Annali di Matematica (2008) 187:369–384
DOI 10.1007/s10231-007-0047-0

From Brunn–Minkowski to sharp Sobolev inequalities

S. G. Bobkov · M. Ledoux

Received: 12 October 2006 / Revised: 14 December 2006 / Published online: 21 March 2007
© Springer-Verlag 2007

Abstract We present a simple direct proof of the classical Sobolev inequality in $\mathbb{R}^n$ with best constant from the geometric Brunn–Minkowski–Lusternik inequality.

Mathematics Subject Classification 46-XX

1 Introduction

The classical Sobolev inequality in $\mathbb{R}^n$, $n \geq 3$, indicates that there is a constant $C_n > 0$ such that for all smooth enough (locally Lipschitz) functions $f : \mathbb{R}^n \to \mathbb{R}$ vanishing at infinity,

$$\|f\|_q \leq C_n \|\nabla f\|_2 \tag{1}$$

where $\frac{1}{q} = \frac{1}{2} - \frac{1}{n}$. Here $\|f\|_q$ denotes the usual L^q-norm of f with respect to Lebesgue measure on $\mathbb{R}^n$, and, for $p \geq 1$,

$$\|\nabla f\|_p = \left(\int_{\mathbb{R}^n} |\nabla f|^p \mathrm{d}x \right)^{1/p}$$

where $|\nabla f|$ is the Euclidean norm of the gradient ∇f of f.

Research supported in part by NSF Gr. No. 0405587 and B. Zegarlinski's Pierre de Fermat Grant 2006 from the Région Midi-Pyrénées, France.

S. G. Bobkov (✉)
School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
e-mail: bobkov@math.umn.edu

M. Ledoux
Institut de Mathématiques, Université Paul-Sabatier, 31062 Toulouse, France
e-mail: ledoux@math.ups-tlse.fr

Inequality (1) goes back to Sobolev [41], as a consequence of a Riesz type rearrangement inequality and the Hardy–Littlewood–Sobolev fractional-integral convolution inequality. Other approaches, including the elementary Gagliardo–Nirenberg argument [20,35], are discussed in classical textbooks (cf. e.g. [1,33,43] ...). The best possible constant in the Sobolev inequality (1) was established independently by Aubin [4] and Talenti [42] in 1976 using symmetrization methods of isoperimetric flavor, together with the study of the one-dimensional extremal problem. Rearrangements arguments have been developed extensively in this context (cf. [11,29] ...). The optimal constant C_n is achieved on the extremal functions $f(x) = (\sigma + |x|^2)^{(2-n)/2}$, $x \in \mathbb{R}^n$, $\sigma > 0$. Building on early ideas by Rosen [38], Lieb [28] determined the best constant and the extremal functions in dimension 3. According to [39], the result seems to have been known before, at least back to the early sixties, in unpublished notes by Rodemich.

The geometric Brunn–Minkowski inequality, and its isoperimetric consequence, is a well-known argument to reach Sobolev type inequalities. It states that for every non-empty Borel measurable bounded sets A, B in $\mathbb{R}^n$,

$$\mathrm{vol}_n(A+B)^{1/n} \geq \mathrm{vol}_n(A)^{1/n} + \mathrm{vol}_n(B)^{1/n} \tag{2}$$

where $\mathrm{vol}_n(\cdot)$ denotes Euclidean volume. The Brunn–Minkowski inequality classically implies the isoperimetric inequality in $\mathbb{R}^n$. Choose namely for B a ball with radius $\varepsilon > 0$ and let then $\varepsilon \to 0$ to get that for any bounded measurable set A in $\mathbb{R}^n$,

$$\mathrm{vol}_{n-1}(\partial A) \geq n\omega_n^{1/n} \mathrm{vol}_n(A)^{(n-1)/n}$$

where $\mathrm{vol}_{n-1}(\partial A)$ is understood as the outer-Minkowski content of the boundary of A and ω_n is the volume of the Euclidean unit ball in $\mathbb{R}^n$. By means of the co-area formula [19,33], the isoperimetric inequality may then be stated equivalently on functions as the L^1-Sobolev inequality

$$\|f\|_q \leq \frac{1}{n\omega_n^{1/n}} \|\nabla f\|_1 \tag{3}$$

where $\frac{1}{q} = 1 - \frac{1}{n}$. Changing $f \geq 0$ into f^r for some suitable r and applying Hölder's inequality yields the L^2- Sobolev inequality (1), however not with its best constant. In the same way, the argument describes the full scale of Sobolev inequalities

$$\|f\|_q \leq C_n(p) \|\nabla f\|_p, \tag{4}$$

$1 \leq p < n$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, $f : \mathbb{R}^n \to \mathbb{R}$ smooth and vanishing at infinity. According to Gromov [34], the L^1-case of the Sobolev inequality appears in Brunn's work from 1887.

The purpose of this note is to show that the Brunn–Minkowski inequality may actually be used to also reach the optimal constants in the Sobolev inequalities (1) and (4). This new approach thus completely bridges the geometric Brunn–Minkowski inequalities and the functional Sobolev inequalities.

Inequality (2) was first proved by Brunn in 1887 for convex sets in dimension 3, then extended by Minkowski (cf. [40]). Lusternik [30] generalized the result in 1935 to arbitrary measurable sets. Lusternik's proof was further analyzed and extended in the works of Hadwiger and Ohmann [24] and Henstock and Macbeath [25] in the fifties. Note in particular that the one-dimensional case is immediate: assume that A and B

are non-empty compact sets in $\mathbb{R}$, and after a suitable shift, that $\sup A = 0 = \inf B$. Then $A \cap B = \{0\}$ and $A + B \supset A \cup B$.

Starting with the contribution [25], integral inequalities have been developed throughout the last century in the investigation of the geometric Brunn–Minkowski–Lusternik theorem. The idea of the following elementary, but fundamental, lemma goes back to Bonnesen's proof of the Brunn–Minkowski inequality (cf. [10]) and may be found already in the paper by Henstock and Macbeath [25]. The result appears in this form independently in the works of Dancs and Uhrin [14] and Das Gupta [15]. We enclose a proof for completeness. As a result, the proof below only relies on the one-dimensional Brunn–Minkowski–Lusternik inequality, which is the only basic ingredient in the argument. All the further developments and applications to Sobolev inequalities are consequences of this elementary lemma.

Lemma *Let $\theta \in [0,1]$ and u, v, w be non-negative measurable functions on $\mathbb{R}$ such that for all $x, y \in \mathbb{R}$,*

$$w\big(\theta x + (1-\theta)y\big) \geq \min\big(u(x), v(y)\big).$$

Then, if $\sup_{x\in\mathbb{R}} u(x) = \sup_{x\in\mathbb{R}} v(x) = 1$,

$$\int w \mathrm{d}x \geq \theta \int u \mathrm{d}x + (1-\theta) \int v \mathrm{d}x.$$

Proof Define, for $t > 0$, $E_u(t) = \{x \in \mathbb{R}; u(x) > t\}$ and similarly $E_v(t)$, $E_w(t)$. Since $\sup_{x\in\mathbb{R}} u(x) = \sup_{x\in\mathbb{R}} v(x) = 1$, for $0 < t < 1$, both $E_u(t)$ and $E_v(t)$ are non-empty, and $E_w(t) \supset \theta E_u(t) + (1-\theta)E_v(t)$. By the one-dimensional Brunn–Minkowski–Lusternik inequality (2), for every $0 < t < 1$,

$$\lambda\big(E_w(t)\big) \geq \theta\lambda\big(E_u(t)\big) + (1-\theta)\lambda\big(E_v(t)\big)$$

where λ denotes Lebesgue measure on $\mathbb{R}$. Hence,

$$\begin{aligned}\int w \mathrm{d}x &\geq \int_0^1 \lambda\big(E_w(t)\big)\mathrm{d}t \\ &\geq \theta \int_0^1 \lambda\big(E_u(t)\big)\mathrm{d}t + (1-\theta)\int_0^1 \lambda\big(E_v(t)\big)\mathrm{d}t \\ &= \theta \int u \mathrm{d}x + (1-\theta)\int v \mathrm{d}x\end{aligned}$$

which is the conclusion. □

As discussed in [14], the preceding lemma may be extended to more general means by elementary changes of variables. For $\alpha \in [-\infty, +\infty]$, denote by $M_\alpha^{(\theta)}(a,b)$ the α-mean of the non-negative numbers a, b with weights $\theta, 1-\theta \in [0,1]$ defined as

$$M_\alpha^{(\theta)}(a,b) = \big(\theta a^\alpha + (1-\theta)b^\alpha\big)^{1/\alpha}$$

(with the convention that $M_\alpha^{(\theta)}(a,b) = \max(a,b)$ if $\alpha = +\infty$, $M_\alpha^{(\theta)}(a,b) = \min(a,b)$ if $\alpha = -\infty$ and $M_\alpha^{(\theta)}(a,b) = a^\theta b^{1-\theta}$ if $\alpha = 0$) if $ab > 0$, and $M_\alpha^{(\theta)}(a,b) = 0$ if $ab = 0$.

Note the extension of the usual arithmetic-geometric mean inequality as

$$M_{\alpha_1}^{(\theta)}(a_1, b_1) M_{\alpha_2}^{(\theta)}(a_2, b_2) \geq M_{\alpha}^{(\theta)}(a_1 a_2, b_1 b_2) \tag{5}$$

if $\frac{1}{\alpha} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2}$, $\alpha_1 + \alpha_2 > 0$.

Corollary 1 *Let $-\infty \leq \alpha \leq +\infty$, $\theta \in [0,1]$ and u, v, w be non-negative measurable functions on $\mathbb{R}$ such that for all $x, y \in \mathbb{R}$,*

$$w\big(\theta x + (1-\theta)y\big) \geq M_{\alpha}^{(\theta)}\big(u(x), v(y)\big).$$

Then, if $a = \sup_{x\in\mathbb{R}} u(x) < \infty$, $b = \sup_{x\in\mathbb{R}} v(x) < \infty$,

$$\int w \mathrm{d}x \geq M_{\alpha}^{(\theta)}(a,b) M_{1}^{(\theta)}\bigg(\frac{1}{a}\int u \mathrm{d}x, \frac{1}{b}\int v \mathrm{d}x\bigg).$$

The statement still holds if a or $b = +\infty$ with the convention that $0 \times \infty = 0$.

Proof Assume first that $-\infty < \alpha < +\infty$. For $\rho = M_{\alpha}^{(\theta)}(a,b) > 0$, set

$$U(x) = \frac{1}{a}\, u\Big(\frac{a^{\alpha} x}{\rho^{\alpha}}\Big) \quad \text{and} \quad V(y) = \frac{1}{b}\, v\Big(\frac{b^{\alpha} y}{\rho^{\alpha}}\Big).$$

Then, if $\eta = \theta a^{\alpha}/\rho^{\alpha} (\in [0,1])$,

$$w\big(\eta x + (1-\eta)y\big) \geq M_{\alpha}^{(\theta)}(a,b) \min\big(U(x), V(y)\big)$$

for all $x, y \in \mathbb{R}$. Since $\sup_{x\in\mathbb{R}} U(x) = \sup_{x\in\mathbb{R}} V(x) = 1$, by the lemma,

$$\begin{aligned}\int w \mathrm{d}x &\geq M_{\alpha}^{(\theta)}(a,b)\bigg(\eta \int U \mathrm{d}x + (1-\eta)\int V \mathrm{d}x\bigg)\\ &= M_{\alpha}^{(\theta)}(a,b)\bigg(\frac{\theta}{a}\int u \mathrm{d}x + \frac{1-\theta}{b}\int v \mathrm{d}x\bigg)\end{aligned}$$

by definition of η. The cases $\alpha = -\infty$ and $\alpha = +\infty$ may be proved by standard limit considerations. The corollary is thus established. □

By the Hölder inequality (5), the preceding corollary implies the more classical Prékopa–Leindler theorem [27,36,37], as well as its generalized form put forward by Borell [8] and Brascamp and Lieb [9], in which the supremum norms of u and v do not appear. Namely, under the assumption of Corollary 1 and provided that $-1 \leq \alpha \leq +\infty$,

$$\begin{aligned}\int w \mathrm{d}x &\geq M_{\alpha}^{(\theta)}(a,b) M_{1}^{(\theta)}\bigg(\frac{1}{a}\int u \mathrm{d}x, \frac{1}{b}\int v \mathrm{d}x\bigg)\\ &\geq M_{\beta}^{(\theta)}\bigg(\int u \mathrm{d}x, \int v \mathrm{d}x\bigg)\end{aligned}$$

where $\beta = \alpha/(1+\alpha)$.

The preceding generalized Prékopa–Leindler theorem is easily tensorisable in $\mathbb{R}^n$ by induction on the dimension to yield that whenever $-\frac{1}{n} \leq \alpha \leq +\infty$, $\theta \in [0,1]$ and $u, v, w : \mathbb{R}^n \to \mathbb{R}_+$ are measurable such that

$$w\big(\theta x + (1-\theta)y\big) \geq M_{\alpha}^{(\theta)}\big(u(x), v(y)\big)$$

for all $x, y \in \mathbb{R}^n$, then

$$\int w\mathrm{d}x \geq M_\beta^{(\theta)}\Big(\int u\mathrm{d}x, \int v\mathrm{d}x\Big)$$

where $\beta = \alpha/(1+\alpha n)$. Namely, assuming the result in dimension $n-1$, for $x_1, y_1, z_1 = \theta x_1 + (1-\theta)y_1 \in \mathbb{R}$ fixed,

$$\int_{\mathbb{R}^{n-1}} w(z_1,t)\mathrm{d}t \geq M^{(\theta)}_{\alpha/(1+\alpha(n-1))}\Big(\int_{\mathbb{R}^{n-1}} u(x_1,t)\mathrm{d}t, \int_{\mathbb{R}^{n-1}} v(y_1,t)\mathrm{d}t\Big).$$

Since $\alpha \geq -\frac{1}{n}$ implies that $\widetilde{\alpha} = \alpha/(1+\alpha(n-1)) \geq -1$, the one-dimensional result applied to $\int_{\mathbb{R}^{n-1}} u(x_1,t)\mathrm{d}t$, $\int_{\mathbb{R}^{n-1}} v(y_1,t)\mathrm{d}t$, $\int_{\mathbb{R}^{n-1}} w(z_1,t)\mathrm{d}t$ yields the conclusion since $\widetilde{\alpha}/(1+\widetilde{\alpha}) = \beta$. The case $\alpha = 0$ corresponds to the Prékopa–Leindler theorem. When applied to the characteristic functions $u = \chi_A$, $v = \chi_B$ of the bounded non-empty sets A, B in $\mathbb{R}^n$ with $\alpha = +\infty$, we immediately recover the Brunn–Minkowski–Lusternik inequality (2).

Most of the proofs of the preceding integral inequalities rely in one way or another on integral parametrizations. They may be proved either first in dimension one together with induction on the dimension as above, or by suitable versions of the parametrizations by multidimensional measure transportation. We refer to the surveys [6,15,21,32] for complete accounts on these various approaches and precise historical developments.

As presented in [14], Corollary 1 may also be turned in dimension n, as a consequence of the generalized Prékopa–Leindler theorem. The resulting statement will be the essential step in the proof of the sharp Sobolev inequalities. In particular, the possibility to use α up to $-\frac{1}{n-1}$ will turn out to be crucial.

For a non-negative function $f : \mathbb{R}^n \to \mathbb{R}$, and $i = 1, \ldots, n$, set

$$m_i(f) = \sup_{x_i \in \mathbb{R}} \int_{\mathbb{R}^{n-1}} f(x)\mathrm{d}x_1 \cdots \mathrm{d}x_{i-1}\mathrm{d}x_{i+1} \cdots \mathrm{d}x_n.$$

Corollary 2 *Let* $-\frac{1}{n-1} \leq \alpha \leq +\infty$, $\theta \in [0,1]$ *and* u, v, w *be non-negative measurable functions on* $\mathbb{R}^n$ *such that for all* $x, y \in \mathbb{R}^n$,

$$w\big(\theta x + (1-\theta)y\big) \geq M_\alpha^{(\theta)}\big(u(x), v(y)\big).$$

If, for some $i = 1, \ldots, n$, $m_i(u) = m_i(v) < \infty$, *then*

$$\int w\mathrm{d}x \geq \theta \int u\mathrm{d}x + (1-\theta)\int v\mathrm{d}x.$$

Proof Apply the generalized Prékopa–Leindler theorem in $\mathbb{R}^{n-1}$ (thus with $-\frac{1}{n-1} \leq \alpha \leq +\infty$) to the functions $u(x), v(y), w(z)$ with $x_i, y_i, z_i = \theta x_i + (1-\theta)y_i$ fixed, and conclude with the lemma applied to $\tilde{u}(x_i) = \int_{\mathbb{R}^{n-1}} u(x)\mathrm{d}x_1 \cdots \mathrm{d}x_{i-1}\mathrm{d}x_{i+1} \cdots \mathrm{d}x_n$, $\tilde{v}(y_i)$ and $\tilde{w}(z_i)$ being defined similarly. □

Under the assumption $m_i(u) = m_i(v)$, the conclusion of Corollary 2 does not depend on α and is thus sharpest for $\alpha = -\frac{1}{n-1}$ (the statement for $-\frac{1}{n-1} < \alpha \leq +\infty$ being actually a consequence of this case). Following the proof of Corollary 1, the complete form of Corollary 2 actually states that (cf. [14]), for every $i = 1, \ldots, n$,

$$\int w\mathrm{d}x \geq M_\beta^{(\theta)}\big(m_i(u), m_i(v)\big) M_1^{(\theta)}\left(\frac{1}{m_i(u)}\int u\mathrm{d}x, \frac{1}{m_i(v)}\int v\mathrm{d}x\right)$$

with $\beta = \alpha/(1+\alpha(n-1))$.

Recently, mass transportation arguments have been developed to simultaneously reach the Brunn–Minkowski–Lusternik inequality and the sharp Sobolev inequalities (cf. [21] [6,32,44,45] ...). In particular, Cordero-Erausquin et al. [13] provide a complete treatment of the classical Sobolev inequalities with their best constants by this tool (see also [2]). Their approach covers in the same way the family of Gagliardo–Nirenberg inequalities put forward by Del Pino and Dolbeault [16] in the context of non-linear diffusion equations (see also [44]). More precisely, by means of Hölder's inequality, the Sobolev inequality (1) implies the family of so-called Gagliardo–Nirenberg inequalities [20,35],

$$\|f\|_r \leq C\|\nabla f\|_2^\lambda \|f\|_s^{1-\lambda} \tag{6}$$

for some constant $C > 0$ and all smooth enough functions $f : \mathbb{R}^n \to \mathbb{R}$ where $r, s > 0$ and $\frac{1}{r} = \frac{\lambda}{q} + \frac{1-\lambda}{s}$, $\lambda \in [0,1]$. The optimal constants are not preserved through Hölder's inequality. However, it was shown by Del Pino and Dolbeault [16] that optimal constants and extremal functions may be described for a sub-family of Gagliardo–Nirenberg inequalities, namely the one for which $r = 2(s-1)$ when $r, s > 2$ and $s = 2(r-1)$ when $r, s < 2$. The extremal functions turn out to be of the form $f(x) = (\sigma + |x|^2)^{2/(2-r)}$ in the first case, whereas in the second case they are given by $f(x) = ([\sigma - |x|^2]_+)^{1/(2-r)}$ (being thus compactly supported). The limiting case $r, s \to 2$ gives rise to the logarithmic Sobolev inequality (in its Euclidean formulation) with the Gaussian kernels as extremals.

While mass transport arguments may be offered to directly reach the n-dimensional Prékopa–Leindler theorem (cf. [6,44] ...), we do not know if Corollary 2 admits an n-dimensional optimal transportation proof.

On the other hand, the Prékopa–Leindler theorem was shown in [7], following the early ideas by Maurey [31] (cf. [26]), to imply the logarithmic Sobolev inequality for Gaussian measures [23] which, in its Euclidean version [12], corresponds to the limiting case $r, s \to 2$ in the scale of Gagliardo–Nirenberg inequalities. In this note, we demonstrate that the extended Prékopa–Leindler theorem in the form of Corollary 2 above may be used to prove in a simple direct way the classical Sobolev inequality (1) with sharp constant. The argument only relies on a suitable choice of functions u, v, w. The varying parameter α in Corollary 2 allows us to cover in the same way precisely the preceding sub-family of Gagliardo–Nirenberg inequalities with optimal constants, justifying thus this particular subset of functional inequalities. As in [13], we may deal as simply with the L^p-versions of the Sobolev and Gagliardo–Nirenberg inequalities (cf. (4)), and even replace the Euclidean norm on $\mathbb{R}^n$ by some arbitrary norm. The extension of the Sobolev inequalities to arbitrary norms on $\mathbb{R}^n$ was known previously [3] by symmetrization methods. With respect to earlier developments (notably the recent [13], which provides a new and complete treatment in this respect), the approach presented here does not provide any type of characterization of extremal functions and their uniqueness, which have to be hinted in the choice of the functions u, v, w.

The next section presents an outline of the direct proof of the sharp Sobolev inequality (1) from Corollary 2. We then discuss variations on the basic principle which lead to the sharp Sobolev and Gagliardo–Nirenberg inequalities (4) and (6).

The last section describes, with standard technical arguments, the rigorous and detailed proof of the Sobolev inequality.

2 Outline of the proof of the Sobolev inequality

We follow the strategy put forward in [7] (see also [22]) on the basis of Corollary 2 rather than the more classical Prékopa–Leindler theorem. For $g : \mathbb{R}^n \to \mathbb{R}$ and $t > 0$, recall the infimum-convolution of g with the quadratic cost defined by

$$Q_t g(x) = \inf_{y \in \mathbb{R}} \left\{ g(y) + \tfrac{1}{2t} |x - y|^2 \right\}, \quad x \in \mathbb{R}^n$$

(with $Q_0 g = g$). It is a standard fact (cf. e.g. [5,18] ...) that, for suitable C^1 functions g,

$$\partial_t Q_t g \,\big|_{t=0} = -\frac{1}{2} |\nabla g|^2. \tag{7}$$

Actually, if g is Lipschitz continuous, the family $\rho = \rho(x,t) = Q_t g(x)$, $t > 0$, $x \in \mathbb{R}^n$, represents the solution of the Hamilton–Jacobi initial value problem $\partial_t \rho + \frac{1}{2} |\nabla \rho|^2 = 0$ in $\mathbb{R}^n \times (0, \infty)$, $\rho = g$ on $\mathbb{R}^n \times \{t = 0\}$.

For $\sigma > 0$, set

$$v_\sigma(x) = \sigma + \tfrac{|x|^2}{2}, \quad x \in \mathbb{R}^n.$$

Let $\sigma > 0$ to be determined and let $g : \mathbb{R}^n \to \mathbb{R}_+$ be smooth and such that $m_1(g^{1-n}) < \infty$. In order not to obscure the main idea, we refer to the appendix for a precise description of the class of functions g that should be considered in order to justify the technical differential arguments freely used below.

By definition of the infimum-convolution operator, we may apply Corollary 2 with $\alpha = -\frac{1}{n-1}$ to the set of (positive) functions

$$\begin{aligned} u(x) &= g(\theta x)^{1-n}, \\ v(y) &= v_\sigma\left(\sqrt{\theta}\, y\right)^{1-n}, \\ w(z) &= \left[(1-\theta)\sigma + \theta Q_{1-\theta} g(z)\right]^{1-n}. \end{aligned}$$

Note that $m_1(u) = \theta^{1-n} m_1(g^{1-n})$ and $m_1(v) = (\sigma\theta)^{(1-n)/2} m_1(v_1^{1-n}) < \infty$. Choose thus $\sigma = \kappa\,\theta > 0$ such that $m_1(u) = m_1(v)$ where $\kappa = \kappa(n, g) = (m_1(v_1^{1-n})/m_1(g^{1-n}))^{2/(n-1)}$. Set $s = 1 - \theta \in (0,1)$. Hence, by Corollary 2, for every $s \in (0,1)$,

$$\int \left(\kappa s + Q_s g\right)^{1-n} \mathrm{d}x \geq \int g^{1-n} \mathrm{d}x + s\,\kappa^{(2-n)/2} \int v_1^{1-n} \mathrm{d}x.$$

Taking the derivative at $s = 0$ yields, by (7),

$$(1-n) \int g^{-n} \left(\kappa - \frac{1}{2} |\nabla g|^2\right) \mathrm{d}x \geq \kappa^{(2-n)/2} \int v_1^{1-n} \mathrm{d}x. \tag{8}$$

Set $g = f^{2/(2-n)}$ so that

$$\frac{2}{(n-2)^2} \int |\nabla f|^2 \mathrm{d}x \geq \kappa \int f^q \mathrm{d}x + \frac{1}{(n-1)\kappa^{(n-2)/2}} \int v_1^{1-n} \mathrm{d}x$$

where we recall that $q = 2n/(n-2)$. In particular,

$$\int |\nabla f|^2 \mathrm{d}x \geq \inf_{\kappa>0} \frac{(n-2)^2}{2}\left(\kappa \int f^q \mathrm{d}x + \frac{1}{(n-1)\kappa^{(n-2)/2}} \int v_1^{1-n} \mathrm{d}x\right). \tag{9}$$

This infimum is precisely $C_n^{-2}\|f\|_q^2$ where C_n is the optimal constant in the Sobolev inequality (1). Actually, if $g(x) = v_1(x) = 1 + \frac{|x|^2}{2}$, the preceding argument develops with equalities at each step with $\kappa = \kappa(n,g) = 1$. Moreover, the infimum on the right-hand side of (9) is attained at $\kappa = 1$ if and only if

$$\int f^q \mathrm{d}x = \int v_1^{-n} \mathrm{d}x = \frac{n-2}{2(n-1)} \int v_1^{1-n} \mathrm{d}x$$

which is easily checked by elementary calculus. Thus (9) is an equality in this case and the conclusion follows.

3 Extensions and comments

As emphasized in the introduction, the same proof, with the varying parameter α in Corollary 2, yields the sub-family of Gagliardo–Nirenberg inequalities recently put forward in [16]. Let us briefly emphasize the modifications in the argument. (It is somewhat surprising that these optimal Gagliardo–Nirenberg inequalities follow from Corollary 2 with $-\frac{1}{n-1} < \alpha \leq +\infty$ which is a consequence of the $\alpha = -\frac{1}{n-1}$ case, whereas they are not direct consequences of the sharp Sobolev inequality.)

For $-\frac{1}{n-1} \leq \alpha < 0$, apply Corollary 2 to

$$u(x) = g(\theta x)^{1/\alpha},$$
$$v(y) = v_\sigma\big(\sqrt{\theta}\, y\big)^{1/\alpha},$$
$$w(z) = \big[(1-\theta)\sigma + \theta Q_{1-\theta} g(z)\big]^{1/\alpha}$$

to get that for all $s \in (0,1)$,

$$\int \big[\kappa s(1-s)^a + (1-s)Q_s g\big]^{1/\alpha} \mathrm{d}x \geq (1-s)^{1-n} \int g^{1/\alpha} \mathrm{d}x + \kappa^c s(1-s)^b \int v_1^{1/\alpha} \mathrm{d}x.$$

Here $a > 0, b, c < 0, \kappa > 0$ depending on n and α (and g), are such that $m_1(u) = m_1(v)$ for some suitable choice of σ. Taking the derivative at $s = 0$,

$$\frac{1}{\alpha} \int g^{(1/\alpha)-1}\Big(\kappa - g - \frac{1}{2}|\nabla g|^2\Big) \mathrm{d}x \geq (n-1) \int g^{1/\alpha} \mathrm{d}x + \kappa^c \int v_1^{1/\alpha} \mathrm{d}x.$$

Set $f = g^p$, $2p - 2 = \frac{1}{\alpha} - 1$, so that

$$-\frac{1}{2\alpha p^2} \int |\nabla f|^2 \mathrm{d}x - \Big[(n-1) + \frac{1}{\alpha}\Big] \int f^r \mathrm{d}x \geq -\frac{\kappa}{\alpha} \int f^s \mathrm{d}x + \kappa^c \int v_1^{1/\alpha} \mathrm{d}x$$

where $r = 2(1-\alpha)/(1+\alpha)$ and $s = 2/(1+\alpha)$. Note that $r, s > 2$, $r = 2(s-1)$. Take the infimum over $\kappa > 0$ on the right-hand side, and rewrite then the inequality by homogeneity to get the Gagliardo–Nirenberg inequality

$$\|f\|_r \leq C \|\nabla f\|_2^\lambda \|f\|_s^{1-\lambda},$$

$\frac{1}{r} = \frac{\lambda}{q} + \frac{1-\lambda}{s}$, with optimal constant C.

To reach the sub-family $r, s < 2$, $s = 2(r-1)$, work now with $0 < \alpha < +\infty$ and replace v_σ by the compactly supported function $[\sigma - \frac{|x|^2}{2}]_+$, $|x| < \sqrt{2\sigma}$. Actually, only the values $0 < \alpha < 1$ are concerned in the argument. We do not know what type of functional information is contained in the interval $\alpha \geq 1$. The case $\alpha = 0$ leading to the logarithmic Sobolev inequality has been studied in [7,22] and follows here as a limiting case.

We can work more generally with the L^p-Sobolev inequalities (4), $1 < p < n$, and similarly with the corresponding sub-family of Gagliardo–Nirenberg inequalities. It is also possible to equip $\mathbb{R}^n$ with an arbitrary norm $\|\cdot\|$ instead of the Euclidean one $|\cdot|$, and to consider

$$\|\nabla f\|_p^p = \int_{\mathbb{R}^n} \|\nabla f(x)\|_*^p \,\mathrm{d}x$$

where $\|\cdot\|_*$ is the dual norm to $\|\cdot\|$. To these tasks, consider as in [22],

$$Q_t g(x) = \inf_{y\in\mathbb{R}^n} \left\{ g(y) + t\, V^*\left(\tfrac{x-y}{t}\right)\right\}, \quad t > 0,\ x \in \mathbb{R}^n,$$

where $V^*(x) = \frac{1}{p^*}\|x\|^{p^*}$ with p^* is the Hölder conjugate of p, i.e. $(1/p) + (1/p^*) = 1$. Then $\rho = \rho(x,t) = Q_t g(x)$ is the solution of the Hamilton–Jacobi equation $\partial_t \rho + V(\nabla \rho) = 0$ with initial condition g, where $V(x) = \frac{1}{p}\|x\|_*^p$ is the Legendre transform of V^* (cf. [18]). The proof then follows along the same lines as before. See the Appendix for details. The general statement obtained in this way is the following (cf. [13,17]). For $1 < p < n$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, $s < r \leq q$, $\lambda \in [0,1]$,

$$\|f\|_r \leq C_n(p,r) \|\nabla f\|_p^\lambda \|f\|_s^{1-\lambda}$$

with $\frac{1}{r} = \frac{\lambda}{q} + \frac{1-\lambda}{s}$, $p(s-1) = r(p-1)$ if $r, s > p$, $p(r-1) = s(p-1)$ if $r, s < p$, and the optimal constant $C_n(r,p)$ is achieved on the extremal functions $(\sigma + \|x\|^{p^*})^{p/(p-r)}$, $x \in \mathbb{R}^n$, $\sigma > 0$, in the first case and $([\sigma - \|x\|^{p^*}]_+)^{(p-1)/(p-r)}$, $x \in \mathbb{R}^n$, $\sigma > 0$, in the second case. The optimal Sobolev inequality (4) corresponds to the limiting case $\lambda \to 1$, $r \to q$, $s \to r$.

4 Proof of the Sobolev inequality

In this last section, we collect the technical details necessary to fully justify the proof of the Sobolev inequality outlined in Sect. 2. Although the case $p = 2$ is a bit more simple, we can actually easily handle in the same way the more general case of $1 < p < n$ and of an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^n$. The arguments are easily modified so to deal similarly with the Gagliardo–Nirenberg inequalities discussed in Sect. 3.

Consider thus on $\mathbb{R}^n$ the Sobolev inequality

$$\|f\|_q \leq C_n(p) \|\nabla f\|_p \tag{10}$$

in the class of all locally Lipschitz functions f vanishing at infinity, with parameters p, q satisfying $1 < p < n$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. The right-hand side in (10) is understood with respect to the given norm $\|\cdot\|$ on $\mathbb{R}^n$. More precisely,

$$\|\nabla f\|_p^p = \int_{\mathbb{R}^n} \big\|\nabla f(x)\big\|_*^p \, \mathrm{d}x$$

where $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$. Our aim is to show that the best constant $C_n(p)$ in (10) corresponds to the family of extremal functions

$$f(x) = \big(\sigma + \|x\|^{p^*}\big)^{(p-n)/p}, \quad x \in \mathbb{R}^n,\ \sigma > 0,$$

where p^* is the conjugate of p. Without loss of generality, we may assume that the norm $x \mapsto \|x\|$ is continuously differentiable in the region $x \neq 0$. In this case, $\|\nabla\|x\|\|_* = 1$ for all $x \neq 0$, and all the extremal functions belong to the class $C^1(\mathbb{R}^n)$.

The associated infimum-convolution operator is constructed for the cost function $V^*(x) = \frac{1}{p^*}\|x\|^{p^*}$, that is,

$$Q_t g(x) = \inf_{y \in \mathbb{R}^n} \big\{g(y) + t\, V^*\big(\tfrac{x-y}{t}\big)\big\}, \quad t > 0,\ x \in \mathbb{R}^n.$$

The dual (Legendre transform) of V^* is $V(x) = \sup_{y \in \mathbb{R}^n}[\langle x, y\rangle - V^*(y)] = \frac{1}{p}\|x\|_*^p$ (and conversely).

We refer to standard references (such as [5,18] ...) for general facts about infimum-convolution operators and solutions to Hamilton–Jacobi equations, and only concentrate below on the aspects relevant to the proof of the Sobolev inequality. What follows is certainly classical, but we could not find appropriate references.

Lemma A *If a function g on $\mathbb{R}^n$ is bounded from below and is differentiable at the point $x \in \mathbb{R}^n$, then*

$$\lim_{t \to 0} \frac{1}{t}\big[Q_t g(x) - g(x)\big] = -V\big(\nabla g(x)\big) = -\frac{1}{p}\big\|\nabla g(x)\big\|_*^p.$$

Proof Fix $x \in \mathbb{R}^n$. By Taylor's expansion, $g(x-h) = g(x) - \langle \nabla g(x), h\rangle + |h|\varepsilon(h)$ with $\varepsilon(h) = \varepsilon_x(h) \to 0$ as $|h| \to 0$. Hence, for vectors $h_t = th$ with fixed $h \in \mathbb{R}^n$,

$$\lim_{t \to 0} \frac{1}{t}\big[g(x-h_t) - g(x)\big] = -\langle \nabla g(x), h\rangle.$$

Since we always have $Q_t g(x) \le g(x-h_t) + tV^*(h)$,

$$\begin{aligned}\limsup_{t \to 0} \frac{1}{t}\big[Q_t g(x) - g(x)\big] &\le \lim_{t \to 0} \frac{1}{t}\big[g(x-h_t) - g(x)\big] + V^*(h)\\ &= -\langle \nabla g(x), h\rangle + V^*(h).\end{aligned}$$

The left-hand side of the preceding does not depend on h. Hence, taking the infimum on the right-hand side over all $h \in \mathbb{R}^n$, we get

$$\limsup_{t \to 0} \frac{1}{t}\big[Q_t g(x) - g(x)\big] \le -V\big(\nabla g(x)\big).$$

Now, we need an opposite inequality for the liminf. Assume without loss of generality that $g \ge 0$. Since $Q_t g(x) \le g(x)$, it is easy to see that for any $t > 0$,

$$Q_t g(x) \ = \inf_{tV^*(h) \le g(x)} \big\{g(x-h_t) + tV^*(h)\big\}.$$

Hence, recalling Taylor's expansion,

$$\frac{1}{t}\left[Q_t g(x) - g(x)\right] = \inf_{tV^*(h)\le g(x)} \left\{ -\langle \nabla g(x), h\rangle + |h|\varepsilon(th) + V^*(h)\right\}. \tag{11}$$

Note first that the argument in $\varepsilon(\cdot) = \varepsilon_x(\cdot)$ is small uniformly over all admissible h since, as is immediate,

$$\sup\left\{t|h|; tV^*(h) \le g(x)\right\} \to 0 \quad \text{as } t \to 0.$$

Thus removing the condition $tV^*(h) \le g(x)$ in (11), we get that, given $\eta > 0$, for all t small enough,

$$\frac{1}{t}\left[Q_t g(x) - g(x)\right] \ge \inf_h \left\{ -\langle \nabla g(x), h\rangle - |h|\eta + V^*(h)\right\}. \tag{12}$$

Now, to get rid of η on the right-hand side for t approaching zero, note that the infimum in (12) may be restricted to the ball $|h| \le r$ for some large r. Indeed, the left-hand side in (12) is non-positive. But if $|h|$ is large enough and $0 < \eta < 1$, the quantity for which we take the infimum will be positive for $V^*(h) \ge C|h| > \langle \nabla g(x), h\rangle + |h|\eta$ with C taken in advance to be as large as we want. Finally, restricting the infimum to $|h| \le r$, we get that

$$\frac{1}{t}\left[Q_t g(x) - g(x)\right] \ge \inf_{|h|\le r} \left\{ -\langle \nabla g(x), h\rangle + V^*(h)\right\} - r\eta = -V\big(\nabla g(x)\big) - r\eta.$$

It remains to take the liminf on the left for $t \to 0$, and then to send η to 0. The proof of Lemma A is complete. □

Our next step is to complement the above convergence with a bound on $|Q_t g(x) - g(x)|/t$ in terms of $\|\nabla g(y)\|_*$ with vectors y that are not far from x. So, given a C^1 function g on $\mathbb{R}^n$, for every point $x \in \mathbb{R}^n$ and $r > 0$, define $Dg(x,r) = \sup_{\|x-y\|\le r} \|\nabla g(y)\|_*$. Note that $Dg(x,r) \to \|\nabla g(x)\|_*$ as $r \to 0$. Assume $g \ge 0$ and write once more

$$Q_t g(x) = \inf_{h\in\mathbb{R}^n} \left\{g(x-h) + \tfrac{\|h\|^{p^*}}{p^* t^{p^*-1}}\right\}, \quad t > 0.$$

Again, since $Q_t g(x) \le g(x)$, the infimum may be restricted to the ball $(\|h\|^{p^*}/p^* t^{p^*-1}) \le g(x)$. Hence, replacing h with th and applying the Taylor formula in integral form, we get that with $r = (p^* g(x))^{1/p^*}$, for any $t > 0$,

$$\begin{aligned}
\frac{1}{t}\left[g(x) - Q_t g(x)\right] &\le \sup_{t\|h\|\le r} \left\{\frac{1}{t}\left[g(x) - g(x-th)\right] - \big(\|h\|^{p^*}/p^*\big)\right\} \\
&\le \sup_{t\|h\|\le r} \left\{Dg\big(x, t\|h\|\big)\|h\| - \big(\|h\|^{p^*}/p^*\big)\right\} \\
&\le \sup_h \left\{Dg(x,r)\|h\| - \big(\|h\|^{p^*}/p^*\big)\right\} \\
&= \frac{1}{p} Dg(x,r)^p.
\end{aligned} \tag{13}$$

In applications, we need to work with functions $g(x) = O(|x|^{p^*})$ as $|x| \to \infty$. So, let us define the class $\mathcal{F}_{p^*}$, $p^* > 1$, of all C^1 functions g on $\mathbb{R}^n$ such that

$$\limsup_{|x|\to\infty} \frac{|\nabla g(x)|}{|x|^{p^*-1}} < \infty.$$

If $f \in \mathcal{F}_{p^*}$, then, for some C, $|\nabla g(x)| \leq C|x|^{p^*-1}$ as long as $|x|$ is large enough, and hence $|g(x)|^{1/p^*} \leq C'|x|$ for $|x|$ large. It easily follows that $Dg(x, (p^* g(x))^{1/p^*}) \leq C''(1+|x|^{p^*-1})$ for all x. As a consequence of (13), we may conclude that for any $g \geq 0$ in $\mathcal{F}_{p^*}$, $p^* > 1$, there is a constant $C > 0$ such that

$$\sup_{t>0} \frac{1}{t} \big[g(x) - Q_t g(x)\big] \leq C\big(1 + |x|^{p^*}\big), \quad x \in \mathbb{R}^n. \tag{14}$$

We may now start the proof of the Sobolev inequality according to the scheme outlined in Sect. 2. Given a parameter $\sigma > 0$, define

$$v_\sigma(x) = \sigma + \tfrac{\|x\|^{p^*}}{p^*}, \quad x \in \mathbb{R}^n.$$

For a positive C^1 function g on $\mathbb{R}^n$, and $\theta \in (0,1)$, define the three (positive, continuous) functions

$$\begin{aligned} u(x) &= g(\theta x)^{1-n}, \\ v(y) &= v_\sigma(\theta^{1/p^*} y)^{1-n}, \\ w(z) &= \big[(1-\theta)\sigma + \theta Q_{1-\theta} g(z)\big]^{1-n}. \end{aligned}$$

The function w is chosen as the optimal one satisfying

$$w\big(\theta x + (1-\theta)y\big)^\alpha \leq \theta u(x)^\alpha + (1-\theta)v(y)^\alpha$$

for $\alpha = -\frac{1}{n-1}$ and all $x, y \in \mathbb{R}^n$. Assume that

$$m_1(g^{1-n}) \;=\; \sup_{x_1 \in \mathbb{R}} \int_{\mathbb{R}^{n-1}} g(x_1, x_2, \cdots, x_n)^{1-n}\, \mathrm{d}x_2 \ldots \mathrm{d}x_n < \infty.$$

By homogeneity, $m_1(u) = \theta^{1-n} m_1(g^{1-n})$ and $m_1(v) = \theta^{(1-n)/p^*} \sigma^{(1-n)/p}\, m_1(v_1^{1-n})$. Note that $m_1(v_1^{1-n}) < \infty$. Hence, we may choose σ such that $m_1(u) = m_1(v)$, that is,

$$\sigma = \kappa\, \theta, \quad \text{where } \ \kappa = \kappa(n, g) = \left(\frac{m_1(v_1^{1-n})}{m_1(g^{1-n})}\right)^{p/(n-1)}.$$

By Corollary 2 (with $\alpha = -\frac{1}{n-1}$), we have $\int w \mathrm{d}x \geq \theta \int u \mathrm{d}x + (1-\theta) \int v \mathrm{d}x$, that is,

$$\int \big[(1-\theta)\sigma + \theta Q_{1-\theta} g(x)\big]^{1-n} \mathrm{d}x \geq \theta \int g(\theta x)^{1-n}\, \mathrm{d}x + (1-\theta) \int v_\sigma \big(\theta^{1/p^*} x\big)^{1-n} \mathrm{d}x.$$

After a change of variable in the last two integrals, and since $\sigma = \kappa\, \theta$, we get, setting $s = 1 - \theta$,

$$\int (\kappa s + Q_s g)^{1-n} \mathrm{d}x \geq \int g^{1-n} \mathrm{d}x + s\, \kappa^{(p-n)/p} \int v_1^{1-n} \mathrm{d}x. \tag{15}$$

Inequality (15) holds true for all $0 < s < 1$, and formally there is equality at $s = 0$. The next step is to compare the derivatives of both sides at this point. To do this, assume $g \in \mathcal{F}_{p^*}$ and

$$g(x) \geq c\big(1 + \|x\|^{p^*}\big) \tag{16}$$

for some constant $c > 0$. (Recall that the functions in $\mathcal{F}_{p^*}$ satisfy an opposite bound $g(x) \leq C(1 + \|x\|^{p^*})$ which will not be used.) Due to (16), $Q_s g(x) \geq c'(1 + \|x\|^{p^*})$

(where $c' > 0$ is independent of s). In particular, $m_1(g^{1-n}) < \infty$, and the first and second integrals in (15) are finite and uniformly bounded over all $s \in (0, 1)$. Rewrite (15) as

$$\kappa^{(p-n)/p} \int v_1^{1-n} \mathrm{d}x \le \int \frac{1}{s} \left[(\kappa s + Q_s g)^{1-n} - g^{1-n} \right] \mathrm{d}x. \tag{17}$$

Now we can use a general inequality

$$|a^{1-n} - b^{1-n}| \le (n-1)|a-b|(a^{-n} + b^{-n}), \quad a, b > 0,$$

to see that, uniformly in s,

$$\begin{aligned} \frac{1}{s} \left[(\kappa s + Q_s g)^{1-n} - g^{1-n} \right] &\le 2(n-1)\Big(\kappa + \frac{1}{s}\,[g - Q_s g]\Big)(Q_s g)^{-n} \\ &\le C'\big(1 + \|x\|^{p^*}\big)^{1-n} \end{aligned}$$

for some constant $C' > 0$. On the last step, we used that $Q_s g(x) \ge c'(1 + \|x\|^{p^*})$ together with the bound (14) for functions from the class $\mathcal{F}_{p^*}$. Since the function $(1 + \|x\|^{p^*})^{1-n}$ is integrable (for $p < n$), we can apply the Lebesgue dominated convergence theorem in order to insert the limit $\lim_{s \to 0}$ inside the integral in (17), and to thus get together with Lemma A,

$$\kappa^{(p-n)/p} \int v_1^{1-n} \mathrm{d}x \le (1-n) \int g^{-n} \Big(\kappa - \frac{\|\nabla g\|_*^p}{p}\Big) \mathrm{d}x,$$

or equivalently,

$$\frac{1}{p} \int g^{-n} \|\nabla g\|_*^p \mathrm{d}x \ge \kappa \int g^{-n} \mathrm{d}x + \frac{1}{(n-1)\kappa^{(n-p)/p}} \int v_1^{1-n} \mathrm{d}x. \tag{18}$$

Now, let us take a non-negative, compactly supported C^1 function f on $\mathbb{R}^n$, and for $\varepsilon > 0$, define C^1 functions

$$g_\varepsilon(x) = \big(f(x) + \varepsilon \varphi(x)\big)^{p/(p-n)} + \varepsilon\big(1 + \|x\|^{p^*}\big)$$

where $\varphi(x) = (1 + \|x\|^{p^*})^{(p-n)/p}$. Clearly, all g_ε satisfy (16). The first partial derivatives of f are continuous and vanishing for large values of $|x|$. More precisely, $g_\varepsilon(x) = c_\varepsilon(1 + \|x\|^{p^*})$ for $|x|$ large enough, so all g_ε belong to the class $\mathcal{F}_{p^*}$. Thus, we can apply (18) to get

$$\frac{1}{p} \int g_\varepsilon^{-n} \|\nabla g_\varepsilon\|_*^p \mathrm{d}x \ge \kappa \int g_\varepsilon^{-n} \mathrm{d}x + \frac{1}{(n-1)\kappa^{(n-p)/p}} \int v_1^{1-n} \mathrm{d}x. \tag{19}$$

Note that $g_\varepsilon^{-n} \le (f + \varepsilon\varphi)^q$ and $\int \varphi^q \mathrm{d}x < \infty$ (where we recall that $q = pn/(n-p)$). Hence, by the Lebesgue dominated convergence theorem again, $\int g_\varepsilon^{-n} \mathrm{d}x$ is convergent, as $\varepsilon \to 0$, to $\int f^q \mathrm{d}x$. By a similar argument, recalling that $\|\nabla \|x\|^{p^*}\|_* = p^* \|x\|^{p^*-1}$, $x \in \mathbb{R}^n$, we see that there is a finite limit for the left integral in (19). As a result, we arrive at

$$\frac{p^{p-1}}{(n-p)^p} \int \|\nabla f\|_*^p \, \mathrm{d}x \ge \kappa \int f^q \mathrm{d}x + \frac{1}{(n-1)\kappa^{(n-p)/p}} \int v_1^{1-n} \mathrm{d}x, \tag{20}$$

which implies

$$\frac{p^{p-1}}{(n-p)^p} \int \|\nabla f\|_*^p \, \mathrm{d}x \ge \inf_{\kappa > 0} \Big(\kappa \int f^q \mathrm{d}x + \frac{1}{(n-1)\kappa^{(n-p)/p}} \int v_1^{1-n} \mathrm{d}x\Big). \tag{21}$$

As we will see with the case of equality below, this is precisely the desired Sobolev inequality (10) with optimal constant. It is now easy to remove the assumption on the compact support of f and thus to extend (21) to all C^1 and furthermore locally Lipschitz functions f (≥ 0) on $\mathbb{R}^n$ vanishing at infinity.

To conclude the argument, we investigate the case of equality. To this task, let us return to the beginning of the argument and check the steps where equality holds true. Take $g = v_1$ so that $\kappa = \kappa(n,g) = 1$ and $\sigma = \theta$. In addition, the right-hand side of (15) automatically turns into $(1+s)\int v_1^{1-n}\mathrm{d}x$. By direct computation,

$$Q_s v_1(x) = 1 + \frac{\|x\|^{p^*}}{p^*(1+s)^{p^*-1}},$$

so the left-hand side of (15) is

$$\begin{aligned}\int (\kappa s + Q_s g)^{1-n}\mathrm{d}x &= \int \left((1+s) + \frac{\|x\|^{p^*}}{p^*\,(1+s)^{p^*-1}}\right)^{1-n}\mathrm{d}x \\ &= (1+s)\int \left(1 + \frac{\|y\|^{p^*}}{p^*}\right)^{1-n}\mathrm{d}y \\ &= (1+s)\int v_1^{1-n}\mathrm{d}y\end{aligned}$$

where we used the change of the variable $x = (1+s)y$. Thus, for $g = v_1$ there is equality in (15), and hence in (18) and (20) as well.

As for (21), first note that, given parameters $A, B > 0$, the function $A\kappa + B\kappa^{(p-n)/p}$, $\kappa > 0$, attains its minimum on the positive half-axis at $\kappa = 1$ if and only if $A = B(n-p)/p$. In the situation of the particular functions $g = v_1$, $f^q = g^{-n} = v_1^{-n}$, we have

$$A = \int v_1^{-n}\mathrm{d}x, \quad B = \frac{1}{n-1}\int v_1^{1-n}\mathrm{d}x.$$

Hence, the infimum in (20) is attained at $\kappa = 1$ if and only if

$$\int v_1^{-n}\mathrm{d}x = \frac{n-p}{p(n-1)}\int v_1^{1-n}\mathrm{d}x.$$

But this equality is easily checked by elementary calculus.

We may thus summarize our conclusions. In the class of all locally Lipschitz functions f on $\mathbb{R}^n$, vanishing at infinity and such that $0 < \|f\|_q < \infty$, the quantity

$$\frac{\|\nabla f\|_p}{\|f\|_q},$$

$1 < p < n$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, is minimized for the functions

$$f(x) = \left(\sigma + \|x\|^{p^*}\right)^{(p-n)/p}, \quad x \in \mathbb{R}^n,\ \sigma > 0.$$

Here $\frac{1}{p} + \frac{1}{p^*} = 1$ and $\|\cdot\|$ is a given norm on $\mathbb{R}^n$, and

$$\|\nabla f\|_p^p = \int_{\mathbb{R}^n} \|\nabla f(x)\|_*^p\,\mathrm{d}x$$

where $\|\cdot\|_*$ is the dual norm to $\|\cdot\|$.

Acknowledgement We thank D. Cordero–Erausquin for suggesting some years ago that the approach of [7] could lead to the sharp Sobolev inequality. We also thank E. Lieb for his references to the early history of the sharp constant.

References

1. Adams, R.A.: Sobolev spaces. Academic, New York (1975)
2. Agueh, M., Ghoussoub, N., Kang, X.: Geometric inequalities via a general comparison principle for interacting gases. Geom. Funct. Anal. **14**, 215–244 (2004)
3. Alvino, A., Ferone, V., Trombetti, G., Lions, P.-L.: Convex symmetrization and applications. Ann. Inst. H. Poincaré Anal. Non Linéaire **14**, 275–293 (1997)
4. Aubin, Th.: Problèmes isopérimétriques et espaces de Sobolev. J. Differ. Geometry **11**, 573–598 (1976)
5. Barles, G.: Solutions de viscosité des équations de Hamilton-Jacobi. Springer, Heidelberg (1994)
6. Barthe, F.: Autour de l'inégalité de Brunn–Minkowski. Ann. Fac. Sci. Toulouse Math. **12**, 127–178 (2003)
7. Bobkov, S., Ledoux, M.: From Brunn–Minkowski to Brascamp–Lieb and to logarithmic Sobolev inequalities. Geom. Funct. Anal. **10**, 1028–1052 (2000)
8. Borell, C.: Convex functions in d-spaces. Period. Math. Hungar. **6**, 111–136 (1975)
9. Brascamp, H.J., Lieb, E.H.: On extensions of the Brunn–Minkovski and Prékopa–Leindler theorems, including inequalities for log-concave functions, and with an application to the diffusion equation. J. Funct. Anal. **22**, 366–389 (1976)
10. Bonnesen, T., Fenchel, W.: Theorie der konvexen Körper. Springer, Heidelberg (1934)
11. Brothers, J., Ziemer, W.: Minimal rearrangements of Sobolev functions. J. Reine Angew. Math. **384**, 153–179 (1988)
12. Carlen, E.: Superadditivity of Fisher's information and logarithmic Sobolev inequalities. J. Funct. Anal. **101**, 194–211 (1991)
13. Cordero-Erausquin, D., Nazaret, B., Villani, C.: A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities. Adv. Math. **182**, 307–332 (2004)
14. Dancs, S., Uhrin, B.: On a class of integral inequalities and their measure-theoretic consequences. J. Math. Anal. Appl. **74**, 388–400 (1980)
15. Das Gupta, S.: Brunn–Minkowski inequality and its aftermath. J. Multivariate Anal. **10**, 296–318 (1980)
16. Del Pino, M., Dolbeault, J.: Best constants for Gagliardo–Nirenberg inequalities and applications to nonlinear diffusions. J. Math. Pures Appl. **81**, 847–875 (2002)
17. Del Pino, M., Dolbeault, J.: The optimal Euclidean L^p-Sobolev logarithmic inequality. J. Funct. Anal. **197**, 151–161 (2003)
18. Evans, L.C.: Partial differential equations. Graduate Studies in Math. 19. Amer. Math. Soc. (1997)
19. Federer, H.: Geometric measure theory. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer, Heidelberg (1969)
20. Gagliardo, E.: Proprietà di alcune classi di funzioni in più variablili. Ric. Mat. **7**, 102–137 (1958)
21. Gardner, R.J.: The Brunn–Minkowski inequality. Bull. Amer. Math. Soc. **39**, 355–405 (2002)
22. Gentil, I.: The general optimal L^p-Euclidean logarithmic Sobolev inequality by Hamilton-Jacobi equations. J. Funct. Anal. **202**, 591–599 (2003)
23. Gross, L.: Logarithmic Sobolev inequalities. Am. J. Math. **97**, 1061–1083 (1975)
24. Hadwiger, H., Ohmann, D.: Brunn–Minkowskischer Satz und Isoperimetrie. Math. Zeit. **66**, 1–8 (1956)
25. Henstock, R., Macbeath, A.M.: On the measure of sum-sets (I). The Theorem of Brunn, Minkowksi and Lusternik. Proc. London Math. Soc. **3**, 182–194 (1953)
26. Ledoux, M.: The concentration of measure phenomenon. Math. Surv. Monogr. 89. Amer. Math. Soc. (2001)
27. Leindler, L.: On a certain converse of Hölder's inequality II. Acta Sci. Math. Szeged **33**, 217–223 (1972)
28. Lieb, E.: The stability of matter. Rev. Mod. Phys. **48**, 553–569 (1976)
29. Lieb, E., Loss, M.: Analysis. Graduate Studies in Mathematics. AMS, New York (1996)
30. Lusternik, L.: Die Brunn–Minkowskiche Ungleichung für beliebige messbare Mengen. Doklady Akad. Nauk. SSSR **3**, 55–58 (1935)
31. Maurey, B.: Some deviations inequalities. Geom Funct. Anal. **1**, 188–197 (1991)

32. Maurey, B.: Inégalité de Brunn–Minkowski–Lusternik, et autres inégalités géométriques et fonctionnelles. Séminaire Bourbaki, vol. 2003/2004, Astérisque **299**, 95–113 (2005)
33. Maz'ja, V.G.: Sobolev spaces. Springer, Heidelberg (1985)
34. Milman, V.D., Schechtman, G.: Asymptotic theory of finite dimensional normed spaces. Lecture Notes in Math. 1200 Springer, Heidelberg (1986)
35. Nirenberg, L.: On elliptic partial differential equations. Ann. Sc. Norm. Pisa **13**, 116–162 (1959)
36. Prékopa, A.: Logarithmic concave measures with applications to stochastic programming. Acta Sci. Math. Szeged **32**, 301–316 (1971)
37. Prékopa, A.: On logarithmic concave measures and functions. Acta Sci. Math. Szeged **34**, 335–343 (1973)
38. Rosen, G.: Minimum value of c in the Sobolev inequality $\|\phi^3\| \leq c\|\nabla\phi\|^3$. Siam J. Appl. Math. **21**, 30–32 (1971)
39. Ross, J.: Mathreviews MR1312686 (96d:46032) on the paper The geometric Sobolev embedding for vector fields and the isoperimetric inequality. Comm. Anal. Geom. **2**, 203–215 (1994) by L. Capogna, D. Danielli, N. Garofalo
40. Schneider, R.: Convex bodies: the Brunn–Minkowski theory. Encyclopedia of Mathematics and its Applications 44. Cambridge University Press, Cambridge (1993)
41. Sobolev, S.L.: On a theorem in functional analysis. Amer. Math. Soc. Translations **34**(2), 39–68 (1963); translated from Mat. Sb. (N.S.) **4**(46), 471–497 (1938)
42. Talenti, G.: Best constants in Sobolev inequality. Ann. di Matem. Pura Ed Appl. **110**, 353–372 (1976)
43. Triebel, H.: Theory of function spaces II. Birkhäuser, (1992)
44. Villani, C.: Topics in optimal transportation. Graduate Studies in Mathematics 58. AMS, New York (2003)
45. Villani, C.: Optimal transport, old and new. École d'Été de Probabilités de St-Flour 2005. Lect. Notes in Math (to appear)