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A Dirichlet problem for polyharmonic functions

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Abstract In this article, the Dirichlet problem of polyharmonic functions is considered. As well the explicit expression of the unique solution to the simple Dirichlet problem for polyharmonic functions is obtained by using the decomposition of polyharmonic functions and turning the problem into an equivalent Riemann boundary value problem for polyanalytic functions, as the approach to find the kernel functions of the solution for the general Dirichlet problem is given.

Keywords Polyharmonic function · Polyanalytic function · Analytic function · Kernel function · Dirichlet problem · Riemann boundary value problem

Mathematics Subject Classification (2000) 30G30 · 45E05

1 Introduction

As is well-known, analytic functions are closely connected with the Cauchy–Riemann operator and generalized analytic functions are defined by the generalized Cauchy–Riemann operator [12]. In this sense, boundary value problems (BVPs) for them are connected with the respective differential operators. Moreover, the Cauchy–Riemann operator has many generalizations. They lead to polyanalytic and

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metaanalytic functions [1]. In recent time, various kinds of BVPs for polyanalytic functions and generalized polyanalytic functions have widely been investigated, see for example [4,7–9,13,14]. In general, two methods are used to deal with BVPs of polyanalytic functions. One approach is to make use of the so-called poly-Cauchy operator [4,7]. The other is to transform the BVPs for polyanalytic functions into equivalent BVPs of analytic functions or systems of analytic functions by the decomposition theorem for polyanalytic functions [13].

Suppose $\mathbb{D}$ is the unit disc in the complex plane $\mathbb{C}$. If $f \in C^{2n}(\mathbb{D})$ satisfies the polyharmonic equation $(\partial_{\bar{z}}\partial_z)^n f = 0$ in $\mathbb{D}$, then f is called an n-harmonic function in $\mathbb{D}$, or simply, a polyharmonic function, where $\partial_{\bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ is the Cauchy–Riemann operator and $\partial_z = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ its adjoint operator. The class of such functions is simply denoted by $Har_n(\mathbb{D})$. Clearly, $Har_1(\mathbb{D})$ is the set of all harmonic functions in $\mathbb{D}$. In fact, the differential operators used to define polyanalytic functions and metaanalytic functions are only connected with the Cauchy–Riemann operator but the one used to define polyharmonic functions is connected with both the Cauchy–Riemann operator and its adjoint operator. $(\partial_{\bar{z}}\partial_z)^n$ is called the polyharmonic operator, in particular $\partial_{\bar{z}}\partial_z$ the harmonic operator.

In addition, the Cauchy–Pompeiu formula including the higher order Pompeiu formula is fairly important for solving general partial differential equations connected with the Cauchy–Riemann operator and its adjoint operator [4]. It will be seen from this article that they also play an important role in finding the solution of a Dirichlet problem for polyharmonic functions, see also [6].

As in [7], the set of all polyanalytic functions of order n on $\mathbb{D}$ is simply denoted by $H_n(\mathbb{D})$. Especially, $H_1(\mathbb{D})$ is the set of all the analytic functions on $\mathbb{D}$. The following decomposition theorem holds for the class of polyanalytic functions.

Theorem 1.1 (see [7]) *Let $\mathbb{D}$ be the unit disc on the complex plane, then*

$$H_n(\mathbb{D}) = H_1(\mathbb{D}) \oplus \bar{z}H_1(\mathbb{D}) \oplus \cdots \oplus \bar{z}^{n-1}H_1(\mathbb{D}).$$

Similarly, we denote the class of all functions satisfying $\partial_z^n f = 0$ in $\mathbb{D}$ by $\overline{H}_n(\mathbb{D})$. Noting $\overline{\partial_z f} = \partial_{\bar{z}}\bar{f}$, we easily see that $\overline{H}_n(\mathbb{D}) = \{\overline{f(z)} : f \in H_n(\mathbb{D})\}$. Thus we also have a similar decomposition theorem.

Theorem 1.2 *Let $\mathbb{D}$ be the unit disc in the complex plane, then*

$$\overline{H}_n(\mathbb{D}) = \overline{H}_1(\mathbb{D}) \oplus z\overline{H}_1(\mathbb{D}) \oplus \cdots \oplus z^{n-1}\overline{H}_1(\mathbb{D}).$$

Corresponding to the decomposition of polyanalytic functions, one also has the following decomposition theorem for the class of polyharmonic functions.

Theorem 1.3 (weak decomposition theorem) *Let $\mathbb{D}$ be the unit disc of the complex plane, then $Har_n(\mathbb{D}) = H_n(\mathbb{D}) + \overline{H_n}(\mathbb{D})$.*

Proof Obviously, if $g(z) = f(z) + \overline{h(z)}$ with $f, h \in H_n(\mathbb{D})$ then $(\partial_{\bar{z}}\partial_z)^n g = 0$ by the rules of differentiation.

Conversely, if $(\partial_{\bar{z}}\partial_z)^n g = 0$ then $\partial_z^n g = \sum_{j=0}^{n-1} \bar{z}^j f_{j+1}(z)$ where the f_j's are analytic in $\mathbb{D}$. Thus f_j has a primitive function φ_j of n-th order, say

$$f_j(z) = \sum_{\ell=0}^{\infty} a_{j,\ell}\, z^\ell \quad \text{then} \quad \varphi_j(z) = \sum_{\ell=0}^{\infty} \frac{a_{j,\ell}\, \ell\,!}{(n+\ell)!} z^{n+\ell},$$

so, $\partial_z^{n-1}\left[g-\sum_{j=0}^{n-1}\bar{z}^j\varphi_{j+1}(z)\right]=0$. By Theorem 1.2, we have

$$g(z)=\sum_{j=0}^{n-1}\bar{z}^j\varphi_{j+1}(z)+\sum_{j=0}^{n-1}z^j\overline{\psi}_{j+1}(z)\quad\text{with }\ \varphi_j,\psi_j\in H_1(\mathbb{D}).\tag{1.1}$$

Remark 1.1 Theorem 1.3 may also directly be derived from the higher order Pompeiu formula in [4].

Remark 1.2 It must be pointed out that φ_j and ψ_j in the decomposition (1.1) are not unique. In fact, denoting the class of all bi-polynomials $\sum_{j=0,\ell=0}^{n-1}c_{j,\ell}z^j\bar{z}^\ell$ by $B\Pi_{n-1}$, then we easily see $B\Pi_n=H_n(\mathbb{D})\bigcap\overline{H}_n(\mathbb{D})$.

Theorem 1.4 (decomposition theorem) *Let $\mathbb{D}$ be the unit disc in the complex plane, then*

$$Har_n(\mathbb{D})=B\Pi_n\oplus z^nH_n(\mathbb{D})\oplus\bar{z}^n\overline{H}_n(\mathbb{D}),$$

i.e., if $g\in Har_n(\mathbb{D})$ then

$$g(z)=\sum_{j=0,\ell=0}^{n-1}c_{j,\ell}z^j\bar{z}^\ell+z^n\sum_{j=0}^{n-1}\bar{z}^jf_{j+1}(z)+\bar{z}^n\sum_{j=0}^{n-1}z^j\overline{h_{j+1}(z)},\tag{1.2}$$

where $c_{j,\ell}$'s are some constants and $f_\ell,h_\ell\in H_1(\mathbb{D})$.

Proof From the weak decomposition theorem we know that the decomposition form is possible. If

$$p(z)+z^nf(z)+\bar{z}^n\overline{h(z)}=0\tag{1.3}$$

where

$$p\in B\Pi_{n-1},\quad f\in H_n(\mathbb{D}),\quad \bar{h}\in\overline{H}_n(\mathbb{D}),\tag{1.4}$$

then, using the operator ∂_z^n to the two hand-sides of (1.2) and letting $F_j(z)=\frac{\partial^n}{\partial z^n}[z^nf_j(z)]$ we get

$$\sum_{j=1}^{n}\bar{z}^{j-1}F_j(z)=0,\tag{1.5}$$

which implies $F_j(z)=0$ by Theorem 1.1. Thus $f_j(z)=0$, i.e., $f(z)=0$. Similarly, we get $h(z)=0$. Now from (1.2), we also get by Theorem 1.1 $p(z)=0$. The proof is completed.

Remark 1.3 In fact we may know by applying the operator $\partial_z^j\partial_{\bar{z}}^\ell$ to (1.2) at $z=0$ that

$$c_{j,\ell}=\frac{\partial_z^j\partial_{\bar{z}}^\ell g(0)}{j!\ell!}.\tag{1.6}$$

Remark 1.4 We call (1.1) the general decomposition of g while we call (1.2) the normal decomposition of g. By Theorem 1.4 we know that

$$z^nf_j(z)=\varphi_j(z)-\sum_{\ell=0}^{n-1}\frac{\varphi_j^{(\ell)}(0)}{\ell!}z^\ell,\quad z^nh_j(z)=\psi_j(z)-\sum_{\ell=0}^{n-1}\frac{\psi_j^{(\ell)}(0)}{\ell!}z^\ell.\tag{1.7}$$

Remark 1.5 In the decomposition (1.1) of g we write

$$\varphi(z) = \sum_{j=0}^{n-1} \bar{z}^j \varphi_{j+1}(z), \quad \psi(z) = \sum_{j=0}^{n-1} \bar{z}^j \psi_{j+1}(z). \tag{1.8}$$

φ and $\overline{\psi}$ are respectively called the components of the n-harmonic function g with respect to $H_n(\mathbb{D})$ and $\overline{H}_n(\mathbb{D})$, while φ_j and ψ_j are respectively called j-holomorphic components of the n-harmonic function g with respect to $H_n(\mathbb{D})$ and $\overline{H}_n(\mathbb{D})$. Let $\varphi(z) = (\mathbf{P}g)(z)$ and $\psi(z) = (\overline{\mathbf{P}}g)(z)$, see Sect. 2, then they are unique up to a summand from $B\Pi_{n-1}$.

In this article, we will discuss the Dirichlet problem of the polyharmonic functions. The unique solution for the Dirichlet problem is obtained by firstly turning the problem into the boundary value problems for polyanalytic functions, and then into several equivalent Riemann boundary value problems of analytic functions. During the transformation, the decomposition theorem for polyharmonic functions and the principle of the symmetric extension for analytic function will be used. The approach here, which is also called the reflection method, is different from the iterating method used in [5]. Besides, the unique solution of the general Dirichlet problem for polyharmonic functions is determined by a class of kernel functions, which may be obtained by some indirect recursive relations between two neighboring kernel functions.

In the following, the main analytic branch of $\log z$ is chosen in the complex plane cut along the negative real axis and assuming $\log 1 = 0$.

2 A Dirichlet problem for polyharmonic functions

Now our problem is to find a function $w \in Har_3(\mathbb{D})$ satisfying the following Dirichlet type boundary condition

$$[(\partial_z \partial_{\bar{z}})^j w]^+(t) = \gamma_j(t), \quad t \in \partial\mathbb{D}, \; j = 0, 1, 2, \tag{2.1}$$

where $\mathbb{D} \subset \mathbb{C}$ is the unit disc and $\partial\mathbb{D}$ its boundary, $f^+(t)$ denotes the boundary value of $f(z)$ when $z \to t$ from inside of $\mathbb{D}$, and $\gamma_j \in H(\partial\mathbb{D})$ i.e. Hölder continuous on $\partial\mathbb{D}$ for $j = 0, 1, 2$. This problem is simply called PHD problem.

Lemma 2.1 *Let $f \in Har_1(\mathbb{D}) \cap C^1(\overline{\mathbb{D}})$ and $\mathbb{D}$ be the unit disc, then for $t \in \partial\mathbb{D}$ the boundary values $(\mathbf{P}f)^+(t)$ and $\left(\overline{\mathbf{P}}f\right)^+(t)$ exist of*

$$\begin{cases} (\mathbf{P}f)(z) = -\dfrac{1}{2\pi i} \displaystyle\int_{\partial\mathbb{D}} (\partial_z f)(\tau) \log(1 - z\bar{\tau}) \mathrm{d}\tau, \\ \left(\overline{\mathbf{P}}f\right)(z) = \dfrac{1}{2\pi i} \displaystyle\int_{\partial\mathbb{D}} \overline{f(\tau)} \dfrac{\mathrm{d}\tau}{\tau - z} + \dfrac{1}{2\pi i} \displaystyle\int_{\partial\mathbb{D}} \overline{(\partial_z f)(\tau)} \log(1 - z\bar{\tau}) \mathrm{d}\bar{\tau}. \end{cases}$$

Proof If $f \in Har_1(\mathbb{D}) \cap C^1(\overline{\mathbb{D}})$, then, by the higher order Pompeiu formula [4],

$$f(z) = -\frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\tau)\frac{d\bar{\tau}}{\tau - z} - \frac{1}{2\pi i}\int_{\partial\mathbb{D}} (\partial_z f)(\tau)\log|\tau - z|^2 d\tau$$
$$= \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\tau)\frac{1}{1-\bar{z}\tau}\frac{d\tau}{\tau} - \frac{1}{2\pi i}\int_{\partial\mathbb{D}} (\partial_z f)(\tau)\log|1 - \bar{z}\tau|^2 d\tau$$

It follows that $(\mathbf{P}f)^+(t)$ and $\left(\overline{\mathbf{P}}f\right)^+(t)$ exist.

Lemma 2.2 *If* $\frac{\partial\varphi}{\partial z} \in H_1(\mathbb{D}) \cap C^1(\overline{\mathbb{D}})$*, then* $\varphi \in H_1(\mathbb{D}) \cap C^1(\overline{\mathbb{D}})$*.*

Proof Since

$$\varphi(z) = \int_0^z \frac{\partial\varphi}{\partial z}(\zeta)d\zeta - \varphi(0), \quad z \in \overline{\mathbb{D}},$$

the lemma is obvious.

Let

$$w(z) = \varphi(z) + \overline{\psi(z)} \tag{2.2}$$

with

$$\begin{cases} \varphi(z) = \varphi_1(z) + \bar{z}\varphi_2(z) + \bar{z}^2\varphi_3(z), \\ \psi(z) = \psi_1(z) + \bar{z}\psi_2(z) + \bar{z}^2\psi_3(z), \end{cases} \tag{2.3}$$

where $\varphi_j, \psi_j, j = 1, 2, 3$, are analytic functions. By Lemma 2.1, $\left[\frac{\partial^2\varphi_3}{\partial z^2}\right]^+(t)$ and $\left[\frac{\partial^2\psi_3}{\partial z^2}\right]^+(t)$ exist. By Lemma 2.2, $\left[\frac{\partial\varphi_k}{\partial z}\right]^+(t), \left[\frac{\partial\psi_k}{\partial z}\right]^+(t)$ for $k = 2, 3$ and $\varphi_j^+(t), \psi_j^+(t)$ for $j = 1, 2, 3$ also exist. So, putting (2.2) into the boundary condition (2.1), we immediately obtain the following boundary conditions for the polyanalytic functions $\varphi(z)$ and $\psi(z)$

$$\varphi_1^+(t) + \bar{t}\varphi_2^+(t) + \bar{t}^2\varphi_3^+(t) + \overline{\psi_1^+(t) + \bar{t}\psi_2^+(t) + \bar{t}^2\psi_3^+(t)} = \gamma_0(t), \quad t \in \partial\mathbb{D}, \tag{2.4}$$

$$\left[\frac{\partial\varphi_2}{\partial z}\right]^+(t) + 2\bar{t}\left[\frac{\partial\varphi_3}{\partial z}\right]^+(t) + \overline{\left[\frac{\partial\psi_2}{\partial z}\right]^+(t) + 2\bar{t}\left[\frac{\partial\psi_3}{\partial z}\right]^+(t)} = \gamma_1(t), \quad t \in \partial\mathbb{D}, \tag{2.5}$$

$$\left[\frac{\partial^2\varphi_3}{\partial z^2}\right]^+(t) + \overline{\left[\frac{\partial^2\psi_3}{\partial z^2}\right]^+(t)} = \frac{1}{2}\gamma_2(t), \; t \in \partial\mathbb{D}. \tag{2.6}$$

Firstly, by the principle of symmetric extension for analytic functions, setting

$$\Phi(z) = \begin{cases} \dfrac{\partial^2\varphi_3}{\partial z^2}(z), & |z| < 1, \\ -\overline{\dfrac{\partial^2\psi_3}{\partial z^2}\left(\dfrac{1}{\bar{z}}\right)}, & |z| > 1, \end{cases} \tag{2.7}$$

then the boundary condition (2.6) is changed into the jump condition for an analytic function

$$\Phi^+(t) = \Phi^-(t) + \frac{1}{2}\gamma_2(t), \quad t \in \partial\mathbb{D}. \tag{2.8}$$

Solving the Riemann problem (2.8) in R_0, one gets [10,11]

$$\Phi(z) = \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} \frac{\gamma_2(\tau)}{\tau - z}\mathrm{d}\tau + C_1 = \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} \frac{\bar{\tau}\gamma_2(\tau)}{1 - z\bar{\tau}}\mathrm{d}\tau + C_1, \tag{2.9}$$

where C_1 is an arbitrary constant. In the following of this article, without additional explanation, C_j and D_k are also free constants. Thus we obtain

$$\frac{\partial^2\varphi_3}{\partial z^2}(z) = \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} \frac{\gamma_2(\tau)}{\tau - z}\mathrm{d}\tau + C_1 = \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} \frac{\bar{\tau}\gamma_2(\tau)}{1 - z\bar{\tau}}\mathrm{d}\tau + C_1, \quad |z| < 1, \tag{2.10}$$

which leads to

$$\frac{\partial\varphi_3}{\partial z}(z) = \frac{-1}{4\pi i}\int\limits_{\partial\mathbb{D}} \gamma_2(\tau)\log(1 - z\bar{\tau})\mathrm{d}\tau + C_1 z + C_2, \quad |z| < 1 \tag{2.11}$$

and

$$\varphi_3(z) = \frac{-1}{4\pi i}\int\limits_{\partial\mathbb{D}} \gamma_2(\tau)[(z-\tau)\log(1-z\bar{\tau}) - z]\mathrm{d}\tau + \frac{1}{2}C_1 z^2 + C_2 z + C_3, \quad |z| < 1. \tag{2.12}$$

On the other hand, one also has

$$-\overline{\frac{\partial^2\psi_3}{\partial z^2}\left(\frac{1}{\bar{z}}\right)} = \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} \frac{\gamma_2(\tau)}{\tau - z}\mathrm{d}\tau + C_1, \quad |z| > 1. \tag{2.13}$$

Thus we obtain

$$\frac{\partial^2\psi_3}{\partial z^2}(z) = \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} \frac{\overline{\gamma_2(\tau)}}{\bar{\tau} - \frac{1}{z}}\mathrm{d}\bar{\tau} - \overline{C}_1 = \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} \frac{\bar{\tau}\overline{\gamma_2(\tau)}}{1 - z\bar{\tau}}\mathrm{d}\tau - \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} \overline{\gamma_2(\tau)}\frac{\mathrm{d}\tau}{\tau} - \overline{C}_1, \quad |z| < 1, \tag{2.14}$$

which also gives

$$\frac{\partial\psi_3}{\partial z}(z) = \frac{-1}{4\pi i}\int\limits_{\partial\mathbb{D}} \overline{\gamma_2(\tau)}[z\bar{\tau} + \log(1 - z\bar{\tau})]\mathrm{d}\tau - \overline{C}_1 z + C_4, \quad |z| < 1, \tag{2.15}$$

and

$$\psi_3(z) = \frac{-1}{4\pi i}\int\limits_{\partial\mathbb{D}} \overline{\gamma_2(\tau)}\left[\frac{1}{2}z^2\bar{\tau} - z + (z - \tau)\log(1 - z\bar{\tau})\right]\mathrm{d}\tau - \frac{1}{2}\overline{C}_1 z^2 + C_4 z + C_5, \ |z| < 1. \tag{2.16}$$

Secondly, observing $t\bar{t} = 1$ on the boundary $\partial\mathbb{D}$ and by the boundary condition (2.5), one has

$$t\left[\frac{\partial\varphi_2}{\partial z}\right]^+(t) + 2\left[\frac{\partial\varphi_3}{\partial z}\right]^+(t) + t^2\overline{\left\{t\left[\frac{\partial\psi_2}{\partial z}\right]^+(t) + 2\left[\frac{\partial\psi_3}{\partial z}\right]^+(t)\right\}} = t\gamma_1(t), \quad t \in \partial\mathbb{D}. \tag{2.17}$$

Setting

$$\Psi(z)=\begin{cases} z\,\dfrac{\partial\varphi_2}{\partial z}(z)+2\dfrac{\partial\varphi_3}{\partial z}(z), & |z|<1,\\[2mm] -z^2\left\{\overline{\dfrac{1}{\bar z}\,\dfrac{\partial\psi_2}{\partial z}\left(\dfrac{1}{\bar z}\right)+2\dfrac{\partial\psi_3}{\partial z}\left(\dfrac{1}{\bar z}\right)}\right\}, & |z|>1,\end{cases} \tag{2.18}$$

then the boundary condition (2.17) is equivalent to the following jump condition

$$\Psi^+(t)=\Psi^-(t)+t\,\gamma_1(t),\quad t\in\partial\mathbb{D}. \tag{2.19}$$

Let Π_n denote the set of all polynomials of degree not more than n. If $n<0$, then we assume $\Pi_n=\{0\}$. Then one gets the general solution for the R_2 problem (2.19) as [10,11]

$$\Psi(z)=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\frac{\tau\,\gamma_1(\tau)}{\tau-z}\mathrm{d}\tau+P_2(z) \tag{2.20}$$

with

$$P_2(z)=D_1+D_2z+D_3z^2\in\Pi_2.$$

So we obtain

$$z\,\frac{\partial\varphi_2}{\partial z}(z)+2\frac{\partial\varphi_3}{\partial z}(z)=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\frac{\tau\,\gamma_1(\tau)}{\tau-z}\mathrm{d}\tau+P_2(z)=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\frac{\gamma_1(\tau)}{1-z\bar\tau}\mathrm{d}\tau+P_2(z),\quad |z|<1. \tag{2.21}$$

Thus, by (2.11) and (2.21), one has

$$\frac{\partial\varphi_2}{\partial z}(z)=\frac{1}{z}\left\{\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\frac{\gamma_1(\tau)}{1-z\bar\tau}\mathrm{d}\tau+P_2(z)+\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\gamma_2(\tau)\log(1-z\bar\tau)\mathrm{d}\tau-2C_1z-2C_2\right\}. \tag{2.22}$$

Clearly, $\frac{\partial\varphi_2}{\partial z}(z)$ given by (2.22) is analytic in $\mathbb{D}$ if and only if

$$D_1-2C_2=-\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\gamma_1(\tau)\mathrm{d}\tau. \tag{2.23}$$

Now (2.22) and (2.23) give

$$\frac{\partial\varphi_2}{\partial z}(z)=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\frac{\bar\tau\gamma_1(\tau)}{1-z\bar\tau}\mathrm{d}\tau+\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\gamma_2(\tau)\frac{\log(1-z\bar\tau)}{z}\mathrm{d}\tau+D_3z+D_2-2C_1, \tag{2.24}$$

which implies

$$\varphi_2(z)=\frac{-1}{2\pi i}\int\limits_{\partial\mathbb{D}}\gamma_1(\tau)\log(1-z\bar\tau)\mathrm{d}\tau-\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\gamma_2(\tau)\sum_{k=1}^{\infty}\frac{(z\bar\tau)^k}{k^2}\mathrm{d}\tau$$
$$+\frac{1}{2}D_3z^2+(D_2-2C_1)z+C_6. \tag{2.25}$$

On the other hand, one has

$$-z^2\left\{\overline{\frac{1}{\bar z}\,\frac{\partial\psi_2}{\partial z}\left(\frac{1}{\bar z}\right)+2\frac{\partial\psi_3}{\partial z}\left(\frac{1}{\bar z}\right)}\right\}=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\frac{\tau\,\gamma_1(\tau)}{\tau-z}\mathrm{d}\tau+P_2(z),\quad |z|>1, \tag{2.26}$$

which gives

$$z\,\frac{\partial\psi_2}{\partial z}(z)+2\frac{\partial\psi_3}{\partial z}(z)=-\bar{z}^2\,\overline{\left[\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\frac{\tau\,\gamma_1(\tau)}{\tau-\frac{1}{\bar{z}}}\mathrm{d}\tau+P_2\left(\frac{1}{\bar{z}}\right)\right]}$$
$$=\frac{-z^3}{2\pi i}\int\limits_{\partial\mathbb{D}}\frac{\bar{\tau}\,\overline{\gamma_1(\tau)}}{1-z\bar{\tau}}\mathrm{d}\bar{\tau}-(\overline{D}_1z^2+\overline{D}_2z+\overline{D}_3),\quad |z|<1. \tag{2.27}$$

Thus

$$\frac{\partial\psi_2}{\partial z}(z)=\frac{1}{z}\Bigg[\frac{-z^3}{2\pi i}\int\limits_{\partial\mathbb{D}}\frac{\bar{\tau}\,\overline{\gamma_1(\tau)}}{1-z\bar{\tau}}\mathrm{d}\bar{\tau}-(\overline{D}_1z^2+\overline{D}_2z+\overline{D}_3)$$
$$+\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\overline{\gamma_2(\tau)}[z\bar{\tau}+\log(1-z\bar{\tau})]\mathrm{d}\tau+2\overline{C}_1z-2C_4\Bigg] \tag{2.28}$$

and

$$\overline{D}_3+2C_4=0. \tag{2.29}$$

So, one gets from (2.28) and (2.29)

$$\frac{\partial\psi_2}{\partial z}(z)=\frac{-z^2}{2\pi i}\int\limits_{\partial\mathbb{D}}\frac{\bar{\tau}\,\overline{\gamma_1(\tau)}}{1-\bar{\tau}z}\mathrm{d}\bar{\tau}$$
$$+\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\overline{\gamma_2(\tau)}\left[\bar{\tau}+\frac{\log(1-z\bar{\tau})}{z}\right]\mathrm{d}\tau-\overline{D}_1z-(\overline{D}_2-2\overline{C}_1), \tag{2.30}$$

which leads to

$$\psi_2(z)=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\overline{\gamma_1(\tau)}\left[\frac{z^2}{2}+\frac{z}{\bar{\tau}}+\frac{1}{\bar{\tau}^2}\log(1-z\bar{\tau})\right]\mathrm{d}\bar{\tau}$$
$$-\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\overline{\gamma_2(\tau)}\sum_{k=2}^{\infty}\frac{(z\bar{\tau})^k}{k^2}\mathrm{d}\tau-\frac{1}{2}\overline{D}_1z^2-(\overline{D}_2-2\overline{C}_1)z+C_7. \tag{2.31}$$

Finally, the boundary condition (2.4) is equivalent to

$$t^2\varphi_1^+(t)+t\varphi_2^+(t)+\varphi_3^+(t)+t^4\,\overline{t^2\psi_1^+(t)+t\psi_2^+(t)+\psi_3^+(t)}=t^2\gamma_0(t),\quad t\in\partial\mathbb{D}. \tag{2.32}$$

Let

$$\Omega(z)=\begin{cases} z^2\,\varphi_1(z)+z\,\varphi_2(z)+\varphi_3(z), & |z|<1,\\ -z^4\,\overline{\left\{\frac{1}{\bar{z}^2}\,\psi_1\left(\frac{1}{\bar{z}}\right)+\frac{1}{\bar{z}}\,\psi_2\left(\frac{1}{\bar{z}}\right)+\psi_3\left(\frac{1}{\bar{z}}\right)\right\}}, & |z|>1,\end{cases} \tag{2.33}$$

then the boundary condition (2.32) is equivalent to the following condition

$$\Omega^+(t)=\Omega^-(t)+t^2\,\gamma_0(t),\quad t\in\partial\mathbb{D}. \tag{2.34}$$

Then the R_4 problem (2.34) has the solution [10,11]

$$\Omega(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\tau^2\gamma_0(\tau)}{\tau - z}\mathrm{d}\tau + P_4(z) \tag{2.35}$$

with

$$P_4(z) = D_4 + D_5 z + D_6 z^2 + D_7 z^3 + D_8 z^4 \in \Pi_4.$$

Thus we get

$$z^2\varphi_1(z) + z\varphi_2(z) + \varphi_3(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\tau^2\gamma_0(\tau)}{\tau - z}\mathrm{d}\tau + P_4(z), \quad |z| < 1. \tag{2.36}$$

By (2.12), (2.25) and (2.36), one has

$$\begin{aligned}\varphi_1(z) = \frac{1}{z^2}\Bigg\{&\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\tau^2\gamma_0(\tau)}{\tau - z}\mathrm{d}\tau + P_4(z) + \frac{z}{2\pi i}\int_{\partial\mathbb{D}} \gamma_1(\tau)\log(1 - z\bar{\tau})\mathrm{d}\tau \\ &+ \frac{z}{2\pi i}\int_{\partial\mathbb{D}} \gamma_2(\tau)\sum_{k=1}^{\infty}\frac{(z\bar{\tau})^k}{k^2}\mathrm{d}\tau - \left[\frac{1}{2}D_3 z^3 + (D_2 - 2C_1)z^2 + C_6 z\right] \\ &+ \frac{1}{4\pi i}\int_{\partial\mathbb{D}} \gamma_2(\tau)\left[(z - \tau)\log(1 - z\bar{\tau}) - z\right]\mathrm{d}\tau - \frac{1}{2}C_1 z^2 - C_2 z - C_3\Bigg\}, \quad |z| < 1.\end{aligned} \tag{2.37}$$

Obviously, $\varphi_1(z)$ is analytic in $\mathbb{D}$ if and only if

$$\begin{cases} C_3 - D_4 = \dfrac{1}{2\pi i}\displaystyle\int_{\partial\mathbb{D}} \tau\,\gamma_0(\tau)\mathrm{d}\tau, \\ C_2 + C_6 - D_5 = \dfrac{1}{2\pi i}\displaystyle\int_{\partial\mathbb{D}} \gamma_0(\tau)\mathrm{d}\tau. \end{cases} \tag{2.38}$$

Thus

$$\begin{aligned}\varphi_1(z) = &\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\gamma_0(\tau)}{\tau - z}\mathrm{d}\tau + D_6 + D_7 z + D_8 z^2 + \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_1(\tau)\frac{\log(1 - z\bar{\tau})}{z}\mathrm{d}\tau \\ &+ \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_2(\tau)\sum_{k=1}^{\infty}\frac{z^{k-1}\bar{\tau}^k}{k^2}\mathrm{d}\tau - \left[\frac{1}{2}D_3 z + D_2 - 2C_1\right] \\ &+ \frac{1}{4\pi i}\int_{\partial\mathbb{D}} \gamma_2(\tau)\frac{(z - \tau)\log(1 - z\bar{\tau}) - z}{z^2}\mathrm{d}\tau - \frac{1}{2}C_1, \quad |z| < 1.\end{aligned} \tag{2.39}$$

Similarly,

$$-z^4\left[\overline{\frac{1}{\bar{z}^2}\psi_1\left(\frac{1}{\bar{z}}\right) + \frac{1}{\bar{z}}\psi_2\left(\frac{1}{\bar{z}}\right) + \psi_3\left(\frac{1}{\bar{z}}\right)}\right] = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\tau^2\gamma_0(\tau)}{\tau - z}\mathrm{d}\tau + P_4(z), \quad |z| > 1, \tag{2.40}$$

which leads to

$$z^2\,\psi_1(z){+}z\,\psi_2(z){+}\psi_3(z){=}{-}\frac{z^5}{2\pi i}\int\limits_{\partial\mathbb{D}}\frac{\bar\tau^2\,\overline{\gamma_0(\tau)}}{1-z\bar\tau}\mathrm{d}\bar\tau{-}(\overline{D}_4z^4{+}\overline{D}_5z^3{+}\overline{D}_6z^2{+}\overline{D}_7z{+}\overline{D}_8),\ |z|<1. \tag{2.41}$$

Thus, by (2.16), (2.31) and (2.41),

$$\begin{aligned}\psi_1(z)=\frac{1}{z^2}\Bigg\{&-\frac{z^5}{2\pi i}\int\limits_{\partial\mathbb{D}}\frac{\bar\tau^2\,\overline{\gamma_0(\tau)}}{1-z\bar\tau}d\bar\tau-(\overline{D}_4z^4+\overline{D}_5z^3+\overline{D}_6z^2+\overline{D}_7z+\overline{D}_8)\\&-\frac{z}{2\pi i}\int\limits_{\partial\mathbb{D}}\overline{\gamma_1(\tau)}\left[\frac{z^2}{2}+\frac{z}{\bar\tau}+\frac{1}{\bar\tau^2}\log(1-z\bar\tau)\right]\mathrm{d}\bar\tau\\&+\frac{z}{2\pi i}\int\limits_{\partial\mathbb{D}}\overline{\gamma_2(\tau)}\sum_{k=2}^{\infty}\frac{(z\bar\tau)^k}{k^2}\mathrm{d}\tau+\frac{1}{2}\overline{D}_1z^3+(\overline{D}_2-2\overline{C}_1)z^2-C_7z\\&+\frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\overline{\gamma_2(\tau)}\left[\frac{1}{2}z^2\bar\tau-z+(z-\tau)\log(1-z\bar\tau)\right]\mathrm{d}\tau\\&+\frac{1}{2}\overline{C}_1z^2-C_4z-C_5\Bigg\}.\end{aligned} \tag{2.42}$$

Clearly if and only if

$$\begin{cases}C_5+\overline{D}_8=0,\\ C_4+C_7+\overline{D}_7=0\end{cases} \tag{2.43}$$

is satisfied, ψ_1 given by (2.42) is analytic in $\mathbb{D}$. Therefore, substituting (2.43) into (2.42), one has

$$\begin{aligned}\psi_1(z)=&-\frac{z^3}{2\pi i}\int\limits_{\partial\mathbb{D}}\frac{\bar\tau^2\,\overline{\gamma_0(\tau)}}{1-z\bar\tau}\mathrm{d}\bar\tau-(\overline{D}_4z^2+\overline{D}_5z+\overline{D}_6)\\&-\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\overline{\gamma_1(\tau)}\left[\frac{z}{2}+\frac{1}{\bar\tau}+\frac{1}{\bar\tau^2z}\log(1-z\bar\tau)\right]\mathrm{d}\bar\tau\\&+\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\overline{\gamma_2(\tau)}\sum_{k=2}^{\infty}\frac{z^{k-1}\bar\tau^k}{k^2}\mathrm{d}\tau+\frac{1}{2}\overline{D}_1z+\overline{D}_2-2\overline{C}_1\\&-\frac{1}{8\pi i}\int\limits_{\partial\mathbb{D}}\overline{\gamma_2(\tau)}\,\tau\,\mathrm{d}\bar\tau-\frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\overline{\gamma_2(\tau)}\,\tau^2\frac{(z-\tau)\log(1-z\bar\tau)-z}{z^2}\mathrm{d}\bar\tau+\frac{1}{2}\overline{C}_1.\end{aligned} \tag{2.44}$$

Until now, we obtain $\varphi_j(z)\,(j=1,2,3)$ and $\psi_j(z)\,(j=1,2,3)$ by solving the three boundary value problems for analytic functions (2.8), (2.19) and (2.34), which are so-called jump problems, the simplest BVPs for analytic functions.

Next, substituting $\varphi_j(z)$, $j=1,2,3$, given by (2.39), (2.25) and (2.12) respectively and $\psi_j(z), j=1,2,3$, given by (2.44), (2.31) and (2.16) respectively into (2.2), one obtains the general expression for $w(z)$, where the terms with $C_j\,(1\le j\le 7)$ and

D_k $(1 \le k \le 8)$ are rewritten as

$$\begin{aligned}
&D_6 + D_7 z + D_8 z^2 - \left[\frac{1}{2}D_3 z + (D_2 - 2C_1)\right] \\
&\quad -\frac{1}{2}C_1 + \bar{z}\left[\frac{1}{2}D_3 z^2 + (D_2 - 2C_1)z + C_6\right] + \bar{z}^2\left[\frac{1}{2}C_1 z^2 + C_2 z + C_3\right] \\
&\quad -(D_4\bar{z}^2 + D_5\bar{z} + D_6) + \frac{1}{2}D_1\bar{z} + (D_2 - 2C_1) + \frac{1}{2}C_1 \\
&\quad +z\left[-\frac{1}{2}D_1\bar{z}^2 - (D_2 - 2C_1)\bar{z} + \overline{C}_7\right] + z^2\left[-\frac{1}{2}C_1\bar{z}^2 + \overline{C}_4\bar{z} + \overline{C}_5\right] \\
&= \left(D_7 - \frac{1}{2}D_3 + \overline{C}_7\right)z + (D_8 + \overline{C}_5)z^2 + \left(\frac{1}{2}D_3 + \overline{C}_4\right)z^2\bar{z} \\
&\quad + \left(C_6 - D_5 + \frac{1}{2}D_1\right)\bar{z} + \left(C_2 - \frac{1}{2}D_1\right)z\bar{z}^2 + (C_3 - D_4)\bar{z}^2. \qquad (2.45)
\end{aligned}$$

By (2.23), (2.29), (2.38) and (2.43), (2.45) equals

$$\begin{aligned}
&\frac{\bar{z}^2}{2\pi i}\int_{\partial\mathbb{D}} \gamma_0(\tau)\tau d\tau + \frac{z\bar{z}^2}{4\pi i}\int_{\partial\mathbb{D}} \gamma_1(\tau)d\tau + \frac{\bar{z}}{2\pi i}\int_{\partial\mathbb{D}} \gamma_0(\tau)d\tau - \frac{\bar{z}}{4\pi i}\int_{\partial\mathbb{D}} \gamma_1(\tau)d\tau \\
&= \frac{\bar{z}}{2\pi i}\int_{\partial\mathbb{D}} \gamma_0(\tau)(1 + \bar{z}\tau)d\tau - \frac{\bar{z}(1 - |z|^2)}{4\pi i}\int_{\partial\mathbb{D}} \gamma_1(\tau)d\tau. \qquad (2.46)
\end{aligned}$$

Further computation gives

$$w(z) = \sum_{k=1}^{3} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_{k-1}(\tau)g_k(z,\tau)\frac{d\tau}{\tau} \qquad (2.47)$$

with

$$g_1(z,\tau) = \frac{1}{1 - z\bar{\tau}} + \frac{1}{1 - \bar{z}\tau} - 1, \qquad (2.48)$$

$$g_2(z,\tau) = (1 - |z|^2)\left[\frac{1}{z\bar{\tau}}\log(1 - z\bar{\tau}) + \frac{1}{\bar{z}\tau}\log(1 - \bar{z}\tau) + 1\right], \qquad (2.49)$$

$$\begin{aligned}
g_3(z,\tau) &= (|z|^2 - 1)\left[\frac{1}{z\bar{\tau}}\int_0^{\zeta} \frac{\log(1 - z\bar{\tau})}{z}dz + \frac{1}{\bar{z}\tau}\int_0^{\bar{z}} \frac{\log(1 - \bar{z}\tau)}{\bar{z}}d\bar{z} + 1\right] \\
&\quad + \frac{1 - |z|^4}{2}\left\{\frac{1}{(z\bar{\tau})^2}[(z\bar{\tau} - 1)\log(1 - z\bar{\tau}) - z\bar{\tau}]\right. \\
&\quad \left. + \frac{1}{(\bar{z}\tau)^2}[(\bar{z}\tau - 1)\log(1 - \bar{z}\tau) - \bar{z}\tau] + \frac{1}{2}\right\} \\
&= (1 - |z|^2)\left[\sum_{k=2}^{\infty} \frac{1}{k^2}((z\bar{\tau})^{k-1} + (\bar{z}\tau)^{k-1}) + 1\right] \\
&\quad - \frac{1 - |z|^4}{2}\left[\sum_{k=2}^{\infty} \frac{1}{k(k+1)}((z\bar{\tau})^{k-1} + (\bar{z}\tau)^{k-1}) + \frac{1}{2}\right]. \qquad (2.50)
\end{aligned}$$

To sum up the above discussion, one has the following result.

Theorem 2.1 *The PHD problem* (2.1) *is solvable and its unique solution is given as* (2.47).

Proof The above process indicates, if PHD problem (2.1) is solvable, then its solution may be written as (2.47). On the other hand, since $z = \tau$ is a singular point of order less than 1 for the function

$$\tilde{g}_2(z,\tau) = \frac{1}{z\bar{\tau}}\log(1 - z\bar{\tau}) + \frac{1}{\bar{z}\tau}\log(1 - \bar{z}\tau) + 1,$$

then

$$\lim_{z\to t, |t|=1, |z|<1} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_1(\tau)\tilde{g}_2(z,\tau)\frac{d\tau}{\tau}$$

exists. Thus

$$\lim_{z\to t, |t|=1, |z|<1} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_1(\tau)g_2(z,\tau)\frac{d\tau}{\tau} = 0.$$

And since

$$-\sum_{k=1}^{\infty}\frac{(z\bar{\tau})^{k-1}}{k^2} = \frac{1}{z\bar{\tau}}\int_0^z \frac{\log(1-\zeta\bar{\tau})}{\zeta}d\zeta \quad \text{and} \quad \frac{1}{(z\bar{\tau})^2}[(z\bar{\tau}-1)\log(1-z\bar{\tau}) - z\bar{\tau}]$$

are analytic in the unit disc and continuous up to its boundary with respect to z, then

$$\lim_{z\to t, |t|=1, |z|<1} g_3(z,\tau) = 0,$$

which leads to

$$\lim_{z\to t, |t|=1, |z|<1} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_2(\tau)g_3(z,\tau)\frac{d\tau}{\tau} = 0.$$

Obviously,

$$\lim_{z\to t, |t|=1, |z|<1} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_0(\tau)g_1(z,\tau)\frac{d\tau}{\tau} = \gamma_0(t).$$

Therefore, one has $w^+(t) = \gamma_0(t), t \in \partial\mathbb{D}$. Similarly (2.1) can be verified for $j = 1, 2$. Also because

$$\begin{cases} (\partial_{\bar{z}}\partial_z)g_k(z,\tau) = g_{k-1}(z,\tau), & k = 2, 3, \\ (\partial_{\bar{z}}\partial_z)g_1(z,\tau) = 0, \end{cases}$$

$w(z)$ given by (2.47) is the solution for the PHD problem (2.1).

3 Dirichlet problem for the harmonic equation

In this section, we will investigate the simplest Dirichlet problem for the polyharmonic equation, see also [3]. The result will be used in the sequel. The problem is to find a function w satisfying the inhomogeneous harmonic equation in the unit disc $\mathbb{D}$ and the corresponding Dirichlet type boundary condition

$$\begin{cases} (\partial_{\bar{z}}\partial_z)w(z) = f(z), & z \in \mathbb{D}, \\ w^+(t) = \gamma_0(t), & t \in \partial\mathbb{D}, \end{cases} \tag{3.1}$$

where $\partial\mathbb{D}$ is the boundary, and $\gamma_0 \in H(\partial\mathbb{D}), f \in C^1(\overline{\mathbb{D}})$.

Let

$$(T_{1,1}f)(z) = \frac{1}{\pi}\int\limits_{\mathbb{D}} f(\zeta)\log|z-\zeta|^2 \mathrm{d}\xi\mathrm{d}\eta \tag{3.2}$$

with $f \in C^1(\bar{\mathbb{D}})$. Then

$$[\partial_{\bar{z}}\partial_z(T_{1,1}f)](z) = f(z). \tag{3.3}$$

It follows from (3.1) and (3.3) that $w - T_{1,1}f \in Har_1(\mathbb{D})$. In addition, $T_{1,1}f \in H(\overline{\mathbb{D}})$ (see [4]). So the Dirichlet problem for the harmonic equation (3.1) is equivalent to the simplest PHD problem: finding a function $u \in Har_1(\mathbb{D})$ satisfying the Dirichlet type boundary condition

$$u^+(t) = \gamma_0(t) - (T_{1,1}f)(t), \quad t \in \partial\mathbb{D}. \tag{3.4}$$

Therefore, we have the following result.

Lemma 3.1 *The solution to the Dirichlet problem* (3.1) *is given by*

$$w(z) = u(z) + (T_{1,1}f)(z),$$

where $u(z)$ is the solution for PHD problem (3.4).

Similar to Sect. 2, the unique solution for PHD problem (3.4) is

$$u(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}[\gamma_0(\tau) - (T_{1,1}f)(\tau)]g_1(z,\tau)\frac{\mathrm{d}\tau}{\tau}, \tag{3.5}$$

where $g_1(z,\tau)$ is given by (2.48). By Lemma 3.1 and (3.5), one has

$$\begin{aligned} w(z) = &\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\gamma_0(\tau)g_1(z,\tau)\frac{\mathrm{d}\tau}{\tau} \\ &-\frac{1}{\pi}\int\limits_{\mathbb{D}} f(\zeta)\left[\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\log|\tau-\zeta|^2 g_1(z,\tau)\frac{\mathrm{d}\tau}{\tau}\right]\mathrm{d}\xi\mathrm{d}\eta + (T_{1,1}f)(z). \end{aligned} \tag{3.6}$$

Noticing

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\log|\tau-\zeta|^2 g_1(z,\tau)\frac{\mathrm{d}\tau}{\tau} = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\log|1-\bar{\tau}\zeta|^2 g_1(z,\tau)\frac{\mathrm{d}\tau}{\tau} = \log|1-\bar{z}\zeta|^2, \quad z \in \mathbb{D},$$

then $w(z)$ given by (3.6) may be rewritten as

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_0(\tau) g_1(z,\tau)\frac{d\tau}{\tau} + \frac{1}{\pi}\int_{\mathbb{D}} f(\zeta)\log\left|\frac{\zeta - z}{1-\bar{z}\zeta}\right|^2 d\xi d\eta, \tag{3.7}$$

where the kernel $\log\left|\frac{1-\bar{z}\zeta}{\zeta-z}\right|^2$ is the harmonic Green function.

In general, the following result holds.

Theorem 3.1 *The Dirichlet problem for the harmonic equation* (3.1) *is solvable and its unique solution may be given as* (3.7).

4 A general PHD problem

In this section, our first problem is to find a function $w \in Har_4(\mathbb{D})$ satisfying the corresponding Dirichlet type condition

$$[(\partial_z \partial_{\bar{z}})^j w]^+(t) = \gamma_j(t), \quad t \in \partial\mathbb{D}, \; j = 0,1,2,3, \tag{4.1}$$

where $\mathbb{D}, \partial\mathbb{D}$ are defined as in Sect. 2, and $\gamma_j \in H(\partial\mathbb{D})$ for $j = 0,1,2,3$. This problem is also called PHD problem.

By the same way as in Sect. 2, one gets the unique solution for PHD problem (4.1) as

$$w(z) = \sum_{k=1}^{4} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_{k-1}(\tau) g_k(z,\tau)\frac{d\tau}{\tau} \tag{4.2}$$

with

$$\begin{aligned}
g_4(z,\tau) = (1-|z|^2)&\left[\frac{1}{\bar{\tau}z}\int_0^z \frac{1}{z_1}\int_0^{z_1} \frac{\log(1-\bar{\tau}z_2)}{z_2} dz_2 dz_1\right.\\
&\left.+ \frac{1}{\tau\bar{z}}\int_0^{\bar{z}} \frac{1}{\bar{z}_1}\int_0^{\bar{z}_1} \frac{\log(1-\tau\bar{z}_2)}{\bar{z}_2} d\bar{z}_2 d\bar{z}_1 + 1\right]\\
&+ \frac{|z|^4-1}{2!}\left[\frac{1}{\bar{\tau}z^2}\int_0^z\int_0^{z_1} \frac{\log(1-\bar{\tau}z_2)}{z_2} dz_2 dz_1\right.\\
&\left.+ \frac{1}{\tau\bar{z}^2}\int_0^{\bar{z}}\int_0^{\bar{z}_1} \frac{\log(1-\tau\bar{z}_2)}{\bar{z}_2} d\bar{z}_2 d\bar{z}_1 + \frac{1}{2!}\right]\\
&+ \frac{|z|^2-1}{2!}\left[\frac{1}{\bar{\tau}z}\int_0^z \frac{1}{z_1^2}\int_0^{z_1} \log(1-\bar{\tau}z_2) dz_2 dz_1\right.\\
&\left.+ \frac{1}{\tau\bar{z}}\int_0^{\bar{z}} \frac{1}{\bar{z}_1^2}\int_0^{\bar{z}_1} \log(1-\tau\bar{z}_2) d\bar{z}_2 d\bar{z}_1 + \frac{1}{2!}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1-|z|^6}{3!}\left[\frac{1}{\bar\tau z^3}\int_0^z\int_0^{z_1}\log(1-\bar\tau z_2)\mathrm{d}z_2\mathrm{d}z_1\right.\\
&\left.+\frac{1}{\tau\bar z^3}\int_0^{\bar z}\int_0^{\bar z_1}\log(1-\tau\bar z_2)\mathrm{d}\bar z_2\mathrm{d}\bar z_1+\frac{1}{3!}\right]\\
&=(|z|^2-1)\left[\sum_{k=2}^{\infty}\frac{(\bar\tau z)^{k-1}+(\tau\bar z)^{k-1}}{k^2}+1\right]\\
&+\frac{1-|z|^4}{2!}\left[\sum_{k=2}^{\infty}\frac{(\bar\tau z)^{k-1}+(\tau\bar z)^{k-1}}{k^2(k+1)}+\frac{1}{2!}\right]\\
&+\frac{1-|z|^2}{2!}\left[\sum_{k=2}^{\infty}\frac{(\bar\tau z)^{k-1}+(\tau\bar z)^{k-1}}{k^2(k+1)}+\frac{1}{2!}\right]\\
&+\frac{|z|^6-1}{3!}\left[\sum_{k=2}^{\infty}\frac{(\bar\tau z)^{k-1}+(\tau\bar z)^{k-1}}{k(k+1)(k+2)}+\frac{1}{3!}\right]
\end{aligned}
\tag{4.3}
$$

and $g_j(z,\tau)$ $(j=1,2,3)$ are given by (2.48), (2.49) and (2.50), respectively.
In general, the following result holds.

Theorem 4.1 *PHD problem* (4.1) *is solvable and its unique solution may be written as* (4.2).

Remark 4.1 $g_k(z,\tau)$ $(k=1,2,3,4)$ in the expression (4.2) are called the kernel functions of the solution for PHD problem (4.1). Moreover, the kernel functions of the solution have the following basic properties

1. $\partial_{\bar z}\partial_z g_1(z,\tau)=0$ and $\partial_{\bar z}\partial_z g_k(z,\tau)=g_{k-1}(z,\tau)$ for $k>1$.
2. $\lim_{z\to t,|t|=1,|z|<1}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma(\tau)g_1(z,\tau)\frac{\mathrm{d}\tau}{\tau}=\gamma(t)$ for $\gamma\in H(\partial D)$.
3. $\lim_{z\to t,|t|=1,|z|<1}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma(\tau)g_2(z,\tau)\frac{\mathrm{d}\tau}{\tau}=0$ for $\gamma\in H(\partial D)$.
4. $\lim_{z\to t,|t|=1,|z|<1}g_k(z,\tau)=0$ for $k>2$ and $|\tau|=1$.

Especially, $g_1(z,\tau)$ is the so-called Schwarz kernel.

Next we will consider the general Dirichlet problem for the polyharmonic function: find a function $w\in Har_n(\mathbb{D})$ satisfying the Dirichlet type boundary condition

$$
[(\partial_z\partial_{\bar z})^j w]^+(t)=\gamma_j(t),\quad t\in\partial\mathbb{D},\ 0\le j<n,
\tag{4.4}
$$

where $\mathbb{D}\subset\mathbb{C}$ is also the unit disc and $\partial\mathbb{D}$ its boundary, and $\gamma_j\in H(\partial\mathbb{D})$ for $0\le j<n$. This problem is also called PHD problem.

Theorem 4.2 *The PHD problem* (4.4) *is solvable and its unique solution is*

$$
w(z)=\sum_{k=1}^{n}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_{k-1}(\tau)g_k(z,\tau)\frac{\mathrm{d}\tau}{\tau},
\tag{4.5}
$$

where the kernel functions $g_k(z,\tau)$ $(0<k\le n)$ possess the basic properties given in Remark 4.1.

Proof By Theorem 3.1, one has

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_0(\tau) g_1(z,\tau)\frac{d\tau}{\tau}$$

when $n = 1$. When $n > 1$, the boundary condition (4.4) may be rewritten as

$$[(\partial_z\partial_{\bar{z}})^{j-1}(\partial_z\partial_{\bar{z}}w)]^+(t) = \gamma_j(t), \quad t\in\partial\mathbb{D},\ 1\le j<n, \tag{4.6}$$

and

$$w^+(t) = \gamma_0(t), \quad t\in\partial\mathbb{D}. \tag{4.7}$$

By the inductive assumption, PHD problem (4.6) has its unique solution

$$(\partial_z\partial_{\bar{z}}w)(z) = \sum_{k=1}^{n-1}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_k(\tau)g_k(z,\tau)\frac{d\tau}{\tau}. \tag{4.8}$$

By Theorem 3.1, the Dirichlet problem for the polyharmonic equation (4.8) with (4.7) has the unique solution

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_0(\tau)g_1(z,\tau)\frac{d\tau}{\tau} + \frac{1}{\pi}\int_{\mathbb{D}}\left[\sum_{k=1}^{n-1}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_k(\tau)g_k(\zeta,\tau)\frac{d\tau}{\tau}\right] \\ &\quad\times\log\left|\frac{\zeta-z}{1-\bar{z}\zeta}\right|^2 d\xi\, d\eta \\ &= \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_0(\tau)g_1(z,\tau)\frac{d\tau}{\tau} + \sum_{k=1}^{n-1}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_k(\tau)\left[\frac{1}{\pi}\int_{\mathbb{D}}g_k(\zeta,\tau)\right. \\ &\quad\left.\times\ \log\left|\frac{\zeta-z}{1-\bar{z}\zeta}\right|^2 d\xi\, d\eta\right]\frac{d\tau}{\tau}. \end{aligned}$$

With, see [3],

$$g_{k+1}(z,\tau) = \frac{1}{\pi}\int_{\mathbb{D}} g_k(\zeta,\tau)\log\left|\frac{\zeta-z}{1-\bar{z}\zeta}\right|^2 d\xi\, d\eta$$

this is

$$w(z) = \sum_{k=1}^{n}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_{k-1}(\tau)g_k(z,\tau)\frac{d\tau}{\tau}.$$

Therefore, the PHD problem (4.4) has a unique solution. On the other hand, it is easy to verify $w(z)$ given by (4.5) being the solution for the PHD problem (4.4). This completes the proof.

Let

$$g_{k,r}(z,\tau) = \frac{1-|z|^{2k}}{kr}\left[\frac{f(z)}{\bar{\tau}z^k} + \frac{\overline{f(z)}}{\tau\bar{z}^k} + \frac{1}{kr}\right] \tag{4.9}$$

and

$$\Delta_{k,r}(z,\tau) = \frac{|z|^2-1}{kr}\left[\frac{1}{\bar{\tau} z}\int_0^{z}\frac{f(z)}{z^k}\mathrm{d}z + \frac{1}{\tau\bar{z}}\int_0^{\bar{z}}\frac{\overline{f(z)}}{\bar{z}^k}\mathrm{d}\bar{z} + \frac{1}{kr}\right]$$
$$+\frac{1-|z|^{2(k+1)}}{(k+1)kr}\left[\frac{1}{\bar{\tau} z^{k+1}}\int_0^{\zeta} f(z)\mathrm{d}z + \frac{1}{\tau\bar{z}^{k+1}}\int_0^{\bar{\zeta}}\overline{f(z)}\mathrm{d}\bar{z} + \frac{1}{(k+1)kr}\right], \tag{4.10}$$

where $k = 1, 2, \ldots$ and $r = \pm 1, \pm 2, \ldots$, and $f \in H_1(\mathbb{D})$ being continuous up to the boundary $\partial\mathbb{D}$ except at most at finitely many singular points of order less than 1 on $\partial\mathbb{D}$ and $z = 0$ being a zero at least of order k for f. To obtain the explicit expression of $g_k(z,\tau)$, we need the following lemma.

Lemma 4.1 *$(\partial_z\partial_{\bar{z}}\Delta_{k,r})(z,\tau) = g_{k,r}(z,\tau)$ and $\lim_{z\to t,|t|=1,|z|<1}\Delta_{k,r}(z,\tau) = 0$, where $g_{k,r}$, $\Delta_{k,r}$ are given by (4.9) and (4.10) respectively.*

Proof By a simple computation, one has

$$(\partial_z\partial_{\bar{z}})\left\{\frac{|z|^2-1}{kr}\left[\frac{1}{\bar{\tau} z}\int_0^{\zeta}\frac{f(z)}{z^k}\mathrm{d}z + \frac{1}{\tau\bar{z}}\int_0^{\bar{\zeta}}\frac{\overline{f(z)}}{\bar{z}^k}\mathrm{d}\bar{z} + \frac{1}{kr}\right]\right\}$$
$$= \frac{1}{kr}\left[\frac{f(z)}{\bar{\tau} z^k} + \frac{\overline{f(z)}}{\tau\bar{z}^k} + \frac{1}{kr}\right]$$

and

$$(\partial_z\partial_{\bar{z}})\left\{\frac{1-|z|^{2(k+1)}}{(k+1)kr}\left[\frac{1}{\bar{\tau} z^{k+1}}\int_0^{\zeta} f(z)\mathrm{d}z + \frac{1}{\tau\bar{z}^{k+1}}\int_0^{\bar{\zeta}}\overline{f(z)}\mathrm{d}\bar{z} + \frac{1}{(k+1)kr}\right]\right\}.$$
$$= \frac{-|z|^{2k}}{kr}\left[\frac{f(z)}{\bar{\tau} z^k} + \frac{\overline{f(z)}}{\tau\bar{z}^k} + \frac{1}{kr}\right]$$

It follows that $(\partial_z\partial_{\bar{z}}\Delta_{k,r})(z,\tau) = g_{k,r}(z,\tau)$. Obviously $\lim_{z\to t,|t|=1,|z|<1}\Delta_{k,r}(z,\tau) = 0$. This completes the proof.

Remark 4.2 By Lemma 4.1, one may easily get $g_3(z,\tau)$ from $g_2(z,\tau)$, and $g_4(z,\tau)$ from $g_3(z,\tau)$. However, it is fairly complicated to obtain them by the reflection method used in Sect. 2

Let

$$g_4(z,\tau) = g_{4,1}(z,\tau) + g_{4,2}(z,\tau) + g_{4,3}(z,\tau) + g_{4,4}(z,\tau).$$

By Lemma 4.1, one immediately obtains

$$g_5(z,\tau) = \Delta_{5,1}(z,\tau) + \Delta_{5,2}(z,\tau) + \Delta_{5,3}(z,\tau) + \Delta_{5,4}(z,\tau) \tag{4.11}$$

with

$$\Delta_{5,1}(z,\tau) = (|z|^2-1)\Bigg[\frac{1}{\bar{\tau} z}\int_0^{z}\frac{1}{z_1}\int_0^{z_1}\frac{1}{z_2}\int_0^{z_2}\frac{\log(1-\bar{\tau} z_3)}{z_3}\mathrm{d}z_3\mathrm{d}z_2\mathrm{d}z_1$$

$$
\begin{aligned}
&+\frac{1}{\tau\bar{z}}\int_0^{\bar{z}}\frac{1}{\bar{z}_1}\int_0^{\bar{z}_1}\frac{1}{\bar{z}_2}\int_0^{\bar{z}_2}\frac{\log(1-\tau\bar{z}_3)}{\bar{z}_3}d\bar{z}_3d\bar{z}_2d\bar{z}_1+1\Bigg]\\
&+\frac{1-|z|^4}{2!}\Bigg[\frac{1}{\bar{\tau}z^2}\int_0^{z}\int_0^{z_1}\frac{1}{z_2}\int_0^{z_2}\frac{\log(1-\bar{\tau}z_3)}{z_3}dz_3dz_2dz_1\\
&+\frac{1}{\tau\bar{z}^2}\int_0^{\bar{z}}\int_0^{\bar{z}_1}\frac{1}{\bar{z}_2}\int_0^{\bar{z}_2}\frac{\log(1-\tau\bar{z}_3)}{\bar{z}_3}d\bar{z}_3d\bar{z}_2d\bar{z}_1+\frac{1}{2!}\Bigg]\\
&=(1-|z|^2)\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{k-1}+(\tau\bar{z})^{k-1}}{k^4}+1\right]\\
&-\frac{1-|z|^4}{2!}\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{(k-1)}+(\tau\bar{z})^{k-1}}{k^3(k+1)}+\frac{1}{2!}\right],
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
\Delta_{5,2}(z,\tau)&=\frac{1-|z|^2}{2!}\Bigg[\frac{1}{\bar{\tau}z}\int_0^{z}\frac{1}{z_1^2}\int_0^{z_1}\int_0^{z_2}\frac{\log(1-\bar{\tau}z_3)}{z_3}dz_3dz_2dz_1\\
&+\frac{1}{\tau\bar{z}}\int_0^{\bar{z}}\frac{1}{\bar{z}_1^2}\int_0^{\bar{z}_1}\int_0^{\bar{z}_2}\frac{\log(1-\tau\bar{z}_3)}{\bar{z}_3}d\bar{z}_3d\bar{z}_2d\bar{z}_1+\frac{1}{2!}\Bigg]\\
&+\frac{|z|^6-1}{3!}\Bigg[\frac{1}{\bar{\tau}z^3}\int_0^{z}\int_0^{z_1}\int_0^{z_2}\frac{\log(1-\bar{\tau}z_3)}{z_3}dz_3dz_2dz_1\\
&+\frac{1}{\tau\bar{z}^3}\int_0^{\bar{z}}\int_0^{\bar{z}_1}\int_0^{\bar{z}_2}\frac{\log(1-\tau\bar{z}_3)}{\bar{z}_3}d\bar{z}_3d\bar{z}_2d\bar{z}_1+\frac{1}{3!}\Bigg]\\
&=-\frac{1-|z|^2}{2!}\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{k-1}+(\tau\bar{z})^{k-1}}{k^3(k+1)}+\frac{1}{2!}\right]\\
&+\frac{1-|z|^6}{3!}\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{k-1}+(\tau\bar{z})^{k-1}}{k^2(k+1)(k+2)}+\frac{1}{3!}\right],
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
\Delta_{5,3}(z,\tau)&=\frac{1-|z|^2}{2!}\Bigg[\frac{1}{\bar{\tau}z}\int_0^{z}\frac{1}{z_1}\int_0^{z_1}\frac{1}{z_2^2}\int_0^{z_2}\log(1-\bar{\tau}z_3)dz_3dz_2dz_1\\
&+\frac{1}{\tau\bar{z}}\int_0^{\bar{z}}\frac{1}{\bar{z}_1}\int_0^{\bar{z}_1}\frac{1}{\bar{z}_2^2}\int_0^{\bar{z}_2}\log(1-\tau\bar{z}_3)d\bar{z}_3d\bar{z}_2d\bar{z}_1+\frac{1}{2!}\Bigg]\\
&+\frac{|z|^4-1}{2\cdot 2!}\Bigg[\frac{1}{\bar{\tau}z^2}\int_0^{z}\int_0^{z_1}\frac{1}{z_2^2}\int_0^{z_2}\log(1-\bar{\tau}z_3)dz_3dz_2dz_1
\end{aligned}
$$

$$+\frac{1}{\tau\bar{z}^2}\int_0^{\bar{z}}\int_0^{\bar{z}_1}\frac{1}{\bar{z}_2^2}\int_0^{\bar{z}_2}\log(1-\tau\bar{z}_3)d\bar{z}_3d\bar{z}_2d\bar{z}_1+\frac{1}{2\cdot 2!}\Bigg]$$

$$=-\frac{1-|z|^2}{2!}\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{k-1}+(\tau\bar{z})^{k-1}}{k^3(k+1)}+\frac{1}{2!}\right]$$

$$+\frac{1-|z|^4}{2\cdot 2!}\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{k-1}+(\tau\bar{z})^{k-1}}{k^2(k+1)^2}+\frac{1}{2\cdot 2!}\right], \tag{4.14}$$

and

$$\Delta_{5,4}(z,\tau)=\frac{|z|^2-1}{3!}\Bigg[\frac{1}{\bar{\tau}z}\int_0^{z}\frac{1}{z_1^3}\int_0^{z_1}\int_0^{z_2}\log(1-\bar{\tau}z_3)dz_3dz_2dz_1$$

$$+\frac{1}{\tau\bar{z}}\int_0^{\bar{z}}\frac{1}{\bar{z}_1^3}\int_0^{\bar{z}_1}\int_0^{\bar{z}_2}\log(1-\tau\bar{z}_3)d\bar{z}_3d\bar{z}_2d\bar{z}_1+\frac{1}{3!}\Bigg]$$

$$+\frac{1-|z|^8}{4!}\Bigg[\frac{1}{\bar{\tau}z^4}\int_0^{z}\int_0^{z_1}\int_0^{z_2}\log(1-\bar{\tau}z_3)dz_3dz_2dz_1$$

$$+\frac{1}{\tau\bar{z}^4}\int_0^{\bar{z}}\int_0^{\bar{z}_1}\int_0^{\bar{z}_2}\log(1-\tau\bar{z}_3)d\bar{z}_3d\bar{z}_2d\bar{z}_1+\frac{1}{4!}\Bigg]$$

$$=\frac{1-|z|^2}{3!}\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{k-1}+(\tau\bar{z})^{k-1}}{k^2(k+1)(k+2)}+\frac{1}{3!}\right]$$

$$-\frac{1-|z|^8}{4!}\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{k-1}+(\tau\bar{z})^{k-1}}{k(k+1)(k+2)(k+3)}+\frac{1}{4!}\right]. \tag{4.15}$$

The relations $(\partial_{\bar{z}}\partial_z)\Delta_{5,j}(z,\tau)=g_{4,j}(z,\tau)$ remain valid for $j=1,2,3,4$.
Also writing

$$\begin{cases}\Delta_{5,1}(z,\tau)=g_{5,1}(z,\tau)+g_{5,2}(z,\tau),\\ \Delta_{5,2}(z,\tau)=g_{5,3}(z,\tau)+g_{5,4}(z,\tau),\\ \Delta_{5,3}(z,\tau)=g_{5,5}(z,\tau)+g_{5,6}(z,\tau),\\ \Delta_{5,4}(z,\tau)=g_{5,7}(z,\tau)+g_{5,8}(z,\tau),\end{cases}$$

one has

$$g_6(z,\tau)=\sum_{k=1}^{8}\Delta_{6,k}(z,\tau) \tag{4.16}$$

with

$$\Delta_{6,1}(z,\tau)=(1-|z|^2)\Bigg[\frac{1}{\bar{\tau}z}\int_0^{z}\frac{1}{z_1}\int_0^{z_1}\frac{1}{z_2}\int_0^{z_2}\frac{1}{z_3}\int_0^{z_3}\frac{\log(1-\bar{\tau}z_4)}{z_4}dz_4dz_3dz_2dz_1$$

$$+\frac{1}{\tau\bar{z}}\int_0^{\bar{z}}\frac{1}{\bar{z}_1}\int_0^{\bar{z}_1}\frac{1}{\bar{z}_2}\int_0^{\bar{z}_2}\frac{1}{\bar{z}_3}\int_0^{\bar{z}_3}\frac{\log(1-\tau\bar{z}_4)}{\bar{z}_4}d\bar{z}_4 d\bar{z}_3 d\bar{z}_2 d\bar{z}_1+1\Bigg]$$

$$+\frac{|z|^4-1}{2!}\Bigg[\frac{1}{\bar{\tau}z^2}\int_0^{z}\int_0^{z_1}\frac{1}{z_2}\int_0^{z_2}\frac{1}{z_3}\int_0^{z_3}\frac{\log(1-\bar{\tau}z_4)}{z_4}dz_4 dz_3 dz_2 dz_1$$

$$+\frac{1}{\tau\bar{z}^2}\int_0^{\bar{z}}\int_0^{\bar{z}_1}\frac{1}{\bar{z}_2}\int_0^{\bar{z}_2}\frac{1}{\bar{z}_3}\int_0^{\bar{z}_3}\frac{\log(1-\tau\bar{z}_4)}{\bar{z}_4}d\bar{z}_4 d\bar{z}_3 d\bar{z}_2 d\bar{z}_1+\frac{1}{2!}\Bigg]$$

$$=-(1-|z|^2)\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{k-1}+(\tau\bar{z})^{k-1}}{k^5}+1\right]$$

$$+\frac{1-|z|^4}{2!}\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{k-1}+(\tau\bar{z})^{k-1}}{k^4(k+1)}+\frac{1}{2!}\right], \tag{4.17}$$

$$\Delta_{6,2}(z,\tau)=\frac{|z|^2-1}{2!}\Bigg[\frac{1}{\bar{\tau}z}\int_0^{z}\frac{1}{z_1^2}\int_0^{z_1}\int_0^{z_2}\frac{1}{z_3}\int_0^{z_3}\frac{\log(1-\bar{\tau}z_4)}{z_4}dz_4 dz_3 dz_2 dz_1$$

$$+\frac{1}{\tau\bar{z}_2}\int_0^{\bar{z}_1}\frac{1}{\bar{z}_2^2}\int_0^{\bar{z}_2}\int_0^{\bar{z}_3}\frac{1}{\bar{z}_3}\int_0^{\bar{z}_3}\frac{\log(1-\tau\bar{z}_4)}{\bar{z}_4}d\bar{z}_4 d\bar{z}_3 d\bar{z}_2 d\bar{z}_1+\frac{1}{2!}\Bigg]$$

$$+\frac{1-|z|^6}{3!}\Bigg[\frac{1}{\bar{\tau}z^3}\int_0^{z}\int_0^{z_1}\int_0^{z_2}\frac{1}{z_3}\int_0^{z_3}\frac{\log(1-\bar{\tau}z_4)}{z_4}dz_4 dz_3 dz_2 dz_1$$

$$+\frac{1}{\tau\bar{z}^3}\int_0^{\bar{z}}\int_0^{\bar{z}_1}\int_0^{\bar{z}_2}\frac{1}{\bar{z}_3}\int_0^{\bar{z}_3}\frac{\log(1-\tau\bar{z}_4)}{\bar{z}_4}d\bar{z}_4 d\bar{z}_3 d\bar{z}_2 d\bar{z}_1+\frac{1}{3!}\Bigg]$$

$$=\frac{1-|z|^2}{2!}\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{k-1}+(\tau\bar{z})^{k-1}}{k^4(k+1)}+\frac{1}{2!}\right]$$

$$-\frac{1-|z|^6}{3!}\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{k-1}+(\tau\bar{z})^{k-1}}{k^3(k+1)(k+2)}+\frac{1}{3!}\right], \tag{4.18}$$

$$\Delta_{6,3}(z,\tau)=\frac{|z|^2-1}{2!}\Bigg[\frac{1}{\bar{\tau}z}\int_0^{z}\frac{1}{z_1}\int_0^{z_1}\frac{1}{z_2^2}\int_0^{z_2}\int_0^{z_3}\frac{\log(1-\bar{\tau}z_4)}{z_4}dz_4 dz_3 dz_2 dz_1$$

$$+\frac{1}{\tau\bar{z}}\int_0^{\bar{z}}\frac{1}{\bar{z}_1}\int_0^{\bar{z}_1}\frac{1}{\bar{z}_2^2}\int_0^{\bar{z}_2}\int_0^{\bar{z}_3}\frac{\log(1-\tau\bar{z}_4)}{\bar{z}_4}d\bar{z}_4 d\bar{z}_3 d\bar{z}_2 d\bar{z}_1+\frac{1}{2!}\Bigg]$$

$$+\frac{1-|z|^4}{2\cdot 2!}\Bigg[\frac{1}{\bar{\tau}z^2}\int_0^{z}\int_0^{z_1}\frac{1}{z_2^2}\int_0^{z_2}\int_0^{z_3}\frac{\log(1-\bar{\tau}z_4)}{z_4}dz_4 dz_3 dz_2 dz_1$$

$$+\frac{1}{\tau\bar{z}^2}\int_0^{\bar{z}}\int_0^{\bar{z}_1}\frac{1}{\bar{z}_2^2}\int_0^{\bar{z}_2}\int_0^{\bar{z}_3}\frac{\log(1-\tau\bar{z}_4)}{\bar{z}_4}d\bar{z}_4 d\bar{z}_3 d\bar{z}_2 d\bar{z}_1+\frac{1}{2\cdot 2!}\Bigg]$$

$$=\frac{1-|z|^2}{2!}\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{k-1}+(\tau\bar{z})^{k-1}}{k^4(k+1)}+\frac{1}{2!}\right]$$

$$-\frac{1-|z|^4}{2\cdot 2!}\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{k-1}+(\tau\bar{z})^{k-1}}{k^3(k+1)^2}+\frac{1}{2\cdot 2!}\right], \tag{4.19}$$

$$\Delta_{6,4}(z,\tau)=\frac{1-|z|^2}{3!}\Bigg[\frac{1}{\bar{\tau}z}\int_0^{z}\frac{1}{z_1^3}\int_0^{z_1}\int_0^{z_2}\int_0^{z_3}\frac{\log(1-\bar{\tau}z_4)}{z_4}dz_4dz_3dz_2dz_1$$

$$+\frac{1}{\tau\bar{z}}\int_0^{\bar{z}}\frac{1}{\bar{z}_1^3}\int_0^{\bar{z}_1}\int_0^{\bar{z}_2}\int_0^{\bar{z}_3}\frac{\log(1-\tau\bar{z}_4)}{\bar{z}_4}d\bar{z}_4 d\bar{z}_3 d\bar{z}_2 d\bar{z}_1+\frac{1}{3!}\Bigg]$$

$$+\frac{|z|^8-1}{4!}\Bigg[\frac{1}{\bar{\tau}z^4}\int_0^{z}\int_0^{z_1}\int_0^{z_2}\int_0^{z_3}\frac{\log(1-\bar{\tau}z_4)}{z_4}dz_4dz_3dz_2dz_1$$

$$+\frac{1}{\tau\bar{z}^4}\int_0^{\bar{z}}\int_0^{\bar{z}_1}\int_0^{\bar{z}_2}\int_0^{\bar{z}_3}\frac{\log(1-\tau\bar{z}_4)}{\bar{z}_4}d\bar{z}_4 d\bar{z}_3 d\bar{z}_2 d\bar{z}_1+\frac{1}{4!}\Bigg]$$

$$=-\frac{1-|z|^2}{3!}\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{k-1}+(\tau\bar{z})^{k-1}}{k^3(k+1)(k+2)}+\frac{1}{3!}\right]$$

$$+\frac{1-|z|^8}{4!}\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{k-1}+(\tau\bar{z})^{k-1}}{k^2(k+1)(k+2)(k+3)}+\frac{1}{4!}\right], \tag{4.20}$$

$$\Delta_{6,5}(z,\tau)=\frac{|z|^2-1}{2!}\Bigg[\frac{1}{\bar{\tau}z}\int_0^{z}\frac{1}{z_1}\int_0^{z_1}\frac{1}{z_2}\int_0^{z_2}\frac{1}{z_3^2}\int_0^{z_3}\log(1-\bar{\tau}z_4)dz_4dz_3dz_2dz_1$$

$$+\frac{1}{\tau\bar{z}}\int_0^{\bar{z}}\frac{1}{\bar{z}_1}\int_0^{\bar{z}_1}\frac{1}{\bar{z}_2}\int_0^{\bar{z}_2}\frac{1}{\bar{z}_3^2}\int_0^{\bar{z}_3}\log(1-\tau\bar{z}_4)d\bar{z}_4 d\bar{z}_3 d\bar{z}_2 d\bar{z}_1+\frac{1}{2!}\Bigg]$$

$$+\frac{1-|z|^4}{2\cdot 2!}\Bigg[\frac{1}{\bar{\tau}z^2}\int_0^{z}\int_0^{z_1}\frac{1}{z_2}\int_0^{z_2}\frac{1}{z_3^2}\int_0^{z_3}\log(1-\bar{\tau}z_4)dz_4dz_3dz_2dz_1$$

$$+\frac{1}{\tau\bar{z}^2}\int_0^{\bar{z}}\int_0^{\bar{z}_1}\frac{1}{\bar{z}_2}\int_0^{\bar{z}_2}\frac{1}{\bar{z}_3^2}\int_0^{\bar{z}_3}\log(1-\tau\bar{z}_4)d\bar{z}_4 d\bar{z}_3 d\bar{z}_2 d\bar{z}_1+\frac{1}{2\cdot 2!}\Bigg]$$

$$=\frac{1-|z|^2}{2!}\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{k-1}+(\tau\bar{z})^{k-1}}{k^4(k+1)}+\frac{1}{2!}\right]$$

$$-\frac{1-|z|^4}{2\cdot 2!}\left[\sum_{k=2}^{\infty}\frac{(\bar{\tau}z)^{k-1}+(\tau\bar{z})^{k-1}}{k^3(k+1)^2}+\frac{1}{2\cdot 2!}\right], \tag{4.21}$$

$$\Delta_{6,6}(z,\tau) = \frac{1-|z|^2}{2\cdot 2!}\Bigg[\frac{1}{\bar\tau z}\int_0^z \frac{1}{z_1^2}\int_0^{z_1}\int_0^{z_2}\frac{1}{z_3^2}\int_0^{z_3}\log(1-\bar\tau z_4)\mathrm{d}z_4\mathrm{d}z_3\mathrm{d}z_2\mathrm{d}z_1$$

$$+\frac{1}{\tau\bar z}\int_0^{\bar z}\frac{1}{\bar z_1^2}\int_0^{\bar z_1}\int_0^{\bar z_2}\frac{1}{\bar z_3^2}\int_0^{\bar z_3}\log(1-\tau\bar z_4)\mathrm{d}\bar z_4\mathrm{d}\bar z_3\mathrm{d}\bar z_2\mathrm{d}\bar z_1+\frac{1}{2\cdot 2!}\Bigg]$$

$$+\frac{|z|^6-1}{2\cdot 3!}\Bigg[\frac{1}{\bar\tau z^3}\int_0^z\int_0^{z_1}\int_0^{z_2}\frac{1}{z_3^2}\int_0^{z_3}\log(1-\bar\tau z_4)\mathrm{d}z_4\mathrm{d}z_3\mathrm{d}z_2\mathrm{d}z_1$$

$$+\frac{1}{\tau\bar z^3}\int_0^{\bar z}\int_0^{\bar z_1}\int_0^{\bar z_2}\frac{1}{\bar z_3^2}\int_0^{\bar z_3}\log(1-\tau\bar z_4)\mathrm{d}\bar z_4\mathrm{d}\bar z_3\mathrm{d}\bar z_2\mathrm{d}\bar z_1+\frac{1}{2\cdot 3!}\Bigg]$$

$$=-\frac{1-|z|^2}{2\cdot 2!}\left[\sum_{k=2}^{\infty}\frac{(\bar\tau z)^{k-1}+(\tau\bar z)^{k-1}}{k^3(k+1)^2}+\frac{1}{2\cdot 2!}\right]$$

$$+\frac{1-|z|^2}{2\cdot 3!}\left[\sum_{k=2}^{\infty}\frac{(\bar\tau z)^{k-1}+(\tau\bar z)^{k-1}}{k^2(k+1)^2(k+2)}+\frac{1}{2\cdot 3!}\right], \tag{4.22}$$

$$\Delta_{6,7}(z,\tau) = \frac{1-|z|^2}{3!}\Bigg[\frac{1}{\bar\tau z}\int_0^z \frac{1}{z_1}\int_0^{z_1}\frac{1}{z_2^3}\int_0^{z_2}\int_0^{z_3}\log(1-\bar\tau z_4)\mathrm{d}z_4\mathrm{d}z_3\mathrm{d}z_2\mathrm{d}z_1$$

$$+\frac{1}{\tau\bar z}\int_0^{\bar z}\frac{1}{\bar z_1}\int_0^{\bar z_1}\frac{1}{\bar z_2^3}\int_0^{\bar z_2}\int_0^{\bar z_3}\log(1-\tau\bar z_4)\mathrm{d}\bar z_4\mathrm{d}\bar z_3\mathrm{d}\bar z_2\mathrm{d}\bar z_1+\frac{1}{3!}\Bigg]$$

$$+\frac{|z|^4-1}{2\cdot 3!}\Bigg[\frac{1}{\bar\tau z^2}\int_0^z\int_0^{z_1}\frac{1}{z_2^3}\int_0^{z_2}\int_0^{z_3}\log(1-\bar\tau z_4)\mathrm{d}z_4\mathrm{d}z_3\mathrm{d}z_2\mathrm{d}z_1$$

$$+\frac{1}{\tau\bar z^2}\int_0^{\bar z}\int_0^{\bar z_1}\frac{1}{\bar z_2^3}\int_0^{\bar z_2}\int_0^{\bar z_3}\log(1-\tau\bar z_4)\mathrm{d}\bar z_4\mathrm{d}\bar z_3\mathrm{d}\bar z_2\mathrm{d}\bar z_1+\frac{1}{2\cdot 3!}\Bigg]$$

$$=-\frac{1-|z|^2}{3!}\left[\sum_{k=2}^{\infty}\frac{(\bar\tau z)^{k-1}+(\tau\bar z)^{k-1}}{k^3(k+1)(k+2)}+\frac{1}{3!}\right]$$

$$+\frac{1-|z|^4}{2\cdot 3!}\left[\sum_{k=2}^{\infty}\frac{(\bar\tau z)^{k-1}+(\tau\bar z)^{k-1}}{k^2(k+1)^2(k+2)}+\frac{1}{2\cdot 3!}\right], \tag{4.23}$$

and

$$\Delta_{6,8}(z,\tau) = \frac{1-|z|^2}{4!}\Bigg[\frac{1}{\bar\tau z}\int_0^z \frac{1}{z_1^4}\int_0^{z_1}\int_0^{z_2}\int_0^{z_3}\log(1-\bar\tau z_4)\mathrm{d}z_4\mathrm{d}z_3\mathrm{d}z_2\mathrm{d}z_1$$

$$+\frac{1}{\tau\bar z}\int_0^{\bar z}\frac{1}{\bar z_1^4}\int_0^{\bar z_1}\int_0^{\bar z_2}\int_0^{\bar z_3}\log(1-\tau\bar z_4)\mathrm{d}\bar z_4\mathrm{d}\bar z_3\mathrm{d}\bar z_2\mathrm{d}\bar z_1+\frac{1}{4!}\Bigg]$$

$$
\begin{aligned}
&+\frac{|z|^{10}-1}{5!}\left[\frac{1}{\bar\tau z^5}\int_0^z\int_0^{z_1}\int_0^{z_2}\int_0^{z_3}\log(1-\bar\tau z_4)\mathrm{d}z_4\mathrm{d}z_3\mathrm{d}z_2\mathrm{d}z_1\right.\\
&\left.+\frac{1}{\tau\bar z^5}\int_0^{\bar z}\int_0^{\bar z_1}\int_0^{\bar z_2}\int_0^{\bar z_3}\log(1-\tau\bar z_4)\mathrm{d}\bar z_4\mathrm{d}\bar z_3\mathrm{d}\bar z_2\mathrm{d}\bar z_1+\frac{1}{5!}\right]\\
=&-\frac{1-|z|^2}{4!}\left[\sum_{k=2}^{\infty}\frac{(\bar\tau z)^{k-1}+(\tau\bar z)^{k-1}}{k^2(k+1)(k+2)(k+3)}+\frac{1}{4!}\right]\\
&+\frac{1-|z|^{10}}{5!}\left[\sum_{k=2}^{\infty}\frac{(\bar\tau z)^{k-1}+(\tau\bar z)^{k-1}}{k^2(k+1)(k+2)(k+3)(k+4)}+\frac{1}{5!}\right]. \qquad (4.24)
\end{aligned}
$$

Remark 4.3 We only give the explicit expression for the kernel functions $g_5(z,\tau)$ and $g_6(z,\tau)$ by Lemma 4.1. Moreover $g_k(z,\tau)$ $(k>6)$ may also be expressed explicitly by Lemma 4.1. In fact, Lemma 4.1 indirectly gives a recursive relation between the neighboring kernel functions $g_k(z,\tau)$ and $g_{k+1}(z,\tau)$ for $k>1$.

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