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Generalized inflection points of very general line bundles on smooth curves

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Abstract Let L be an invertible sheaf on a smooth curve C. A generalized inflection point of L is an inflection point of $L^{\otimes n}$ for some integer $n > 0$. A generalized inflection point P of L is called strongly normal if there is a unique integer $n > 0$ such that P is an inflection point of $L^{\otimes n}$ and moreover its inflection weight is equal to 1. In case L is a very general invertible sheaf of degree x on C then all generalized inflection points of L are strongly normal.

Keywords Inflection point · Linear system · Divisor · Curve · Invertible sheaf

Mathematics Subject Classification (2000) 14H55

1 Introduction

Let C be a smooth irreducible projective curve of genus g defined over the field $\mathbb{C}$ of the complex numbers. A divisor D on C is a formal linear combination $\sum_i n_i P_i$ with n_i integers and P_i points on C (assume $P_i \neq P_j$ for $i \neq j$). We call n_i the *multiplicity* of D at P_i denoted by $\text{mult}_{P_i}(D)$. Let g^r_d be a linear system on C and let P be a point on C. The *ramification sequence* of g^r_d at P is the sequence of integers $0 \leq \alpha_0(P) \leq \cdots \leq \alpha_r(P) \leq d - r$ such that (see [1, p. 37])

$$\{\alpha_i(P) + i : 0 \leq i \leq r\} = \{\text{mult}_P(D) : D \in g^r_d\}.$$

We call $\text{w}(P) = \sum_{i=0}^r \alpha_i(P)$ the *inflection weight* of g^r_d at P. The point P is called an *inflection point* for g^r_d if and only if $\text{w}(P) > 0$. The linear system g^r_d has finitely many inflection points. An inflection point P of g^r_d is called a *normal inflection point* if and only if $\text{w}(P) = 1$.

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Let L be an invertible sheaf on C. We write $|L|$ to denote the complete linear system associated to L [in case $h^0(C, L) = 0$ then $|L| = \emptyset$]. We say P is a (normal) inflection point of L if P is a (normal) inflection point of $|L|$.

Definition Let L be an invertible sheaf on C and let P be a point on C. We say that P is a *generalized inflection point* of L if and only if there exists an integer $n > 0$ such that P is an inflection point of $L^{\otimes n}$.

Let P be a generalized inflection point of L. We say that P is a *normal generalized inflection point* of L if and only if $\mathrm{w}(P) \leq 1$ for $L^{\otimes n}$ for all integers $n > 0$.

Let P be a normal generalized inflection point of L. We say that P is a *strongly normal generalized inflection point* of L if and only if there is a unique integer $n > 1$ such that P is an inflection point of $L^{\otimes n}$.

In this paper we prove the following theorem.

Theorem *Let $x > 0$ be an integer. If L is a very general invertible sheaf of degree x on C then all generalized inflection points of L are strongly normal.*

The meaning of very general is as follows. Let $\mathrm{Pic}^x(C)$ be the irreducible g-dimensional projective variety parameterizing the isomorphism classes of invertible sheaves of degree x on C. Then there exists a countable union Y of properly [i.e., different from $\mathrm{Pic}^x(C)$] closed subsets of $\mathrm{Pic}^x(C)$ such that the statement in the theorem holds for all invertible sheaves of degree x not corresponding to a point of Y. In particular the statement of the theorem holds for the generic invertible sheaf of degree x on C.

2 Some related results and questions

In [6] one investigates so-called *multiple Weierstrass points*. Those are generalized inflection points P of some invertible sheaf L such that P is an inflection point of $L^{\otimes n}$ for infinitely many integers $n > 0$. This corresponds to very special situations, it is the opposite of the definition of strongly normal generalized inflection points.

In case $L = K$, the canonical line bundle on C, then a (generalized) inflection point of K is called a *(generalized) Weierstrass point* on C. In [5, Theorem 3] it is proved that for the generic smooth curve C of genus $g \geq 3$ all generalized Weierstrass points are normal. (In this statement the condition $g \geq 3$ is necessary.)

Question 1 Are the generalized Weierstrass points on a generic curve C of genus $g \geq 3$ strongly normal?
See also the remark at the end of the paper.

A very interesting special class of smooth projective curves are the smooth plane curves $C \subset \mathbb{P}^2$ (say of degree d, hence genus $g = (d-1)(d-2)/2$). Let L be the invertible sheaf associated to the embedding $C \subset \mathbb{P}^2$. In this case $K = L^{\otimes(d-3)}$

Question 2 Are the generalized inflection points of the generic smooth plane curve C of degree d (strongly) normal?

Question 2 contains the question whether the Weierstrass points on a general smooth plane curve C are normal.

Question 1 is a question on the linear system g^{g-1}_{2g-2} on the generic curve C of genus g. The following question is a generalization from that point of view.

Question 3 Let $\rho_d^r(g) = g - (r+1)(g-d+r) \geq 0$ and assume L corresponds to a very general g_d^r on the generic curve C of genus g. Are the generalized inflection points of L (strongly) normal?

Let L be the invertible sheaf corresponding to a very general g_k^1 on a generic k-gonal curve C of genus g. In a forthcoming paper we investigate the generalized inflection points of L. In particular we prove that they are normal in case they are not inflection points of g_k^1 itself.

Notations

Let L be an invertible sheaf of degree d on C and let $a > 0$ be an integer. We write $|aL|$ to denote the complete linear system associated to $L^{\otimes a}$. We write $[L]$ to denote the corresponding point of $\mathrm{Pic}^d(C)$. If D is a divisor on C of degree d then we also write $[D]$ instead of $[\mathcal{O}_C(D)]$. In case a point of $\mathrm{Pic}^d(C)$ is denoted by $[L]$ (resp. $[D]$) then L (resp. D) is a suited line bundle (resp. divisor). As usual we write W_d^r to denote the set of points $[L] \in \mathrm{Pic}^d(C)$ satisfying $h^0(C, L) \geq r+1$. In case $X_i \subset \mathrm{Pic}^{d_i}(C)$ and $n_i > 0$ is an integer for $i = 1, 2$ then we write $n_1X_1 + n_2X_2$ to denote the image of $X_1 \times X_2$ under the map $\mathrm{Pic}^{d_1}(C) \times \mathrm{Pic}^{d_2}(C) \to \mathrm{Pic}^d(C)$ (with $d = n_1d_1 + n_2d_2$) mapping $([L_1], [L_2])$ to $[L_1^{\otimes n_1} \otimes L_2^{\otimes n_2}]$.

3 Proof of the theorem

Since we are going to prove a statement concerning a very general element of $\mathrm{Pic}^x(C)$ we can assume that for each integer $a \geq 1$ the invertible sheaf $L^{\otimes a}$ is non-special, hence $|aL|$ is empty if $ax < g$ and $\dim(|aL|) = ax - g$ if $ax \geq g$.

In case a point P is an inflection point of $L^{\otimes a}$ then $ax \geq g$ and $|L \otimes \mathcal{O}_C(-(ax - g + 1)P)| \neq \emptyset$. Hence a point P on C is an inflection point for $|aL|$ if and only if $ax \geq g$ and there exists an effective divisor D of degree $g-1$ such that

$$D + (ax - g + 1)P \in |aL|.$$

In case P is an inflection point of $L^{\otimes a}$ that is not normal then either $\alpha_{ax-g}(P) \geq 2$ or $\alpha_{ax-g-1}(P) \geq 1$ (ramification sequence of $L^{\otimes a}$ at P). In the first case there exists an effective divisor D of degree $g-2$ such that

$$D + (ax - g + 2)P \in |aL|.$$

In the second case $\dim(|L \otimes \mathcal{O}_C(-(ax-g)P)|) \geq 1$ hence

$$[L \otimes \mathcal{O}_C(-(ax-g)P)] \in W_g^1.$$

So, there exists a special effective divisor D of degree g (hence $\dim(|D|) \geq 1$) such that

$$D + (ax - g)P \in |aL|.$$

Consider the multiplication morphism

$$m_a : \mathrm{Pic}^x(C) \to \mathrm{Pic}^{ax}(C) : M \to M^{\otimes a}.$$

It is an unramified finite covering. The two situations implying that $L^{\otimes a}$ has a non-normal generalized inflection point can be written as

$$m_a(L) \in W_{g-2}^0 + (ax - g + 2)W_1^0$$
$$m_a(L) \in W_g^1 + (ax - g)W_1^0.$$

Since $W^0_{g-2} + (ax - g + 2)W^0_1$ and $W^1_g + (ax - g)W^0_1$ are $(g-1)$-dimensional properly closed subsets of $\mathrm{Pic}^{ax}(C)$, their inverse images under m_a are $(g-1)$-dimensional properly closed subsets of $\mathrm{Pic}^x(C)$. Let Y be the union of those closed subsets for all integers $a \geq 1$. For a very general $L \in \mathrm{Pic}^x(C)$ we can assume $L \notin Y$ hence all generalized inflection points of L are normal.

A normal generalized inflection point P of L is not strongly normal if and only if there exist integers $a, b \geq 1$ and divisors D, D' of degree $g-1$ such that the following two conditions do hold

$$D + (ax - g + 1)P \in |aL| \tag{1}$$
$$D' + ((a+b)x - g + 1)P = |(a+b)L|. \tag{2}$$

This is equivalent to the following two conditions

$$D + (ax - g + 1)P \in |aL| \tag{3}$$
$$|D + bL| = |D' + bxP| \tag{4}$$

Fix integer $a, b \geq 1$. Let V be the subspace of $W^0_{g-1} \times C \times \mathrm{Pic}^x(C)$ of triples $([D], P, L)$ such that (3) holds. The projection $V \to W^0_{g-1} \times C$ is an unramified finite covering, hence all components of V do have dimension g and do dominate $W^0_{g-1} \times C$. Consider the morphisms $\mu : V \to \mathrm{Pic}^{bx+g-1}(C) \times C$ and $\nu : W^0_{g-1} \times C \to \mathrm{Pic}^{bx+g-1}(C) \times C$ defined by

$$\mu([D], P, L) = ([D] + b[L], P)$$
$$\nu([D'], P') = ([D' + bxP'], P').$$

The condition that L has a normal generalized inflection point that is not strongly normal is equivalent to the condition that there exist integers $a, b \geq 1$; an effective divisor D of degree $g-1$ and a point P on C such that $([D], P, L) \in V$ [i.e. (3)] and $\mu([D], P, L) \in \mathrm{im}(\nu)$ [i.e. (4)]. In order to finish the proof of the theorem it is enough to prove that, for all integers $a, b \geq 1$ there exists an effective divisor D of degree $g-1$ and a point P on C such that for each $L \in \mathrm{Pic}^x(C)$ satisfying $([D], P, L) \in V$ one has $\mu([D], P, L) \notin \mathrm{im}(\nu)$. Indeed from the description of V it would follow that $\mu^{-1}(\mathrm{im}(\nu))$ has dimension at most $g-1$ and its image in $\mathrm{Pic}^x(C)$ (under the projection) is a properly closed subset of $\mathrm{Pic}^x(C)$. Let Y be the union of all those properly closed subsets for all choices of $a, b \geq 1$. For a very general $L \in \mathrm{Pic}^x(C)$ we can also assume $L \notin Y$ hence all generalized inflection points of L are strongly normal.

So now assume fixed integers $a, b \geq 1$. In case $([D], P, L) \in V$ satisfying $\mu([D], P, L) \in \mathrm{im}(\nu)$ there exists an effective divisor D' of degree $g-1$ such that (3) and (4) do hold. From (4) we obtain

$$|aD + abL| = |aD' + abxP|.$$

Since $D + (ax - g + 1)P \in |aL|$ it implies

$$aD' \in |(a+b)D - b(g-1)P|. \tag{5}$$

Consider the morphism $\mu' : W^0_{g-1} \times C \to \mathrm{Pic}^{a(g-1)}(C)$ defined by

$$\mu'([D], P) = (a+b)[D] - b(g-1)[P]$$

then (5) is equivalent to $\mu'([D], P) \in aW^0_{g-1}$. Hence it is enough to prove that there exists an effective divisor D of degree $g-1$ and a point P such that $\mu'([D], P) \notin aW^0_{g-1}$.

Take $A \in \mathrm{Pic}^{a(g-1)}(C)$ satisfying $A \notin aW^0_{g-1}$ and consider the curve $A + b(g-1)W^0_1$ inside $\mathrm{Pic}^{(a+b)(g-1)}(C)$. Let $\Gamma \subset \mathrm{Pic}^{g-1}(C)$ be a component of the inverse image of $A + b(g-1)W^0_1$ under m_{a+b}. Inside $\mathrm{Pic}^{g-1}(C)$ the divisor W^0_{g-1} is ample (it defines the principal polarization on the Jacobian of C), hence $\Gamma \cap W^0_{g-1}$ is not empty.

Take $[D] \in \Gamma \cap W^0_{g-1}$. Because $(a+b)[D] \in A + b(g-1)W^0_1$ there exists $P \in C$ such that

$$(a+b)[D] - b(g-1)[P] = \mu'([D], P) = A.$$

Because of the choice of A one has $\mu'([D], P) \notin aW^0_{g-1}$. This implies there is no $L \in \mathrm{Pic}^x(C)$ such that $([D], L, P) \in V$ and $\mu([D], L, P) \in \mathrm{im}(\nu)$. This finishes the proof of the theorem.

Remark Some time after finishing the paper the preprint [3] appeared containing a proof of our result (see Theorem 0.10). The first part of that proof is exactly the same as ours, the second part uses the connectedness theorem from [4].
G. Farkas also communicated to me that a very general curve of genus $g \geq 4$ has only strongly normal generalized Weierstrass points (answering Question 1). In [2, Sect. 6] it is proved that the locus on the universal family of curves of genus g of inflection points of nK is irreducible for all $n \geq 1$. Since that locus is different for different values of n no intersection of two of them dominates the moduli space of curves of genus g.

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