Annali di Matematica (2008) 187:555–562
DOI 10.1007/s10231-007-0055-0

Analytic non-integrable Hamiltonian systems and irregular singularity

Masafumi Yoshino

Received: 29 August 2006 / Revised: 9 April 2007 / Published online: 24 October 2007

Abstract We study a C^∞-Liouville-integrable and analytic non-integrable Hamiltonian system. We will show that an irregular singular character plays a crucial role in the analytic non-integrability of the system.

Keywords Liouville integrability · Analytic non-integrability · Irregular singularity

Mathematics Subject Classification (2000) 37J30 · 35C10 · 37F50

1 Introduction

A Hamiltonian system in n degrees of freedom is called C^∞-Liouville-integrable if there are n smooth first integrals in involution which are independent on an open dense set. If a first integral is analytic, then we say that it is analytic-integrable. There are many works which study the integrability and the normal form theory. (cf. [3], [4] and [5]). Recently, Gorni-Zampieri, [2] gave a simple and interesting example of a C^∞-Liouville-integrable Hamiltonian system which is not analytic-integrable in any neighborhood of an equilibrium point. This example shows that when one studies the non-integrability of a Hamiltonian system, it is necessary to show the non-integrability not only in an analytic class but also in a C^∞ class.

The object of this paper is to study the analytic non-integrability of a C^∞-Liouville-integrable Hamiltonian system from the viewpoint of the irregular singular character of a system. We will give a general class of C^∞-Liouville-integrable and analytic non-integrable Hamil-

Partially supported by Grant-in-Aid for Scientific Research (No. 11640183), Ministry of Education, Science and Culture, Japan.

M. Yoshino (✉)
Department of Mathematics, Graduate School of Science, Hiroshima University,
1-3-1 Kagamiyama, Higashi-Hiroshima, Hiroshima 739-8526, Japan
e-mail: yoshino@math.sci.hiroshima-u.ac.jp

tonian systems containing Gorni-Zampieri's example and a simple elementary proof of the non-integrability, which shows the role of an irregular singular behavior of a system.

2 Analytic non-integrability

Let $\sigma \geq 1$ be an integer. Let $r(q_1, q_2)$ be an analytic function of $(q_1, q_2) \in \mathbb{R}^2$ in some neighborhood of the origin $0 \in \mathbb{R}^2$ such that

$$r \equiv r(q_1, q_2) = cq_1^{2\sigma} + a(q_1^{2\sigma})q_2^2 + \tilde{r}(q_1, q_2)q_2^3, \quad c > 0, \tag{1}$$

where $\tilde{r}(q_1, q_2)$ is analytic at the origin and $a(t)(t = q_1^{2\sigma})$ is a polynomial of t such that $a(0) > 0$. We are interested in the following analytic Hamiltonian in $\mathbb{R}^4$ with two degrees of freedom

$$\mathcal{H} = -q_2 p_2 \partial_{q_1} r(q_1, q_2) + \left(r(q_1, q_2)^2 + q_2 \partial_{q_2} r(q_1, q_2)\right) p_1, \tag{2}$$

where $\partial_{q_1} = \frac{\partial}{\partial q_1}$ and $\partial_{q_2} = \frac{\partial}{\partial q_2}$. The associated Hamiltonian system is given by

$$\begin{cases} \dot{q}_1 = \partial \mathcal{H}/(\partial p_1) = r^2 + q_2 \partial_{q_2} r, \\ \dot{q}_2 = \partial \mathcal{H}/(\partial p_2) = -q_2 \partial_{q_1} r, \\ \dot{p}_1 = -\partial \mathcal{H}/(\partial q_1) = q_2 p_2 \partial_{q_1}^2 r - \left(2r \partial_{q_1} r + q_2 \partial_{q_1} \partial_{q_2} r\right) p_1, \\ \dot{p}_2 = -\partial \mathcal{H}/(\partial q_2) = p_2 \partial_{q_1} r + q_2 p_2 \partial_{q_1} \partial_{q_2} r - \left(2r \partial_{q_2} r + \partial_{q_2} r + q_2 \partial_{q_2}^2 r\right) p_1. \end{cases} \tag{3}$$

We need a definition in order to state our theorem.

Definition 1 We say that a polynomial $a(t)$ satisfies the monodromy condition (M) if the following equation has a polynomial solution $U(t)$

$$ct^2 U' - 2U + c((2\sigma)^{-1} - 3)tU = c(ct + 1)a(t). \tag{4}$$

Then we have

Theorem 1 *Suppose that (1) is satisfied. Assume that $a(t)$ does not satisfy (M). Then the Hamiltonian system (3) is C^∞-Liouville-integrable, while it is not analytic-integrable in any neighborhood of the origin. More precisely, for any analytic first integral $u = u(q_1, q_2, p_1, p_2)$ of (3) in $\mathbb{R}^4$, there exists a function ϕ of one-variable, being analytic at $0 \in \mathbb{R}$ such that $u = \phi \circ \mathcal{H}$.*

By Lemma 2 we have

Corollary 1 *Suppose that (1) is satisfied. Assume that $a(t) \equiv a_0 > 0$. Then the Hamiltonian system (3) is C^∞-Liouville-integrable, while it is not analytic-integrable in any neighborhood of the origin.*

Example 1 If we set $\sigma = 1, a \equiv 2, c = 2$ and $r = 2(q_1^2 + q_2^2)$, then we have the Hamiltonian $\mathcal{H} = 4(-q_1 q_2 p_2 + (q_1^2 + q_2^2)^2 p_1 + q_2^2 p_1)$ studied in [2] apart from the constant 4. Our proof shows that the analytic non-integrability is closely related with the irregular singular character of the Hamiltonian system corresponding to $\mathcal{H}$. We also remark that a similar divergence phenomenon due to the irregular singularity was also noted in [1].

Remark 1 We recall that (4) has an irregular singularity at $t = 0$. Such an equation has no analytic solution at the origin except for the pathological case where (M) is fulfilled. (cf. Lemmas 3 and 6).

As Gorni and Zampieri observed in [2], it is important that the system has the set $\{q_2 = 0\}$ as an invariant manifold on which every analytic first integral is functionally dependent on $\mathcal{H}$. The essential point of the proof of the analytic non-integrability of (3) lies in the unique continuation of the relation on the invariant manifold to its neighborhood. In [2], this was carried out by the power series method. We will show that the unique continuation is closely related with the monodromy structure of the hidden subsystem (4) of the corresponding system (15). Although the system for which (M) holds is not a generic one, it gives a new phenomenon.

3 Preliminary lemma

Lemma 1 *The polynomial $a(t)$ of degree m ($m \geq 0$ or $m = -\infty$) satisfies (M) if and only if (4) has a unique polynomial solution $U(t)$ of degree m.*

Here we use the convention that $a(t) \equiv 0$ is the polynomial of degree $-\infty$.

Proof The sufficiency is clear. We will prove the necessity. Suppose that $U(t)$ is a polynomial solution of degree k of (4). We insert the expansions

$$a(t)(ct+1) = \sum_{\nu} b_\nu t^\nu, \quad U(t) = \sum_{\nu} U_\nu t^\nu, \tag{5}$$

into (4) and compare the coefficients of t^ν. Then we have $-2U_0 = cb_0$ and

$$c(\nu - 1)U_{\nu-1} - 2U_\nu + c((2\sigma)^{-1} - 3)U_{\nu-1} = cb_\nu, \quad \nu \geq 1. \tag{6}$$

It follows that the U_ν's are uniquely determined. Hence $U(t)$ is unique, if it exists.

Next we will show that the degree of $U(t)$ is equal to m. Let k be the degree of $U(t)$. Suppose that $k > m$. If $m = -\infty$, then we have $a(t) \equiv 0$. By what we have proved in the above we have $U(t) \equiv 0$, i.e., $k = -\infty$, a contradiction to the condition, $k > m$. Hence we have $m \geq 0$. Then, by setting $\nu = k + 1$ in (6) and noting that $b_{k+1} = 0$, we have $kU_k + ((2\sigma)^{-1} - 3)U_k = 0$. Because $k + (2\sigma)^{-1} - 3 \neq 0$, we have $U_k = 0$. This contradicts to the assumption that U is a polynomial of degree k. Hence we have $k \leq m$.

Suppose that $k < m$. Then the left-hand side of (4) is a polynomial of degree at most m. Because the right-hand side of (4) is of degree $m + 1$, we have a contradiction. Hence we have $k = m$. □

Lemma 2 *Suppose that $a(t)$ is a constant, $a(t) \equiv a_0$. Then $a(t)$ satisfies (M) if and only if $a_0 = 0$.*

Proof Assume that $a_0 = 0$. Because $U = 0$ is a polynomial solution of (4), a satisfies (M). Conversely, suppose that a satisfies (M), and let $U(t)$ be a polynomial solution of (4). By the preceeding lemma we may assume that $U(t) \equiv \alpha$ for some constant α. By (4) we have $ca_0 = -2\alpha, ca_0 = (1/(2\sigma) - 3)\alpha$. It follows that $(1/(2\sigma) - 1)\alpha = 0$. Because $1/(2\sigma) - 1 \neq 0$ by the assumption $\sigma \in \mathbb{N}$, we obtain $\alpha = 0$, and hence $a_0 = 0$. □

Lemma 3 *The set of polynomials of degree $m (m \geq 0)$ satisfying (M) is contained in a manifold of codimension 1 in the set of polynomials of degree m.*

Proof If $m = 0$, then the assertion follows from the preceeding lemma. Hence we assume $m \geq 1$. We use the same notation as in Lemma 1. Let $a(t) = \sum_{\nu=0}^{m} a_\nu t^\nu$ satisfy (M). By the definition of b_ν we have $b_0 = a_0$ and $b_\nu = a_\nu + ca_{\nu-1}$ $(\nu \geq 1)$.

We put $\nu = m+1$ and $\nu = m$ in (6). Because $b_{m+1} = ca_m$ and $U_{m+1} = 0$, we have

$$2U_m = -c(a_m + ca_{m-1}) + c\left((2\sigma)^{-1} + m - 4\right) U_{m-1}, \tag{7}$$

$$\left((2\sigma)^{-1} + m - 3\right) U_m = ca_m. \tag{8}$$

We can easily see that U_{m-1} is a linear function of $a_0, \ldots, a_{m-1}$ by (6). If we eliminate U_m from (7) and (8), then we can easily see that the coefficient of a_m in the resultant relation does not vanish because $\sigma \in \mathbb{N}$. Hence we obtain a nontrivial linear relation among $a_0, \ldots, a_m$. □

Lemma 4 *Let $a(t)$ be a polynomial and assume that $c > 0$. Then (4) has an analytic solution $U(t)$ in some neighborhood of the origin if and only if $a(t)$ satisfies (M).*

Proof The sufficiency is trivial. In order to show the necessity, let $U(t)$ be an analytic solution of (4). If $a(t) \equiv 0$, then $a(t)$ satisfies (M) by Lemma 2. Hence we may assume $a(t) \not\equiv 0$. By expanding $U(t) = \sum U_\nu t^\nu$, we consider (6). If $a(t)$ is of degree m, then we have $b_\nu = 0$ for $\nu > m+1$. By (6) we obtain

$$2U_\nu = c\left((2\sigma)^{-1} + \nu - 4\right) U_{\nu-1}, \quad \nu > m+1. \tag{9}$$

If $U_{m+1} = 0$, then we have $U_\nu = 0$ $(\nu > m+1)$. Hence U is a polynomial, which implies that a satisfies (M). If $U_{m+1} \neq 0$, then we have

$$2U_{m+2} = c\left((2\sigma)^{-1} + m - 2\right) U_{m+1}. \tag{10}$$

Similarly, we have

$$U_{m+k} = (c/2)^{k-1}\left((2\sigma)^{-1} + m + k - 4\right) \cdots \left((2\sigma)^{-1} + m - 2\right) U_{m+1}. \tag{11}$$

Because the right-hand side grows like $k!$ as $k \to \infty$, $U(t)$ does not converge in any neighborhood of the origin, which contradicts to the analyticity of U. □

Lemma 5 *Let α be a constant. Then every solution $v = v(q_1, p_1)$ of the next equation*

$$q_1^{2\sigma+1}\frac{\partial v}{\partial q_1} - 4\sigma q_1^{2\sigma} p_1 \frac{\partial v}{\partial p_1} + \alpha v = 0, \tag{12}$$

which is analytic in some neighborhood of $q_1 = p_1 = 0$ vanishes if and only if $\alpha \neq 0$. If $\alpha = 0$, then v has the expression $v = \phi(p_1 q_1^{4\sigma})$ for some analytic function ϕ of one variable in some neighborhood of the origin.

Proof We assume $\alpha \neq 0$. Let $v = \sum_{k=0}^{\infty} v_k(q_1) p_1^k$ be the Taylor expansion of the solution v of (12). Then v_k satisfies the equation $q_1^{2\sigma+1} v_k' - 4\sigma q_1^{2\sigma} k v_k + \alpha v_k = 0$. If we substitute the expansion of v_k, $v_k = \sum_{j=0}^{\infty} v_{k,j} q_1^j$ into the equation, then we have $v_{k,j} = 0$ for $j = 0, 1, 2, \ldots$. Hence we have $v_k = 0$ $(k = 0, 1, 2, \ldots)$, and $v = 0$.

Next, assume that $\alpha = 0$. Then (12) can be written in $q_1(\partial v/\partial q_1) - 4\sigma p_1(\partial v/\partial p_1) = 0$. The analytic solution of the equation is given by $v = \phi(p_1 q_1^{4\sigma})$ for some analytic function ϕ of one variable. This especially implies that (12) has a nontrivial analytic solution v. □

Although we do not use the next lemma in the proof of the main theorem, we state it in order to make it clear that the condition (M) on $a(t)$ is related to the vanishing of a certain monodromy. First we prepare some notation. We set $\alpha = -2/c$, $\beta = 1/(2\sigma) - 3$ and $b(t) = a(t)(ct+1)$ in (4). Then the general solution of (4) is given by $U(t) = Bt^{-\beta}e^{\alpha/t} + U_0(t)$, where B is a constant and $U_0(t)$ is given by

$$U_0(t) := t^{-\beta}\mathrm{e}^{\alpha/t}\int_{\gamma_t} s^{\beta-2}\mathrm{e}^{-\alpha/s}b(s)\mathrm{d}s. \tag{13}$$

Here the path γ_t is the line segment from the origin to t when t lies in the domain $\Re t < 0$ on a fixed Riemann sheet. If t is outside the domain, then one first goes to some point t_0, $\Re t_0 < 0$ from the origin, then one goes from t_0 to t along some simple closed curve in some neighborhood of the origin which passes t_0 and t, that contains 0 inside. The integral (13) converges because $\alpha < 0$. Let $U_0(te^{2\pi i})$ be an analytic continuation of $U_0(t)$ along a simple closed curve which encircles the origin. Because $U_0(te^{2\pi i})$ is also a solution of (4), we have the expression

$$U_0(t\mathrm{e}^{2\pi i}) = U_0(t) + At^{-\beta}\mathrm{e}^{\alpha/t}, \tag{14}$$

for some constant A, where A is a monodromy constant. Then we have

Lemma 6 *The function $a(t)$ satisfies (M) if and only if $A = 0$.*

Proof First we assume that $U_0(t)$ is bounded when $t \to 0$ on the first sheet. Assume that $a(t)$ satisfy (M). By Lemma 4 (4) has a holomorphic solution $U(t)$. By the formula in the above we have $U(t) = U_0(t) + Bt^{-\beta}e^{\alpha/t}$. If we let $t \to 0$, $\Re t < 0$, then $e^{\alpha/t}$ tends to infinity because $\alpha < 0$. It follows that $B = 0$ and $U_0(t)$ is holomorphic and single-valued. Hence we have $A = 0$.

Conversely, if $A = 0$, then $U_0(t)$ is a solution of (4) which is single-valued, holomorphic and bounded outside the origin. By Riemann's theorem, $U_0(t)$ is holomorphic in some neighborhood of the origin. By Lemma 4, $a(t)$ satisfies (M).

Therefore it remains to prove the boundedness of $U_0(t)$. Let t be on the first sheet such that $\Re t \le 0$. Then the points 0, s, t lie on the same line γ_t in this order. It follows that $\Re(1/t - 1/s) \ge 0$. Hence we have $|e^{\alpha(1/t-1/s)}| \le 1$ because $\alpha < 0$. Because $\beta < 0$ we have $|t^{-\beta}|$ is bounded when $t \to 0$, $\Re t \le 0$. It follows that $U_0(t)$ is bounded when $t \to 0$, $\Re t \le 0$. We note that the term s^β in the integrand can be absorbed in $e^{\alpha/s}$ by partial integration. Next, let $\Re t > 0$ on the same sheet. By deforming γ_t, we may assume that the points 0, t, s lie on the straight line in this order near the origin. It follows that $\Re(1/t - 1/s) > 0$, from which we have the same assertion. □

4 Proof of Theorem 2.1

Proof We note that u is the first integral of the Hamiltonian system (3) if and only if u is a solution of the following first order equation

$$\begin{aligned}\{\mathcal{H}, u\} \equiv &\left(q_2p_2\partial_{q_1}^2 r - \left(2r\partial_{q_1}r + q_2\partial_{q_1}\partial_{q_2}r\right)p_1\right)\frac{\partial u}{\partial p_1}\\ &+\left(p_2\partial_{q_1}r + q_2p_2\partial_{q_1}\partial_{q_2}r - \left(2r\partial_{q_2}r + \partial_{q_2}r + q_2\partial_{q_2}^2 r\right)p_1\right)\frac{\partial u}{\partial p_2}\\ &+\left(r^2 + q_2\partial_{q_2}r\right)\frac{\partial u}{\partial q_1} - q_2(\partial_{q_1}r)\frac{\partial u}{\partial q_2} = 0.\end{aligned} \tag{15}$$

We set

$$u = \begin{cases} q_2 \exp\left(-\frac{1}{r}\right) & \text{if } (q_1, q_2) \neq (0, 0), \\ 0 & \text{if } (q_1, q_2) = (0, 0). \end{cases} \tag{16}$$

By the assumptions (1) we can easily see that u is C^∞ in some neighborhood of the origin. Moreover, we can easily verify, by simple computations, that u is a solution of (15). Hence u is a C^∞ first integral of (3). We can easily see that u and $\mathcal{H}$ are functionally independent on the open dense set in some neighborhood of the origin. Hence (3) is C^∞-Liouville-integrable in some neighborhood of the origin.

Next we will show that (3) is not analytic-integrable. Let $u = u(q_1, q_2, p_1, p_2)$ be any analytic first integral of (3). We define

$$v \equiv v(q_1, p_1, p_2) := u(q_1, 0, p_1, p_2). \tag{17}$$

By setting $q_2 = 0$ in (15) and noting that $\partial_{q_2} r(q_1, 0) \equiv 0$ and $r(q_1, 0) = cq_1^{2\sigma}$ by (1), we obtain

$$2\sigma p_2 \frac{\partial v}{\partial p_2} - 4c\sigma q_1^{2\sigma} p_1 \frac{\partial v}{\partial p_1} + cq_1^{2\sigma+1} \frac{\partial v}{\partial q_1} = 0. \tag{18}$$

We expand v into the power series of p_2, $v = \sum_{j=0}^\infty v_j(q_1, p_1)p_2^j$. Then we see that $v_j(q_1, p_1)(j = 0, 1, \ldots)$ satisfy (12) with $\alpha = 2\sigma j/c$. It follows from Lemma 5 that $v_j = 0$ if $j \neq 0$ and $v_0 = \tilde{\phi}(p_1 q_1^{4\sigma}) = \phi(c^2 p_1 q_1^{4\sigma})$ for some analytic function $\tilde{\phi}(t)$ and $\phi(t) := \tilde{\phi}(t/c^2)$. It follows from (2) that $v = v_0 = \phi(c^2 p_1 q_1^{4\sigma}) = \phi(\mathcal{H}|_{q_2=0})$. We define

$$g(q_1, q_2, p_1, p_2) := u(q_1, q_2, p_1, p_2) - \phi(\mathcal{H}). \tag{19}$$

By (17) and by recalling that $\mathcal{H}$ is a first integral we see that g is an analytic solution of (15) such that $g(q_1, 0, p_1, p_2) \equiv 0$. In order to prove Theorem 1 we shall show $g(q_1, q_2, p_1, p_2) \equiv 0$ in some neighborhood of the origin.

First we will show that

$$g(q_1, q_2, p_1, p_2) = \phi_1(p_1 q_1^{4\sigma}) p_2 q_2 + h_2(q_1, p_1, p_2) q_2^2 + \tilde{h}_3(q_1, q_2, p_1, p_2) q_2^3, \tag{20}$$

for some analytic function ϕ_1 of one variable and analytic functions h_2 and $\tilde{h}_3$. Because g is analytic we have the expansion

$$g(q_1, q_2, p_1, p_2) = g_1(q_1, p_1, p_2) q_2 + h_2(q_1, p_1, p_2) q_2^2 + \tilde{h}_3(q_1, q_2, p_1, p_2) q_2^3. \tag{21}$$

We substitute (21) with $u = g$ into (15) and compare the coefficients of q_2. By (1) we have

$$-4c\sigma q_1^{2\sigma} p_1 \frac{\partial g_1}{\partial p_1} + 2\sigma \left(p_2 \frac{\partial g_1}{\partial p_2} - g_1 \right) + cq_1^{2\sigma+1} \frac{\partial g_1}{\partial q_1} = 0. \tag{22}$$

By substituting the expansion $g_1(q_1, p_1, p_2) = \sum_{m=0}^\infty g_{1,m}(q_1, p_1) p_2^m$ into (22) and comparing the coefficients of p_2^m we obtain

$$-4c\sigma q_1^{2\sigma} p_1 \frac{\partial g_{1,m}}{\partial p_1} + 2\sigma(m-1) g_{1,m} + cq_1^{2\sigma+1} \frac{\partial g_{1,m}}{\partial q_1} = 0. \tag{23}$$

By Lemma 5 we have $g_{1,m} = 0$ if $m \neq 1$, and $g_{1,1}(q_1, p_1) = \phi_1(p_1 q_1^{4\sigma})$ for some analytic function ϕ_1 of one variable. Hence we have $g_1(q_1, p_1, p_2) = g_{1,1}(q_1, p_1) p_2 = \phi_1(p_1 q_1^{4\sigma}) p_2$, which proves (20).

Next we suppose that

$$g(q_1,q_2,p_1,p_2)=\phi_{n-1}(p_1q_1^{4\sigma})p_2^{n-1}q_2^{n-1} \\ +h_n(q_1,p_1,p_2)q_2^n+\tilde{h}_{n+1}(q_1,q_2,p_1,p_2)q_2^{n+1}, \tag{24}$$

for some $n\geq 2$, some analytic function ϕ_{n-1} of one variable and analytic functions $h_n(q_1, p_1, p_2)$ and $\tilde{h}_{n+1}(q_1,q_2,p_1,p_2)$. We substitute (24) into (15) with $u=g$ and we compare the coefficients of q_2^n. By (1) we have

$$2cp_2^n\sigma(2\sigma-1)q_1^{6\sigma-2}\phi'_{n-1} \\ -4\sigma c^2q_1^{4\sigma-1}p_1\frac{\partial h_n}{\partial p_1}-4a(q_1^{2\sigma})(cq_1^{2\sigma}+1)(n-1)p_1p_2^{n-2}\phi_{n-1} \\ +2\sigma cq_1^{2\sigma-1}\left(p_2\frac{\partial h_n}{\partial p_2}-nh_n\right)+c^2q_1^{4\sigma}\frac{\partial h_n}{\partial q_1}=0. \tag{25}$$

By substituting the expansion $h_n(q_1,p_1,p_2)=\sum_{m=0}^{\infty}h_{n,m}(q_1,p_1)p_2^m$ into (25) and comparing the coefficients of p_2^{n-2} we obtain

$$-4c^2\sigma q_1^{4\sigma-1}p_1\frac{\partial h_{n,n-2}}{\partial p_1}-4a(q_1^{2\sigma})(cq_1^{2\sigma}+1)(n-1)p_1\phi_{n-1} \\ -4c\sigma q_1^{2\sigma-1}h_{n,n-2}+c^2q_1^{4\sigma}\frac{\partial h_{n,n-2}}{\partial q_1}=0. \tag{26}$$

We will show that

$$h_{n,n-2}=0,\quad \phi_{n-1}=0. \tag{27}$$

If we can prove $\phi_{n-1}=0$, then it follows from (26) that $v:=h_{n,n-2}$ satisfies (12) with $\alpha=-4\sigma/c$. Hence, by Lemma 5 we have $h_{n,n-2}=0$. In order to show $\phi_{n-1}=0$ we insert the expansions

$$\phi_{n-1}(p_1q_1^{4\sigma})=\sum_{k=0}^{\infty}\phi_{n-1,k}\,p_1^kq_1^{4\sigma k},\quad h_{n,n-2}(q_1,p_1)=\sum_{k=0}^{\infty}h_{n,n-2,k}(q_1)p_1^k \tag{28}$$

into (26) and compare the coefficients of p_1^k. Then we obtain, for $k\geq 0$

$$-4c^2\sigma q_1^{4\sigma-1}kh_{n,n-2,k}-4c\sigma q_1^{2\sigma-1}h_{n,n-2,k}+c^2q_1^{4\sigma}\frac{\partial h_{n,n-2,k}}{\partial q_1} \\ =4a(q_1^{2\sigma})(cq_1^{2\sigma}+1)(n-1)\phi_{n-1,k-1}q_1^{4\sigma(k-1)}, \tag{29}$$

where we set $\phi_{n-1,-1}=0$. If we set $q_1=0$ and $k=1$ in (29), then we obtain $0=4a(0)(n-1)\phi_{n-1,0}$. Because $a(0)\neq 0$ by the assumption, we have $\phi_{n-1,0}=0$.

Suppose that $\phi_{n-1,k-1}\neq 0$ for some $k\geq 2$. We divide both sides of (29) by $q_1^{2\sigma-1}$. Then the right-hand side of (29) is divisible by q_1^N, $N=4\sigma(k-1)+1-2\sigma\geq 2\sigma+1$. Because the operator $-4c^2\sigma kq_1^{2\sigma}+c^2q_1^{2\sigma+1}(d/dq_1)$ in the left-hand side of the equation increases the power of q_1, it follows that $h_{n,n-2,k}$ is divisible by q_1^N. We set $h_{n,n-2,k}(q_1)=q_1^NW(q_1)$. Then we have $q_1(d/dq_1)h_{n,n-2,k}=q_1^N(N+q_1(d/dq_1))W$. It follows from (29) that W satisfies

$$(N-4\sigma k)c^2q_1^{2\sigma}W-4c\sigma W+c^2q_1^{2\sigma+1}\frac{dW}{dq_1} \\ =4(n-1)\phi_{n-1,k-1}a(q_1^{2\sigma})(cq_1^{2\sigma}+1). \tag{30}$$

We set $W = \sum_{j=0}^{2\sigma-1} q_1^j W_j(q_1^{2\sigma})$. Because the right-hand side of (30) is a function of $q_1^{2\sigma}$, W_j $(1 \le j < 2\sigma)$ satisfy

$$c^2 q_1^{2\sigma}(N - 4\sigma k + j)W_j - 4c\sigma W_j + c^2 q_1^{2\sigma+1}\frac{\mathrm{d}W_j}{\mathrm{d}q_1} = 0. \tag{31}$$

By a similar argument as in the proof of Lemma 5 we have $W_j = 0$ for $1 \le j < 2\sigma$. Hence we have $W(q_1) = W_0(q_1^{2\sigma}) =: V(t)$ $(t = q_1^{2\sigma})$. Because $q_1(d/dq_1)V = 2\sigma t(d/dt)V$, it follows from (30) that

$$\begin{aligned}(1-6\sigma)c^2 tV - 4c\sigma V + 2c^2\sigma t^2\frac{\mathrm{d}V}{\mathrm{d}t} \\ = 4(n-1)\phi_{n-1,k-1}a(t)(ct+1).\end{aligned} \tag{32}$$

Then the function $U := c^2\sigma V(t)/(2(n-1)\phi_{n-1,k-1})$ is an analytic solution of (4). On the other hand, by the assumption on $a(t)$ and Lemma 4, U is not analytic. This is a contradiction. Hence we have $\phi_{n-1,k-1} = 0$. Because k is arbitrary we have $\phi_{n-1} = 0$.

Next we set $\phi_{n-1} = 0$ in (25) and consider the coefficients of p_2^m $(m \ne n)$. Then we see that $v := h_{n,m}$ satisfies (12) with $\alpha = 2\sigma(m-n)/c$. By Lemma 5 we have $h_{n,m} = 0$ if $n \ne m$, and $h_{n,n} = \phi_n(p_1 q_1^{4\sigma})$ for some analytic function ϕ_n of one variable. It follows that $h_n(q_1, p_1, p_2) = h_{n,n}(q_1, p_1)p_2^n = \phi_n(p_1 q_1^{4\sigma})p_2^n$. Hence we have (24) with n replaced by $n+1$. By induction we obtain (24) for an arbitrary integer $n \ge 2$.

It follows from (24) with n replaced by $n+2$ that, for every $n \ge 0$ we have $\partial_{q_2}^n g(q_1, 0, p_1, p_2) \equiv 0$, where (q_1, p_1, p_2) is in some neighborhood of the origin which may depend on n. On the other hand $\partial_{q_2}^n g(q_1, 0, p_1, p_2)$ is analytic in some neighborhood of the origin independent of n. By analytic continuation, $\partial_{q_2}^n g(q_1, 0, p_1, p_2) \equiv 0$ in some neighborhood of the origin independent of n. By the partial Taylor expansion $g = \sum_n \partial_{q_2}^n g(q_1, 0, p_1, p_2)q_2^n/n!$, we have $g = 0$. □

Acknowledgments The author would like to express his gratitude to an anonymous referee for pointing out useful suggestions towards the improvement of the paper.

References

1. Cicogna, G., Walcher, S.: Convergence of normal form transformations: the role of symmetries. Acta Appl. Math. **70**, 95–111 (2002)
2. Gorni, G., Zampieri, G.: Analytic-non-integrability of an integrable analytic Hamiltonian system. Diff. Geom. Appl. **22**, 287–296 (2005)
3. Ito, H.: Integrability of Hamiltonian systems and Birkoff normal forms in the simple resonance case. Math. Ann. **292**, 411–444 (1992)
4. Stolovitch, L.: Singular complete integrability. Publ. Math. I.H.E.S. **91**, 134–210 (2000)
5. Zung, N.T.: Convergence versus integrability in Poincaré–Dulac normal form. Math. Res. Lett. **9**(2–3), 217–228 (2002)