. In Table 1 we list the conjugated subgroups together with the respective isometry φ (the proof may be found in [9]). We will use complex coordinates to represent a point $w \in \mathbb{H}^2$, that is $w = x + iy$.

In Table 1 G_{12}^+ (respectively G_{12}^-) is the one-parameter subgroup of isometries generated by the vector field $X_{12}^+ = X_1 + aX_2$, with $a > 0$ (respectively $X_{12}^- = X_1 + aX_2$, with $a < 0$), and G_{12}^* is generated by X_{12}^*. Moreover, it is easy to check that given two Killing vector fields $X = \sum_{i=1}^3 a_i X_i$ and $Y = \sum_{i=1}^3 b_i X_i$, which are conjugated by φ, then $\widetilde{X} = X + a_4 X_4$ and $\widetilde{Y} = Y + a_4 X_4$, with $a_4 \in \mathbb{R}$, are also conjugated by φ. Therefore, denoting by "$\sim$" conjugation of groups, we have:

$$G_{14} \sim G_{24}, \quad G_{34} \sim G_{134}, \quad G_{34} \sim G_{234},$$

$$G_{34} \sim G_{124}^-, \quad G_{1234} \sim G_{124}, \quad G_{124}^+ \sim G_{124}^*,$$

where $G_{124}^{\pm}$ is the one-parameter subgroup generated by $X_{12}^{\pm}$ and X_4, while the group G_{124}^* is generated by X_{12}^* and X_4. The above discussion leads to the following:

Proposition 2 *Let X be a Killing vector field in $\mathbb{H}^2 \times \mathbb{R}$. Then any G_X-invariant surface in $\mathbb{H}^2 \times \mathbb{R}$ is congruent to a surface which is invariant under the action of one of the following groups: G_{24}, G_{34} or G^*_{124}.*

As the group G^*_{124} mixes rotation with vertical translation will be called *helicoidal group* and a G^*_{124}-invariant surface will be called *helicoidal surface*. In [5] it is given the complete classification of the helicoidal CMC surfaces by using the disk model for the hyperbolic plane, and in [6] by means of the upper half-plane model. In the above papers it is also studied the case of the G_4-invariant CMC surfaces. Therefore, from Proposition 2, one needs to study the remaining cases: G_{24} and G_{34}. These subgroups are now called *parabolic* and *hyperbolic*.

4 G_{24}-invariant (parabolic) surfaces with constant mean curvature

At first we observe that, since the group G_{24} acts freely (without fixed points) on $\mathbb{H}^2 \times \mathbb{R}$, the regular part is $(\mathbb{H}^2 \times \mathbb{R})_r = \mathbb{H}^2 \times \mathbb{R}$. A set of two functionally independent invariant functions is given by $u(x, y, z) = bx - az$ and $v(x, y, z) = y$ and, with respect to these functions, the orbit space and the orbital metric are given by:

$$\mathscr{B} = \{(u, v) \in \mathbb{R}^2 \mid v > 0\}, \qquad \widetilde{g} = \frac{\mathrm{d}u^2}{a^2 + b^2 v^2} + \frac{\mathrm{d}v^2}{v^2}.$$

Let now $\gamma(s) = (u(s), v(s))$ be a curve in $\mathscr{B}$, parametrized by arc-length, that generates, under the action of G_{24}, the surface $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$.

As

$$\omega(\xi) = \sqrt{\frac{a^2 + b^2 v^2}{v^2}} \qquad \text{and} \qquad D_{\mathbf{n}} \log(\omega(\xi)) = -\frac{a^2 \cos\sigma}{a^2 + b^2 v^2},$$

from Theorem 1, the mean curvature of Σ can be written as $H = \cos\sigma + \dot{\sigma}$, where σ is the angle that γ makes with the $\frac{\partial}{\partial u}$ direction. Thus γ generates a G_{24}-invariant CMC surface if and only if u and v satisfy the following system of ODE's:

$$\begin{cases} \dot{u} = \sqrt{a^2 + b^2 v^2} \cos\sigma \\ \dot{v} = v \sin\sigma \\ \dot{\sigma} = H - \cos\sigma. \end{cases} \tag{2}$$

From now on we will assume that the mean curvature H is constant. Before starting the study of (2) we make some elementary remarks (see [9]).

Proposition 3 *Solutions of* (2) *are invariant under*:

1. *Translations in the u direction*;
2. *Reflections across the line $u = u_0$.*

As a consequence of the second item of Proposition 3 we have the following:

Corollary 1 *Let $\gamma(s) = (u(s), v(s))$ be a solution of* (2) *defined for $s \in (s_0 - \epsilon, s_0]$ with $\sigma(s_0) \in \{0, \pi\}$. Then $\gamma(s)$ can be extended to the interval $(s_0 - \epsilon, s_0 + \epsilon)$ by a reflection across the line $u = u(s_0)$.*

Proposition 4 (The first integral) *The function* $J(s) = \sigma/v$ *is constant along any curve which is a solution of* (2). *Thus the solutions of* (2) *are characterized by the equation*

$$\dot{\sigma} = k\,v, \qquad k \in \mathbb{R}. \tag{3}$$

Theorem 2 *Let* $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ *be a* G_{24}*-invariant CMC surface and let* γ *be the profile curve in the orbit space. Then we have the following characterization of* γ *according to the value of* H *and* k:

(A) $H \equiv 0$ (*minimal surfaces*). *The profile curve is*
  (A_1) *for* $k = 0$, *a* vertical straight line;
  (A_2) *for* $k \neq 0$, *of* semi-circle *type.*
(B) $H > 1$. *In this case* $k > 0$ *and the profile curve is of* nodal *type* (*see Table* 6).
(C) $H = 1$. *The profile curve is*
  (C_1) *for* $k = 0$, *a* horizontal straight line;
  (C_2) *for* $k > 0$, *of* folium *type* (*see Table* 6).
(D) $0 < H < 1$. *The profile curve is*
  (D_1) *for* $k = 0$, *of the type showed in Table* 6;
  (D_2) *for* $k > 0$, *of the type showed in Table* 6;
  (D_3) *for* $k < 0$, *of* semi-circle *type* (*see Table* 6).

Proof Observe that

$$\frac{\mathrm{d}u}{\mathrm{d}v} = \frac{(H - kv)\sqrt{a^2 + b^2 v^2}}{v\sqrt{1 - (H - kv)^2}}. \tag{4}$$

We can assume, without loss of generality, that $H \geq 0$ and we shall study the two cases, $H = 0$ and $H > 0$, separately.

1. *Minimal surfaces* ($H = 0$)

(A_1) If $k = 0$ we have that $u = c$, with $c \in \mathbb{R}$, and Σ is the plane $bx - az = c$.
(A_2) If $k \neq 0$ the integration of (4) leads to an elliptic integral of the second kind[1] and

$$u(v) = -|a|E\left(\arcsin(kv), -\frac{b^2}{a^2k^2}\right).$$

We can reflect the curve only once obtaining a global curve of *semi-circle* type.

2. Surfaces with $H > 0$

- If $k = 0$, then $\cos\sigma = H$ and σ is constant. In particular:

  (C_1) If $H = 1$, as $\mathrm{d}v/\mathrm{d}u = 0$, we have $v = c$, $c > 0$, and Σ is the plane $y = c$.
  (D_1) If $H < 1$, integrating (4), we have:

$$u(v) = \frac{H}{\sqrt{1 - H^2}}\left[\sqrt{a^2 + b^2v^2} - a\ln\left(\frac{2(a + \sqrt{a^2 + b^2v^2})}{a^2 v}\right)\right] + c, \quad v > 0,$$

[1] The elliptic integral of the second kind is defined by

$$E(\phi, m) = \int_0^{\phi} \sqrt{1 - m\sin^2\theta}\,\mathrm{d}\theta.$$

where $c \in \mathbb{R}$. Substituting the expressions of the invariant functions we find that

$$az = bx - \frac{H}{\sqrt{1-H^2}}\left[\sqrt{a^2+b^2y^2} - a\ln\left(\frac{2(a+\sqrt{a^2+b^2y^2})}{a^2y}\right)\right] + c, \quad y > 0.$$

This surface is an example of a CMC complete graph of type $z = f(x) + g(y)$.

- If $k > 0$, as $\lim_{v\to 0^+}\cos\sigma = H$, then, depending on the value of H, we have:
  - if $H > 1$, the curve γ does not approach the line $v = 0$;
  - if $H = 1$, the curve γ tends asymptotically to the line $v = 0$;
  - if $H < 1$, the curve γ tends to the line $v = 0$ with an angle $\sigma = \arccos(H)$.

We will study these three cases separately.
(B) *Surfaces with $H > 1$.* In this case $v_m \leq v \leq v_M$, where $v_m = (H-1)/k$ and $v_M = (H+1)/k$. Choosing initial conditions $v(0) = v_m$ and $u(0) = 0$, it results that $\sigma(0) = 0$ and $\dot\sigma(0) = H - 1 > 0$. Also $\cos\sigma(s_2) = -1$, where $v(s_2) = v_M$ for some $s_2 > 0$. This means that there exists a certain $s_1 \in (0, s_2)$ with $\sigma(s_1) = \pi/2$. Therefore, in $v(s_1) = H/k$ there is a local minimum for $u(v)$. According to Corollary 1, we can reflect the curve infinitely many times. The resulting curve is of *nodal* type.
(C_2) *Surfaces with $H = 1$.* Here $u(v)$ is defined for $0 < v \leq v_M$, where $v_M = 2/k$. If $v(s_1) = v_M$ we have $\cos\sigma(s_1) = -1$. Also, $\mathrm{d}u/\mathrm{d}v > 0$ if and only if $v < 1/k$ and γ tends asymptotically to the line $v = 0$. The profile curve is of *folium* type.
(D_2) *Surfaces with $H < 1$.* The curve $u(v)$ is defined for $0 < v \leq v_M$, where $v_M = (H+1)/k$. When $v(s_1) = v_M$ it results that $\cos\sigma(s_1) = -1$ and, also, in H/k there is a local maximum for $u(v)$. The curve tends to the line $v = 0$ with an angle $\sigma = \arccos(H)$ and can be reflected only one time across the line $u = u(s_1)$.

- (D_3) If $k < 0$, we have that $H < 1$ and $u(v)$ is defined for $0 < v \leq v_M$, where $v_M = (H-1)/k$. When v assumes the value $v_M = v(s_1)$, since $\cos\sigma(s_1) = 1$, the curve is parallel to the u direction. Also $\dot\sigma(s) = kv < 0$. Reflecting the curve only one time across the line $u = u(s_1)$, we obtain a curve of *semi-circle* type. □

Remark 4 Among the G_{24}-invariant surfaces, the G_2-invariant ones are very interesting because we can give the explicit parametrizations of their profile curves. In this way, we find new explicit examples of CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$. For the action of G_2, the metric on the orbital space $\mathscr{B} = \{(u, v) \in \mathbb{R}^2 \mid v > 0\}$ is given by $\tilde g = du^2 + dv^2/v^2$. In Table 2 we gather the parametrizations of the profile curves in terms of k and H. For $k \neq 0$ we only give the case when $k > 0$. In fact, for $k < 0$ (when the surface exists) we obtain the same parametrization to the case $k > 0$.

5 G_{34}-invariant (hyperbolic) surfaces with constant mean curvature

Let G_{34} be the isometry subgroup generated by the Killing vector field $X_{34} = aX_3 + bX_4$, with $a, b \in \mathbb{R}$. Observe that, since the action of this group on $\mathbb{H}^2 \times \mathbb{R}$ is free, then $(\mathbb{H}^2 \times \mathbb{R})_r = \mathbb{H}^2 \times \mathbb{R}$. Introducing cylindrical coordinates (r, θ, z), with $r > 0$ and $\theta \in (0, \pi)$, we have:

$$g = \frac{\mathrm{d}r^2}{r^2\sin^2\theta} + \frac{\mathrm{d}\theta^2}{\sin^2\theta} + \mathrm{d}z^2 \quad \text{and} \quad X_{34} = ar\frac{\partial}{\partial r} + b\frac{\partial}{\partial z}.$$

Table 2 Profile curves of G_2-invariant surfaces

	$k = 0$	$k > 0$
$H = 0$	$u(v) = c \in \mathbb{R}$	$u(v) = -\arcsin(kv), \quad 0 < v \le 1/\lvert k\rvert$
$H = 1$	$v(u) = c > 0$	$u(v) = \arcsin(1 - kv) - \sqrt{\frac{2-kv}{kv}}, \quad 0 < v \le \frac{2}{k}$
$H > 1$		$u(v) = \arcsin(H - kv) - \frac{H}{\sqrt{H^2-1}} \tan^{-1}\left[\frac{H^2 - 1 - Hkv}{\sqrt{(H^2-1)(1-(kv-H)^2)}}\right], \quad \frac{H-1}{k} \le v \le \frac{H+1}{k}$
$H < 1$	$u(v) = \frac{H \ln v}{\sqrt{1-H^2}}$	$u(v) = \arcsin(H - kv) - \frac{H}{\sqrt{1-H^2}} \ln\left[\frac{2(1-H^2+Hkv+\sqrt{(1-H^2)(1-(kv-H)^2)})}{H\sqrt{1-H^2}v}\right], \quad 0 < v \le \frac{H+1}{k}$

Taking as invariant functions $u(r, \theta, z) = \theta$ and $v(r, \theta, z) = az - b \ln r$, the orbit space is $\mathscr{B} = \{(u, v) \in \mathbb{R}^2 \mid u \in (0, \pi)\}$ and the orbital metric becomes

$$\tilde{g} = \frac{du^2}{\sin^2 u} + \frac{dv^2}{a^2 + b^2 \sin^2 u}.$$

Let $\gamma(s) = (u(s), v(s))$ be a profile curve, parametrized by arc-length, of a G_{34}-invariant surface Σ. According to Theorem 1, the mean curvature of Σ along a principal orbit is given by $H = \dot{\sigma} - \cos u \sin \sigma$, where σ is the angle that the curve makes with the u-axis. Therefore, γ is a solution of the following system:

$$\begin{cases} \dot{u} = \sin u \cos \sigma \\ \dot{v} = \sqrt{(a^2 + b^2 \sin^2 u)}\ \sin \sigma \\ \dot{\sigma} = H + \cos u \sin \sigma. \end{cases} \tag{5}$$

As in the previous case, solutions of (5) possess some features as shown in

Proposition 5 *Solutions of* (5) *are invariant under*:

1. *translations in the* v *direction*;
2. *reflections across the line* $v = v_0$.

Corollary 2 *Let* $\gamma(s) = (u(s), v(s))$ *be a solution of* (5) *defined on* $(s_0 - \epsilon, s_0]$ *with* $\sigma(s_0) = \pm\pi/2$. *Then* $\gamma(s)$ *may be extended to the interval* $(s_0 - \epsilon, s_0 + \epsilon)$ *by a reflection across the line* $v = v(s_0)$.

Proposition 6 (The first integral) *If* H *is constant on* Σ, *then the function*

$$J(s) = \frac{H \cos u + \sin \sigma}{\sin u}$$

is constant along a solution of (5). *Therefore, the solutions of* (5) *are characterized by the equation*

$$\frac{H \cos u + \sin \sigma}{\sin u} = k, \qquad k \in \mathbb{R}. \tag{6}$$

We observe that, using (6), the third equation of (5) can be rewritten as

$$\dot{\sigma} = \tan u\,(k - \sin u\,\sin\sigma). \tag{7}$$

Remark 5 For a fixed $k \in \mathbb{R}$, we indicate by $\gamma_k(s) = (u(s), v(s), \sigma(s))$, $s \in (s_1, s_2)$, the profile curve corresponding to k and $H > 0$. Then the profile curve γ_{-k} is the reflected of γ_k across the line $u = \pi/2$, runned in the opposite direction. In fact

$$\gamma_{-k}(s) = (\tilde{u}(s), \tilde{v}(s), \tilde{\sigma}(s)) = (\pi - u(s_1 + s_2 - s), v(s_1 + s_2 - s), 2\pi - \sigma(s_1 + s_2 - s)),$$

with $s \in (s_1, s_2)$. Therefore,

$$\begin{cases} \dot{\tilde{u}} = \sin\tilde{u}\cos\tilde{\sigma} \\ \dot{\tilde{v}} = \sqrt{(a^2 + b^2\,\sin^2\tilde{u})}\,\sin\tilde{\sigma} \\ \dot{\tilde{\sigma}} = -\tan\tilde{u}\,(k + \sin\tilde{u}\,\sin\tilde{\sigma}), \end{cases} \tag{8}$$

which implies that γ_{-k} is a solution of (5) for $-k$ (see (7)). Consequently, the profile curves of the G_{34}-invariant surfaces with mean curvature $H > 0$, for $k = 0$, are symmetric with respect to the line $u = \pi/2$.

The qualitative study of (5) yields the following:

Theorem 3 *Let $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ be a G_{34}-invariant CMC surface and let $\gamma = \Sigma/G_{34}$ be its profile curve in the orbit space. Then we have the following characterization of γ according to the value of the mean curvature H and of k.*

(A) $(\mathbf{H} = 0)$ - *The profile curve is*

- (A_1) *for $k = 0$, a* horizontal straight line *and generates the funnel surface*;
- (A_2) *for $|k| > 1$, plotted in Table 7*;
- (A_3) *for $|k| = 1$, of* hyperbole *type* (*Table 7*);
- (A_4) *for $0 < |k| < 1$, plotted in Table 7.*

(B) $(\mathbf{H} > \mathbf{1})$ - *The profile curve is*

- (B_1) *for $k = 0$, of* ellipse *type*;
- (B_2) *for $k \neq 0$, of* nodal *type.*

(C) $(\mathbf{H} = \mathbf{1})$ - *The profile curve is*

- (C_1) *for $k = 0$, of* parabola *type* (*see Table 8*);
- (C_2) *for $k \neq 0$, of* folium *type* (*see Table 8*).

(D) $(\mathbf{0} < \mathbf{H} < \mathbf{1})$ - *The profile curve is*

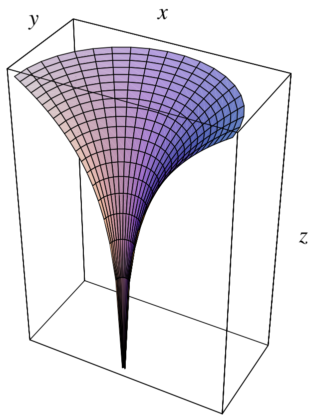
- (D_1) *for $k = 0$, plotted in Table 7*;
- (D_2) *for $|k| > \sqrt{1 - H^2}$, represented in Table 7*;
- (D_3) *for $|k| = \sqrt{1 - H^2}$, represented in Table 7*;
- (D_4) *for $0 < |k| < \sqrt{1 - H^2}$, represented in Table 7.*

Proof From (5), we have that:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{(k\sin u - H\cos u)\sqrt{a^2 + b^2\sin^2 u}}{\sin u\sqrt{1 - (k\sin u - H\cos u)^2}}. \tag{9}$$

As in the previous case, we can assume, without loss of generality, that $H \geq 0$ and we shall study the two cases, $H = 0$ and $H > 0$, separately.

Fig. 2 Funnel surface



(1) *Minimal surfaces* ($H = 0$)

(A_1) If $k = 0$, then $v = c \in \mathbb{R}$ and Σ is given by $az = b \ln r + c$ (see Fig. 2). This surface, called the *funnel surface*, is a complete graph ruled by its level curves.

If $k \neq 0$, (9) yields an elliptic integral of the first kind.[2] When $a = 1$ and $b = 0$ we have that $v(u) = k\, F(u, k^2)$. Note that it is enough to study the case $k > 0$. In fact, the profile curves for $k < 0$ can be obtained reflecting across the u-axis the solutions relative to the case $k > 0$. Moreover, we have:

$$\dot{\sigma}(s) = \sin\sigma \ \cos u = k \sin u \ \cos u, \tag{10}$$

and $\lim_{u\to 0^+} \sin\sigma = 0 = \lim_{u\to\pi^-} \sin\sigma$.

(A_2) If $k > 1$, then $u \in (0, u_0] \cup [\pi - u_0, \pi)$, where $u_0 = \arcsin(1/k)$. If $u = u_0$ or $u = \pi - u_0$, it results that $\sin\sigma = 1$. From (10) we have that the function $\sigma(s)$ is always increasing in $(0, u_0]$ and it is always decreasing in $[\pi - u_0, \pi)$.

(A_3) If $k = 1$, then $u \neq \pi/2$ and

$$v(u) = \frac{\cos u}{|\cos u|}\left[\sqrt{a^2+b^2}\tanh^{-1}\left[\frac{\sqrt{a^2+b^2}\sin u}{\sqrt{a^2+b^2\sin^2 u}}\right] - \frac{ba}{|a|}\sinh^{-1}\left(\frac{b\sin u}{a}\right)\right].$$

(A_4) If $0 < k < 1$, then $u \in (0, \pi)$. Also, from $\sin\sigma = k < 1$, it follows that in $u = \pi/2$ there is an inflection point of γ with oblique tangent.

(2) *Surfaces with* $H > 0$

From Remark 5 it is sufficient to study the profile curves for $k \geq 0$.

- For $k = 0$, as $\lim_{u\to 0^+} \sin\sigma = -H$ and $\lim_{u\to\pi^-} \sin\sigma = H$, we have that:

[2] The elliptic integral of the first kind is defined by:

$$F(\phi, m) = \int_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{1 - m^2 \sin^2\theta}}.$$

- if $H > 1$, the curve γ does not reach the lines $u = 0$ and $u = \pi$;
- if $H = 1$, the curve γ tends asymptotically to the lines $u = 0$ and $u = \pi$;
- if $H < 1$, the curve γ tends to the lines $u = 0$ and $u = \pi$ with an angle $\sigma = \arcsin(-H)$ and $\sigma = \arcsin(H)$, respectively.

Also $\dot{\sigma}(s) = H \sin^2 u > 0$, so that $\sigma(s)$ is always increasing.
(B_1) *Surfaces with* $H > 1$. In this case $u_m \leq u \leq u_M$, where $u_m = \arccos(1/H)$ and $u_M = \pi - u_m$. Choosing the initial conditions $u(0) = u_m$ and $v(0) = 0$, we have $\sigma(0) = 3\pi/2$ and $\dot{\sigma}(0) > 0$. Also, if $u_M = u(s_2)$ for some $s_2 > 0$, it results $\sigma(s_2) = \pi/2$ and there exists $s_1 \in (0, s_2)$ so that $\sigma(s_1) = 2\pi$. The curve is symmetric with respect to the line $u = \pi/2$ (see Remark 5) and can be reflected only once across the u-axis, obtaining a curve of *ellipse* type. Note that if $a = 1$ and $b = 0$, by integrating we find

$$v(u) = -\frac{H}{\sqrt{H^2-1}} \arctan\left[\sqrt{\frac{1 - H^2\cos^2 u}{H^2 - 1}}\right] + c, \qquad u \in [u_m, u_M],$$

and thus

$$z(r, \theta) = -\frac{H}{\sqrt{H^2-1}} \arctan\left[\sqrt{\frac{1 - H^2\cos^2 \theta}{H^2 - 1}}\right] + c,$$

with $\theta \in [\arccos(1/H), \arccos(-1/H)]$ and $c \in \mathbb{R}$.
(C_1) *Surfaces with* $H = 1$. In this case, integrating (9), we have that

$$v(u) = \frac{\sqrt{a^2 + b^2\sin^2 u}}{\sin u} - b\tanh^{-1}\left[\frac{b \sin u}{\sqrt{a^2 + b^2 \sin^2 u}}\right] + c, \qquad u \in (0, \pi),$$

where $c \in \mathbb{R}$. Therefore, the corresponding surface is given by:

$$a\, z(r, \theta) = b \ln r + \frac{\sqrt{a^2 + b^2\sin^2 \theta}}{\sin \theta} - b\tanh^{-1}\left[\frac{b \sin \theta}{\sqrt{a^2 + b^2 \sin^2 \theta}}\right] + c, \quad a \neq 0,$$

with $\theta \in (0, \pi)$. For $a = 1$ we have the family also obtained by R. Sá Earp ([12]).
(D_1) *Surfaces with* $H < 1$. In this case $0 < u < \pi$ and in $u = \pi/2$ there is a minimum of $v(u)$. For $a = 1$ and $b = 0$ we get:

$$v(u) = \frac{H}{\sqrt{1 - H^2}} \ln\left(2\frac{\sqrt{1-H^2} + \sqrt{1 - H^2\cos^2 u}}{\sin u}\right) + c, \quad c \in \mathbb{R}.$$

The corresponding G_3-invariant complete surface is given by:

$$z(r, \theta) = \frac{H}{\sqrt{1 - H^2}} \ln\left(2\frac{\sqrt{1-H^2} + \sqrt{1 - H^2\cos^2 \theta}}{\sin \theta}\right) + c, \quad \theta \in (0, \pi).$$

- For $k > 0$, we have $\lim_{u \to 0^+} \sin\sigma = -H$ and $\lim_{u \to \pi^-} \sin\sigma = H$, hence:
  - if $H > 1$, the curve γ does not reach the lines $u = 0$ and $u = \pi$;
  - if $H = 1$, the curve γ tends asymptotically to the line $u = 0$ or $u = \pi$;
  - if $H < 1$, the curve γ tends to the lines $u = 0$ and $u = \pi$ with an angle $\sigma = \arcsin(-H)$ and $\sigma = \arcsin(H)$, respectively.

Table 3 Limiting values for u in case (B_2)

	$k \geq 1$	$0 < k < 1$
u_m	$\arcsin\left(\frac{-k+H\sqrt{H^2+k^2-1}}{H^2+k^2}\right)$	$\arcsin\left(\frac{-k+H\sqrt{H^2+k^2-1}}{H^2+k^2}\right)$
u_M	$\arcsin\left(\frac{k+H\sqrt{H^2+k^2-1}}{H^2+k^2}\right)$	$\pi - \arcsin\left(\frac{k+H\sqrt{H^2+k^2-1}}{H^2+k^2}\right)$

Table 4 Limiting values for u in case (D_2)

	$k \geq 1$	$\sqrt{1-H^2} < k < 1$
u_m	$\arcsin\left(\frac{k+H\sqrt{H^2+k^2-1}}{H^2+k^2}\right)$	$\pi - \arcsin\left(\frac{k+H\sqrt{H^2+k^2-1}}{H^2+k^2}\right)$
u_M	$\pi - \arcsin\left(\frac{k-H\sqrt{H^2+k^2-1}}{H^2+k^2}\right)$	$\pi - \arcsin\left(\frac{k-H\sqrt{H^2+k^2-1}}{H^2+k^2}\right)$

Moreover, if $\cos u \neq 0$, then $\dot{\sigma}(s) = \sin u\,(H \sin u + k \cos u)$. Also, if $\cos u = 0$, then $\dot{\sigma} = H > 0$. Therefore, $\dot{\sigma} = 0$ implies that $\cos u \neq 0$. Consequently

$$\dot{\sigma}(s) = 0 \Leftrightarrow u = \widetilde{u} = \arctan\left(\frac{-k}{H}\right) = \pi - \arcsin\left(\frac{k}{\sqrt{H^2+k^2}}\right), \tag{11}$$

and $\dot{\sigma}(s) > 0$ if and only if $u \in (0, \widetilde{u})$

(B_2) *Surfaces with* $H > 1$. In this case $u_m \leq u \leq u_M$ (see Table 3). As $\widetilde{u} > u_M$ the function $\sigma(s)$ is always increasing and in $\arcsin(H/\sqrt{H^2+k^2})$ there is a local minimum for $v(u)$. We can reflect the curve infinitely many times obtaining a curve of *nodal* type.

(C_2) *Surfaces with* $H = 1$. Now $0 < u \leq u_M$, where if $k \geq 1$, it results that $u_M = \arcsin(2k/(1+k^2))$, while if $k < 1$, then $u_M = \pi - \arcsin(2k/(1+k^2))$. The v-axis is a vertical asymptotic line for the curve $v(u)$ and when $u = u_M$ the curve is parallel to the v direction. Also, as $\tilde{u} > u_M$ then $\sigma(s)$ is always increasing. For $u = \arctan(1/k)$ the curve $v(u)$ has a local minimum. We can reflect the profile curve only one time obtaining the curve of *folium* type.

Surfaces with $H < 1$

For this case there exist three different subcases depending on k.

(D_2) If $k > \sqrt{1-H^2}$, it results that $u \in (0, u_m] \cup [u_M, \pi)$ (see Table 4). When $u = u_m$ or $u = u_M$, we have $\sigma = \pi/2$. Since $u_m < \tilde{u} < u_M$, in $(0, u_m]$ the function $\sigma(s)$ is always increasing and in $[u_M, \pi)$ is always decreasing. Moreover in $\arcsin(H/\sqrt{H^2+k^2})$ there is a local minimum for $v(u)$.

(D_3) If $k = \sqrt{1-H^2}$, we obtain that

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{(k - H\cot u)\sqrt{a^2 + b^2\sin^2 u}}{|k\cos u + H \sin u|}, \qquad u \neq \tilde{u} = \arcsin(k).$$

It's easy to check that $\sigma(s)$ is increasing in $(0, \tilde{u})$ and decreasing in $(\tilde{u}, \pi)$. Also, $u = \tilde{u}$ is a vertical asymptotic line of the curve $v(u)$ and in $\arcsin(H) \in (0, \tilde{u})$ there is a local minimum for $v(u)$.

(D_4) If $0 < k < \sqrt{1-H^2}$, we have $u \in (0, \pi)$. From (11) it results that $\sigma(s)$ is increasing if and only if $u \in (0, \tilde{u})$. Moreover in $\tilde{u}$ the curve $v(u)$ has an inflection point with oblique tangent and in the point $\arctan(H/k) \in (0, \tilde{u})$ it has a local minimum. ◻

6 Tables of the plots of the profile curves

In this section we gather the plots of the profile curves for all the invariant surfaces in $\mathbb{H}^2 \times \mathbb{R}$. We have divided the pictures in tables according to the one-parameter subgroups and, for completeness, we have added the profile curves of the translational and helicoidal surfaces studied in [5,6,11]. The profile curves of the helicoidal surfaces with $H < 1$ are similar to those with $H = 1$. When in a table a figure is missing means that the invariant surface does not exist in the corresponding case (Tables 5, 6, 7, 8, 9).

Table 5 Profile curves of G_4-invariant (translational) surfaces

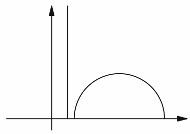
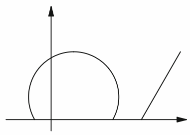
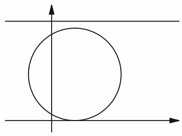
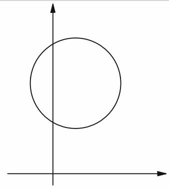
$H = 0$	$H < 1$	$H = 1$	$H > 1$
			

Table 6 Profile curves of G_{24}-invariant (parabolic) surfaces

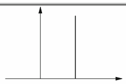
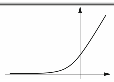
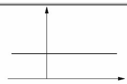
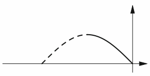
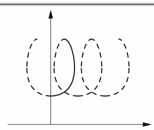
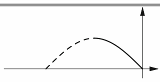
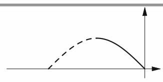
	$H = 0$	$H < 1$	$H = 1$	$H > 1$
$k = 0$				
$k > 0$				
$k < 0$				

Table 7 Profile curves of G_{34}-invariant (hyperbolic) surfaces ($H < 1$)

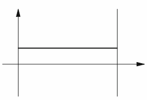
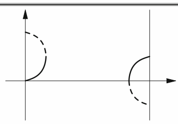
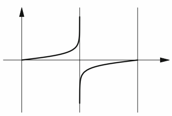
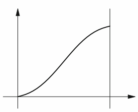
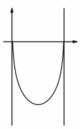
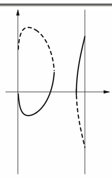
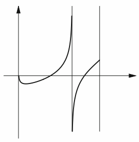
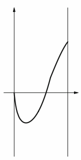
	$k = 0$	$k > \sqrt{1 - H^2}$	$k = \sqrt{1 - H^2}$	$k < \sqrt{1 - H^2}$
$H = 0$				
$H < 1$				

Table 8 Profile curves of G_{34}-invariant (hyperbolic) surfaces ($H \geq 1$)

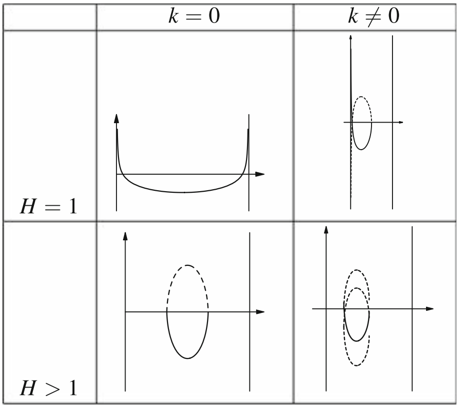
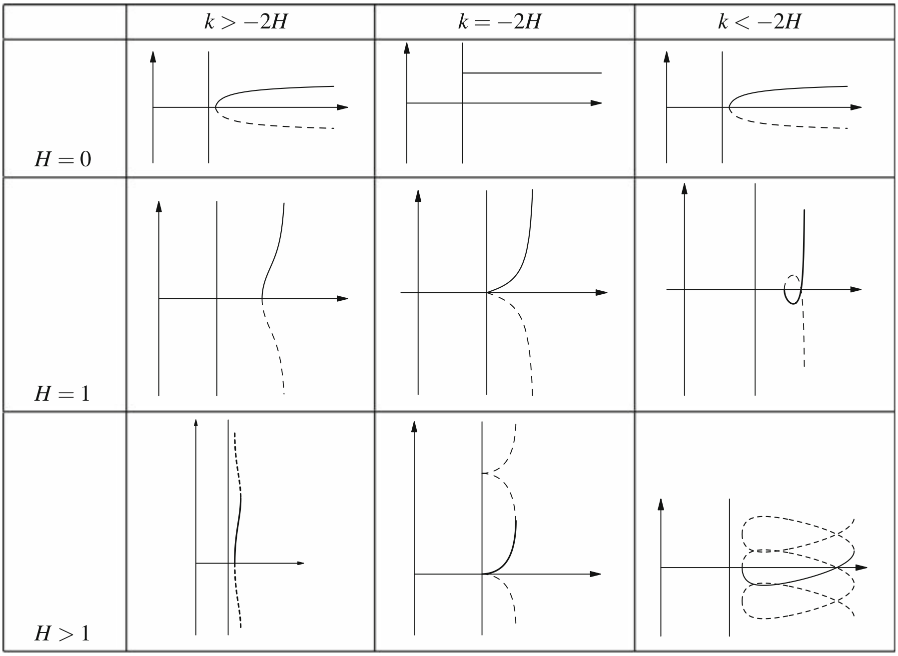


Table 9 Profile curves of G^*_{124}-invariant (helicoidal) surfaces



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