Annali di Matematica (2008) 187:323–352
DOI 10.1007/s10231-007-0046-1

Anti-selfdual Lagrangians II: unbounded non self-adjoint operators and evolution equations

Nassif Ghoussoub · **Leo Tzou**

Received: 17 March 2006 / Revised: 5 June 2006 / Published online: 30 March 2007

Abstract New variational principles based on the concept of anti-selfdual (ASD) Lagrangians were recently introduced in "AIHP-Analyse non linéaire, 2006". We continue here the program of using such Lagrangians to provide variational formulations and resolutions to various basic equations and evolutions which do not normally fit in the Euler-Lagrange framework. In particular, we consider stationary boundary value problems of the form $-Au \in \partial\varphi(u)$ as well ass dissipative initial value evolutions of the form $-\dot{u}(t) - Au(t) + \omega u(t) \in \partial\varphi(l, u(l))$ where φ is a convex potential on an infinite dimensional space, A is a linear operator and ω is any scalar. The framework developed in the above mentioned paper reformulates these problems as $0 \in \bar{\partial}L(u)$ and $\dot{u}(t) \in \bar{\partial}L(t, u(t))$ respectively, where $\bar{\partial}L$ is an "ASD" vector field derived from a suitable Lagrangian L. In this paper, we extend the domain of application of this approach by establishing existence and regularity results under much less restrictive boundedness conditions on the anti-selfdual Lagrangian L so as to cover equations involving unbounded operators. Our main applications deal with various nonlinear boundary value problems and parabolic initial value equations governed by transport operators with or without a diffusion term.

Nassif Ghoussoub research was partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada. The author gratefully acknowledges the hospitality and support of the Centre de Recherches Mathématiques in Montréal where this work was initiated. Leo Tzou's research was partially supported by a doctoral postgraduate scholarship from the Natural Science and Engineering Research Council of Canada.

N. Ghoussoub (✉)
Department of Mathematics, University of British Columbia,
Vancouver, BC, Canada V6T 1Z2
e-mail: nassif@math.ubc.ca

L. Tzou
Department of Mathematics, University of Washington,
Seattle, WA 98105, USA
e-mail: lleo@math.washington.edu

Mathematics Subject Classification (2000) 35B05 · 47H20 · 58J35

1 Introduction

This paper is a continuation of [10], where the concept of *anti-selfdual* (*ASD*) *Lagrangians* was shown to be inherent in many basic boundary-value and initial-value problems. Motivated by a conjecture of Brezis-Ekeland [4,5], and eventually by the subsequent related work of Auchmuty [1,2], a new variational framework was established where, solutions of various equations which are not normally of Euler–Lagrange type, can still be obtained as minima of functionals of the form

$$I(u) = L(u, Au) + \ell(b_1(x), b_2(x)) \quad \text{or } I(u) = \int_0^T L(t, u(t), \dot{u}(t) + Au(t))\mathrm{d}t + \ell(u(0), u(T)).$$

where L is an anti-selfdual Lagrangian, A is essentially a skew-adjoint operator modulo boundary terms represented by a pair of operators (b_1, b_2), and where ℓ is a compatible boundary Lagrangian which insures that I is a selfdual functional in a sense to be recalled later. For such Lagrangians, the minimal value will always be zero and—just like the self (and antiself) dual equations of quantum field theory (e.g. Yang-Mills and others)—the equations associated to such minima are not derived from the fact they are critical points of the functional I, but because they are also zeroes of derived Lagrangians obtained by "completing the squares". In other words, the infimum will satisfy

$$L(u, Au) + \langle u, Au\rangle = 0 \quad \text{resp., } L(t, u(t), \dot{u}(t) + Au(t)) + \langle u(t), \dot{u}(t)\rangle = 0,$$

which are functional reformulations of the equations at hand. It is also shown in [10] that ASD Lagrangians possess remarkable permanence properties making them more prevalent than expected and quite easy to construct and/or identify. The variational game changes from the analytical proofs of existence of extremals for general action functionals, to a more algebraic search of an appropriate ASD Lagrangian for which the value of the corresponding minimization problem is always equal to zero. This makes them efficient new tools for proving existence and uniqueness results for a large array of differential equations.

We tackle here again the question of providing a variational formulation and resolution of boundary value problems of the form:

$$\begin{cases} -Au + f \in \partial\varphi(u) \\ b_1(u) = u_1, \end{cases} \tag{1}$$

as well as parabolic evolution equations of the form:

$$\begin{cases} -Au(t) - \dot{u}(t) + \omega u(t) \in \partial\varphi(t, u(t)) & \text{a.e. } t \in [0, T] \\ b_1(u(t)) = b_1(u_0) & \text{a.e } t \in [0, T] \\ u(0) = u_0, \end{cases} \tag{2}$$

where φ is a convex lower semicontinuous functional, A is a linear operator, b_1 is a boundary operator, u_1 is a prescribed boundary value, $w \in \mathbb{R}$, and u_0 is a given initial value. We shall use the formalism developed in [10] to rewrite these problems as

$$0 \in \bar{\partial}L(u) \quad \text{and } -\dot{u}(t) \in \bar{\partial}L(t, u(t)) \tag{3}$$

respectively, where $\bar{\partial}L$ is an "ASD" vector field derived—in an appropriate way—from the "potential" L. In this paper, we extend the domain of application of this approach by establishing existence and regularity results for the equations in (3) under much less restrictive boundedness conditions on the anti-selfdual Lagrangian L. In particular, we will be able to associate—via an integral representation formula—a semi-group of nonlinear contractions $(T_t)_{t\in\mathbb{R}^+}$ to any ASD Lagrangian L in such a way that $x(t) = T_t x_0$ is a solution of $-\dot{u}(t) \in \bar{\partial}L(u(t))$ starting at x_0.

As applications, we provide—under minimal hypothesis—a variational resolution to equations involving non self-adjoint operators such as the following *transport equation*:

$$\begin{cases} -\vec{a}(x)\cdot\vec{\nabla}u(x) + a_0 u = |u(x)|^{p-1}u + f & \text{on } \Omega \subset \mathbb{R}^n \\ u(x) = 0 & \text{on } \Sigma_-, \end{cases} \tag{4}$$

where $\vec{a} = (a_i)_i : \Omega \to \mathbf{R}^n$ is a vector field, $f \in L^2(\Omega)$, and where $\Sigma_- = \{x \in \partial\Omega;\ \vec{a}(x)\hat{n}(x) < 0\}$, $\hat{n}$ being the outer normal vector. We also provide a variational resolution to general dissipative initial value problems such as the following convection-diffusion evolutions.

$$\begin{aligned} -\frac{\partial u}{\partial t}(x,t) + \vec{a}(x)\cdot\vec{\nabla}u(x,t) &= \Delta_p u(x,t) + a_0(x)u(x,t) + \omega u(x,t) \quad \text{on } [0,T]\times\Omega \\ u(x,0) &= u_0(x) \quad \text{on } \Omega \\ u(x,t) &= 0 \quad \text{on } [0,T]\times\partial\Omega. \end{aligned} \tag{5}$$

But more importantly, we also deal with the more delicate case where the equation is purely non-self-adjoint such as:

$$\begin{aligned} -\frac{\partial u}{\partial t}(x,t) + \vec{a}(x)\cdot\vec{\nabla}u(x,t) &= a_0(x)u(x,t) + u(x,t)|u(x,t)|^{p-2} + \omega u(x,t) \quad \text{on } [0,T]\times\Omega \\ u(x,0) &= u_0(x) \quad \text{on } \Omega \\ u(x,t) &= u_0(x) \quad \text{on } [0,T]\times\Sigma_- \end{aligned} \tag{6}$$

Needless to say, these equations have been the subject of many studies starting from the fundamental paper of Bardos [3]. They are however not normally solved by the methods of the calculus of variations since they do not correspond to Euler–Lagrange equations of action functionals of the form $\int_\Omega F(x, u(x), \nabla u(x)\,\mathrm{d}x$ or $\int_0^T L(t, x(t), \dot{x}(t)\mathrm{d}t$. They turned out to be ideal "elementary" examples of nonself-adjoint equations on which to illustrate the applicability of our method including many of its subtleties.

The paper, though sufficiently self-contained, is better read in conjunction with [10]. It is organized as follows: In Sect. 2, we isolate the conditions under which the composition of an anti-selfdual Lagrangian with an unbounded linear operator yields a global Lagrangian that is also anti-selfdual. Section 3 gives the first applications to non-homogeneous transport equations. In Sect. 4, we prove our main result which associates a semi-group of nonlinear operators to a general ASD Lagrangian, including those involving semi-convex terms. This result is applied in Sect. 5 to provide variational resolutions to several parabolic initial-value problems. Some of the results were announced in [7,8]. The case involving nonlinear operators is dealt with in [11].

Added in proof: Recently, the first named author proved in [13] that every maximal monotone operator is actually an anti-selfdual vector field, which means that ASD-Lagrangians can be seen as the potentials of maximal monotone operators leading to a variational formulation and resolution of most equations involving such nonlinear operators. This "selfdual variational theory" and its applications is developed in detail in the upcoming monograph [12].

2 Anti-selfdual Lagrangians and unbounded operators

Given a reflexive Banach space X, we consider the class $\mathcal{L}(X)$ of Lagrangians $L : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ which are convex and lower semi-continuous (in both variables) and which are not identically $+\infty$. The Legendre-Fenchel dual (in both variables) of L is defined at any pair $(q, y) \in X^* \times X$ by:

$$L^*(q,y) = \sup\{\langle q,x\rangle + \langle y,p\rangle - L(x,p);\, x \in X, p \in X^*\}.$$

We refer to Ekeland–Temam [6] for the basics of convex analysis, and we recall from [10] the following notions.

Definition 2.1 (1) Say that L is an *anti-selfdual Lagrangian* on $X \times X^*$, if

$$L^*(p,x) = L(-x,-p) \quad \text{for all } (p,x) \in X^* \times X. \tag{7}$$

(2) L is *partially anti-selfdual*, if

$$L^*(0,x) = L(-x,0) \quad \text{for all } x \in X. \tag{8}$$

Denote by $\mathcal{L}_{\mathrm{AD}}(X)$ the class of anti-selfdual (ASD) Lagrangians on a given Banach space X. This is a quite interesting and natural class of Lagrangians as they appear in several basic PDEs and evolution equations. The basic example of an anti-selfdual Lagrangian is given by a function L on $X \times X^*$, of the form

$$L(x,p) = \varphi(x) + \varphi^*(-p) \tag{9}$$

where φ is a convex and lower semi-continuous function on X and φ^* is its Legendre conjugate on X^*. But the class $\mathcal{L}_{\mathrm{AD}}(X)$ was shown in [10] to be much richer as it goes well beyond convex functions and their conjugates, especially because it is stable under composition with skew-symmetric operators. Indeed if $\Lambda : X \to X^*$ is a bounded linear skew-symmetric (i.e., $\Lambda^* = -\Lambda$), and if L is an ASD Lagrangian, it is then easy to see that the Lagrangian

$$M(x,p) = L(x, \Lambda x + p) \tag{10}$$

is also anti-selfdual. However, in various applications, we are often faced with an unbounded operator Λ which may still satisfy various aspects of anti-symmetry. In the sequel we study to what extent the composition formula (10) above remains valid for such operators.

Let A be a linear—not necessarily bounded—map from its domain $D(A) \subset X$ into X^*. Assuming $D(A)$ dense in X, we consider the domain of its adjoint A^* which is defined as:

$$D(A^*) = \{x \in X; \sup\{\langle x, Ay\rangle; y \in D(A), \|y\|_X \le 1\} < +\infty\}.$$

Definition 2.2 Let X be a reflexive Banach space and let A be a linear map from its domain $D(A) \subset X$ into X^*. Say that

1. A is antisymmetric if $D(A) \subset D(A^*)$ and if $A^* = -A$ on $D(A)$.
2. A is skew-adjoint if it is antisymmetric and if $D(A) = D(A^*)$.

The following easy lemma shows that the composition formula still holds for anti-symmetric operators, provided they have a large enough domain.

Lemma 2.3 *Let φ be a convex proper lower semi-continuous functional on X with a symmetric domain $D(\varphi)$, and let $A : D(A) \subset X \to X^*$ be an anti-symmetric operator such that $D(\varphi) \subseteq D(A)$. Then the Lagrangian*

$$M(x,p) = \begin{cases} \varphi(x) + \varphi^*(Ax - p) & x \in D(\varphi) \\ +\infty & \text{elsewhere} \end{cases}$$

is anti-selfdual on X.

Proof Let $(\tilde{x},\tilde{p}) \in X \times X^*$ and suppose first that $\tilde{x} \in D(\varphi)$. Then

$$\begin{aligned} M^*(\tilde{p},\tilde{x}) &= \sup_{x\in D(\varphi)} \sup_{p\in X^*} \left\{\langle x,\tilde{p}\rangle + \langle \tilde{x},p\rangle - \left(\varphi(x) + \varphi^*(Ax-p)\right)\right\} \\ &= \sup_{x\in D(\varphi)} \sup_{q\in X^*} \left\{\langle x,\tilde{p}\rangle + \langle \tilde{x},Ax-q\rangle - \varphi(x) - \varphi^*(q)\right\} \\ &= \sup_{x\in D(\varphi)} \sup_{q\in X^*} \left\{\langle x,\tilde{p}\rangle - \langle A\tilde{x},x\rangle - \langle \tilde{x},q\rangle - \varphi(x) - \varphi^*(q)\right\} \\ &= \varphi(-\tilde{x}) + \varphi^*(\tilde{p} - A\tilde{x}) \\ &= M(-\tilde{x},-\tilde{p}). \end{aligned}$$

If now $\tilde{x} \notin D(\varphi)$, then

$$\begin{aligned} M^*(\tilde{p},\tilde{x}) &\geq -\varphi(0) + \sup_{p\in X^*} \left\{\langle \tilde{x},p\rangle - \varphi^*(-p)\right\} \\ &= -\varphi(0) + \varphi(-\tilde{x}) = +\infty = M(-\tilde{x},-\tilde{p}). \end{aligned}$$

We shall also deal with situations where operators are skew-adjoint provided one takes into account certain boundary terms. We introduce the following notion □

Definition 2.4 Let A be a linear map from its domain $D(A)$ in a reflexive Banach space X into X^* and consider (b_1,b_2) to be a pair of linear maps from its domain $D(b_1,b_2)$ in X into the product of two Hilbert spaces $H_1 \times H_2$. We say that A is *skew-adjoint modulo the boundary operators* (b_1,b_2) if the following properties are satisfied:

1. The space $X_0 := Ker(b_1,b_2) \cap D(A)$ is dense in X
2. The image of $S := D(A) \cap D(b_1,b_2)$ by (b_1,b_2) is dense in $H_1 \times H_2$.
3. An element y in X belongs to S if and only if

$$\sup\left\{\langle y,Ax\rangle - \frac{1}{2}(\|b_1(x)\|^2_{H_1} + \|b_2(x)\|^2_{H_2}); x \in S, \|x\|_X < 1\right\} < \infty.$$

4. For every $x,y \in S$, we have

$$\langle y,Ax\rangle = -\langle Ay,x\rangle + \langle b_1(x),b_1(y)\rangle_{H_1} - \langle b_2(x),b_2(y)\rangle_{H_2}.$$

It is clear that if b_1, b_2 are the zero operators on X, then our definition coincides with the notion of skew-adjoint operator in Definition 2.2.2). Here is our main result concerning the composition of ASD Lagrangians with non-necessarily bounded skew-adjoint operators.

Proposition 2.1 *Let $L : X \times X^* \to \mathbb{R}$ be an ASD Lagrangian on a reflexive Banach space X such that for ever $p \in X^*$, the function $x \to L(x,p)$ is bounded on the bounded sets of X. Let $A : D(A) \to X^*$ be skew-adjoint modulo the boundary operators $(b_1, b_2) : X \to H_1 \times H_2$. Then the Lagrangian defined by*

$$M(x,p) = \begin{cases} L(x, Ax+p) + \frac{\|b_1(x)\|_{H_1}^2}{2} + \frac{\|b_2(x)\|_{H_2}^2}{2} & \text{if } x \in D(A) \cap D(b_1,b_2) \\ +\infty & \text{if } x \notin D(A) \cap D(b_1,b_2) \end{cases}$$

is anti-selfdual on X.

Proof The idea is to use the density of $\text{Ker}(b_1, b_2)$ and the continuity of L in the first variable to split the space X in such a way that the supremum over the main term and the supremum over the boundary term are independent of each other. Indeed, if $\tilde{x} \in S := D(A) \cap D(b_1, b_2)$, then

$$M^*(\tilde{p},\tilde{x}) = \sup_{\substack{x\in S\\ p\in X^*}} \left\{ \langle \tilde{x}, p\rangle + \langle x, \tilde{p}\rangle - L(x, Ax+p) - \frac{\|b_1(x)\|_{H_1}^2}{2} - \frac{\|b_2(x)\|_{H_2}^2}{2} \right\}$$

Substituting $q = Ax + p$, we get

$$M^*(\tilde{p},\tilde{x}) = \sup_{\substack{x\in S\\ q\in X^*}} \left\{ \langle \tilde{x}, q - Ax\rangle + \langle x, \tilde{p}\rangle - L(x, q) - \frac{\|b_1(x)\|_{H_1}^2}{2} - \frac{\|b_2(x)\|_{H_2}^2}{2} \right\}$$

Since $\tilde{x} \in D(A)$, we have that:

$$\langle \tilde{x}, Ax\rangle = -\langle x, A\tilde{x}\rangle + \langle b_1(x), b_1(\tilde{x})\rangle - \langle b_2(x), b_2(\tilde{x})\rangle.$$

which yields

$$\begin{aligned} M^*(\tilde{p},\tilde{x}) = \sup_{\substack{x\in S\\ q\in X^*}} \Big\{ & \langle x, A\tilde{x}\rangle - \langle b_1(x), b_1(\tilde{x})\rangle + \langle b_2(x), b_2(\tilde{x})\rangle + \langle \tilde{x}, q\rangle \\ & + \langle x, \tilde{p}\rangle - L(x,q) - \frac{\|b_1(x)\|_{H_1}^2}{2} - \frac{\|b_2(x)\|_{H_2}^2}{2} \Big\} \end{aligned}$$

Now for all $x_0 \in \text{Ker}(b_1, b_2)$, we obviously have $b_1(x) = b_1(x + x_0)$ and $b_2(x) = b_2(x + x_0)$, so that for all $x_0 \in \text{Ker}(b_1, b_2) \cap D(A) \subseteq S$,

$$\begin{aligned} M^*(\tilde{p},\tilde{x}) = \sup_{\substack{x\in S\\ q\in X^*}} \Big\{ & \langle x, A\tilde{x}\rangle - \langle b_1(x+x_0), b_1(\tilde{x})\rangle + \langle b_2(x+x_0), b_2(\tilde{x})\rangle + \langle \tilde{x}, q\rangle \\ & + \langle x, \tilde{p}\rangle - L(x,q) - \frac{\|b_1(x+x_0)\|_{H_1}^2}{2} - \frac{\|b_2(x+x_0)\|_{H_2}^2}{2} \Big\} \end{aligned}$$

It follows that

$$\begin{aligned} M^*(\tilde{p},\tilde{x}) = \sup \Big\{ & \langle x, A\tilde{x} + \tilde{p}\rangle - \langle b_1(x+x_0), b_1(\tilde{x})\rangle + \langle b_2(x+x_0), b_2(\tilde{x})\rangle \\ & + \langle \tilde{x}, q\rangle + L(x,q) - \frac{\|b_1(x+x_0)\|_{H_1}^2}{2} - \frac{\|b_2(x+x_0)\|_{H_2}^2}{2}; \\ & x \in S, q \in X^*, x_0 \in \text{Ker}(b_1, b_2) \cap D(A) \Big\} \end{aligned}$$

Since S is a linear space, we may set $w = x + x_0$ and write

$$M^*(\tilde{p}, \tilde{x}) = \sup\Big\{ \langle w - x_0, A\tilde{x} + \tilde{p}\rangle - \langle b_1(w), b_1(\tilde{x})\rangle + \langle b_2(w), b_2(\tilde{x})\rangle + \langle \tilde{x}, q\rangle$$
$$- L(w - x_0, q) - \frac{\|b_1(w)\|^2_{H_1}}{2} - \frac{\|b_2(w)\|^2_{H_2}}{2}; w \in S, q \in X^*,$$
$$x_0 \in \text{Ker }(b_1, b_2) \cap D(A) \Big\}$$

Now, for each fixed $w \in S$ and $q \in X^*$, the supremum over $x_0 \in \text{Ker}(b_1, b_2) \cap D(A)$ can be taken as a supremum over $x_0 \in X$ since $\text{Ker}(b_1, b_2) \cap D(A)$ is dense in X and all terms involving x_0 are continuous in that variable. Furthermore, for each fixed $w \in S$ and $q \in X^*$, the supremum over $x_0 \in X$ of the terms $w - x_0$ can be written as supremum over $v \in X$ where $v = w - x_0$. So setting $v = w - x_0$ we get

$$M^*(\tilde{p}, \tilde{x}) = \sup\Big\{ \langle v, A\tilde{x} + \tilde{p}\rangle - \langle b_1(w), b_1(\tilde{x})\rangle + \langle b_2(w), b_2(\tilde{x})\rangle + \langle \tilde{x}, q\rangle - L(v, q)$$
$$- \frac{\|b_1(w)\|^2_{H_1}}{2} - \frac{\|b_2(w)\|^2_{H_2}}{2}; v \in X, q \in X^*, w \in S \Big\}$$
$$= \sup_{v\in X}\sup_{q\in X^*} \big\{ \langle v, A\tilde{x} + \tilde{p}\rangle + \langle \tilde{x}, q\rangle - L(v, q)\big\}$$
$$+ \sup_{w\in S}\Big\{ \langle b_1(w), -b_1(\tilde{x})\rangle + \langle b_2(w), b_2(\tilde{x})\rangle - \frac{\|b_1(w)\|^2_{H_1}}{2} - \frac{\|b_2(w)\|^2_{H_2}}{2}\Big\}$$

Since the range of $(b_1, b_2) : S \to H_1 \times H_2$ is dense in the $H_1 \times H_2$ topology, the boundary term can be written as

$$\sup_{a\in H_1}\sup_{b\in H_2}\Big\{ \langle a, -b_1(\tilde{x})\rangle + \langle b, b_2(\tilde{x})\rangle - \frac{\|a\|^2_{H_1}}{2} - \frac{\|b\|^2_{H_2}}{2}\Big\} = \frac{\|b_1(\tilde{x})\|^2_{H_1}}{2} + \frac{\|b_2(\tilde{x})\|^2_{H_2}}{2}$$

while the main term is clearly equal to $L^*(A\tilde{x} + \tilde{p}, \tilde{x}) = L(-\tilde{x}, -A\tilde{x} - \tilde{p})$ in such a way that $M^*(p, \tilde{x}) = M(-\tilde{x}, -\tilde{p})$ if $\tilde{x} \in D(A) \cap D(b_1, b_2)$.

Now if $\tilde{x} \notin S = D(A) \cap D(b_1, b_2)$ then

$$M^*(\tilde{p}, \tilde{x}) = \sup_{\substack{x\in S\\ q\in X^*}} \Big\{ \langle \tilde{x}, q - Ax\rangle + \langle x, \tilde{p}\rangle - L(x, q) - \frac{\|b_1(x)\|^2_{H_1}}{2} - \frac{\|b_2(x)\|^2_{H_2}}{2}\Big\}$$
$$\geq \sup_{\substack{x\in S\\ \|x\|_X<1}} \Big\{ \langle -\tilde{x}, Ax\rangle + \langle x, \tilde{p}\rangle - L(x, 0) - \frac{\|b_1(x)\|^2_{H_2}}{2} - \frac{\|b_2(x)\|^2_{H_2}}{2}.\Big\}$$

Since by assumption $L(x, 0) < C$ whenever $\|x\|_X < 1$, we finally obtain that

$$M^*(\tilde{p}, \tilde{x}) \geq \sup_{\substack{x\in S\\ \|x\|_X<1}} \Big\{ \langle -\tilde{x}, Ax\rangle + \langle x, \tilde{p}\rangle - C - \frac{\|b(x)\|^2_{H_2}}{2} - \frac{\|b_2(x)\|^2_{H_2}}{2}\Big\}$$
$$= +\infty$$
$$= M(-\tilde{x}, -\tilde{p})$$

since $\tilde{x} \notin S$ whenever $-\tilde{x} \notin S$. Therefore $M^*(\tilde{p}, \tilde{x}) = M(-\tilde{x}, -\tilde{p})$ for all $(\tilde{x}, \tilde{p}) \in X \times X^*$ and M is an anti-selfdual Lagrangian. □

2.1 The example of the transport operator $A = \vec{a} \cdot \vec{\nabla} u + \frac{(\vec{\nabla} \cdot \vec{a})}{2} u$

Consider the transport operator $u \mapsto \vec{a} \cdot \vec{\nabla} u + \frac{(\vec{\nabla} \cdot \vec{a})}{2} u$ on the space $X = L^p(\Omega)$, in conjunction with two trace operators (restrictions) onto two appropriate subsets of $\partial\Omega$. We show that this operator is skew-adjoint modulo the corresponding boundary operators, in the sense of Definition 2.2. These properties will be crucial for the next sections where we establish existence results for both stationary and evolution equations involving first order transport operators.

We shall adopt the framework of [3], and in particular all conditions imposed there on the smooth vector field $\vec{a}$ defined on a neighborhood of a C^∞ bounded open set Ω in $\mathbb{R}^n$. Set $X = L^p(\Omega)$ for $1 < p < +\infty$ and define

$$\Sigma_\pm = \{x \in \partial\Omega; \pm\vec{a}(x) \cdot \hat{n}(x) \geq 0\} \text{ be the entrance and exit sets of the transport operator } \vec{a} \cdot \vec{\nabla},$$

the corresponding Hilbert spaces:

$$H_1 = L^2(\Sigma_+; |\vec{a} \cdot \hat{n}| d\sigma), \quad H_2 = L^2(\Sigma_-; |\vec{a} \cdot \hat{n}| d\sigma),$$

as well as the boundary (trace) operators $(b_1 u, b_2 u) = (u|_{\Sigma_+}, u|_{\Sigma_-})$ whose domain is

$$D(b_1, b_2) = \{u \in L^p(\Omega); (u|_{\Sigma_+}, u|_{\Sigma_-}) \in H_1 \times H_2\}.$$

We shall consider the operator $Au = \vec{a} \cdot \vec{\nabla} u + \frac{(\vec{\nabla} \cdot \vec{a})}{2} u$ with domain

$$D(A) = \{u \in L^p(\Omega); \vec{a} \cdot \vec{\nabla} u + \frac{(\vec{\nabla} \cdot \vec{a})}{2} u \in L^q(\Omega)\} \text{ into } L^q(\Omega),$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Observe that $D(A)$ is a Banach space under the norm

$$\|u\|_{D(A)} = \|u\|_p + \|\vec{a} \cdot \vec{\nabla} u\|_q$$

and that $S := D(A) \bigcap D(b_1) \bigcap D(b_2)$ is also a Banach space under the norm

$$\|u\|_S = \|u\|_p + \|\vec{a} \cdot \vec{\nabla} u\|_q + \|u|_{\Sigma_+}\|_{L^2(\Sigma_+; |\vec{a} \cdot \hat{n}| d\sigma)}$$

Under the assumptions listed above, $C^\infty(\bar{\Omega})$ is dense in both spaces (c.f. Bardos [3]).

Lemma 2.5 *The unbounded operator A is skew-adjoint modulo the boundary* (b_1, b_2) *on the space* $L^p(\Omega)$.

Proof We check the four criteria of Definition 2.4 (See also [16]). For 1) it suffices to note that $C_0^\infty(\Omega) \subset Ker(b_1, b_2) \bigcap D(A)$, in such a way that $Ker(b_1, b_2) \bigcap D(A)$ is dense in X. Criteria (2) follows by a simple argument with coordinate charts, as it is easy to show that for all $(v_+, v_-) \in C_0^\infty(\Sigma_+) \times C_0^\infty(\Sigma_-)$ there exists $u \in C^\infty(\bar{\Omega})$ such that $(u|_{\Sigma_+}, u|_{\Sigma_-}) = (v_+, v_-)$. The embedding of $C_0^\infty(\Sigma_\pm) \subset L^2(\Sigma_\pm; |\vec{a} \cdot \hat{n}| d\sigma)$ is dense, and therefore the image of $C^\infty(\bar{\Omega}) \subset S$ under (b_1, b_2) is dense in $H_1 \times H_2$.

For criteria 3), we need to check that, if $u \in X$, then it belongs to S if and only if

$$\sup \left\{ \langle u, Av \rangle - \frac{1}{2}(\|b_1(v)\|_{H_1}^2 + \|b_1(v)\|_{H_2}^2); v \in S, \|v\|_X < 1 \right\} < \infty. \tag{11}$$

The "if" direction follows directly from Green's theorem and the fact that $C^\infty(\bar{\Omega})$ is dense in the Banach space S under the norm $\|u\|_S$. For the reverse implication, suppose that (11) holds, then obviously

$$\sup\left\{\langle u, Av\rangle; v \in C_0^\infty(\Omega), \|v\|_X < 1\right\}$$

which means that $\vec{a}\cdot\vec{\nabla}u + \frac{\vec{\nabla}\cdot\vec{a}}{2}u \in L^q(\Omega)$ in the sense of distribution, therefore $u \in D(A)$. Now to show that $u \in D(b_1)\bigcap D(b_2)$, we observe that if $u \in X$ and $u \in D(A)$, then $u|_{\Sigma_+} \in L^2_{\rm loc}(\Sigma_+; |\vec{a}\cdot\hat{n}|d\sigma)$. To check that $u|_{\Sigma_+} \in L^2(\Sigma_+; |\vec{a}\cdot\hat{n}|d\sigma)$ a simple argument using Green's Theorem shows that (11) implies that

$$\sup\left\{\int_{\Sigma_+} uv|\vec{a}\cdot\hat{n}|\mathrm{d}\sigma; v \in C_o^\infty(\Sigma_+), \int_{\Sigma_+} |v|^2|\vec{a}\cdot\hat{n}|\mathrm{d}\sigma \le 1\right\} < +\infty$$

which means that $u|_{\Sigma_+} \in L^2(\Sigma_+; |\vec{a}\cdot\hat{n}|\mathrm{d}\sigma)$ and $u \in D(b_1)$. The same argument works for $u|_{\Sigma_-}$ and criterion 3) is therefore satisfied.
For 4), note that by Green's theorem we have

$$\int_\Omega ((\mathbf{a}\cdot\nabla u)v + \frac{1}{2}\mathrm{div}\,\mathbf{a})uv\mathrm{d}x = -\int_\Omega ((\mathbf{a}\cdot\nabla v)u + \frac{1}{2}\mathrm{div}\,\mathbf{a})uv\mathrm{d}x + \int_{\partial\Omega} uv\mathbf{n}\cdot\mathbf{a}\mathrm{d}\sigma$$

for all $u, v \in C^\infty(\bar{\Omega})$ and the identity on S follows since $C^\infty(\bar{\Omega})$ is dense in S for the norm $\|u\|_S$. □

3 Anti-selfdual vector fields and PDEs involving the transport operators

We now recall the following framework from [10].

Definition 3.1 Let X be a reflexive Banach space and let $L : X \times X^* \to \mathbb{R}\cup\{+\infty\}$ be a convex lower semi-continuous function, that is not identically equal to $+\infty$. The *derived vector fields of* L at $x \in X$ (resp., $p \in X^*$) are the –possibly empty– sets

$$\overline{\partial}L(x) := \{p \in X^*;\ L(x,-p) - \langle x,p\rangle = 0\} \quad \text{resp.,}$$
$$\tilde{\partial}L(p) := \{x \in X;\ L(x,-p) - \langle x,p\rangle = 0\}.$$

The domains of the vector field $\overline{\partial}L$ are the sets

$$\mathrm{Dom}(\overline{\partial}L) = \{x \in X; \overline{\partial}L(x) \neq \emptyset\} \quad \text{resp.,} \quad \mathrm{Dom}(\tilde{\partial}L) = \{p \in X^*; \tilde{\partial}L(p) \neq \emptyset\}.$$

The above defined potentials should not be confused with the subdifferential ∂L of L as a convex function on $X \times X^*$. Actually, it is easy to see that if L is an anti-selfdual Lagrangian, then $p \in \overline{\partial}L(x)$ if and only if $x \in \tilde{\partial}L(p)$ if and only if $(p,-x) \in \partial L(x,-p)$. This is also equivalent to say that $0 \in \overline{\partial}L_p(x)$ where L_p is the translated Lagrangian $L_p(x,q) = L(x,p+q) + \langle x,p\rangle$.

For a basic ASD Lagrangian $L(x,p) = \varphi(x) + \varphi^*(-p)$, it is clear that

$$\overline{\partial}L(x) = \partial\varphi(x) \quad \text{while } \tilde{\partial}L(p) = \partial\varphi^*(p).$$

Recalling that an ASD Lagrangian L satisfies $L(x,p) + \langle x,p\rangle \ge 0$ for every $(x,p) \in X \times X^*$, the problem of proving that $\tilde{\partial}L(p)$ is non-empty for a given $p \in X^*$ amounts to showing that the infimum of the functional $I_p(x) = L(x,-p) - \langle x,p\rangle$ over $x \in X$

is equal to zero and is attained. The following proposition from [10] gives sufficient conditions for this to happen.

Proposition 3.1 *Let L be an anti-selfdual Lagrangian L on a reflexive Banach space $X \times X^*$. If for some $x_0 \in X$, the map $q \mapsto L(x_0, q)$ is bounded on a ball of X^*, then for each $p \in X^*$ the infimum of $I_p(x) = L(x, -p) - \langle x, p\rangle$ over X is equal to zero and is attained at some point $\bar{x} \in \tilde{\partial} L(p)$.*

We have also the following easy observations, some of which are proven in Sect. 4.

Proposition 3.2 *The following assertions hold whenever L is an anti-selfdual Lagrangian L on a reflexive Banach space $X \times X^*$.*

1. *The vector field $x \to \bar{\partial} L(p)$ is monotone in the sense that $\langle x - y, p - q\rangle \geq 0$ for $x \in \bar{\partial} L(p)$ and $y \in \bar{\partial} L(q)$.*
2. *If L is strictly convex in the second variable, then the vector field $x \to \bar{\partial} L(x)$ is single-valued.*
3. *If L is uniformly convex in the second variable (i.e., if $L(x,p) - \epsilon \frac{\|x\|^2}{2}$ is convex in x for some $\epsilon > 0$) then the vector field $x \to \bar{\partial} L(x)$ is a Lipschitz maximal monotone map.*

As noted above, non-trivial examples of anti-selfdual Lagrangians are of the form

$$L(x,p) = \varphi(x) + \varphi^*(-Ax - p) \tag{12}$$

where φ is a convex and lower semi-continuous function on X, φ^* is its Legendre conjugate on X^* and where $A : X \to X^*$ is bounded and skew-symmetric. In this case, it is easy to see that

$$\bar{\partial} L(x) = \Lambda x + \partial\varphi(x) \quad \text{while } \tilde{\partial} L(p) = (\Lambda + \partial\varphi)^{-1}(p).$$

This suggests that ASD vector fields are natural extensions of operators of the form $A + \partial\varphi$, where A is positive and φ is convex. This is an important subclass of maximal monotone operators which can now be resolved variationally.[1] More importantly, the stability of this subclass under various operations including composition with unbounded skew-adjoint operators and the adjoining of boundary operators makes it very functional in applications. The following Lax-Milgram type result for unbounded operators is one of them.

Proposition 3.3 *Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be proper convex and lower semi-continuous and assume that $A : D(A) \subset X \to X^*$ is a skew-adjoint operator modulo the boundary $(b_1, b_2) : D(b_1, b_2) \to H_1 \times H_2$ where H_1, H_2 are two Hilbert spaces. Suppose there exists a constant $C > 0$ such that for every $x \in X$,*

$$\frac{1}{C}\left(\|x\|_X^{p_1} - 1\right) \leq \varphi(x) \leq C\left(\|x\|_X^{p_2} + 1\right), \tag{13}$$

where $p_1, p_2 > 0$. Then there exists $\bar{x} \in D(A) \cap D(b_1, b_2)$ such that the functional

$$I(x) = \varphi(x) + \varphi^*(Ax) + \frac{\|b_1(x)\|_{H_1}^2}{2} + \frac{\|b_2(x)\|_{H_2}^2}{2}$$

[1] See what was *added in proof* at the end of the introduction.

attains its infimum at $\bar{x}$ in such a way that

$$I(\bar{x}) = \inf_{x \in X} I(x) = 0, \tag{14}$$

$$A\bar{x} \in \partial\varphi(\bar{x}) \tag{15}$$

$$b_2(\bar{x}) = 0. \tag{16}$$

Proof The Lagrangian $M(x,p) : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$M(x,p) = \begin{cases} \varphi(x) + \varphi^*(Ax - p) + \frac{\|b_1(x)\|_{H_1}^2}{2} + \frac{\|b_2(x)\|_{H_2}^2}{2} & \text{if } x \in D(A) \cap D(b_1, b_2) \\ +\infty & \text{otherwise} \end{cases}$$

is anti-selfdual on X. Indeed, $L(x,p) := \varphi(x) + \varphi^*(-p)$ is clearly ASD, and since $x \mapsto L(x,p)$ is bounded on the bounded sets of X for all $p \in X^*$, we apply Proposition 2.1 to conclude that M is ASD. Moreover, since $p \to M(0,p)$ is bounded on the bounded sets of X^*, Proposition 3.1 applies to $I(x) = M(x,0)$ and we obtain $\bar{x}$ such that $0 = \inf_{x \in X} M(x,0) = M(\bar{x},0)$. Since $M(\bar{x},0) < \infty$, we get that $\bar{x} \in D(A) \cap D(b_1,b_2)$ and $\varphi(\bar{x}) + \varphi^*(A\bar{x}) + \frac{\|b_1(\bar{x})\|_{H_1}^2}{2} + \frac{\|b_2(\bar{x})\|_{H_2}^2}{2} = 0$. Now observe that

$$\begin{aligned} 0 &= \varphi(\bar{x}) + \varphi^*(A\bar{x}) + \frac{\|b_1(\bar{x})\|_{H_1}^2}{2} + \frac{\|b_2(\bar{x})\|_{H_2}^2}{2} \\ &= \varphi(\bar{x}) + \varphi^*(A\bar{x}) - \langle \bar{x}, A\bar{x} \rangle + \langle \bar{x}, A\bar{x} \rangle + \frac{\|b_1(\bar{x})\|_{H_1}^2}{2} + \frac{\|b_2(\bar{x})\|_{H_2}^2}{2} \\ &= \varphi(\bar{x}) + \varphi^*(A\bar{x}) - \langle \bar{x}, A\bar{x} \rangle + \|b_2(\bar{x})\|_{H_2}^2 \geq 0, \end{aligned}$$

and therefore $A\bar{x} \in \partial\varphi(\bar{x})$ and $b_2(\bar{x}) = 0$ simultaneously. □

3.1 Applications to PDEs involving the transport operator

We will deal with two types of equations some of which where dealt with in [9] via a slightly different variational formulation.

1. Transport equation:

$$\begin{aligned} \vec{a}(x) \cdot \vec{\nabla} u(x) + a_0(x)u(x) + u(x)|u(x)|^{p-2} &= f(x) \quad \text{for } x \in \Omega \\ u(x) &= 0 \quad \text{for } x \in \Sigma_+. \end{aligned}$$

2. Transport equation with viscosity:

$$\begin{aligned} -\Delta_p u(x) + \vec{a}(x) \cdot \vec{\nabla} u(x) + a_0(x)u(x) + u|u(x)|^{m-2} &= f(x) \quad x \in \Omega \\ u(x) &= 0 \quad x \in \partial\Omega. \end{aligned}$$

1. Transport equations without diffusion terms

In this case we will assume that the domain Ω and the vector field $\vec{a}(\cdot)$ satisfy all the assumption in Sect. 2. We distinguish two cases:

Case 1 $p \geq 2$.

The Banach space is then $X = L^p(\Omega)$ since by Lemma 2.4 the operator $A : D(A) \to X^*$ defined by $A : u \mapsto \vec{a} \cdot \vec{\nabla} u + \frac{1}{2}(\vec{\nabla} \cdot \vec{a})u$ with domain $D(A) = \{u \in L^p(\Omega) \mid \vec{a} \cdot \vec{\nabla} u +$

$\frac{\vec{\nabla}\vec{a}}{2}u \in L^q(\Omega)\}$. is then skew-adjoint modulo the boundary operators $(b_1 u, b_2 u) = (u|_{\Sigma_+}, u|_{\Sigma_-})$ whose domain is

$$D(b_1, b_2) = \{u \in L^p(\Omega); (u|_{\Sigma_+}, u|_{\Sigma_-}) \in L^2(\Sigma_+; |\vec{a}\cdot\hat{n}|d\sigma) \times L^2(\Sigma_-; |\vec{a}\cdot\hat{n}|d\sigma)\}$$

In this case we get the following

Theorem 3.2 *Assume $p \geq 2$, and let $f \in L^q \left(\frac{1}{p}+\frac{1}{q}=1\right)$ and $a_0 \in L^\infty(\Omega)$. Suppose there exists $\tau \in C^1(\bar{\Omega})$ such that*

$$\vec{a}\cdot\vec{\nabla}\tau + \tfrac{1}{2}(\vec{\nabla}\cdot\vec{a}+a_0) \geq 0 \text{ on } \Omega. \tag{17}$$

Define on $X = L^p(\Omega)$ the following convex function

$$\varphi(u) := \frac{1}{p}\int_\Omega e^{2\tau}|e^{-\tau}u|^p \mathrm{d}x + \frac{1}{2}\int_\Omega \vec{a}\cdot\vec{\nabla}\tau|u|^2\mathrm{d}x + \frac{1}{4}\int_\Omega(\vec{\nabla}\cdot\vec{a}+a_0)|u|^2\mathrm{d}x + \int_\Omega uf\mathrm{d}x,$$

as well as the functional

$$I(u) := \varphi(u) + \varphi^*(\vec{a}\cdot\vec{\nabla}u + \frac{1}{2}(\vec{\nabla}\cdot\vec{a})u) + \frac{1}{2}\int_{\Sigma_+}|u|^2|\vec{a}\cdot\hat{n}|\mathrm{d}\sigma + \frac{1}{2}\int_{\Sigma_-}|u|^2|\vec{a}\cdot\hat{n}|\mathrm{d}\sigma$$

on the $S := D(A)\cap D(b_1,b_2)$ and $+\infty$ elsewhere on $L^p(\Omega)$.

1. *Then there exists $\bar{u} \in S$ such that $I(\bar{u}) = \inf\{I(u) \mid u \in L^p(\Omega)\} = 0$.*
2. *The function $\bar{v} := e^{-\tau}\bar{u}$ satisfies the nonlinear transport equation*

$$\begin{aligned} \vec{a}\cdot\vec{\nabla}\bar{v} - \frac{a_0}{2}\bar{v} &= \bar{v}|\bar{v}|^{p-2} + f \quad \text{on } \Omega,\\ \bar{v} &= 0 \quad\quad\quad\quad\quad \text{on } \Sigma_+. \end{aligned} \tag{18}$$

Proof Consider

$$M(u,p) = \begin{cases} \varphi(u) + \varphi^*(Au+p) + \ell(b_1(u), b_2(u)) & \text{if } u \in S\\ +\infty & \text{if } u \notin S \end{cases} \tag{19}$$

where $\ell(\cdot,\cdot) : L^2(\Sigma_+; |\vec{a}\cdot\hat{n}|d\sigma) \times L^2(\Sigma_-; |\vec{a}\cdot\hat{n}|d\sigma) \to \mathbb{R}$ is defined by

$$\ell(h,k) := \frac{1}{2}\int_{\Sigma_+}|h|^2|\vec{a}\cdot\hat{n}|\mathrm{d}\sigma + \frac{1}{2}\int_{\Sigma_-}|k|^2|\vec{a}\cdot\hat{n}|\mathrm{d}\sigma.$$

Since A is skew-adjoint modulo the boundary, we conclude from Proposition 2.1 that M is an ASD Langrangian on the space $L^p(\Omega)\times L^q(\Omega)$ and satisfies all the hypothesis of Proposition 3.1. There exists then $\bar{u} \in L^p(\Omega)$ such that $0 = M(\bar{u},0) = \inf\{M(u,0); u \in L^p\}$ which means that $0 = I(\bar{u}) = \inf\{I(u) \mid u \in S\}$ and assertion 1) is verified.

To get 2) we observe again that

$$0 = I(\bar{u}) = \varphi(\bar{u}) + \varphi^*(A\bar{u}) + \frac{1}{2}\int_{\Sigma_+} |\bar{u}|^2|\vec{a}\cdot\hat{n}|\mathrm{d}\sigma + \frac{1}{2}\int_{\Sigma_-} |\bar{u}|^2|\vec{a}\cdot\hat{n}|\mathrm{d}\sigma$$

$$\geq \langle \bar{u}, A\bar{u}\rangle + \frac{1}{2}\int_{\Sigma_+} |\bar{u}|^2|\vec{a}\cdot\hat{n}|\mathrm{d}\sigma + \frac{1}{2}\int_{\Sigma_-} |\bar{u}|^2|\vec{a}\cdot\hat{n}|\mathrm{d}\sigma$$

$$= \int_{\Sigma_+} |\bar{u}|^2|\vec{a}\cdot\hat{n}|\mathrm{d}\sigma \geq 0$$

In particular, $\varphi(\bar{u}) + \varphi^*(A\bar{u}) = \langle u, Au\rangle$ and $\int_{\Sigma_+} |\bar{u}|^2|\vec{a}\cdot\hat{n}|\mathrm{d}\sigma = 0$, in such a way that $A\bar{u} \in \partial\varphi(\bar{u})$ and $u|_{\Sigma_+} = 0$. In other words,

$$\vec{a}\cdot\vec{\nabla}\bar{u} + \frac{1}{2}(\vec{\nabla}\cdot\vec{a})\bar{u} = e^{\tau}\bar{u}|e^{-\tau}\bar{u}|^{p-2} + (\vec{a}\cdot\vec{\nabla}\tau)\bar{u} + \frac{1}{2}(\vec{\nabla}\cdot\vec{a} + a_0)\bar{u} + f$$

and $\bar{u}|_{\Sigma_+} = 0$. Multiply now both equations by $e^{-\tau}$ and use the product rule for differentiation to get $\vec{a}\cdot\vec{\nabla}\bar{v} - \frac{a_0}{2}\bar{v} = \bar{v}|\bar{v}|^{p-2} + f$ and $\bar{v}|_{\Sigma_+} \equiv 0$, where $\bar{v} := e^{-\tau}\bar{u}$. □

Case 2 $1 < p \leq 2$

In this case, the right space is $X = L^2(\Omega)$ and $A : D(A) \to X^*$ is defined as in the first case but with domain $D(A) = \{u \in L^2(\Omega) \mid \vec{a}\cdot\vec{\nabla}u \in L^2(\Omega)\}$. By Lemma 2.4, A is again skew-adjoint modulo the boundary operators $(b_1u, b_2u) = (u|_{\Sigma_+}, u|_{\Sigma_-})$ whose domain is

$$D(b_1, b_2) = \{u \in L^2(\Omega) | (u|_{\Sigma_+}, u|_{\Sigma_-}) \in L^2(\Sigma_+; |\vec{a}\cdot\hat{n}|\mathrm{d}\sigma) \times L^2(\Sigma_-; |\vec{a}\cdot\hat{n}|\mathrm{d}\sigma)\}.$$

We then obtain the following result:

Theorem 3.3 *Assume* $1 < p \leq 2$ *and let* $f \in L^2(\Omega)$ *and* $a_0 \in L^\infty(\Omega)$. *Suppose there exists* $\tau \in C^1(\bar{\Omega})$ *such that for some* $\epsilon > 0$ *we have*:

$$\vec{a}\cdot\vec{\nabla}\tau + \frac{1}{2}(\vec{\nabla}\cdot\vec{a} + a_0) \geq \epsilon > 0 \text{ on } \Omega. \tag{20}$$

Define on $L^2(\Omega)$ *the convex functional*

$$\varphi(u) := \frac{1}{p}\int_\Omega e^{2\tau}|e^{-\tau}u|^p\mathrm{d}x + \frac{1}{2}\int_\Omega \vec{a}\cdot\vec{\nabla}\tau|u|^2\mathrm{d}x + \frac{1}{4}\int_\Omega (\vec{\nabla}\cdot\vec{a} + a_0)|u|^2\mathrm{d}x + \int_\Omega uf\mathrm{d}x$$

as well as the functional

$$I(u) := \varphi(u) + \varphi^*(\vec{a}\cdot\vec{\nabla}u + \frac{1}{2}(\vec{\nabla}\cdot\vec{a})u) + \frac{1}{2}\int_{\Sigma_+} |u|^2|\vec{a}\cdot\hat{n}|\mathrm{d}\sigma + \frac{1}{2}\int_{\Sigma_-} |u|^2|\vec{a}\cdot\hat{n}|\mathrm{d}\sigma$$

on $S := D(A) \cap D(b_1, b_2)$ *and* $+\infty$ *elsewhere in* $L^2(\Omega)$.

1. *There exists then* $\bar{u} \in S$ *such that* $I(\bar{u}) = \inf\{I(u) \mid u \in L^2(\Omega)\} = 0$.
2. *The function* $\bar{v} := e^{-\tau}\bar{u}$ *satisfies the nonlinear transport equation* (18).

Proof Define M as in (19) and again by Lemma 2.4, A is skew-adjoint modulo the boundary. Since now φ is bounded on the bounded sets of L^2, we can now invoke Proposition 2.1 to conclude that $M(u,p)$ is an ASD Langrangian on the space $L^2(\Omega)\times L^2(\Omega)$. But in this case, φ is coercive because of condition (20), and therefore φ^* is bounded on bounded sets. All the hypothesis of Proposition 3.1 are now satisfied so there exists $\bar{u}\in L^2(\Omega)$ such that $0=M(\bar{u},0)=\inf\{M(u,0)\mid u\in L^2\}$. The rest follows as in the case when $p\geq 2$. □

2. Transport equations with a diffusion term

In this case, the conditions on the smooth vector field $\vec{a}$ and Ω need not be as restrictive as in the case where the equation is purely governed by the transport operator. This is because the setting will only require that the operator $A: D(A)\to X^*$ defined by $u\mapsto \vec{a}\cdot\vec{\nabla}u+\frac{1}{2}(\vec{\nabla}\cdot\vec{a})u$ be only anti-symmetric. The setup is as follows: Let $X=L^2(\Omega)$ and consider the above operator A on the domain

$$D(A)=\{u\in L^2(\Omega)\mid \vec{a}\cdot\vec{\nabla}u+\frac{1}{2}(\vec{\nabla}\cdot\vec{a})u\in L^2(\Omega)\}.$$

We then have the following

Theorem 3.4 *Let $\vec{a}\in C^\infty(\bar{\Omega})$ be a smooth vector field on $\Omega\subset\mathbb{R}^n$, and let $a_0\in L^\infty(\Omega)$. Let $p\geq 2$ and $1\leq m\leq\frac{np}{n-p}$, and assume the following coercivity condition:*

$$\vec{\nabla}\cdot\vec{a}+a_0\geq 0 \text{ on } \Omega. \tag{21}$$

Consider the following convex and lower semi-continuous functional on $L^2(\Omega)$:

$$\varphi(u):=\begin{cases}\frac{1}{p}\int_\Omega|\nabla u|^p\mathrm{d}x+\frac{1}{4}\int_\Omega(\vec{\nabla}\cdot\vec{a}+a_0)|u|^2\mathrm{d}x+\frac{1}{m}\int_\Omega|u|^m\mathrm{d}x+\int_\Omega uf\mathrm{d}x\\ \quad\text{if } u\in W_0^{1,p}(\Omega)\\ +\infty\quad\text{if } u\notin W_0^{1,p}(\Omega)\end{cases}$$

and define the functional $I(u):=\varphi(u)+\varphi^*(\vec{a}\cdot\vec{\nabla}u+\frac{1}{2}(\vec{\nabla}\cdot\vec{a})u)$ on $W_0^{1,p}(\Omega)$ and $+\infty$ elsewhere on $L^2(\Omega)$.

1. *There exists then $\bar{u}\in W_0^{1,p}(\Omega)$ such that $I(\bar{u})=\inf\{I(u);\,u\in L^2(\Omega)\}=0$.*
2. *The minimizer $\bar{u}$ satisfies the equation*

$$\begin{aligned}-\Delta_p\bar{u}+\frac{1}{2}a_0\bar{u}+\bar{u}|\bar{u}|^{m-2}+f=\vec{a}\cdot\vec{\nabla}\bar{u}&\qquad\text{on }\Omega\\ \bar{u}=0&\qquad\text{on }\partial\Omega.\end{aligned}$$

Proof The functional φ has a symmetric domain that is contained in the domain of A. So Lemma 2.5 applies and the Lagrangian

$$M(u,p):=\begin{cases}\varphi(u)+\varphi^*(Au-p) & \text{if } u\in W_0^{1,p}(\Omega)\\ +\infty & \text{if } u\notin W_0^{1,p}(\Omega)\end{cases}$$

is ASD. Now φ is obviously coercive on $L^2(\Omega)$ and therefore φ^* is bounded on bounded sets of $L^2(\Omega)$. Proposition 3.1 applies and we find $\bar{u}\in L^2(\Omega)$ such that $0=M(\bar{u},0)=\inf\{M(u,0)\mid u\in L^2(\Omega)\}$. Clearly $\bar{u}\in W_0^{1,p}(\Omega)$ and the rest follows as in the preceeding cases. □

4 Semi-groups associated to anti-selfdual Lagrangians

In this section we develop further the variational theory for dissipative evolution equations via the theory of ASD Lagrangians. The goal is to extend the variational theory of gradient flows [14,15] and the one for other parabolic equations developed in [10] so as to include evolutions of the form

$$-\dot{x}(t) \in \partial\varphi(x(t)) + Ax(t) + \omega x(t), \tag{22}$$

where A is an unbounded positive operator and ω is any real number. The framework proposed in [10] leads to the formulation of (22) as

$$-\dot{x}(t) \in \bar{\partial} L(t, x(t)) \text{ for all } t \in [0, T], \tag{23}$$

where the anti-selfdual Lagrangians $L(t, \cdot, \cdot)$ on the state space are associated to the convex functional φ, the operator A, and the scalar ω in the following way:

$$L(t, x, p) = e^{2wt}\left\{\varphi(e^{-wt}x) + \varphi^*(-e^{-wt}(Ax + p))\right\}. \tag{24}$$

Consider now the path space $A_H^2 = \{u : [0, T] \to H; \dot{u} \in L_H^2\}$ consisting of all absolutely continuous arcs $u : [0, T] \to H$, equipped with the norm $\|u\|_{A_H^2} = (\|u(0)\|_H^2 + \int_0^T \|\dot{u}\|^2 \mathrm{d}t)^{\frac{1}{2}}$.

The next step is based on the fact that under appropriate boundedness conditions, a (time-dependent) anti-selfdual Lagrangian $L : [0, T] \times H \times H \to \mathbb{R}$ on a Hilbert space H, “lifts” to a (partially) anti-selfdual Lagrangian $\mathcal{L}$ on path space $A_H^2 = \{u : [0, T] \to H; \dot{u} \in L_H^2\}$ via the formula

$$\mathcal{L}(x, p) = \int_0^T L(t, x(t) + p(t), \dot{x}(t))\mathrm{d}t + \ell(x(0) + a, x(T)) \tag{25}$$

where ℓ is an appropriate time-boundary Lagrangian and where $(p(t), a) \in L_H^2 \times H$ which happens to be a convenient representation for the dual of A_H^2. Equation (23) can then be formulated as a stationary equation on path space of the form

$$0 \in \bar{\partial}\mathcal{L}(x), \tag{26}$$

hence reducing the dynamic problem to the stationary case already considered above.

However, the boundedness condition on the Lagrangian L used in Theorem 4.2 of [10] is too stringent for most applications, especially when unbounded operators are involved. The rest of the section consists of showing that these conditions can be considerably relaxed through a λ-regularization procedure reminiscent of the standard theory for convex functions, but also the Yosida theory for operators, which seems to apply as naturally to ASD Lagrangians, a concept that allow for their superposition. This procedure is however made more complicated by the presence of the linear term—hence of $e^{\omega t}$ in the Lagrangian which prevents it from being autonomous. This factor will however allow—among other things—to relax the convexity assumptions on φ.

Here is the main result of this section.

Theorem 4.1 *Let L be an anti-selfdual Lagrangian on a Hilbert space $H \times H$ that is uniformly convex in the first variable. Assuming* $\mathrm{Dom}(\bar{\partial}L)$ *is non-empty, then for any $\omega \in \mathbf{R}$ there exists a semi-group of maps $(T_t)_{t \in \mathbf{R}^+}$ on* $\mathrm{Dom}(\bar{\partial}L)$ *such that*:

1. *$T_0 x = x$ and $\|T_t x - T_t y\| \leq e^{-\omega t}\|x - y\|$ for any $x, y \in \mathrm{Dom}(\bar{\partial} L)$.*
2. *The semi-group is defined for any $x_0 \in \mathrm{Dom}(\bar{\partial} L)$ by $T_t x_0 = x(t)$ where $x(t)$ is the unique path that minimizes on A^2_H the functional*

$$I(u) = \int_0^T e^{2\omega t} L(u(t), \omega u(t) + \dot{u}(t))\mathrm{d}t + \frac{1}{2}\|u(0)\|^2 - 2\langle x_0, u(0)\rangle + \|x_0\|^2 + \frac{1}{2}\|e^{\omega T} u(T)\|^2$$

in such a way that $I(x) = \inf_{u \in A^2_H} I(u) = 0$.

3. *For any $x_0 \in \mathrm{Dom}(\bar{\partial} L)$ the path $x(t) = T_t x_0$ satisfies the following:*

$$\begin{aligned} -\dot{x}(t) - \omega x(t) &\in \bar{\partial} L(x(t)) \\ x(0) &= x_0. \end{aligned} \tag{27}$$

First we shall prove the following improvement of Theorem 4.2 of [10] provided L is autonomous. The boundedness condition is still there, but we first cover the semi-convex case.

Proposition 4.1 *Assume $L : H \times H \to \mathbb{R} \cup \{+\infty\}$ is an anti-selfdual Lagrangian that is uniformly convex and suppose*

$$L(x, 0) \leq C(\|x\|^2 + 1) \quad \text{for all } x \in H. \tag{28}$$

Then, for any $w \in \mathbb{R}$ and any $x_0 \in H$, there exists $\hat{x} \in C^1\big([0, T] : H\big)$ such that $\hat{x}(0) = x_0$ and

$$\int_0^T e^{2\omega t} L(e^{-\omega t}\hat{x}(t), e^{-\omega t}\dot{\hat{x}}(t))\mathrm{d}t + \frac{1}{2}\|\hat{x}(0)\|^2 - 2\langle x_0, \hat{x}(0)\rangle + \|x_0\|^2 + \frac{1}{2}\|\hat{x}(T)\|^2 = 0 \tag{29}$$

$$-e^{-wt}\big(\dot{\hat{x}}(t), \hat{x}(t)\big) \in \partial L\big(e^{-wt}\hat{x}(t), e^{-wt}\dot{\hat{x}}(t)\big) \quad \forall t \in [0, T], \tag{30}$$

$$\|\dot{\hat{x}}(t)\| \leq C(w, T)\|\dot{\hat{x}}(0)\| \quad \forall t \in [0, T], \tag{31}$$

where $C(w, T)$ is a positive constant.

The following lemma shows how the uniform convexity of the Lagrangian yield certain regularity properties of the solutions.

Lemma 4.2 *Assume $L : H \times H \to \mathbb{R} \cup \{+\infty\}$ is an anti-selfdual Lagrangian that is uniformly convex in both variables. Then, for all $x, u \in H$, there exists a unique $v \in H$ –denoted $v = R(u, x)$– such that $x = \partial_2 L(u, v)$, and the map $(u, x) \to R(u, x)$ is jointly Lipschitz on $H \times H$.*

In particular, if $v, x, u : [0, T] \to H$ are paths such that $x, u \in C([0, T]; H)$ and $-x(t) = \partial_2 L\big(u(t), v(t)\big)$ for almost all $t \in [0, T]$, then $v \in C([0, T]; H)$ and $-x(t) = \partial_2 L\big(u(t), v(t)\big)$ for all $t \in [0, T]$.

Proof We shall use the standard fact that if $F : H \to \mathbb{R} \cup \{+\infty\}$ is convex and lower semi-continuous and if its Legendre dual F^* is uniformly convex, then for every $x \in H$, the subdifferential $\partial F(x)$ is nonempty, single-valued and the map $x \to \partial F(x)$ is Lipschitz on H.

Since now the Lagrangian L is uniformly convex, then $L(x,p) = M(x,p) + \varepsilon\left(\frac{\|x\|^2}{2} + \frac{\|p\|^2}{2}\right)$, where M is convex lower semi-continuous, in such a way that $x = \partial_2 L(u,v)$ if and only if $0 \in \partial_2 M(u,v) + \varepsilon v - x$ if and only if v is the solution to the following minimization problem

$$\min_p \left\{ M(u,p) + \frac{\varepsilon\|p\|^2}{2} - \langle x,p\rangle \right\}.$$

But for each fixed u and x, the map $p \mapsto M(u,p) - \langle x,p\rangle$ majorizes a linear functional and therefore the minimum is attained uniquely at v by strict convexity and obviously $x = \partial_2 L(u,v)$.

To establish the Lipschitz property, write

$$R(u_1,x_1) - R(u_2,x_2) = R(u_1,x_1) - R(u_1,x_2) + R(u_1,x_2) - R(u_2,x_2).$$

We first bound $\|R(u_1,x_1) - R(u_1,x_2)\|$ as follows:

Since $x_1 = \partial_2 L\big(u_1, R(u_1,x_1)\big)$, $x_2 = \partial_2 L\big(u_1, R(u_1,x_2)\big)$ and $L(u_1,v) = M(u_1,v) + \frac{\varepsilon\|v\|^2}{2}$ for some M convex and lower semi-continuous, it follows that $x_j = \partial_2 M\big(u_1, R(u_1,x_j)\big) + \varepsilon R(u_1,x_j)$ for $j = 1,2$, so by monotonicity we get

$$\begin{aligned} 0 &\le \big\langle R(u_1,x_1) - R(u_1,x_2), \partial M\big(u_1, R(u_1,x_1)\big) - \partial M\big(u_1, R(u_1,x_2)\big)\big\rangle \\ &= \langle R(u_1,x_1) - R(u_1,x_2), x_1 - \varepsilon R(u_1,x_1) - x_2 + \varepsilon R(u_1,x_2)\rangle \end{aligned}$$

which yields that

$$\varepsilon\|R(u_1,x_1) - R(u_1,x_2)\|^2 \le \|R(u_1,x_1) - R(u_1,x_2)\|\,\|x_1 - x_2\|$$

and therefore

$$\|R(u_1,x_1) - R(u_1,x_2)\| \le \frac{1}{\varepsilon}\|x_1 - x_2\|. \tag{32}$$

Now we bound $\|R(u_1,x_2) - R(u_2,x_2)\|$.

Let $x_2 = \partial_2 L\big(u_j, R(u_j,x_2)\big) = \partial_2 M\big(u_j, R(u_j,x_2)\big) + \varepsilon R(u_j,x_2)$ for $j = 1,2$. and write by monotonicity that

$$0 \le \big\langle R(u_1,x_2) - R(u_2,x_2), \partial_2 M\big(u_1, R(u_1,x_2)\big) - \partial_2 M\big(u_1, R(u_2,x_2)\big)\big\rangle$$

Setting $p_j = R(u_j,x_2)$, we have with this notation

$$\begin{aligned} \langle p_1 - p_2, \partial_2 M(u_1,p_2) - \partial_2 M(u_2,p_2)\rangle &\le \langle p_1 - p_2, \partial_2 M(u_1,p_1) - \partial_2 M(u_2,p_2)\rangle \\ &= \langle p_1 - p_2, x_2 - \varepsilon p_1 - x_2 + \varepsilon p_2\rangle \\ &= -\varepsilon\|p_1 - p_2\|^2. \end{aligned}$$

so that $\varepsilon\|p_1 - p_2\|^2 \le \|p_1 - p_2\|\,\|\partial_2 M(u_1,p_2) - \partial_2 M(u_2,p_2)\|$, and since $\partial_2 M(u_j,p_2) = \partial_2 L(u_j,p_2) - \varepsilon p_2$, we get that

$$\varepsilon\|p_1 - p_2\| \le \|\partial_2 L(u_1,p_2) - \partial_2 L(u_2,p_2)\| \le \|\partial L(u_1,p_2) - \partial L(u_2,p_2)\|$$

Here we use the fact that L is both anti-selfdual and uniformly convex, to deduce that L^* is also uniformly convex. Use now the remark at the beginning of this proof to get:

$$\|\partial L(u,p) - \partial L(u',p')\| \le C\big(\|u - u'\| + \|p - p'\|\big)$$

from which follows that $\|p_1 - p_2\| \leq \frac{C}{\epsilon}\|u_1 - u_2\|$, hence

$$\|R(u_1,x_2) - R(u_2,x_2)\| \leq \frac{C}{\epsilon}\|u_1 - u_2\|. \tag{33}$$

Combining estimates (32) and (33), we finally get

$$\|R(u_1,x_1) - R(u_2,x_2)\| \leq \frac{1}{\epsilon}(1+C)\left(\|u_1 - u_2\| + \|x_1 - x_2\|\right).$$

□

Proof of Proposition 4.1 Apply Theorem 4.2 of [10] to the Lagrangian $M(t,x,p) = e^{2\omega t}L(e^{-\omega t}x, e^{-\omega t}p)$ which is also anti-selfdual ([10]). There exists then $\hat{x} \in A_H^2$ such that $\big(\hat{x}(t), \dot{\hat{x}}(t)\big) \in \mathrm{Dom}(M)$ for almost all $t \in [0,T]$ and $I(\hat{x}) = \inf_{u \in A_H^2} I(u) = 0$, where

$$I(u) = \int_0^T M(t,u(t),\dot{u}(t))\mathrm{d}t + \frac{1}{2}\|u(0)\|^2 - 2\langle x_0, u(0)\rangle + \|x_0\|^2 + \frac{1}{2}\|u(T)\|^2.$$

The path $\hat{x}$ then satisfies: $\hat{x}(0) = x_0$ and for almost all $t \in [0,T]$, $-\big(\dot{\hat{x}}(t), \hat{x}(t)\big) \in \partial M\big(t, \hat{x}(t), \dot{\hat{x}}(t)\big)$. The chain rule $\partial M(t,x,p) = e^{wt}\partial L\big(e^{-wt}x, e^{-wt}p\big)$ then yields that for almost all $t \in [0,T]$

$$-e^{-wt}\big(\dot{\hat{x}}, \hat{x}(t)\big) \in \partial L\big(e^{-wt}\hat{x}(t), e^{-wt}\dot{\hat{x}}(t).$$

Apply Lemma 4.2 to $x(t) = u(t) = e^{-wt}\hat{x}(t)$ and $v(t) = e^{-wt}\dot{\hat{x}}(t)$ to conclude that $\dot{\hat{x}} \in C\big([0,T] : H\big)$. Thus $\hat{x} \in C^1\big([0,T] : H\big)$. Since L is anti-selfdual and uniformly convex, we get from from the remark at the beginning of the proof of Lemma 4.2 that $(x,p) \mapsto \partial L(x,p)$ is Lipschitz. So by continuity, we have now for all $t \in [0,T]$

$$-e^{-wt}\big(\dot{\hat{x}}(t), \hat{x}(t)\big) \in \partial L\big(e^{-wt}\hat{x}(t), e^{-wt}\dot{\hat{x}}(t)\big)$$

and (30) is verified. □

To establish (31), we first differentiate to obtain:

$$e^{-2wt}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{x}(t) - e^{-wh}\hat{x}(t+h)\|^2 = 2e^{-2wt}\langle \hat{x}(t) - e^{-wh}\hat{x}(t+h), \dot{\hat{x}}(t) - e^{-wh}\dot{\hat{x}}(t+h)\rangle.$$

Setting now $v_1(t) = \partial_1 L\big(e^{-wt}\hat{x}(t), e^{-wt}\dot{\hat{x}}(t)\big)$ and $v_2(t) = \partial_2 L\big(e^{-wt}\hat{x}(t), e^{-wt}\dot{\hat{x}}(t)\big)$, we obtain from (30) and monotonicity that

$$\begin{aligned} e^{-2wt}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{x}(t) - e^{-wh}\hat{x}(t+h)\|^2 &= \langle e^{-wt}\hat{x}(t) - e^{-w(t+h)}\hat{x}(t+h), -v_1(t) + v_1(t+h)\rangle \\ &\quad + \langle e^{-wt}\dot{\hat{x}}(t) - e^{-w(t+h)}\dot{\hat{x}}(t+h), -v_2(t) + v_2(t+h)\rangle \\ &\leq 0. \end{aligned}$$

We conclude from this that $\frac{\|\hat{x}(t) - e^{-hw}\hat{x}(t+h)\|}{h} \leq \frac{\|\hat{x}(0) - e^{-hw}\hat{x}(h)\|}{h}$ and as we let $h \to 0$, we get $\|w\hat{x}(t) + \dot{\hat{x}}(t)\| \leq \|wx_0 + \dot{\hat{x}}(0)\|$. Therefore $\|\dot{\hat{x}}(t)\| \leq \|\dot{\hat{x}}(0)\| + |w|\|\hat{x}(t)\|$ and

$$\|\hat{x}(t)\| \leq \int_0^t \|\dot{\hat{x}}(s)\|\,\mathrm{d}s \leq \|\dot{\hat{x}}(0)\|T + |w|\int_0^t \|\hat{x}(s)\|\,\mathrm{d}s.$$

It follows from Grönwall's inequality that $\|\hat{x}(t)\| \leq \|\dot{\hat{x}}(0)\|\big(C+|w|e^{|w|T}\big)$ for all $t \in [0,T]$ and finally that $\|\dot{\hat{x}}(t)\| \leq \|\dot{\hat{x}}(0)\|\big(C+|w|+|w|^2e^{|w|T}\big)$.

We now proceed with the proof of Theorem 4.1. For that we associate a Yosida-type λ-regularization of the Lagrangian so that the boundedness condition in Proposition 4.1 is satisfied, then we make sure that all goes well when we take the limit as λ goes to 0. First, we need the following lemmas relating the properties of a Lagrangian to those of its λ-regularization.

Lemma 4.3 *For a Banach space X and a convex functional $L : X \times X^* \to \mathbb{R} \cup \{+\infty\}$, define for each $\lambda > 0$, the following Lagrangian*

$$L_\lambda(x,p) := \inf_{z\in X}\left\{L(z,p) + \frac{\|x-z\|^2}{2\lambda}\right\} + \frac{\lambda\|p\|^2}{2}.$$

1. *If L is anti-selfdual, then L_λ is also anti-selfdual.*
2. *If L is uniformly convex in the first variable, then L_λ is uniformly convex (in both variables) on $X \times X^*$.*

Proof Fix $(q,y) \in X^* \times X$ and write:

$$\begin{aligned}
(L_\lambda)^*(q,y) &= \sup\{\langle q,x\rangle + \langle y,p\rangle - L(z,p) - \frac{\|x-z\|^2}{2\lambda} - \frac{\lambda\|p\|^2}{2}; (z,x,p) \in X\times X\times X^*\}\\
&= \sup\{\langle q,v+z\rangle + \langle y,p\rangle - L(z,p) - \frac{\|v\|^2}{2\lambda} - \frac{\lambda\|p\|^2}{2}; (z,v,p) \in X\times X\times X^*\}\\
&= \sup_{p\in X^*}\left\{\langle y,p\rangle + \sup_{(z,v)\in X\times X}\{\langle q,v+z\rangle - L(z,p) - \frac{\|v\|^2}{2\lambda}\} - \frac{\lambda\|p\|^2}{2}\right\}\\
&= \sup_{p\in X^*}\left\{\langle y,p\rangle + \sup_{z\in X}\{\langle q,z\rangle - L(z,p)\} + \sup_{v\in X}\{\langle q,v\rangle - \frac{\|v\|^2}{2\lambda}\} - \frac{\lambda\|p\|^2}{2}\right\}\\
&= \sup_{p\in X^*}\left\{\langle y,p\rangle + \sup_{z\in X}\{\langle q,z\rangle - L(z,p)\} + \frac{\lambda\|q\|^2}{2} - \frac{\lambda\|p\|^2}{2})\right\}\\
&= \sup_{p\in X^*}\sup_{z\in X}\left\{\langle y,p\rangle + \langle q,z\rangle - L(z,p) - \frac{\lambda\|p\|^2}{2}\right\} + \frac{\lambda\|q\|^2}{2}\\
&= (L+T)^*(q,y) + \frac{\lambda\|q\|^2}{2}
\end{aligned}$$

where $T(z,p) := \frac{\lambda\|p\|^2}{2}$ for all $(z,p) \in X \times X^*$. Note now that

$$T^*(q,y) = \sup_{z,p}\left\{\langle q,z\rangle + \langle y,p\rangle - \frac{\lambda\|p\|^2}{2}\right\} = \begin{cases} +\infty & \text{if } q \neq 0\\ \frac{\|y\|^2}{2\lambda} & \text{if } q = 0\end{cases}$$

in such a way that by using the duality between sums and convolutions in both variables, we get

$$\begin{aligned}
(L+T)^*(q,y) &= \mathrm{conv}(L^*,T^*)(q,y)\\
&= \inf_{r\in X^*, z\in X}\big\{L^*(r,z) + T^*(-r+q,-z+y)\big\}\\
&= \inf_{z\in X}\left\{L^*(q,z) + \frac{\|y-z\|^2}{2\lambda}\right\}.
\end{aligned}$$

and finally

$$\begin{aligned}
L_\lambda^*(q,y) &= (L+T)^*(q,y) + \frac{\lambda\|q\|^2}{2} \\
&= \inf_{z\in X}\left\{L^*(q,z) + \frac{\|y-z\|^2}{2\lambda}\right\} + \frac{\lambda\|q\|^2}{2} \\
&= \inf_{z\in X}\left\{L(-z,-q) + \frac{\lambda\|q\|^2}{2} + \frac{\|y-z\|^2}{2\lambda}\right\} \\
&= L_\lambda(-y,-q).
\end{aligned}$$

(2) For each $\lambda > 0$, there exists $\varepsilon > 0$ such that $M(x,p) := L(x,p) - \frac{\varepsilon\|x\|^2}{\lambda^2}$ is convex. Pick $\delta = \frac{1-\frac{1}{1+\varepsilon}}{\lambda}$ so that $1+\varepsilon = \frac{1}{1-\lambda\delta}$ and write

$$\begin{aligned}
L_\lambda(x,p) - \frac{\lambda\|p\|^2}{2} - \delta\frac{\|x\|^2}{2} &= \inf_{z\in X}\left\{L(z,p) + \frac{\|x-z\|^2}{2\lambda} - \frac{\delta\|x\|^2}{2}\right\} \\
&= \inf_{z\in X}\left\{L(z,p) + \frac{\|x\|^2}{2\lambda} - \frac{\langle x,z\rangle}{\lambda} + \frac{\|z\|^2}{2\lambda} - \frac{\delta\|x\|^2}{2}\right\} \\
&= \inf_{z\in X}\left\{L(x,p) + \frac{\left\|\sqrt{\frac{1}{\lambda}-\delta}\,x\right\|^2}{2} - \frac{\langle x,z\rangle}{\lambda} + \frac{\|z\|^2}{2\lambda}\right\} \\
&= \inf_{z\in X}\left\{L(z,p) + \frac{\left\|\sqrt{1-\lambda\delta}\,x\right\|^2}{2\lambda} - \frac{\langle\sqrt{1-\lambda\delta}\,x, \frac{z}{\sqrt{1-\lambda\delta}}\rangle}{\lambda} + \frac{\|z\|^2}{2\lambda}\right\} \\
&= \inf_{z\in X}\left\{M(z,p) + \frac{\varepsilon\|z\|^2}{2\lambda} + \frac{\left\|\sqrt{1-\lambda\delta}\,x\right\|^2}{2\lambda}\right. \\
&\qquad\left. - \frac{\langle\sqrt{1-\lambda\delta}\,x, \frac{z}{\sqrt{1-\lambda\delta}}\rangle}{\lambda} + \frac{\|z\|^2}{2\lambda}\right\} \\
&= \inf_{z\in X}\left\{M(z,p) + \frac{(1+\varepsilon)\|z\|^2}{2\lambda} - \frac{\langle\sqrt{1-\lambda\delta}\,x, \frac{z}{\sqrt{1-\lambda\delta}}\rangle}{\lambda}\right. \\
&\qquad\left. + \frac{\left\|\sqrt{1-\lambda\delta}\,x\right\|^2}{2\lambda}\right\} \\
&= \inf_{z\in X}\left\{M(z,p) + \frac{\left\|\frac{z}{\sqrt{1-\lambda\delta}} - \sqrt{1-\lambda\delta}\,x\right\|^2}{2\lambda}\right\}
\end{aligned}$$

which means that $(z,p,x) \mapsto M(z,p) + \frac{\left\| \frac{z}{\sqrt{1-\lambda\delta}} - \sqrt{1-\lambda\delta}x \right\|^2}{2\lambda}$ is convex and therefore the infimum in z is convex, which means that $L_\lambda(x,p) - \frac{\lambda\|p\|^2}{2} - \delta\frac{\|x\|^2}{2}$ is itself convex, and L_λ is uniformly convex. □

Lemma 4.4 *For a given convex functional* $L : H\times H \to \mathbb{R}\cup\{+\infty\}$ *and* $\lambda > 0$, *denote for each* $(p,x) \in H \times H$, *by* $J_\lambda(x,p)$ *the minimizer of the following optimization problem*: $\inf\{L(z,p) + \frac{\|x-z\|^2}{2\lambda}; z \in H\}$.

1. *For each* $(x,p) \in H \times H$, *we have*

$$\partial_1 L_\lambda(x,p) = \frac{x - J_\lambda(x,p)}{\lambda} \in \partial_1 L\big(J_\lambda(x,p),p\big). \tag{34}$$

2. *If* $L : H \times H \to \mathbb{R} \cup \{+\infty\}$ *is an anti-selfdual Lagrangian that is uniformly convex in the first variable, then the map* $(x,p) \to J_\lambda(x,p)$ *is Lipschitz on* $H \times H$.

Proof (1) is straightforward. For (2), use Lemma 4.3 to deduce that L_λ is anti-selfdual and uniformly convex in both variables, which means that L_λ^* is also uniformly convex in both variables. It follows from that $(x,p) \mapsto \partial L_\lambda(x,p)$ is Lipschitz. From (34) above, we see that $J_\lambda(x,p) = x - \lambda\partial_1 L_\lambda(x,p)$ is Lipschitz as well. □

The following lemma will be useful in obtaining a uniform bound on the first derivatives of the family of approximate solutions.

Lemma 4.5 *Assume* $L : H \times H \to \mathbb{R} \cup \{+\infty\}$ *is an anti-selfdual Lagrangian and let* L_λ *be its* λ*-regularization, then the following hold*:

1. *If* $y \in \bar{\partial} L_\lambda(x)$ *for some* $x \in H$, *then* $y \in \bar{\partial} L(J_\lambda(x,-y))$.
2. *If* $y \in \bar{\partial} L(x)$ *for some* $x \in H$, *then* $\|y_\lambda\|_H \le \|y\|_H$ *for any* $y_\lambda \in \bar{\partial} L_\lambda(x)$

Proof (1) If $-(y,x) = \partial L_\lambda(x,y)$ then $L_\lambda(x,y) + L_\lambda^*(-y,-x) = -2\langle x,y\rangle$ and since L_λ is an ASD Lagrangian, we have $L_\lambda(x,y) + L_\lambda(x,y) = -2\langle x,y\rangle$, hence

$$\begin{aligned}
-2\langle x,y\rangle &= L_\lambda(x,y) + L_\lambda(x,y)\\
&= 2\left(L\big(J_\lambda(x,y),y\big) + \frac{\|x - J_\lambda(x,y)\|_H^2}{2\lambda} + \frac{\lambda\|y\|_H^2}{2}\right)\\
&= L^*\big(-y, -J_\lambda(x,y)\big) + L\big(J_\lambda(x,y),y\big) + 2\left(\frac{\|-x + J_\lambda(x,y)\|_H^2}{2\lambda} + \frac{\lambda\|y\|_H^2}{2}\right)\\
&\ge -2\langle y, J_\lambda(x,y)\rangle + 2\langle -x + J_\lambda(x,y), y\rangle\\
&= -2\langle x,y\rangle.
\end{aligned}$$

The second last inequality is deduced by applying Fenchel's inequality to the first two terms and the last two terms. The above chain of inequality shows that all inequalities are equalities. This implies, again by Fenchel's inequality that $-\big(y, J_\lambda(x,y)\big) \in \partial L\big(J_\lambda(x,y),y\big)$.

(2) If $-(y_\lambda,x) = \partial L_\lambda(x,y_\lambda)$, we get from the previous lemma that $-y_\lambda = \frac{x-J_\lambda(x,y_\lambda)}{\lambda} \in \partial_1 L\big(J_\lambda(x,y_\lambda),y_\lambda\big)$, and by the first part of this lemma, that $-\big(y_\lambda, J_\lambda(x,y_\lambda)\big) \in \partial L\big(J_\lambda(x,y_\lambda),y_\lambda\big)$. Now since $(-y,-x) \in \partial L(x,y)$. Setting $v_\lambda = J_\lambda(x,y_\lambda)$, and since $-\big(y_\lambda,v_\lambda\big) \in \partial L\big(v_\lambda,y_\lambda\big)$, we get from monotonicity and by the fact that $y_\lambda = \frac{v_\lambda - x}{\lambda}$,

$$\begin{aligned}0 &\leq \langle (x,y)-(v_\lambda,y_\lambda), \big(\partial_1 L(x,y), \partial_2 L(x,y)\big) - (-y_\lambda, -v_\lambda)\rangle \\ &= \langle (x,y)-(v_\lambda,y_\lambda), (-y,-x) - (\frac{x-v_\lambda}{\lambda}, -v_\lambda)\rangle \\ &= -\frac{\|x-v_\lambda\|_H^2}{\lambda} + \langle v_\lambda - x, y\rangle + \langle y, v_\lambda - x\rangle - \langle y_\lambda, v_\lambda - x\rangle \\ &= -2\frac{\|x-v_\lambda\|_H^2}{\lambda} + 2\langle v_\lambda - x, y\rangle\end{aligned}$$

which yields that $\frac{\|x-v_\lambda\|_H}{\lambda} \leq \|y\|_H$ and finally the desired bound $\|y_\lambda\| \leq \|y\|$ for all $\lambda > 0$. □

End of Proof of Theorem 4.1 Let $M_\lambda(t,x,p) = e^{2\omega t} L_\lambda(e^{-\omega t}x, e^{-\omega t}p)$ which is also anti-selfdual and uniformly convex by Lemma 4.3.

We now have $L_\lambda(t,x,0) \leq L(0,0) + \frac{\|x\|^2}{2\lambda}$, hence Proposition 4.1 applies and we get for all $\lambda > 0$ a solution $x_\lambda \in C^1\big([0,T]:H\big)$ such that $x_\lambda(0) = x_0$,

$$\int_0^T M_\lambda\big(t, x_\lambda(t), \dot{x}_\lambda(t)\big)\,\mathrm{d}t + \ell\big(x_\lambda(0), x_\lambda(T)\big) = 0 \tag{35}$$

$$-e^{-wt}\big(\dot{x}_\lambda(t), x_\lambda(t)\big) \in \partial L_\lambda\big(e^{-wt}x_\lambda(t), e^{-wt}\dot{x}_\lambda(t)\big) \text{ for all } t \in [0,T] \tag{36}$$

$$\|\dot{x}_\lambda(t)\| \leq C(w,T)\|\dot{x}_\lambda(0)\|. \tag{37}$$

Here $\ell\big(x_\lambda(0), x_\lambda(T)\big) = \frac{1}{2}\|x_\lambda(0)\|^2 - 2\langle x_0, x_\lambda(0)\rangle + \|x_0\|^2 + \frac{1}{2}\|u_\lambda(T)\|^2$. By the defintion of $M_\lambda(t,x,p)$, identity (35) can be written as

$$\int_0^T e^{2wt} L_\lambda\big(e^{-wt}x_\lambda(t), e^{-wt}\dot{x}_\lambda(t)\big)\,\mathrm{d}t + \ell\big(x_\lambda(0), x_\lambda(T)\big) = 0, \tag{38}$$

and since

$$L_\lambda(x,p) = L\big(J_\lambda(x,p), p\big) + \frac{\|x - J_\lambda(x,p)\|^2}{2\lambda} + \frac{\lambda\|p\|^2}{2},$$

(38) can be written as

$$\int_0^T e^{2wt}\left(L\big(v_\lambda(t), e^{-wt}\dot{x}_\lambda(t)\big) + \frac{\|e^{-wt}x_\lambda(t) - v_\lambda(t)\|^2}{2\lambda} + \frac{\lambda\|e^{-wt}\dot{x}_\lambda(t)\|^2}{2}\right)\mathrm{d}t$$
$$+\ell\big(x_\lambda(0), x_\lambda(T)) = 0$$

where $v_\lambda(t) = J_\lambda\big(e^{-wt}x_\lambda(t), e^{-wt}\dot{x}_\lambda(t)\big)$. Using Lemma 4.6.(1), we get from (36) that for all t,

$$-e^{-wt}\dot{x}_\lambda(t) = \partial_1 L_\lambda\big(e^{-wt}x_\lambda(t), e^{-wt}\dot{x}_\lambda(t)\big) = \frac{e^{-wt}x_\lambda(t) - v_\lambda(t)}{\lambda}. \tag{39}$$

Setting $t = 0$ in (36) and noting that $x_\lambda(0) = x_0$, we get that $-\big(\dot{x}_\lambda(0), x_0\big) \in \partial L_\lambda\big(x_0, \dot{x}_\lambda(0)\big)$, and since $x_0 \in \mathrm{Dom}(\bar{\partial} L)$, we can apply Lemma 4.7.2 to get that $\|\dot{x}_\lambda(0)\| \leq C$ for all $\lambda > 0$. Now plug this inequality in (37) to obtain:

$$\|\dot{x}_\lambda(t)\| \leq D(w,T) \quad \forall \lambda > 0 \quad \forall t \in [0,T].$$

This yields by (39) that

$$\|e^{-wt}x_\lambda(t) - v_\lambda(t)\| \leq e^{|w|T} D(w,T)\lambda \quad \forall t \in [0,T],$$

hence

$$\frac{\|e^{-wt}x_\lambda(t) - v_\lambda(t)\|^2}{\lambda} \to 0 \text{ uniformly in } t. \tag{40}$$

Moreover, since $\|\dot{x}_\lambda(\cdot)\|_{L^2_H} \leq D(w,T)$ for all $\lambda > 0$, there exists $\hat{x} \in A^2_H$ such that –up to a subsequence–

$$x_\lambda \rightharpoonup \hat{x} \text{ in } A^2_H \tag{41}$$

and again by (39) we have

$$\int_0^T \|v_\lambda(t) - e^{-wt}\hat{x}(t)\|^2_H \mathrm{d}t \to 0, \tag{42}$$

while clearly

$$\lambda \frac{\|e^{-wt}\dot{x}_\lambda(t)\|^2}{2} \to 0 \text{ uniformly}. \tag{43}$$

Now use (38)–(41) and the lower semi-continuity of L, to deduce from (38), that as $\lambda \to 0$ we have

$$I(\hat{x}) = \int_0^T e^{2wt} L\big(e^{-wt}\hat{x}(t), e^{-wt}\dot{\hat{x}}(t)\big)\,\mathrm{d}t + \ell\big(\hat{x}(0), \hat{x}(T)\big) \leq 0.$$

Since we already know that $I\big(x\big) \geq 0$ for all $x \in A^2_H$, we finally get our claim that $0 = I\big(\hat{x}\big) = \inf_{x \in A^2_H} I\big(x\big)$. The rest is straightforward.

Now define $T_t x_0 := e^{-\omega t}\hat{x}(t)$. It is easy to see that $x(t) := T_t x_0$ satisfies equation (27) and that $T_0 x_0 = x_0$. We need to check that $\{T_t\}_{t \in \mathbb{R}^+}$ is a semi-group. By uniqueness of minimizers, this is equivalent to show that for all $s < T$, the function $v(t) := x(t+s)$ satisfies

$$0 = \int_0^{T-s} e^{2\omega t} L\left(v(t), e^{-\omega t}\left(\frac{\mathrm{d}}{\mathrm{d}t} e^{\omega t} v(t)\right)\right)\mathrm{d}t + \frac{1}{2}\|v(0)\|^2 - 2\langle T_s x_0, v(0)\rangle + \|T_s x_0\|^2$$
$$+\frac{1}{2}\|e^{\omega(T-s)} v(T)\|^2.$$

By the definition of $x(t)$ and the fact that $I\big(\hat{x}\big) = 0$ we have,

$$0 = \int_0^T e^{2\omega t} L\left(x(t), e^{-\omega t}\left(\frac{\mathrm{d}}{\mathrm{d}t} e^{\omega t} x(t)\right)\right)\mathrm{d}t + \frac{1}{2}\|x(0)\|^2 - 2\langle x_0, x(0)\rangle + \|x_0\|^2$$
$$+\frac{1}{2}\|e^{\omega T} x(T)\|^2.$$

Since $x(t)$ satisfies equation (27) we have

$$0=\int_0^s e^{2\omega t}L\left(x(t),e^{-\omega t}\left(\frac{\mathrm{d}}{\mathrm{d}t}e^{\omega t}x(t)\right)\right)\mathrm{d}t+\frac{1}{2}\|x(0)\|^2-2\langle x_0,x(0)\rangle+\|x_0\|^2$$
$$+\frac{1}{2}\|e^{\omega s}x(s)\|^2.$$

By subtracting the two equations we get

$$0=\int_s^T e^{2\omega t}L\left(x(t),e^{-\omega t}\left(\frac{\mathrm{d}}{\mathrm{d}t}e^{\omega t}x(t)\right)\right)\mathrm{d}t+\frac{1}{2}\|e^{\omega T}x(T)\|^2-\frac{1}{2}\|e^{\omega s}x(s)\|^2.$$

Make a substitution $r=t-s$ and we obtain

$$0=e^{2\omega s}\left\{\int_0^{T-s} e^{2\omega r}L\left(v(r),e^{-\omega r}\left(\frac{\mathrm{d}}{\mathrm{d}r}e^{\omega r}v(r)\right)\right)\mathrm{d}r+\frac{1}{2}\|e^{\omega(T-s)}x(T)\|^2-\frac{1}{2}\|x(s)\|^2\right\}$$

and finally

$$0=\int_0^{T-s} e^{2\omega r}L\left(v(r),e^{-\omega r}\left(\frac{\mathrm{d}}{\mathrm{d}r}e^{\omega r}v(r)\right)\right)\mathrm{d}r+\frac{1}{2}\|v(0)\|^2-2\langle T_sx_0,v(0)\rangle$$
$$+\|T_sx_0\|^2+\frac{1}{2}\|e^{\omega(T-s)}v(T)\|^2.$$

It follows that $T_s(T_tx_0)=T_{s+t}x_0$.

To check the Lipschitz constant of the semi-group, we differentiate $\|T_tx_0-T_tx_1\|^2$ and use equation (23) in conjunction with monotonicity to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|T_tx_0-T_tx_1\|^2\le-\omega\|T_tx_0-T_tx_1\|^2$$

whenever $x_0,x_1\in\mathrm{Dom}(\bar{\partial}L)$. A simple application of Grönwall's inequality gives the desired conclusions.

5 Variational resolution of parabolic initial-value problems

We now apply the results of the last section to the class of ASD Lagrangian of the form $L(x,p)=\varphi(x)+\varphi^*(Ax-p)$ to obtain variational formulations and proofs of existence for parabolic equations of the form

$$\begin{aligned}-\dot{x}(t)+Ax(t)&\in\partial\varphi(x(t))+\omega x(t)\\ x(0)&=x_0\\ b_1(x(t))&=b_1(x_0).\end{aligned}$$

Here again, we have two cases. The first is dealt with in Sect. 5.1 and requires the operator to be only anti-symmetric while the framework is still purely Hilbertian. The second case requires that the operator be skew-adjoint—and if necessary—modulo a pair of boundary operators. The framework there will be on an evolution triple

$X \subset H \subset X^*$ with X being a Banach space that is anchored on a Hilbert space H. It is dealt with in Sect. 5.2.

5.1 Parabolic equations involving a diffusion term

In the first proposition, we start by assuming the same hypothesis as in Theorem 4.1, that is uniform convexity (in the first variable) of the Lagrangian and a homogeneous initial condition. We will then show how to do away with these conditions in the corollary that follows.

Proposition 5.1 *Let $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semi-continuous and proper function on a Hilbert space H, and let A be an anti-symmetric linear operator into H, whose domain $D(A)$ contains $D(\varphi)$. Assume that φ is uniformly convex, with a symmetric domain such that $\partial\varphi(0)$ is non-empty. For any given $\omega \in \mathbb{R}$ and $T > 0$, define the following functional on $A^2_H([0, T])$*

$$I(u) = \int_0^T e^{2\omega t}\varphi(e^{-\omega t}x(t)) + e^{2\omega t}\varphi^*(e^{-\omega t}(Ax(t) - \dot{x}(t)))\mathrm{d}t + \frac{1}{2}(\|x(0)\|^2 + \|x(T)\|^2).$$

Then, there exists a path $\bar{x} \in A^2_H([0, T])$ such that:

1. $I(\bar{x}) = \inf_{x \in A^2_H([0,T])} I(x) = 0.$
2. *If $\bar{v}(t)$ is defined by $\bar{v}(t) := e^{-\omega t}\bar{x}(t)$ then it satisfies*

$$\begin{aligned} -\dot{\bar{v}}(t) + A\bar{v}(t) - \omega\bar{v}(t) &\in \partial\varphi(\bar{v}(t)) \quad \textit{for a.e. } t \in [0, T] \\ \bar{v}(0) &= 0. \end{aligned} \tag{44}$$

Proof Setting $\varphi_t(x) := e^{2\omega t}\varphi(e^{-\omega t}x)$, the assumptions ensure that

$$L(t, x, p) := \begin{cases} \varphi_t(x) + \varphi_t^*(Ax - p) & \text{if } x \in D(\varphi) \\ +\infty & \text{elsewhere} \end{cases}$$

is an ASD Lagrangian by Lemma 2.3. Since $\partial\varphi(0)$ is non-empty, it is easy to verify that $0 \in \mathrm{Dom}\bar{\partial}L$ and all the hypothesis of Theorem 4.1 are satisfied. There exists then $\bar{x}(\cdot) \in A^2_H([0, T])$ such that

$$0 = \int_0^T L(t, \bar{x}(t), \dot{\bar{x}}(t))\,\mathrm{d}t + \ell\big(\bar{x}(0), \bar{x}(T)\big)$$

where $\ell(a, b) := \frac{1}{2}(\|a\|^2 + \|b\|^2)$. Therefore

$$\begin{aligned} 0 &= \int_0^T \varphi_t(\bar{x}(t)) + \varphi_t^*(A\bar{x}(t) - \dot{\bar{x}}(t))\mathrm{d}t + \frac{1}{2}(\|\bar{x}(0)\|^2 + \|\bar{x}(T)\|^2) \\ &\geq \int_0^T \langle \bar{x}(t), A\bar{x}(t) - \dot{\bar{x}}(t)\rangle \mathrm{d}t + \frac{1}{2}(\|\bar{x}(0)\|^2 + \|\bar{x}(T)\|^2) \\ &= \|\bar{x}(0)\|^2 \geq 0. \end{aligned}$$

It follows that

$$-\dot{\bar{x}}(t) + A\bar{x}(t) \in e^{\omega t}\partial\varphi(e^{-\omega t}\bar{x}(t))$$
$$\bar{x}(0) = 0$$

and by a simple application of the product-rule we see that $\bar{v}(t)$ defined by $\bar{v}(t) := e^{-\omega t}\bar{x}(t)$ satisfies (44). □

Corollary 5.1 *Let $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semi-continuous and proper function on a Hilbert space H, and let A be an anti-symmetric linear operator on H, whose domain $D(A)$ contains $D(\varphi)$. Then for any $x_0 \in D(A) \cap D(\partial\varphi)$, any $\omega \in \mathbb{R}$ and any $T > 0$, there exists $\bar{u} \in A^2_H([0, T])$ such that*

$$-\dot{\bar{u}}(t) + A\bar{u}(t) - \omega\bar{u}(t) \in \partial\varphi(\bar{u}(t)) \quad \textit{for a.e. } t \in [0, T] \tag{45}$$
$$\bar{u}(0) = x_0.$$

Proof Define the convex function $\psi : H \to \bar{\mathbb{R}}$ by

$$\psi(x) := \varphi(x + x_0) + \frac{\|x\|^2}{2} - \langle x, Ax_0\rangle + \langle x, \omega x_0\rangle.$$

By the fact that $\partial\varphi(x_0)$ is non-empty, it is easy to check that ψ satisfies all the conditions of Proposition 5.1. Therefore, there exists $\bar{v}(\cdot) \in A^2_H([0, T])$ satisfying the evolution equation

$$-\dot{\bar{v}}(t) + A\bar{v}(t) \in \partial\psi(\bar{v}(t)) + (\omega - 1)\bar{v}(t) \quad \text{for a.e. } t \in [0, T]$$
$$\bar{v}(0) = 0$$

Since $\partial\psi(x) = \partial\varphi(x + x_0) + x - Ax_0 + \omega x_0$, we get that $\bar{u}(t) := \bar{v}(t) + x_0$ satisfies equation (45). □

5.2 Evolution driven by the transport operator and the p-Laplacian

Consider the following evolution equation on a smooth bounded domain of $\mathbb{R}^n$.

$$\begin{aligned} -u_t(x,t) + \vec{a}(x) \cdot \nabla u(x,t) &= -\Delta_p u(x,t) + \frac{1}{2}a_0(x)u(x,t) + \omega u(x,t) && \text{on } [0,T]\times\Omega \\ u(x,0) &= u_0(x) && \text{on } \Omega \\ u(x,t) &= 0 && \text{on } [0,T]\times\partial\Omega. \end{aligned} \tag{46}$$

We can establish variationally the following

Corollary 5.2 *Let $\vec{a} : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field and $a_0 \in L^\infty(\Omega)$. For $p \geq 2$, $\omega \in \mathbb{R}$, and any u_0 in $W^{1,p}_0(\Omega) \cap \{u; \Delta_p u \in L^2(\Omega)\}$, there exists $\bar{u} \in A^2_{L^2(\Omega)}([0, T])$ that solves (46). Furthermore, $\Delta_p\bar{u}(x, t) \in L^2(\Omega)$ for almost all $t \in [0, T]$.*

Proof The operator $Au = \vec{a} \cdot \nabla u + \frac{1}{2}(\nabla \cdot \vec{a})u$ with domain $D(A) = H^1_0(\Omega)$ is anti-symmetric. In order to apply corollary 5.1 with $H = L^2(\Omega)$ and A, we need to insure convexity of the potential and for that we pick $K > 0$ such that $\nabla\cdot\vec{a}(x) + a_0(x) + K \geq 1$ for all $x \in \Omega$.
Now define $\varphi : H \to \bar{\mathbb{R}}$ by

$$\varphi(u) := \begin{cases} \frac{1}{p}\int_\Omega |\nabla u(x)|^p \mathrm{d}x + \frac{1}{4}\int_\Omega(\nabla\cdot\vec{a}(x) + a_0(x) + K)|u(x)|^2\mathrm{d}x & \text{if } u \in W^{1,p}_0(\Omega) \\ +\infty & \text{elsewhere} \end{cases}$$

By observing that φ is a convex l.s.c. function with symmetric domain and $D(\varphi) \subset D(A)$, we can apply Corollary 5.1 with the linear factor $(\omega - \frac{K}{2})$, to obtain the existence of a $\bar{u}(\cdot) \in A_H^2([0,T])$ such that

$$-\dot{\bar{u}}(t) + A\bar{u}(t) \in \partial\varphi(\bar{u}(t)) + (\omega - \frac{K}{2})\bar{u}(t) \text{ for a.e. } t \in [0,T]$$
$$\bar{u}(0) = x_0$$

and this is precisely the equation (46). Since now $\partial\varphi(\bar{u}(t))$ is a non-empty set in H for almost all $t \in [0,T]$, we have $\Delta_p \bar{u}(x,t) \in L^2(\Omega)$ for almost all $t \in [0,T]$. □

5.3 Parabolic equations driven by first-order operators

In this subsection we deal with parabolic equations of the form:

$$\begin{aligned} -\dot{x}(t) + Ax(t) - wx(t) &\in \partial\varphi(x(t)) \quad \text{for a.e. } t \in [0,T] \\ x(0) &= x_0 \\ b_1(x(t)) &= b_1(x_0) \quad \text{for a.e. } t \in [0,T], \end{aligned} \tag{47}$$

where the operator A is skew-adjoint modulo boundary operators (b_1, b_2). Here we need the framework of an evolution triple, where X is a reflexive Banach space and H is a Hilbert space satisfying $X \subset H \subset X^*$ in such a way that each space is dense in the following one. Again we start with a theorem that assumes all the hypothesis of Theorem 4.1. We will then relax these conditions in the corollary that follows it.

Proposition 5.2 *Let $X \subset H \subset X^*$ be an evolution triple and let $A : D(A) \subset X \to X^*$ be a skew-adjoint operator modulo boundary operators $(b_1, b_2) : D(b_1, b_2) \to H$. Let $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ be a uniformly convex lower semi-continuous function on H, that is bounded on the bounded sets of X and which is also coercive on X. Assume that $\partial\varphi(0) \cap H$ is non-empty, then for any $w \in \mathbb{R}$ and any $T > 0$, then there exists a solution $v \in A_H^2$ for the initial value problem*

$$\begin{aligned} -\dot{v}(t) + Av(t) - wv(t) &\in \partial\varphi\big(v(t)\big) \quad \text{a.e.} \quad t \in [0,T] \\ b_1(v(t)) &= 0 \qquad\qquad \text{for a.e. } t \in [0,T] \\ v(0) &= 0. \end{aligned} \tag{48}$$

It is obtained by minimizing over A_H^2 the functional

$$I(u) = \int_0^T e^{2\omega t}\left\{\varphi(e^{-\omega t}u(t)) + \varphi^*(e^{-\omega t}(-Au(t) - \dot{u}(t))\right\} dt$$
$$+\frac{1}{2}\int_0^T (\|b_1(x(t))\|_{H_1}^2 + \|b_2(x(t))\|_{H_2}^2) dt + \frac{1}{2}\|u(0)\|^2 + \frac{1}{2}\|u(T)\|^2.$$

The minimum of I is then zero and is attained at a path $y(t)$ such that $x(t) = e^{-\omega t}y(t)$ is a solution of (48).

Typical convex functions satisfying the conditions above are ones such that for some $C > 0$, $m, n > 1$ we have the following growth condition:

$$\mathbb{C}\left(\|x\|_X^m - 1\right) \le \varphi(x) \le C\left(\|x\|_X^n + 1\right). \tag{49}$$

The corresponding Lagrangian L is ASD on $X \times X^*$ where $X \subseteq H \subseteq X^*$. Since our theory for evolution equations applies to Hilbert spaces, the following lemma will bridge the gap:

Lemma 5.3 *Let $X \subset H \subset X^*$ be an evolution triple, and suppose $L : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ is ASD on the Banach space X. Assume the following two conditions:*

1. *For all $x \in X$, the map $L(x,\cdot) : X^* \to \mathbb{R} \cup \{+\infty\}$ is continuous on X^*.*
2. *There exists $x_0 \in X$ such that $p \to L(x_0,p)$ is bounded on the bounded sets of X^*.*

Then the Lagrangian defined on H by

$$M(x,p) := \begin{cases} L(x,p) & x \in X \\ +\infty & x \in H \backslash X \end{cases}$$

is anti-selfdual on $H \times H$.

Proof For $(\tilde{x},\tilde{p}) \in X \times H$, write

$$\begin{aligned} M^*(\tilde{p},\tilde{x}) &= \sup_{\substack{x \in X \\ p \in H}} \left\{ \langle \tilde{x},p\rangle_H + \langle \tilde{p},x\rangle_H - L(x,p) \right\} \\ &= \sup_{x\in X} \sup_{p \in X^*} \left\{ \langle \tilde{x},p\rangle_{X,X^*} + \langle x,\tilde{p}\rangle_{X,X^*} - L(x,p) \right\} \\ &= L(-\tilde{x},-\tilde{p}). \end{aligned}$$

If $\tilde{x} \in H \backslash X$, then $M^*(\tilde{p},\tilde{x}) = \sup_{\substack{x \in X \\ p \in H}} \left\{ \langle \tilde{x},p\rangle_H + \langle \tilde{p},x\rangle_H - L(x,p) \right\} \geq \langle \tilde{p},x_0\rangle + \sup_{p\in H} \left\{ \langle \tilde{x},p\rangle_H - L(x_0,p) \right\}$. Since $\tilde{x} \notin X$, we have that $\sup \left\{ \langle \tilde{x},p\rangle ; p \in H, \|p\|_{X^*} \leq 1 \right\} = +\infty$. Since $p \to L(x_0,p)$ is bounded on the bounded sets of X^*, it follows that $M^*(\tilde{p},\tilde{x}) \geq \langle \tilde{p},x_0\rangle + \sup_{p\in H} \left\{ \langle \tilde{x},p\rangle_H - L(x_0,p) \right\} = +\infty$. □

Proof of Proposition 5.2 Again the Lagrangian

$$L(x,p) := \begin{cases} \varphi(x) + \varphi^*(Ax - p) + \frac{1}{2}(\|b_1(x)\|_{H_1}^2 + \|b_2(x)\|_{H_2}^2) & \text{if } x \in D(A) \cap D(b_1,b_2) \\ +\infty & \text{elsewhere} \end{cases}$$

is anti-selfdual on $X \times X^*$ by Proposition 2.1. The coercivity condition on φ ensures—via Lemma 5.3—that $L(x,p)$ lifts to a ASD Lagrangian on $H \times H$ that is uniformly convex in the first variable. It is easy to check that all the conditions of Theorem 4.1 are satisfied by $L(x,p)$. Therefore, there exists $\bar{x}(\cdot) \in A_H^2([0,T])$ such that $I(\bar{u}) = 0$, which yields

$$\begin{aligned} 0 &= \int_0^T \varphi_t(\bar{x}(t)) + \varphi_t^*(A\bar{x}(t) - \dot{\bar{x}}(t)) + \frac{1}{2}(\|b_1(x(t))\|_{H_1}^2 + \|b_2(x(t))\|_{H_2}^2)\mathrm{d}t \\ &\quad + \frac{1}{2}(\|\bar{x}(0)\|^2 + \|\bar{x}(T)\|^2) \\ &\geq \int_0^T \langle \bar{x}(t), A\bar{x}(t) - \dot{\bar{x}}(t) + \frac{1}{2}(\|b_1(x(t))\|_{H_1}^2 + \|b_2(x(t))\|_{H_2}^2)\mathrm{d}t + \frac{1}{2}(\|\bar{x}(0)\|^2 + \|\bar{x}(T)\|^2) \\ &= \int_0^T \|b_1(\bar{x}(t))\|_{H_1}^2 \mathrm{d}t + \|\bar{x}(0)\|^2 \geq 0. \end{aligned}$$

So all inequalities are equalities, and we obtain $-\dot{\bar{x}}(t) + A\bar{x}(t) \in e^{\omega t}\partial\varphi(e^{-\omega t}\bar{x}(t))$, $b_1(\bar{x}(t)) = 0$, and $\bar{x}(0) = 0$. We now set $\bar{v}(t) := e^{-\omega t}\bar{x}(t)$ and the rest is straightforward. □

Corollary 5.4 *Let $X \subset H \subset X^*$ be an evolution triple and let $A : D(A) \subset H \to X^*$ be a skew-adjoint operator modulo boundary operators $(b_1, b_2) : D(b_1, b_2) \to H$. Let $\varphi : X \to \mathbb{R}$ be a convex lower semi-continuous function on X, that is bounded on the bounded sets of X and also coercive on X. Assume*

$$x_0 \in D(A) \cap D(b_1, b_2) \quad \textit{and } \partial\varphi(x_0) \cap H \textit{ is non-empty.} \tag{50}$$

Then, for all $\omega \in \mathbb{R}$ and for all $T > 0$, there exists $\bar{u} \in A^2_H([0, T])$ which solves (47).

Proof Define the convex function $\psi : X \to \mathbb{R}$ by

$$\psi(x) := \varphi(x + x_0) + \frac{\|x\|^2}{2} - \langle x, Ax_0\rangle + \langle x, \omega x_0\rangle$$

It is easy to check that ψ satisfies all the conditions of Proposition 5.2. Therefore, there exists $\bar{v} \in A^2_H([0, T])$ satisfying the evolution equation

$$\begin{aligned} -\dot{\bar{v}}(t) + A\bar{v}(t) &\in \partial\psi(\bar{v}(t)) + (\omega - 1)\bar{v}(t) \quad \text{for a.e. } t \in [0, T] \\ b_1(\bar{v}(t)) &= 0 \quad \text{for a.e. } t \in [0, T] \\ \bar{v}(0) &= 0. \end{aligned}$$

Since $\partial\psi(x) = \partial\varphi(x + x_0) + x - Ax_0 + \omega x_0$, we have that $\bar{u}(t) := \bar{v}(t) + x_0$ solves (47). □

Evolutions driven by transport operators

Consider the evolution equation

$$\begin{aligned} -u_t(x,t) + \vec{a}(x) \cdot \nabla u(x,t) &= \frac{1}{2}a_0(x)u(x,t) + u(x,t)|u(x,t)|^{p-2} + \omega u(x,t) \quad \text{on } [0,T] \times \Omega \\ u(x,0) &= u_0(x) \quad \text{on } \Omega \\ u(x,t) &= u_0(x) \quad \text{on } [0,T] \times \Sigma_+. \end{aligned} \tag{51}$$

We assume that the domain Ω and the vector field $\vec{a}(\cdot)$ satisfies all the assumption in Sect. 2.

Corollary 5.5 *Let $p > 1$, $f \in L^2(\Omega)$ and $a_0 \in L^\infty(\Omega)$. For any $\omega \in \mathbb{R}$ and $u_0 \in L^\infty(\Omega) \cap H^1(\Omega)$ there exists $\bar{u}(\cdot) \in A^2_{L^2(\Omega)}([0, T])$ satisfying (51).*

Proof We distinguish two cases: □

Case 1 $p \geq 2$. We then take $X = L^p(\Omega)$, $H = L^2(\Omega)$. since again the operator $A : D(A) \to X^*$ defined as $Au = \vec{a} \cdot \nabla u + \frac{1}{2}\left(\nabla \cdot \vec{a}\right) u$ with domain

$$D(A) = \left\{u \in L^p(\Omega) \mid \vec{a} \cdot \nabla u + \frac{\nabla\vec{a}}{2}u \in L^q(\Omega)\right\}$$

is skew-adjoint modulo the boundary operators $(b_1u, b_2u) = (u|_{\Sigma_+}, u|_{\Sigma_-})$ whose domain is

$$D(b_1, b_2) = \{u \in L^p(\Omega) | (u|_{\Sigma_+}, u|_{\Sigma_-}) \in L^2(\Sigma_+; |\vec{a} \cdot \hat{n}|\mathrm{d}\sigma) \times L^2(\Sigma_-; |\vec{a} \cdot \hat{n}|\mathrm{d}\sigma)\}$$

Case 2 $1 < p < 2$. The space is then $X = H = L^2(\Omega)$.

In both case, we pick $K > 0$ such that $\nabla \cdot \vec{a}(x) + a_0(x) + K \geq 1$ for all $x \in \Omega$, and define the function $\varphi : X \to \mathbb{R}$ by

$$\varphi(u) := \frac{1}{p} \int_\Omega |u(x)|^p \mathrm{d}x + \frac{1}{4} \int_\Omega (\nabla \cdot \vec{a}(x) + a_0(x) + K)|u(x)|^2 \mathrm{d}x$$

Then φ is a convex, l.s.c. function that is bounded on bounded sets of X and coercive on X. Since $u_0(x) \in L^\infty(\Omega) \cap H^1(\Omega)$, $\partial\varphi(u_0)$ is non-empty and $u_0 \in D(A) \cap D(b_1, b_2)$.

So by corollary 5.4, there exists $\bar{u}(\cdot) \in A^2_H([0, T])$ such that

$$\begin{aligned} -\dot{\bar{u}}(t) + A\bar{u}(t) &\in \partial\varphi(\bar{u}(t)) + (\omega - \frac{K}{2})\bar{u}(t) \quad \text{for } t \in [0, T] \\ b_1(\bar{u}(t)) &= b_1(u_0) \quad \text{for } t \in [0, T] \\ \bar{u}(0) &= u_0 \end{aligned}$$

and this is precisely the equation (51) and we are done.

References

1. Auchmuty, G.: Saddle points and existence-uniqueness for evolution equations. Differ. Integr. Equ. **6**, 1161–1171 (1993)
2. Auchmuty, G.: Variational principles for operator equations and initial value problems. Nonlinear Anal. Theory, Methods Appl. **12**(5), 531–564 (1988)
3. Bardos, C.: Problèmes aux limites pour les equations aux dérivées partielles du premier ordre a coefficients réels; Théorèmes d'approximation; Application à l'équation de transport. Ann. scient. Ec. Norm. Sup., 4^e série, t. **3**, 185–233 (1970)
4. Brezis, H., Ekeland, I.: Un principe variationnel associé à certaines equations paraboliques. Le cas independant du temps, C. R. Acad. Sci. Paris Sér. A **282**, 971–974 (1976)
5. Brezis, H., Ekeland, I.: Un principe variationnel associé à certaines equations paraboliques. Le cas dependant du temps. C. R. Acad. Sci. Paris Sér. A **282**, 1197–1198 (1976)
6. Ekeland, I., Temam, R.: Convex analysis and variational problems. Classics in Applied Mathematics. **28** SIAM (1999 Edition)
7. Ghoussoub, N.: A theory of anti-selfdual Lagrangians: stationary case. C. R. Acad. Sci., Paris, Ser. I **340**, 245–250 (2005)
8. Ghoussoub, N.: A theory of anti-selfdual Lagrangians: dynamical case. C. R. Acad. Sci., Paris, Ser. I **340**, 325–330 (2005)
9. Ghoussoub, N.: A variational principle for non-linear transport equations. Commun. Pure Appl. Anal. **4**(4), 735–742 (2005)
10. Ghoussoub, N.: Anti-selfdual Lagrangians: variational resolutions of non self-adjoint equations and dissipative evolutions. AIHP-Analyse non linéaire, 35 pages (Published online 29 September 2006)
11. Ghoussoub, N.: Anti-symmetric Hamiltonians: variational resolution for Navier-Stokes equations and other nonlinear evolutions. Comm. Pure Appl. Math. 35 pages (Published Online: 28 Dec 2006)
12. Ghoussoub, N.: Self-dual partial differential systems and their variational principles. Monograph, Preprint, 256 pages (2006)
13. Ghoussoub, N.: Maximal monotone operators are selfdual vector fields and vice-versa. Proc. AMS (2006)
14. Ghoussoub, N., McCann, R.: A least action principle for steepest descent in a non-convex landscape. Contemp. Math. **362**, 177–187 (2004)
15. Ghoussoub, N., Tzou, L.: A variational principle for gradient flows. Math. Annalen **30**(3), 519–549 (2004)
16. Lions, J.L., Magenes, E.: Non-homogeneous boundary value problems and applications, vol. 3, Springer, Berlin (1973)