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On the existence of multiple positive solutions of quasilinear elliptic eigenvalue problems

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Abstract We study a quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}(\phi(|\nabla u|)\nabla u) = \lambda f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with nonhomogeneous principal part ϕ. Under the hypothesis $f(x,t) = o(\phi(t)t)$ at $t = 0$ and ∞, the existence of multiple positive solutions is proved by using the variational arguments in the Orlicz–Sobolev spaces.

Keywords Quasilinear elliptic problem · Multiple positive solutions · Orlicz–Sobolev spaces

Mathematics Subject Classification (2000) 35J20; 35J25; 35J70; 47J10; 47J30

1 Introduction

In this paper, we study the existence of positive solutions of quasilinear elliptic equations of the form

$$-\operatorname{div}(\phi(|\nabla u|)\nabla u) = \lambda f(x,u) \quad \text{in } \Omega \tag{1.1}$$

with Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

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Here Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with smooth boundary $\partial\Omega$,

$$\phi(t)t \in C(\mathbb{R}), \quad f(x,t) \in C(\Omega \times \mathbb{R})$$

and λ real parameter.

The multiplicity of positive solutions of the type of (1.1) with (1.2) has been discussed precisely by several authors. Especially Ambrosetti et al. [1] consider a semilinear elliptic boundary value problem with concave and convex nonlinearities, that is,

$$\begin{cases} -\Delta u = \lambda u^q + u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

with $0 < q < 1 < p \leq (N+2)/(N-2)$, and obtain the following result.

There exists a constant $\Lambda > 0$ such that

(i) for all $\lambda \in (0, \Lambda)$, (1.3) has at least two solutions;
(ii) for $\lambda = \Lambda$, (1.3) has at least one solution;
(iii) for all $\lambda > \Lambda$, (1.3) has no solution.

Following this, García Azorero et al. [9] and Fukagai and Narukawa [8] consider the same problems for the p-Laplace equation and more general quasilinear Eq. (1.1) respectively. Under the hypothesis $\phi(t)t = o(f(x,t))$ at $t = 0$ and ∞, which corresponds to the concave and convex nonlinearity, they have shown the same results for Eq. (1.1).

In this paper, we consider the case $f(x,t) = o(\phi(t)t)$ at $t = 0$ and ∞ contrary to the case treated previously. Furthermore, we also consider an Eq. (1.1) with more general principal part.

Let us put

$$\Phi(t) = \int_0^t \phi(s)s \, \mathrm{d}s.$$

Then, there often arise the Eq. (1.1) associated by a nonhomogeneous nonlinearities Φ in the fields of physics, e.g.,

(a) nonlinear elasticity: $\Phi(t) = (1+t^2)^\gamma - 1$, $\gamma > 1/2$,
(b) plasticity: $\Phi(t) = t^\alpha(\log(1+t))^\beta$, $\alpha \geq 1$, $\beta > 0$,
(c) generalized Newtonian fluids: $\Phi(t) = \int_0^t s^{1-\alpha}(\sinh^{-1} s)^\beta \mathrm{d}s$, $0 \leq \alpha \leq 1$, $\beta > 0$.

For details, see [4, 5, 7].

The energy functional corresponding to Eq. (1.1) with general ϕ is not defined appropriately in the usual Sobolev space. In fact, examples (b) and (c) are naturally treated in the general Orlicz–Sobolev spaces defined by the corresponding Φ. Thus, we introduce the appropriate Orlicz–Sobolev space and analyze the functional attached to Eq. (1.1) on this space.

Here, we put the assumption

(ϕ_1) $\phi(t) \in C^1(0,\infty)$, $\phi(t) > 0$, $(\phi(t)t)' > 0$ for $t > 0$,
(ϕ_2) there exist $\ell, m > 1$ such that $\ell \leq \frac{\Phi'(t)t}{\Phi(t)} \leq m$ for any $t > 0$,
(ϕ_3) there exist $a_0, a_1 > 0$ such that $a_0 \leq \frac{\Phi''(t)t}{\Phi'(t)} \leq a_1$ for any $t > 0$.

Throughout this paper, we assume (ϕ_1)–(ϕ_3).

We recall some known properties of Orlicz–Sobolev spaces in Sect. 2 and some basic properties of the operator appearing in the principal part of (1.1) in Sect. 3. In Sect. 4, we put the following assumptions on f:

(f_1) $f(x,t) \in C(\overline{\Omega} \times \mathbb{R})$, $f(x,0) = 0$ for $x \in \Omega$,
(f_2) $f(x,t) \geq 0$ for $x \in \Omega$, $t > 0$, and there exists an open set $\Omega_0 \subset \Omega$ such that $f(x,t) > 0$ for $x \in \Omega_0$, $t > 0$,
(f_3) $f(x,t)t \leq c_0\Phi(t)$ for $x \in \Omega$, $t \geq 0$ with some constant $c_0 > 0$,
(f_4) $f(x,t)t = o(\Phi(t))$ as $t \to \infty$ uniformly in $x \in \Omega$.

Under assumptions (f_1)–(f_4), we discuss the existence of a positive solution. As a simple example satisfying (f_1)–(f_4) for examples (a)–(c), let $f(x,t) = a(x)g(t)$ with a nonnegative nontrivial continuous function $a(x)$ on $\overline{\Omega}$. Here $g(t)$ is a positive continuous function on $(0,\infty)$ such that

$$g(t) \sim \begin{cases} t^p & \text{as } t \to 0 \\ t^q & \text{as } t \to \infty \end{cases}$$

with some constants p, q satisfying $p \geq 1$, $q < 2\gamma - 1$ in example (a), $p \geq \alpha + \beta - 1$, $q < \alpha - 1$ in (b), and $p \geq 1 - \alpha + \beta$, $q < 1 - \alpha$ in (c), respectively. In Sect. 5, we put a slightly restrictive assumption, instead of (f_2),

(f_2') $f(x,t) > 0$ for $x \in \Omega$, $t > 0$,

and additional assumptions

(f_5) $f(x,t)t = o(\Phi(t))$ as $t \to 0$ uniformly in $x \in \Omega$,
(f_6) $f(x,t)$ is nondecreasing in $t > 0$ for each $x \in \Omega$.

Under these assumptions, we construct a second solution in Theorem 5.1. The final section of the paper concerns the existence of the maximal solution.

2 Orlicz–Sobolev spaces

As is stated in Sect. 1, we put

$$\Phi(t) = \int_0^t \phi(s)s\,\mathrm{d}s.$$

Under assumptions (ϕ_1) and (ϕ_2), and $\Phi(t)$ is an N-function satisfying Δ_2-condition, i.e.,

$$\Phi(2t) \leq k\Phi(t), \quad t \geq 0$$

for some constant $k > 0$.

In order to analyze Eq. (1.1), we introduce the Orlicz space $L_\Phi(\Omega)$ and Orlicz–Sobolev space $W_0^1 L_\Phi(\Omega)$. Noting that $\Phi(t)$ satisfies a Δ_2-condition, we see

$$L_\Phi(\Omega) = \left\{ u\text{: measurable on } \Omega;\ \int_\Omega \Phi(|u(x)|)\mathrm{d}x < \infty \right\}$$

with the norm

$$\|u\|_\Phi = \inf\left\{k > 0;\ \int_\Omega \Phi\left(\frac{|u(x)|}{k}\right)\mathrm{d}x \le 1\right\},$$

and $W_0^1 L_\Phi(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with the norm

$$\|u\|_{W_0^1 L_\Phi(\Omega)} = \|\nabla u\|_\Phi + \|u\|_\Phi.$$

In the case when Φ satisfies Δ_2-condition, it is easy to see that a sequence $\{u_n\}$ in $L_\Phi(\Omega)$ converges to some $u \in L_\Phi(\Omega)$ if and only if

$$\int_\Omega \Phi(|u_n - u|)\mathrm{d}x \to 0 \quad \text{as } n \to \infty,$$

and further, $\{u_n\}$ is bounded in $L_\Phi(\Omega)$ if and only if $\{\int_\Omega \Phi(|u_n|)\mathrm{d}x\}$ is bounded. Since the Poincaré inequality

$$\|u\|_\Phi \le C\|\nabla u\|_\Phi, \quad u \in W_0^1 L_\Phi(\Omega)$$

holds with some constant $C > 0$, the norm $\|\nabla u\|_\Phi$ is equivalent to the norm $\|u\|_{W_0^1 L_\Phi(\Omega)}$. Hereafter, we adopt $\|\nabla u\|_\Phi$ as the norm of $W_0^1 L_\Phi(\Omega)$. Using the Δ_2-condition on Φ, we see that the dual space $L_\Phi(\Omega)^*$ is identified with $L_{\widetilde{\Phi}}(\Omega)$. Here $\widetilde{\Phi}$ is the complement function of Φ defined by

$$\widetilde{\Phi}(s) = \max_{t\ge 0}\{st - \Phi(t)\} \quad \text{for } s \ge 0.$$

Then $\widetilde{\Phi}(s) = \int_0^s \eta(t)\mathrm{d}t$, where $\eta(t)$ is the inverse function of $\phi(t)t$. Furthermore, the Hölder inequality

$$\left|\int_\Omega uv\mathrm{d}x\right| \le 2\|u\|_{\widetilde{\Phi}}\|v\|_\Phi$$

holds for any $u \in L_{\widetilde{\Phi}}(\Omega)$, $v \in L_\Phi(\Omega)$. From assumption (ϕ_2), $\widetilde{\Phi}$ also satisfies a Δ_2-condition. Hence, we easily see that the spaces $L_\Phi(\Omega)$ and $W_0^1 L_\Phi(\Omega)$ are reflexive and the functional

$$I(u) = \int_\Omega \Phi(|\nabla u|)\mathrm{d}x \quad \text{on } W_0^1 L_\Phi(\Omega)$$

is Fréchet differentiable. Under assumptions (ϕ_1)–(ϕ_3), some elementary inequalities listed in the following lemma are valid. For the proofs, see [6].

Lemma 2.1 *Let* $\zeta_0(t) = \min\{t^\ell, t^m\}$, $\zeta_1(t) = \max\{t^\ell, t^m\}$, $\zeta_2(t) = \min\{t^{\ell/(\ell-1)}, t^{m/(m-1)}\}$, $\zeta_3(t) = \max\{t^{\ell/(\ell-1)}, t^{m/(m-1)}\}$, $t \ge 0$. *Then*

$$\zeta_0(\rho)\Phi(t) \le \Phi(\rho t) \le \zeta_1(\rho)\Phi(t) \qquad \text{for } \rho, t > 0,$$
$$\zeta_2(\rho)\widetilde{\Phi}(t) \le \widetilde{\Phi}(\rho t) \le \zeta_3(\rho)\widetilde{\Phi}(t) \qquad \text{for } \rho, t > 0,$$
$$\zeta_0(\|u\|_\Phi) \le \int_\Omega \Phi(|u|)\mathrm{d}x \le \zeta_1(\|u\|_\Phi) \quad \text{for } u \in L_\Phi(\Omega),$$
$$\zeta_2(\|u\|_{\widetilde{\Phi}}) \le \int_\Omega \widetilde{\Phi}(|u|)\mathrm{d}x \le \zeta_3(\|u\|_{\widetilde{\Phi}}) \quad \text{for } u \in L_{\widetilde{\Phi}}(\Omega).$$

As in the usual Sobolev space, the following inequality is also valid in the Orlicz–Sobolev space $W_0^1 L_\Phi(\Omega)$.

Lemma 2.2 *Let* $d = \mathrm{diam}(\Omega)$. *Then the inequality*

$$\int_\Omega \Phi\left(\frac{|u|}{d}\right)\mathrm{d}x \le \int_\Omega \Phi(|\nabla u|)\mathrm{d}x$$

holds for any $u \in W_0^1 L_\Phi(\Omega)$.

3 The operator $Lu = -\operatorname{div}(\phi(|\nabla u|)\nabla u)$

In this section, we show some basic properties of the operator $Lu = -\operatorname{div}(\phi(|\nabla u|)\nabla u)$ which will be used afterward.

For a function $h \in W_0^1 L_\Phi(\Omega)^*$, we mean a weak solution of

$$\begin{cases} -\operatorname{div}(\phi(|\nabla u|)\nabla u) = h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

by a function $u \in W_0^1 L_\Phi(\Omega)$ satisfying

$$\int_\Omega \phi(|\nabla u|)\nabla u \cdot \nabla\varphi \mathrm{d}x = \langle h, \varphi\rangle \quad \text{for any } \varphi \in W_0^1 L_\Phi(\Omega),$$

where $\langle\ ,\ \rangle$ denotes the duality between the space $W_0^1 L_\Phi(\Omega)^*$ and $W_0^1 L_\Phi(\Omega)$.

Let us define a operator L on $W_0^1 L_\Phi(\Omega)$ by

$$Lu = -\operatorname{div}(\phi(|\nabla u|)\nabla u), \quad u \in W_0^1 L_\Phi(\Omega).$$

Then, we have the following lemma.

Lemma 3.1 *The operator* L *is a homeomorphism from* $W_0^1 L_\Phi(\Omega)$ *onto* $W_0^1 L_\Phi(\Omega)^*$.

Proof By the definition of $\widetilde{\Phi}$ and Δ_2-condition of Φ, the inequality

$$\widetilde{\Phi}(\phi(t)t) \le \Phi(2t) \le k\Phi(t)$$

holds for any $t \ge 0$. Thus,

$$\int_\Omega \widetilde{\Phi}(\phi(|\nabla u|)|\nabla u|)\mathrm{d}x \le k \int_\Omega \Phi(|\nabla u|)\mathrm{d}x < \infty \quad \text{for any } u \in W_0^1 L_\Phi(\Omega).$$

Hence, by using the convexity of $\widetilde{\Phi}$, we have

$$\|\phi(|\nabla u|)\nabla u\|_{\widetilde{\Phi}} \le \max\left\{1, k\int_\Omega \Phi(|\nabla u|)\mathrm{d}x\right\}.$$

Then, using Hölder's inequality, we have

$$\left|\int_\Omega \phi(|\nabla u|)\nabla u \cdot \nabla v \mathrm{d}x\right| \le 2\|\phi(|\nabla u|)\nabla u\|_{\widetilde{\Phi}}\|\nabla v\|_\Phi$$
$$\le 2\max\left\{1, k\int_\Omega \Phi(|\nabla u|)\mathrm{d}x\right\}\|\nabla v\|_\Phi$$

for any $u,\ v \in W_0^1 L_\Phi(\Omega)$. This shows that $Lu \in W_0^1 L_\Phi(\Omega)^*$ for any $u \in W_0^1 L_\Phi(\Omega)$. For $h \in W_0^1 L_\Phi(\Omega)^*$, it is easy to see that a solution u of $Lu = h$ is unique. In order to show the invertibility of L, let

$$K_h(u) \equiv \int_\Omega \Phi(|\nabla u|)\mathrm{d}x - \langle h, u\rangle, \quad u \in W_0^1 L_\Phi(\Omega),$$

for $h \in W_0^1 L_\Phi(\Omega)^*$. Clearly, a critical point of K_h satisfies the equation $Lu = h$. Thus, the invertibility of the operator L follows from the coercivity and weak lower semi-continuity of the functional $K_h(u)$ by the usual arguments. These facts are easily shown by the convexity of Φ. In fact, take $\varepsilon > 0$ and let $\|\nabla u\|_\Phi \ge 1+\varepsilon$. Then, since Φ is convex, the inequality

$$\frac{1+\varepsilon}{\|\nabla u\|_\Phi}\int_\Omega \Phi(|\nabla u|)\mathrm{d}x \ge \int_\Omega \Phi\left(\frac{1+\varepsilon}{\|u\|_\Phi}|\nabla u|\right)\mathrm{d}x > 1$$

holds. The last inequality follows from the definition of $\|\nabla u\|_\Phi$. Hence,

$$\int_\Omega \Phi(|\nabla u|)\mathrm{d}x > \frac{\|\nabla u\|_\Phi}{1+\varepsilon}.$$

Namely, if $\|\nabla u\|_\Phi > 1$, then

$$\int_\Omega \Phi(|\nabla u|)\mathrm{d}x \ge \|\nabla u\|_\Phi.$$

This shows that the functional K_h is coercive. The weak lower semi-continuity also follows from the convexity of Φ (see e.g. Theorem 1.2 in Chap. 3 of [3]). Finally, the continuity of the inverse operator is obtained by the estimate in Lemma 3.2 stated later. Let us take $h_n,\ h_0 \in W_0^1 L_\Phi(\Omega)$, $h_n \ne h_0$ satisfying $h_n \to h_0$ in $W_0^1 L_\Phi(\Omega)^*$, and $u_n, u_0 \in W_0^1 L_\Phi(\Omega)$ be the solution of

$$-\operatorname{div}(\phi(|\nabla u_n|)\nabla u_n) = h_n,$$

and

$$-\operatorname{div}(\phi(|\nabla u_0|)\nabla u_0) = h_0$$

respectively. Since $\{h_n\}$ is bounded in $W_0^1 L_\Phi(\Omega)^*$, we easily see that the sequence $\{u_n\}$ is bounded in $W_0^1 L_\Phi(\Omega)$ by the coerciveness of the functional K_h. Therefore,

$$\int_\Omega (\phi(|\nabla u_n|)\nabla u_n - \phi(|\nabla u_0|)\nabla u_0)\cdot(\nabla u_n - \nabla u_0)\mathrm{d}x$$
$$= \langle h_n - h_0, u_n - u_0\rangle \to 0 \quad \text{as } n \to \infty.$$

On the other hand, using Hölder's inequality and Lemma 3.2, we have

$$
\begin{aligned}
&\int_\Omega \Phi(|\nabla u_n - \nabla u_0|)\mathrm{d}x \\
&\quad \le \left(\int_\Omega \frac{\Phi(|\nabla u_n - \nabla u_0|)^{(\ell+1)/\ell}}{(\Phi(|\nabla u_n|) + \Phi(|\nabla u_0|))^{1/\ell}} \mathrm{d}x \right)^{\ell/(\ell+1)} \\
&\qquad \times \left(\int_\Omega (\Phi(|\nabla u_n|) + \Phi(|\nabla u_0|)\mathrm{d}x \right)^{1/(\ell+1)} \\
&\quad \le M \left(\frac{1}{k_0} \int_\Omega (\phi(|\nabla u_n|)\nabla u_n - \phi(|\nabla u_0|)\nabla u_0) \cdot (\nabla u_n - \nabla u_0)\mathrm{d}x \right)^{\ell/(\ell+1)} \\
&\quad \to 0 \quad \text{as } n \to \infty
\end{aligned}
$$

for some constant $M > 0$. This shows that $u_n \to u_0$ in $W_0^1 L_\Phi(\Omega)$. □

Lemma 3.2 *There exists $k_0 > 0$ such that*

$$
(\phi(|\xi|)\xi - \phi(|\eta|)\eta) \cdot (\xi - \eta) \ge k_0 \frac{\Phi(|\xi - \eta|)^{(\ell+1)/\ell}}{(\Phi(|\xi|) + \Phi(|\eta|))^{1/\ell}}
$$

for $\xi, \eta \in \mathbb{R}^N$, $\xi \ne 0$.

For the proof of Lemma 3.2, see Sect. 2 in [6].

Lieberman [14, 15] has considered the regularity of weak solutions to the type of boundary value problems

$$
\begin{cases} \operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}
$$

Applying his results to our operator, we have the following lemma.

Lemma 3.3 *Let $h \in L^\infty(\Omega)$ and u be a weak solution in $W_0^1 L_\Phi(\Omega)$ of (3.1). Then u belongs to $C^{1,\alpha}(\overline{\Omega})$ with some $\alpha > 0$ depending on a_0, a_1, ℓ and m. Furthermore, there exists a constant $\kappa(\|h\|_{L^\infty(\Omega)}) > 0$ depending on $\|h\|_{L^\infty(\Omega)}$ such that*

$$
\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le \kappa(\|h\|_{L^\infty(\Omega)}).
$$

This constant $\kappa(\|h\|_{L^\infty(\Omega)})$ tends to 0 as $\|h\|_{L^\infty(\Omega)}$ goes to 0.

By the argument given by Tolksdorf [17, Sect. 3], we see that the strong maximum principle holds for this equation.

Lemma 3.4 *Let $h \in L^\infty(\Omega)$, $h \ge 0$ and $h \ne 0$. Then the solution $u \in W_0^1 L_\Phi(\Omega)$ of (3.1) is positive on Ω.*

Furthermore, we see that the comparison principle holds for the operator L by an argument similar to that by Guedda and Véron [12], in which the p-Laplace operator is discussed. Namely, we have the following lemma.

Lemma 3.5 *Let $h_i \in L^\infty(\Omega)$, $i = 1, 2$, and $u_i \in W_0^1 L_\Phi(\Omega)$ be the solutions of*

$$\begin{cases} -\operatorname{div}(\phi(|\nabla u|)\nabla u) = h_i & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

$i = 1, 2$, respectively. If $0 \le h_1 \le h_2$ and the set

$$C = \{x \in \Omega;\ h_1(x) = h_2(x)\}$$

has an empty interior, then

$$0 \le u_1 < u_2 \text{ in } \Omega \quad \text{and} \quad \frac{\partial u_2}{\partial \nu} < \frac{\partial u_1}{\partial \nu} \le 0 \text{ on } \partial\Omega,$$

where ν denotes the outer unit normal on $\partial\Omega$.

4 Existence of a positive solution for large λ

In this section, we consider a quasilinear boundary value problem

$$(P_\lambda) \begin{cases} -\operatorname{div}(\phi(|\nabla u|)\nabla u) = \lambda f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where ϕ and f satisfy (ϕ_1)–(ϕ_3) and (f_1)–(f_3) respectively. A weak solution is defined by a function $u \in W_0^1 L_\Phi(\Omega)$, which satisfies the equation

$$\int_\Omega \phi(|\nabla u|)\nabla u \cdot \nabla\varphi \mathrm{d}x - \lambda \int_\Omega f(x, u)\varphi \mathrm{d}x = 0 \tag{4.2}$$

for any $\varphi \in W_0^1 L_\Phi(\Omega)$. Note that the value of $f(x, t)$ for $x \in \Omega$, $t < 0$ has no effect on positive solutions of (P_λ). So we replace the value of $f(x, t)$ by 0 for $x \in \Omega$, $t < 0$. For a weak solution of (P_λ), we have the following lemma.

Lemma 4.1 *Let ϕ and f satisfy (ϕ_1)–(ϕ_3) and (f_1)–(f_3) respectively. Then a weak solution $u \in W_0^1 L_\Phi(\Omega)$ of (P_λ) belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha = \alpha(\lambda) > 0$ depending on $\lambda > 0$. Furthermore, $\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le d$ with a constant $d = d(\lambda, \|u\|_\Phi)$ depending on $\|u\|_\Phi$ and λ.*

Proof Substituting $\varphi = u$ in Eq. (4.2), we have

$$\int_\Omega \phi(|\nabla u|)|\nabla u|^2 \mathrm{d}x = \lambda \int_\Omega f(x, u)u \mathrm{d}x.$$

From assumption (f_3), the inequality

$$\int_\Omega \phi(|\nabla u|)|\nabla u|^2 \mathrm{d}x \le c_1 \int_\Omega \Phi(|u|)\mathrm{d}x \tag{4.3}$$

holds with some constant $c_1 > 0$. Noting that $\phi(t)t$ is increasing and using (ϕ_2), (4.2), we have

$$\begin{aligned}\int_\Omega |\nabla[\Phi(|u|)]|\mathrm{d}x &= \int_\Omega \phi(|u|)|u||\nabla u|\mathrm{d}x\\ &\le \int_\Omega \{\phi(|u|)|u|^2 + \phi(|\nabla u|)|\nabla u|^2\}\mathrm{d}x\\ &\le c_2\int_\Omega \Phi(|u|)\mathrm{d}x\end{aligned}$$

with some constant $c_2 > 0$. Then the Poincaré–Sobolev inequality implies the inequality

$$\left(\int_\Omega \Phi(|u|)^\nu \mathrm{d}x\right)^{1/\nu} \le S_1\int_\Omega |\nabla[\Phi(|u|)]|\mathrm{d}x \le c_2 S_1\int_\Omega \Phi(|u|)\mathrm{d}x, \tag{4.4}$$

where $\nu = N/(N-1)$ and S_1 is the Sobolev constant for the imbedding from $W_0^{1,1}(\Omega)$ to $L^\nu(\Omega)$. Take an integer $q > 1$ and assume that $\Phi(|u|)$ belongs to $L^q(\Omega)$. Let $\Phi_L(t) = \Phi(\min\{t, L\})$ for $L > 0$ and put $\varphi = \Phi_L(|u|)^{q-1}u$ in Eq. (4.2). Noting

$$\begin{aligned}&\nabla\varphi = \Psi_L(|u|)\nabla u,\\ &\Psi_L(t) = \begin{cases}\Phi(t)^{q-1} + (q-1)\Phi(t)^{q-2}\phi(t)t^2, & 0 \le t \le L\\ \Phi(L)^{q-1}, & t > L,\end{cases}\end{aligned} \tag{4.5}$$

we see that the inequality

$$\begin{aligned}\int_\Omega \Psi_L(|u|)\phi(|\nabla u|)|\nabla u|^2\mathrm{d}x &= \lambda\int_\Omega f(x,u)\Phi_L(|u|)^{q-1}u\mathrm{d}x\\ &\le \lambda c_0\int_\Omega \Phi(|u|)^q\mathrm{d}x\end{aligned} \tag{4.6}$$

holds. From assumption (ϕ_2),

$$\Psi_L(|u|) \ge \Phi(|u|)^{q-1} + \ell(q-1)\Phi(|u|)^{q-1} \ge c_3 q\Phi(|u|)^{q-1} \tag{4.7}$$

for $|u| \le L$ with some constant $c_3 > 0$. Thus, from (4.5) to (4.7), we have

$$\begin{aligned}&\int_\Omega |\nabla[\Phi_L(|u|)^q]|\mathrm{d}x\\ &\quad\le q\int_{\{x\in\Omega;\ |u|\le L\}} \Phi_L(|u|)^{q-1}\phi(|u|)|u||\nabla u|\mathrm{d}x\\ &\quad\le \frac{1}{c_3}\int_{\{x\in\Omega;\ |u|\le L\}} \Psi_L(|u|)\phi(|u|)|u||\nabla u|\mathrm{d}x\\ &\quad\le \frac{1}{c_3}\int_{\{x\in\Omega;\ |u|\le L\}} \Psi_L(|u|)\{\phi(|u|)|u|^2 + \phi(|\nabla u|)|\nabla u|^2\}\mathrm{d}x\\ &\quad\le \frac{1}{c_3}\int_{\{x\in\Omega;\ |u|\le L\}} \Psi_L(|u|)\phi(|u|)|u|^2\mathrm{d}x + \frac{\lambda c_0}{c_3}\int_\Omega \Phi(|u|)^q\mathrm{d}x.\end{aligned}$$

Here, by assumption (ϕ_2),

$$\begin{aligned}\Psi_L(|u|)\phi(|u|)|u|^2 &\le [\Phi(|u|^{q-1}) + (q-1)m\Phi(|u|)^{q-1}]m\Phi(|u|)\\ &\le c_4 q\Phi(|u|)^q \quad \text{on } \{x \in \Omega;\ |u| \le L\}\end{aligned}$$

for some constant $c_4 > 0$. Thus, we have

$$\int_{\{x\in\Omega;\ |u|\le L\}} \Psi_L(|u|)\phi(|u|)|u|^2 \mathrm{d}x \le c_4 q \int_\Omega \Phi(|u|)^q \mathrm{d}x.$$

Consequently,

$$\int_\Omega |\nabla[\Phi_L(|u|)^q]|\mathrm{d}x \le c_5 q \int_\Omega \Phi(|u|)^q \mathrm{d}x$$

for some constant $c_5 > 0$. From Sobolev's inequality, the inequality

$$\begin{aligned}\left(\int_\Omega \Phi_L(|u|)^{q\nu}\mathrm{d}x\right)^{1/\nu} &\le S_1 \int_\Omega |\nabla[\Phi_L(|u|)^q]|\mathrm{d}x\\ &\le c_6 q \int_\Omega \Phi(|u|)^q \mathrm{d}x\end{aligned}$$

holds with a constant $c_6 > 0$. Letting $L \to \infty$, we see that $\Phi(|u|)$ belongs to $L^{q\nu}(\Omega)$ and the inequality

$$\left(\int_\Omega \Phi(|u|)^{q\nu}\mathrm{d}x\right)^{1/\nu} \le c_6 q \int_\Omega \Phi(|u|)^q \mathrm{d}x$$

holds. Namely, the inequality

$$\left(\int_\Omega \Phi(|u|)^{q\nu} c_6^N \mathrm{d}x\right)^{1/(q\nu)} \le q^{1/q} \left(\int_\Omega \Phi(|u|)^q c_6^N \mathrm{d}x\right)^{1/q}$$

holds. Putting $q = \nu^m$, $m = 0, 1, 2, \ldots$, and taking iterative procedure, we have

$$\|\Phi(|u|)\|_{\nu^{m+1}} \le (\nu^m)^{1/\nu^m} \|\Phi(|u|)\|_{\nu^m}, \quad m = 0, 1, 2, \ldots, \tag{4.8}$$

where $\|\cdot\|_q$ denotes the usual Lebesgue norm with the measure $c_6^N \mathrm{d}x$. By inequality (4.8),

$$\|\Phi(|u|)\|_{\nu^m} \le \nu^{\sum_{j=0}^{m-1} j/\nu^j} \|\Phi(|u|)\|_1 \le \nu^{N^2-1} \|\Phi(|u|)\|_1.$$

Letting $m \to \infty$, we have

$$\|\Phi(|u|)\|_\infty \le c_6^N \nu^{N^2-1} \int_\Omega \Phi(|u|)\mathrm{d}x.$$

This implies that u belongs to $L^\infty(\Omega)$ and $\|u\|_{L^\infty(\Omega)} \le d$ with a constant d depending on λ and $\|u\|_\Phi$. Once we obtain the uniform estimate of u, Hölder estimate of ∇u follows from Lemma 3.3, since $h = f(x, u)$ belongs to $L^\infty(\Omega)$. □

Theorem 4.1 *Let assumptions* (ϕ_1)–(ϕ_3), (f_1)–(f_4) *be satisfied. Then there exists a constant* $\lambda_0 > 0$ *such that* (P_λ) *has a positive solution for any* $\lambda > \lambda_0$.

Proof We consider the functional

$$I_\lambda(u) = \int_\Omega \Phi(|\nabla u|)\mathrm{d}x - \lambda \int_\Omega F(x,u)\mathrm{d}x \quad \text{for } u \in W_0^1 L_\Phi(\Omega),$$

where

$$F(x,t) = \int_0^t f(x,s)\mathrm{d}s.$$

Then the functional I_λ is continuously Fréchet differentiable on $W_0^1 L_\Phi(\Omega)$. See e.g. García-Huidobro et al. [10]. By using the maximum principle, it is easy to see that critical points of I_λ are nonnegative weak solutions of (P_λ). Furthermore, applying Lemmas 3.4 and 4.1, we easily see that nonnegative nontrivial solutions of (P_λ) are positive in Ω. Thus, nonzero critical points of I_λ are positive solutions of (P_λ). Now we give a solution by a global minimizer of I_λ. By assumption (f_4), for any $\lambda > 0$, there exists a constant $T_\lambda > 0$ such that the inequality

$$|f(x,t)| \le \frac{1}{2\lambda}\frac{\Phi(t)}{t}$$

holds for $x \in \Omega$, $t \ge T_\lambda$. Since $\Phi(t)/t$ is increasing, we have the inequality

$$\begin{aligned} F(x,t) &= \int_0^t f(x,s)\mathrm{d}s \\ &\le \frac{1}{2\lambda}\int_0^t \frac{\Phi(s)}{s}\mathrm{d}s + T_\lambda \sup_{(x,t)\in\Omega\times[0,T_\lambda]} |f(x,t)| \\ &\le \frac{1}{2\lambda}\Phi(t) + c_\lambda \end{aligned}$$

for $(x,t) \in \Omega \times [0,\infty)$, where $c_\lambda = T_\lambda \sup_{(x,t)\in\Omega\times[0,T_\lambda]} |f(x,t)|$. Using Poincaré's inequality, Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned} I_\lambda(u) &\ge \frac{1}{2}\int_\Omega \Phi(|\nabla u|)\mathrm{d}x - \lambda c_\lambda |\Omega| \\ &\ge \frac{1}{2}\zeta_0(\|\nabla u\|_\Phi) - \lambda c_\lambda |\Omega| \end{aligned} \tag{4.9}$$

for any $u \in W_0^1 L_\Phi(\Omega)$. Thus, there exists a constant $\rho_\lambda > 0$ such that

$$I_\lambda(u) > 0 \quad \text{on } \|\nabla u\|_\Phi = \rho \tag{4.10}$$

for $\rho \ge \rho_\lambda$. Now, take $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \ge 0$, $\emptyset \ne \operatorname{supp}\varphi \subset \Omega_0$. Then,

$$I_\lambda(\varphi) = \int_\Omega \Phi(|\nabla\varphi|)\mathrm{d}x - \lambda\int_\Omega F(x,\varphi)\mathrm{d}x < 0 \tag{4.11}$$

for sufficiently large $\lambda > 0$. For this λ, take $\rho > \max\{\rho_\lambda, \|\nabla\varphi\|_\Phi\}$. Noting $\varphi \in B_\rho = \{u \in W_0^1 L_\Phi(\Omega);\ \|\nabla u\|_\Phi < \rho\}$ and inequality (4.11), we see

$$m_\lambda \equiv \inf_{u \in B_\rho} I_\lambda(u) \in (-\infty, 0). \tag{4.12}$$

Since Φ is convex, the functional

$$\int_\Omega \Phi(|\nabla u|)\mathrm{d}x$$

is weakly lower semi-continuous. Furthermore, by the usual arguments, we see that the functional

$$\int_\Omega F(x, u)\mathrm{d}x$$

is weakly continuous on $W_0^1 L_\Phi(\Omega)$ under assumption (f_4). For proof, see, e.g. [6]. Hence, the functional I_λ is weakly lower semi-continuous. Since the norm of $W_0^1 L_\Phi(\Omega)$ is weakly lower semi-continuous, it is easy to see that the infimum m_λ is achieved by u_λ in $\overline{B_\rho}$. By (4.10) and (4.12), we see $u_\lambda \notin \partial B_\rho$. This shows that u_λ is a critical point of I_λ. □

By Theorem 4.1, the set

$$\mathcal{M} = \{\lambda \in \mathbb{R}; \text{there exists a positive solution of } (P_\lambda)\}$$

is not empty. Here put

$$\Lambda = \inf \mathcal{M}. \tag{4.13}$$

Then we have the following lemma.

Lemma 4.2 *Under assumptions* (ϕ_1)–(ϕ_3) *and* (f_1)–(f_3), *the constant* Λ *is positive.*

Proof Let λ be a constant for which (P_λ) has a positive solution u. Substituting u for φ in Eq. (4.2), we have the equation

$$\int_\Omega \phi(|\nabla u|)|\nabla u|^2\mathrm{d}x = \lambda \int_\Omega f(x, u)u\mathrm{d}x.$$

By assumptions (ϕ_2) and (f_3) and Poincaré's inequality, we have

$$\begin{aligned}\lambda = \frac{\int_\Omega \phi(|\nabla u|)|\nabla u|^2\mathrm{d}x}{\int_\Omega f(x, u)u\mathrm{d}x} &\geq \frac{\ell \int_\Omega \Phi(|\nabla u|)\mathrm{d}x}{c_0 \int_\Omega \Phi(|u|)\mathrm{d}x} \\ &\geq \frac{\ell}{c_0} \frac{\int_\Omega \Phi(|\nabla u|)\mathrm{d}x}{\int_\Omega \Phi(d|\nabla u|)\mathrm{d}x} \geq \frac{\ell}{c_0 \zeta_1(d)},\end{aligned}$$

where $d = \mathrm{diam}(\Omega)$. This shows that $\Lambda \geq \ell/(c_0\zeta_1(d)) > 0$. □

In order to show the existence of a positive solution of (P_λ) for any $\lambda > \Lambda$, we construct a super-solution of (P_λ).

Let $\eta(t)$ be the inverse function of $\phi(t)t$. Taking $R > 0$ such that $B_R(0) \supset \Omega$, we define the operator $\mathcal{F}$ as

$$(\mathcal{F}w)(r) = \int_r^R \eta\left(\lambda \int_0^t \left(\frac{s}{t}\right)^{N-1} \bar{f}(w(s))\mathrm{d}s\right) \mathrm{d}t, \quad 0 \le r \le R,$$

for $w \in C^0[0, R]$. Here

$$\bar{f}(t) = \sup_{(x,s)\in\Omega\times(0,t)} f(x, s).$$

Then, it is easy to see that $v(x) \equiv (\mathcal{F}w)(r)$, $r = |x|$, is well defined and satisfies the equation

$$-\operatorname{div}(\phi(|\nabla v|)\nabla v) = \lambda \bar{f}(w) \quad \text{in } B_R(0).$$

The function $\bar{f}(t)$ is positive and non-decreasing for $t > 0$. Thus, if $w \ge \mathcal{F}w$ in Ω and $\mathcal{F}w \ge 0$ on $\partial\Omega$, then $v \equiv \mathcal{F}w$ is a super-solution of (4.1). Now we have the following lemma.

Lemma 4.3 *Let assumptions* (ϕ_1)–(ϕ_3), (f_1), (f_2), *and* (f_4) *be satisfied. Then there exists a constant* $c > 1$ *such that the function* $v \equiv \mathcal{F}c$ *is a super-solution of* (P_λ).

Proof As is stated previously, it is sufficient to show

$$c \ge \mathcal{F}c \quad \text{in } \Omega$$

and

$$\mathcal{F}c \ge 0 \quad \text{on } \partial\Omega.$$

For a positive constant c, $(\mathcal{F}c)(r) \ge 0$ on $[0, R]$. Thus, the inequality $\mathcal{F}c \ge 0$ on $\partial\Omega$ is valid. Since η is nondecreasing, the inequality

$$\begin{aligned}(\mathcal{F}c)(r) &= \int_r^R \eta\left(\lambda \int_0^t \left(\frac{s}{t}\right)^{N-1} \bar{f}(c)\mathrm{d}s\right) \mathrm{d}t \\ &\le \int_0^R \eta\left(\lambda \int_0^t \bar{f}(c)\mathrm{d}s\right) \mathrm{d}t \le R\eta(\lambda R \bar{f}(c))\end{aligned}$$

holds for $0 \le r \le R$. By assumptions (f_4) and (ϕ_2), we easily see that, for any $\varepsilon > 0$, there exist $t_0 > 0$ such that

$$|\bar{f}(c)| \le \varepsilon\phi(c)c \quad \text{for } c > t_0.$$

Note $\widetilde{\Phi}(s) = \int_0^s \eta(\sigma)\mathrm{d}\sigma$. Then, from Lemma 2.5 in [6], the inequality

$$\frac{m}{m-1}\widetilde{\Phi}(s) \le \eta(s)s \le \frac{\ell}{\ell-1}\widetilde{\Phi}(s)$$

holds for $s \geq 0$. Applying the second inequality in Lemma 2.1 in addition to this, we have

$$\begin{aligned}
R\eta(\lambda R\bar{f}(c)) &\leq R\eta(\varepsilon\lambda R\phi(c)c) \\
&\leq \frac{\ell R}{\ell-1}\frac{\widetilde{\Phi}(\varepsilon\lambda R\phi(c)c)}{\varepsilon\lambda R\phi(c)c} \\
&\leq \frac{\ell R}{\ell-1}\frac{\zeta_3(\varepsilon\lambda R)\widetilde{\Phi}(\phi(c)c)}{\varepsilon\lambda R\phi(c)c} \\
&\leq \frac{\ell(m-1)R}{(\ell-1)m}\frac{\zeta_3(\varepsilon\lambda R)\eta(\phi(c)c)}{\varepsilon\lambda R} \\
&= \frac{\ell(m-1)R}{(\ell-1)m}\frac{\zeta_3(\varepsilon\lambda R)}{\varepsilon\lambda R}c
\end{aligned}$$

for $c > t_0$. Namely,

$$\frac{(\mathcal{F}c)(r)}{c} \leq \frac{\ell(m-1)R}{(\ell-1)m}\frac{\zeta_3(\varepsilon\lambda R)}{\varepsilon\lambda R} \quad \text{for } c > t_0. \tag{4.14}$$

Taking $\varepsilon > 0$ so small that the right-hand side of inequality (4.14) is smaller than 1, we see that

$$(\mathcal{F}c)(r) \leq c \quad \text{for } 0 \leq r \leq R,$$

that is, $v \equiv \mathcal{F}c$ is a super-solution of (P_λ). □

Theorem 4.2 *Under assumptions* (ϕ_1)–(ϕ_3), (f_1)–(f_4) *there exists a positive solution of* (P_λ) *for any* λ *greater than* Λ.

Proof Let λ be a constant greater than Λ. By the definition of Λ, we can take $\lambda_0 < \lambda$ such that there exists a positive solution u_{λ_0} of (P_{λ_0}). Since

$$\begin{aligned}
-\operatorname{div}(\phi(|\nabla u_{\lambda_0}|)\nabla u_{\lambda_0}) &= \lambda_0 f(x, u_{\lambda_0}) \\
&\leq \lambda f(x, u_{\lambda_0}),
\end{aligned}$$

u_{λ_0} is a sub-solution of (P_λ). Let v be the super-solution of (P_λ) given in Lemma 4.3. By Lemma 4.3, we can take $c = c_R > 1$ for each $R > 0$ satisfying $B_R(0) \supset \Omega$. Thus, taking $R_0 > 0$ so that $\Omega \subset B_{R_0}(0)$, we have

$$\begin{aligned}
v(x) &= \int_r^R \eta\left(\lambda\int_0^t \left(\frac{s}{t}\right)^{N-1}\bar{f}(c)\mathrm{d}s\right)\mathrm{d}t \\
&= \int_r^R \eta\left(\frac{\lambda\bar{f}(c)}{N}t\right)\mathrm{d}t \\
&\geq \int_{R_0}^R \eta\left(\frac{\lambda\bar{f}(1)}{N}R_0\right)\mathrm{d}t \to \infty \quad \text{as } R \to \infty
\end{aligned}$$

for $0 < r < R_0$. Thus, taking $R > 0$ sufficiently large, we see

$$v(x) > u_{\lambda_0}(x) > 0 \quad \text{in } \Omega.$$

Now put

$$Z = \left\{u \in W_0^1 L_\Phi(\Omega);\ u_{\lambda_0}(x) \le u(x) \le v(x) \text{ a.e. in } \Omega\right\}$$

and consider the functional I_λ on Z. It is easy to see that I_λ has a minimizer u_λ on Z. Note that the inequality

$$(\phi(|\xi|)\xi - \phi(|\eta|)\eta) \cdot (\xi - \eta) \ge 0$$

holds for $\xi, \eta \in \mathbb{R}^N$. Then we see that u_λ is a solution of (P_λ), by the same argument as in Theorem 2.4 in [16]. It is clearly positive in Ω. This completes the proof. □

5 Existence of a second solution

In this section, under assumptions (ϕ_1)–(ϕ_3), (f_1), (f_2'), and (f_4)–(f_6), we give the main theorem as follows.

Theorem 5.1 *Let (ϕ_1)–(ϕ_3), (f_1), (f_2'), and (f_4)–(f_6) be satisfied. Then there exists a constant $\Lambda > 0$ such that*

(i) for all $\lambda \in (0, \Lambda)$, (P_λ) has no positive solution;
(ii) for $\lambda = \Lambda$, (P_λ) has at least one positive solution;
(iii) for all $\lambda > \Lambda$, (P_λ) has at least two positive solutions u_λ, v_λ satisfying $v_\lambda \le u_\lambda$ and $v_\lambda \ne u_\lambda$ in Ω.

We show this theorem through some lemmas. Throughout this section, we assume (ϕ_1)–(ϕ_3), (f_1), (f_2'), and (f_4)–(f_6). Furthermore, as in Sect. 4, we assume $f(x,t) = 0$ for $x \in \Omega$, $t < 0$.

To begin with, consider the functional I_λ on the space $X \equiv C^1(\overline{\Omega}) \cap W_0^1 L_\Phi(\Omega)$ with the norm $\|u\|_X = \|\nabla u\|_{L^\infty(\Omega)}$.

Lemma 5.1 *$u = 0$ is a local minimizer of I_λ on X.*

Proof By assumption (f_5), for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x,t)t| \le \varepsilon \Phi(t) \quad \text{for } 0 < t < \delta.$$

Since $\Phi(t)/t$ is increasing by (ϕ_2), the inequality

$$|F(x,t)| = \left|\int_0^t f(x,s)\mathrm{d}s\right| \le \varepsilon \int_0^t \frac{\Phi(s)}{s}\mathrm{d}s \le \varepsilon \Phi(t)$$

holds for $0 < t < \delta$. Since the inequality

$$\|u\|_{L^\infty(\Omega)} \le c\|\nabla u\|_{L^\infty(\Omega)}$$

holds for any $u \in X$ with some constant $c > 0$, the inequality

$$\left|\int_\Omega F(x,u)\mathrm{d}x\right| \le \varepsilon \int_\Omega \Phi(|u|)\mathrm{d}x$$

is valid for any $u \in X$ with $\|u\|_X \leq \delta/c$. Taking $\varepsilon > 0$ sufficiently small and using Poincaré's inequality, we see that

$$\begin{aligned} I_\lambda(u) &= \int_\Omega \Phi(|\nabla u|)\mathrm{d}x - \lambda \int_\Omega F(x,u)\mathrm{d}x \\ &\geq \int_\Omega \Phi(|\nabla u|)\mathrm{d}x - \lambda\varepsilon \int_\Omega \Phi(|u|)\mathrm{d}x \\ &\geq \frac{1}{2}\int_\Omega \Phi(|\nabla u|)\mathrm{d}x > 0 \quad (= I_\lambda(0)) \end{aligned}$$

holds for any $u \in X$ with $0 < \|u\|_X \leq \delta/c$. This shows that $u = 0$ is a local minimizer of I_λ on X. □

In order to find out a second local minimizer of I_λ in the space X for each $\lambda > \Lambda$, let us take $\lambda_0 \in (\Lambda, \lambda)$, the positive solution u_{λ_0} of (P_{λ_0}) and a parameta $\lambda_1 > \lambda$, the solution u_{λ_1} of (P_{λ_1}), which is given in Theorem 4.2 with λ_1 in place of λ. By the construction, we easily see $0 < u_{\lambda_0} \leq u_{\lambda_1}$ in Ω. Hence, by (f_6), we have

$$0 \leq \lambda_0 f(x, u_{\lambda_0}) < \lambda_1 f(x, u_{\lambda_1}).$$

Using Lemma 3.5, we see

$$u_{\lambda_0} < u_{\lambda_1} \text{ in } \Omega \quad \text{and} \quad \frac{\partial u_{\lambda_1}}{\partial \nu} < \frac{\partial u_{\lambda_0}}{\partial \nu} \leq 0 \text{ on } \partial\Omega.$$

Now put

$$\tilde{f}(x,s) = \begin{cases} f(x, u_{\lambda_0}(x)) & \text{for } s \leq u_{\lambda_0}(x) \\ f(x,s) & \text{for } u_{\lambda_0}(x) < s < u_{\lambda_1}(x) \\ f(x, u_{\lambda_1}(x)) & \text{for } s \geq u_{\lambda_1}(x), \end{cases}$$

$$\tilde{F}(x,t) = \int_0^t \tilde{f}(x,s)\mathrm{d}s$$

and

$$\tilde{I}_\lambda(u) = \int_\Omega \Phi(|\nabla u|)\mathrm{d}x - \lambda \int_\Omega \tilde{F}(x,u)\mathrm{d}x.$$

Since $\tilde{f}$ is bounded and continuous, the functional $\tilde{I}_\lambda$ has a global minimizer u_λ on the space $W_0^1 L_\Phi(\Omega)$. Thus, u_λ solves the equation

$$\begin{cases} -\operatorname{div}(\phi(|\nabla u|)\nabla u) = \lambda \tilde{f}(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Applying Lemma 4.1 with $\tilde{f}(x,u)$ in place of $f(x,u)$, we see that u_λ belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$.

Lemma 5.2 *$u = u_\lambda$ is a local minimizer of I_λ on X.*

Proof Since

$$\lambda_0 \tilde{f}(x, u_{\lambda_0})\ (= \lambda_0 f(x, u_{\lambda_0})) < \lambda \tilde{f}(x, u_\lambda) < \lambda_1 \tilde{f}(x, u_{\lambda_1})\ (= \lambda_1 f(x, u_{\lambda_1})) \quad \text{in } \Omega,$$

Lemma 3.5 shows that

$$u_{\lambda_0} < u_\lambda < u_{\lambda_1} \quad \text{in } \Omega$$

and

$$\frac{\partial u_{\lambda_1}}{\partial \nu} < \frac{\partial u_\lambda}{\partial \nu} < \frac{\partial u_{\lambda_0}}{\partial \nu} \quad \text{on } \partial\Omega.$$

Thus, for small $\varepsilon > 0$, if $u \in X$ satisfies $\|u - u_\lambda\|_X \le \varepsilon$, then

$$u_{\lambda_0} < u < u_{\lambda_1} \quad \text{in } \Omega.$$

Since

$$I_\lambda(u) = \tilde{I}_\lambda(u) + \lambda K,$$

$$K = \int_\Omega \{f(x, u_{\lambda_0})u_{\lambda_0} - F(x, u_{\lambda_0})\}\mathrm{d}x$$

for $u_{\lambda_0} < u < u_{\lambda_1}$, u_λ is a local minimizer of I_λ in X. □

Modifying the argument of H^1 versus C^1 local minimizers by Brezis and Nirenberg [2], we have the following lemma.

Lemma 5.3 *Let $u_0 \in X$ be a local minimizer of I_λ in X. Then u_0 is a local minimizer of I_λ on $W_0^1 L_\Phi(\Omega)$.*

Proof Suppose that the claim does not hold. Then there exists a sequence $\{u_n\} \subset W_0^1 L_\Phi(\Omega)$ such that

$$I_\lambda(u_n) < I_\lambda(u_0) \quad \text{and} \quad \lim_{n\to\infty} \|\nabla u_n - \nabla u_0\|_\Phi = 0. \tag{5.2}$$

Put

$$Q(u) = \int_\Omega \{\Phi(|u - u_0|) + F(x, u) - F(x, u_0) - f(x, u_0)(u - u_0)\}\mathrm{d}x$$

for $u \in W_0^1 L_\Phi(\Omega)$ and

$$Z_\varepsilon = \left\{u \in W_0^1 L_\Phi(\Omega);\ Q(u) \le \varepsilon\right\}$$

for $\varepsilon > 0$. Note that assumption (f_6) implies

$$F(x, u) - F(x, u_0) - f(x, u_0)(u - u_0) \ge 0 \quad \text{in } \Omega.$$

Putting $\varepsilon_n = Q(u_n) > 0$, we see, by (5.2), $\varepsilon_n \to 0$. Since the inequality

$$\begin{aligned}\left|\int_\Omega F(x, u)\mathrm{d}x\right| &\le c_1 \int_\Omega \Phi(|u|)\mathrm{d}x + c_2 \\ &\le c_3 \left(\int_\Omega \Phi(|u - u_0|)\mathrm{d}x + \int_\Omega \Phi(|u_0|)\mathrm{d}x\right) + c_2 \\ &\le c_3 Q(u) + c_4\end{aligned}$$

holds with some constants $c_1, c_2, c_3, c_4 > 0$, we have

$$\inf_{u \in Z_{\varepsilon_n}} I_\lambda(u) \geq -\lambda(c_3\varepsilon_n + c_4) \neq -\infty.$$

Thus, using the weak lower semi-continuity and the coercivity of I_λ and the continuity of Q, we see that there exists a minimizer $v_n \in W_0^1 L_\Phi(\Omega)$ of I_λ on Z_{ε_n} for each n. Thus, $v_n \neq u_0$ and

$$I_\lambda(v_n) \leq I_\lambda(u_n) < I_\lambda(u_0).$$

Furthermore, v_n satisfies the equation

$$\begin{cases} -\operatorname{div}(\phi(|\nabla v_n|)\nabla v_n) = h_n & \text{in } \Omega \\ v_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.3}$$

where

$$h_n = \lambda f(x, v_n) - \mu_n(\phi(|v_n - u_0|)(v_n - u_0) + f(x, v_n) - f(x, u_0))$$

with a Lagrange multiplier $\mu_n \geq 0$. Note that u_0 is assumed to be a local minimizer of I_λ on X. Thus, it satisfies the equation

$$\int_\Omega \phi(|\nabla u_0|)\nabla u_0 \cdot \nabla\varphi \mathrm{d}x = \lambda \int_\Omega f(x, u_0)\varphi \mathrm{d}x \tag{5.4}$$

for any $\varphi \in X$, and hence, for any $\varphi \in W_0^1 L_\Phi(\Omega)$. Therefore, from (5.3), (5.4), and Lemma 3.2 we have

$$\begin{aligned} 0 &\leq \int_\Omega (\phi(|\nabla v_n|)\nabla v_n - \phi(|\nabla u_0|)\nabla u_0) \cdot (\nabla v_n - \nabla u_0)\mathrm{d}x \\ &= \int_\Omega (\lambda f(x, v_n) - \mu_n(\phi(|v_n - u_0|)(v_n - u_0) + f(x, v_n) - f(x, u_0)) \\ &\quad - \lambda f(x, u_0))(v_n - u_0)\mathrm{d}x \\ &= -(\mu_n - \lambda) \int_\Omega (f(x, v_n) - f(x, u_0))(v_n - u_0)\mathrm{d}x \\ &\quad - \mu_n \int_\Omega \phi(|v_n - u_0|)|v_n - u_0|^2 \mathrm{d}x. \end{aligned}$$

Since, from $v_n \neq u_0$,

$$\int_\Omega \phi(|v_n - u_0|)|v_n - u_0|^2 \mathrm{d}x > 0,$$

and from (f_6)

$$\int_\Omega (f(x, v_n) - f(x, u_0))(v_n - u_0)\mathrm{d}x \geq 0,$$

we have $0 \le \mu_n < \lambda$. Since $v_n \in Z_{\varepsilon_n}$, the inequality

$$\begin{aligned}\int_\Omega \Phi(|v_n|)\mathrm{d}x &\le c_1 \left(\int_\Omega \Phi(|v_n - u_0|)\mathrm{d}x + \int_\Omega \Phi(|u_0|)\mathrm{d}x \right) \\ &\le c_1 \left(Q(v_n) + \int_\Omega \Phi(|u_0|)\mathrm{d}x \right) \\ &\le c_2\end{aligned}$$

holds with some constants $c_1, c_2 > 0$. Namely, $\{\|v_n\|_\Phi\}$ is bounded. Thus, applying Lemma 4.1, we see that $\{v_n\}$ belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$ and $\{\|v_n\|_{C^{1,\alpha}(\overline{\Omega})}\}$ is bounded. By Ascoli-Arzelà's theorem, there exists a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ and $v_0 \in X$ such that $v_{n_j} \to v_0$ in X. On the other hand, $v_n \in Z_{\varepsilon_n}$ and $\varepsilon_n \to 0$ imply that

$$\int_\Omega \Phi(|v_n - u_0|)\mathrm{d}x \to 0$$

as $n \to \infty$. Hence, $v_0 = u_0$. Therefore, $v_{n_j} \to u_0$ in X and $I_\lambda(v_{n_j}) < I_\lambda(u_0)$, which contradicts the assumption that u_0 is a local minimizer of I_λ on X. □

From Lemmas 5.1–5.3, we have immediately the following corollary.

Corollary 5.1 *$u = 0$ and u_λ are local minimizers of I_λ on $W_0^1 L_\Phi(\Omega)$.*

Using assumption (f_4) and Sobolev's imbedding theorem and applying Lemma 3.1, we easily see that I_λ satisfies Palais-Smale condition on $W_0^1 L_\Phi(\Omega)$, that is, if

$$I'_\lambda(u_n) \to 0 \text{ in } W_0^1 L_\Phi(\Omega)^* \quad \text{and} \quad \{I_\lambda(u_n)\} \text{ is bounded},$$

then $\{u_n\}$ contains a convergent subsequence in $W_0^1 L_\Phi(\Omega)$. In fact, let $\{u_n\}$ be a sequence in $W_0^1 L_\Phi(\Omega)$ satisfying the previous condition. As in (4.9), the inequality

$$\frac{1}{2}\zeta_0(\|\nabla u_n\|_\Phi) \le I_\lambda(u_n) + \lambda c_\lambda |\Omega|$$

holds under assumption (f_4). This shows that $\{u_n\}$ is bounded in $W_0^1 L_\Phi(\Omega)$. By Sobolev's imbedding theorem, $W_0^1 L_\Phi(\Omega)$ is compactly imbedded into $L_\Phi(\Omega)$. Furthermore, by assumption (f_4), $f(x, \cdot)$ is continuous from $W_0^1 L_\Phi(\Omega)$ to $W_0^1 L_\Phi(\Omega)^*$ (see e.g. Sect. 17 in [13]). Hence, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{f(x, u_{n_k})\}$ converges in $W_0^1 L_\Phi(\Omega)^*$. Hence,

$$\begin{aligned}-\operatorname{div}(\phi(|\nabla u_{n_k}|)\nabla u_{n_k}) &= I'_\lambda(u_{n_k}) + \lambda f(x, u_{n_k}) \\ &\to 0 \quad \text{as } k \to \infty\end{aligned}$$

in $W_0^1 L_\Phi(\Omega)^*$. From Lemma 3.1, the sequence $\{u_{n_k}\}$ is convergent in $W_0^1 L_\Phi(\Omega)$. This shows that I_λ satisfies the Palais-Smale condition.

Now we can apply the mountain-pass theorem by Ghoussoub and Preiss [11], which asserts the existence of the third critical point under the existence of two distinct local minimizers of a functional satisfying the Palais-Smale condition. Thus, from Corollary 5.1, there exists a critical point of I_λ which is distinct from

0 and u_λ, which is clearly a positive solution. Consequently, we obtain two distinct positive solutions of (P_λ).

Nevertheless, in order to obtain ordered positive solutions of (P_λ), we consider the functional

$$J_\lambda(w) = \int_\Omega \{\Phi(|\nabla(u_\lambda + w)|) - \phi(|\nabla u_\lambda|)\nabla u_\lambda \cdot \nabla w - \lambda G(x, w)\}\mathrm{d}x,$$

where

$$g(x, s) = \begin{cases} f(x, u_\lambda(x) + s) - f(x, u_\lambda(x)) & \text{for } s < 0 \\ 0 & \text{for } s \geq 0, \end{cases}$$

$$G(x, t) = \int_0^t g(x, s)\mathrm{d}s.$$

Concerning this functional, we have the following lemma.

Lemma 5.4 *$w = 0$ and $-u_\lambda$ are local minimizers of J_λ on $W_0^1 L_\Phi(\Omega)$.*

Proof Noting

$$G(x, t) = \begin{cases} F(x, u_\lambda(x) + t) - F(x, u_\lambda(x)) - f(x, u_\lambda(x))t & \text{for } t < 0 \\ 0 & \text{for } t \geq 0 \end{cases}$$

and

$$g(x, s) \leq f(x, u_\lambda(x) + s) - f(x, u_\lambda(x)) \quad \text{for } s \in \mathbb{R},$$

we see that

$$G(x, t) \leq F(x, u_\lambda(x) + t) - F(x, u_\lambda(x)) - f(x, u_\lambda(x))t \quad \text{for } t \in \mathbb{R}.$$

Hence, by Corollary 5.1, the inequality

$$\begin{aligned} &J_\lambda(w) - J_\lambda(0) \\ &= \int_\Omega \{\Phi(|\nabla(u_\lambda + w)|) - \Phi(|\nabla u_\lambda|) - \phi(|\nabla u_\lambda|)\nabla u_\lambda \cdot \nabla w - \lambda G(x, w)\}\mathrm{d}x \\ &\geq \int_\Omega \{\Phi(|\nabla(u_\lambda + w)|) - \Phi(|\nabla u_\lambda|) - \phi(|\nabla u_\lambda|)\nabla u_\lambda \cdot \nabla w \\ &\quad - \lambda(F(x, u_\lambda(x) + w) - F(x, u_\lambda(x)) - f(x, u_\lambda(x))w)\}\mathrm{d}x \\ &= I_\lambda(u_\lambda + w) - I_\lambda(u_\lambda) - \langle I'_\lambda(u_\lambda), w\rangle \\ &= I_\lambda(u_\lambda + w) - I_\lambda(u_\lambda) \geq 0 \end{aligned} \tag{5.5}$$

is valid for any $w \in W_0^1 L_\Phi(\Omega)$ with small $\|\nabla w\|_\Phi$. This implies that $w = 0$ is a local minimizer of J_λ on $W_0^1 L_\Phi(\Omega)$. Furthermore, since $-u_\lambda \leq 0$ and u_λ is a solution of (P_λ), we have

$$\begin{aligned} &J_\lambda(-u_\lambda) - J_\lambda(0) \\ &= \int_\Omega \{\phi(|\nabla u_\lambda|)|\nabla u_\lambda|^2 - \lambda G(x, -u_\lambda) - \Phi(|\nabla u_\lambda|)\}\mathrm{d}x \\ &= \int_\Omega \{\phi(|\nabla u_\lambda|)|\nabla u_\lambda|^2 + \lambda F(x, u_\lambda) - \lambda f(x, u_\lambda)u_\lambda - \Phi(|\nabla u_\lambda|)\}\mathrm{d}x \\ &= -I_\lambda(u_\lambda). \end{aligned} \tag{5.6}$$

From (5.5) and (5.6), we have

$$J_\lambda(w) - J_\lambda(-u_\lambda) \geq I_\lambda(u_\lambda + w).$$

Since $u = 0$ is a local minimizer of I_λ by Corollary 5.1, the inequality

$$J_\lambda(w) - J_\lambda(-u_\lambda) \geq 0$$

holds for any $w \in W_0^1 L_\Phi(\Omega)$ with sufficiently small $\|\nabla(u_\lambda + w)\|_\Phi$. This implies that $-u_\lambda$ is a local minimizer of J_λ. □

Finally, we come to the proof of Theorem 5.1.

Proof of Theorem 5.1. Let Λ be the constant given by (4.13). Then statement (i) is proved by the definition of Λ and Lemma 4.2. Now we shall give the proof of statement (iii). Note that the functional J_λ is continuously Fréchet differentiable on $W_0^1 L_\Phi(\Omega)$ and

$$\begin{aligned}\langle J_\lambda'(w), \varphi\rangle = &\int_\Omega \phi(|\nabla(u_\lambda + w)|)\nabla(u_\lambda + w)\cdot\nabla\varphi \mathrm{d}x - \int_\Omega \phi(|\nabla u_\lambda|)\nabla u_\lambda \cdot \nabla\varphi \mathrm{d}x \\ &- \lambda \int_\Omega g(x, w)\varphi \mathrm{d}x, \quad \varphi \in W_0^1 L_\Phi(\Omega).\end{aligned} \tag{5.7}$$

Similarly to I_λ, the functional J_λ satisfies the Palais-Smale condition. By Lemma 5.4, $w = 0$ and $-u_\lambda$ are local minimizers of J_λ. Thus, by the mountain-pass theorem by Ghoussoub and Preiss, there exists a critical point w_λ of J_λ which is distinct from 0 and $-u_\lambda$. Put $\varphi = w_\lambda^+ \equiv \max\{w_\lambda, 0\}$ in (5.7). Then $\langle J_\lambda'(w_\lambda), \varphi\rangle = 0$ implies

$$\begin{aligned}&\int_{\Omega^+} \{\phi(|\nabla(u_\lambda + w_\lambda)|)\nabla(u_\lambda + w_\lambda) - \phi(|\nabla u_\lambda|)\nabla u_\lambda\}\cdot(\nabla(u_\lambda + w_\lambda) - \nabla u_\lambda)\mathrm{d}x \\ &\quad = \lambda \int_{\Omega^+} g(x, w_\lambda) w_\lambda \mathrm{d}x\end{aligned} \tag{5.8}$$

where $\Omega^+ = \{x \in \Omega;\ w_\lambda(x) > 0\}$. Since $f(x, t)$ is nondecreasing in $t > 0$ for each $x \in \Omega$, $g(x, t) \leq 0$ for any $(x, t) \in \Omega \times \mathbb{R}$. Thus, the right-hand side of (5.8) is nonpositive. Then Lemma 3.2 implies that $\nabla w_\lambda = 0$ on Ω^+, and hence $w_\lambda^+ = 0$. Namely, $w_\lambda \leq 0$ in Ω. Thus, w_λ satisfies the equation

$$\begin{aligned}&-\mathrm{div}(\phi(|\nabla(u_\lambda + w_\lambda)|)\nabla(u_\lambda + w_\lambda)) + \mathrm{div}(\phi(|\nabla u_\lambda|)\nabla u_\lambda) \\ &\quad = \lambda(f(x, u_\lambda + w_\lambda) - f(x, u_\lambda)).\end{aligned}$$

Put $v_\lambda = u_\lambda + w_\lambda$. Then, v_λ is a solution of (P_λ) and $0 \neq v_\lambda \leq u_\lambda$, $v_\lambda \neq u_\lambda$ in Ω. By Lemma 3.4, $v_\lambda > 0$ in Ω. This completes the proof of (iii).

Finally, we prove (ii). Take a sequence $\{\lambda_n\}$ such that $\Lambda < \lambda_n < \Lambda + 1$ and $\lambda_n \to \Lambda$ as $n \to \infty$. Let u_{λ_n} be a positive solution of (P_λ) for the corresponding parameter $\lambda = \lambda_n$. By (f_4) and (f_5), for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that the inequality

$$f(x, t)t \leq \varepsilon\Phi(t) + C_\varepsilon$$

holds for any $t \geq 0$. Thus, by Eq. (4.1) with $\lambda = \lambda_n$, (ϕ_2) and Poincaré's inequality,

$$\begin{aligned}\ell \int_\Omega \Phi(|\nabla u_{\lambda_n}|)\mathrm{d}x &\leq \int_\Omega \phi(|\nabla u_{\lambda_n}|)|\nabla u_{\lambda_n}|^2 \mathrm{d}x \\ &= \lambda_n \int_\Omega f(x, u_{\lambda_n}) u_{\lambda_n} \mathrm{d}x \\ &\leq (\Lambda + 1)\left(c_1 \varepsilon \int_\Omega \Phi(|\nabla u_{\lambda_n}|)\mathrm{d}x + C_\varepsilon |\Omega|\right)\end{aligned}$$

is valid with some $c_1 > 0$. Taking $\varepsilon > 0$ sufficiently small, we have

$$\int_\Omega \Phi(|\nabla u_{\lambda_n}|)\mathrm{d}x \leq c_2 \tag{5.9}$$

with some constant $c_2 > 0$. Thus, $\{u_{\lambda_n}\}$ is bounded in $W_0^1 L_\Phi(\Omega)$. Then, by Lemma 4.1, this sequence $\{u_{\lambda_n}\}$ is bounded in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$. By Ascoli-Arzelà's theorem, there exists a subsequence of $\{u_{\lambda_n}\}$, which converges to some $u_\Lambda \geq 0$ in $C^1(\overline{\Omega})$. It is clear that u_Λ solves (P_λ) with $\lambda = \Lambda$.

Here we claim $u_\Lambda \neq 0$. In fact, let us assume $u_\Lambda = 0$. Then, since $u_{\lambda_n} \to 0$ uniformly on Ω as $n \to \infty$, for any $\varepsilon > 0$, the inequality

$$|f(x, u_{\lambda_n}) u_{\lambda_n}| \leq \varepsilon \Phi(|u_{\lambda_n}|)$$

holds on Ω for sufficiently large n. Thus, we have

$$\begin{aligned}\ell \int_\Omega \Phi(|\nabla u_{\lambda_n}|)\mathrm{d}x &\leq \int_\Omega \phi(|\nabla u_{\lambda_n}|)|\nabla u_{\lambda_n}|^2 \mathrm{d}x = \lambda_n \int_\Omega f(x, u_{\lambda_n}) u_{\lambda_n} \mathrm{d}x \\ &\leq \lambda_n \varepsilon \int_\Omega \Phi(|u_{\lambda_n}|)\mathrm{d}x \leq C \lambda_n \varepsilon \int_\Omega \Phi(|\nabla u_{\lambda_n}|)\mathrm{d}x\end{aligned}$$

with some constant $C > 0$ for large n. Since $\int_\Omega \Phi(|\nabla u_{\lambda_n}|)\mathrm{d}x \neq 0$, the inequality

$$\lambda_n > \frac{\ell}{C\varepsilon}$$

holds for large n. Taking $n \to \infty$, we have

$$\Lambda \geq \frac{\ell}{C\varepsilon}$$

for any $\varepsilon > 0$, which is a contradiction. Hence, $u_\Lambda \neq 0$. By Lemma 3.4, we see that $u_\Lambda > 0$. This completes the proof of (ii). □

6 Maximal solution

In this section, we show the existence of the maximal solution of (P_λ) by using the super- and sub-solution argument. For this purpose, we recall the Sobolev conjugate function Φ_* of Φ, which is defined by

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{(N+1)/N}} \mathrm{d}s, \tag{6.1}$$

when

$$\int_0^1 \frac{\Phi^{-1}(s)}{s^{(N+1)/N}} \mathrm{d}s < \infty.$$

In the case when the aforementioned integral is divergent, Φ_* is defined by replacing Φ in (6.1) by another N-function equivalent to Φ near infinity. The function Φ_* mentioned earlier is N-function when

$$\int_1^\infty \frac{\Phi^{-1}(s)}{s^{(N+1)/N}} \mathrm{d}s = \infty. \tag{6.2}$$

For simplicity, we assume (6.2) on the function Φ. In the other case, let us see Remark 6.2.

In order to construct the maximal solution of (P_λ), in addition to (f_1)–(f_4), we assume

(f_7) there exist an N-function Ψ equivalent to Φ_* near infinity and a constant $K > 0$ such that $f(x,t) + K\Psi'(t)$ is nondecreasing in $t > 0$ for each $x \in \Omega$.

Theorem 6.1 *Let (ϕ_1)–(ϕ_3) and (f_1)–(f_4), (f_7) be satisfied. Then, for any $\lambda > \Lambda$, the problem (P_λ) has the maximal solution ω_λ, namely the solution ω_λ of (P_λ) which satisfies $\omega_\lambda(x) \geq u(x)$ on Ω for any positive solution u of (P_λ).*

Note that, if (f_6) holds, then (f_7) is trivially satisfied. Thus, Theorem 6.1 is valid under the assumptions in Theorem 5.1.

To show this theorem, consider the boundary value problem

$$\begin{cases} -\operatorname{div}(\phi(|\nabla u|)\nabla u) + \lambda K\psi(|u|)u = h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.3}$$

for h in $L^\infty(\Omega)$, where $\psi(t)t = \Psi'(t)$, $t > 0$.

Lemma 6.1 *For any $h \in L^\infty(\Omega)$, there exists a solution u of (6.3) in $W_0^1 L_\Phi(\Omega)$.*

Proof Let us define the functional E on $W_0^1 L_\Phi(\Omega)$ by

$$E(u) = \int_\Omega \Phi(|\nabla u|)\mathrm{d}x + \lambda K \int_\Omega \Psi(|u|)\mathrm{d}x - \int_\Omega hu\mathrm{d}x, \quad u \in W_0^1 L_\Phi(\Omega).$$

Then we easily see that E is Frechét differentiable on $W_0^1 L_\Phi(\Omega)$ and a critical point of E solves the problem (6.3). Using Hölder's and Poincaré's inequalities, we have

$$\begin{aligned} E(u) &\geq \int_\Omega \Phi(|\nabla u|)\mathrm{d}x - 2\|h\|_{\widetilde{\Phi}}\|u\|_\Phi \\ &\geq \zeta_0(\|\nabla u\|_\Phi) - c\|h\|_{L^\infty(\Omega)}\|\nabla u\|_\Phi \end{aligned} \tag{6.4}$$

for any $u \in W_0^1 L_\Phi(\Omega)$ with some constant $c > 0$. Since $\zeta_0(t) = t^\ell$ for $t \geq 1$ with $\ell > 1$,

$$\inf_{u \in W_0^1 L_\Phi(\Omega)} E(u) > -\infty.$$

Furthermore, inequality (6.4) implies that the functional E is coercive. Noting that Φ and Ψ are convex, we easily see that E is weakly lower semi-continuous in $W_0^1 L_\Phi(\Omega)$. Hence, E has the global minimum on $W_0^1 L_\Phi(\Omega)$ at some point u. This u solves (6.3). □

By a proof similar to that of inequality (5.9), we can show the uniform boundedness of the positive solutions of (P_λ) in $W_0^1 L_\Phi(\Omega)$. Applying Lemma 4.1 we have the following lemma.

Lemma 6.2 *Let λ be in (Λ, ∞). Then there exists a constant $M = M(\lambda) > 0$ depending only on λ such that $\|u\|_{L^\infty(\Omega)} \leq M$ for any positive solution u of (P_λ).*

Proof of Theorem 6.1 Let λ be fixed in (Λ, ∞). Take the super-solution $v(x) \equiv (\mathcal{F}c)(|x|)$ constructed in Lemma 4.3. As in the proof of Theorem 4.2, $v(x)$ can be chosen so that

$$v(x) \geq M \tag{6.5}$$

holds on Ω for the constant M in Lemma 6.2. Now let us define the sequence of functions $\{v_n\}$ successively by

$$v_0 = v,$$
$$\begin{cases} -\operatorname{div}(\phi(|\nabla v_{n+1}|)\nabla v_{n+1}) + \lambda K \psi(|v_{n+1}|)v_{n+1} = h_n & \text{in } \Omega \\ v_{n+1} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.6}$$

$n \geq 0$, where

$$h_n = \lambda K \psi(|v_n|)v_n + \lambda f(x, v_n).$$

Since v_0 (> 0) is a super-solution of (P_λ), the inequality

$$-\operatorname{div}(\phi(|\nabla v_0|)\nabla v_0) + \lambda K \psi(|v_0|)v_0 \geq h_0 \tag{6.7}$$

holds in Ω. Since $h_0 \in L^\infty(\Omega)$, v_1 is well defined in $W_0^1 L_\Phi(\Omega)$ by Lemma 6.2. Noting that $\psi(t)t$ is nondecreasing and using (6.6) with $n = 0$ and (6.7), we see

$$\begin{aligned} &\int_{\Omega^+} (\phi(|\nabla v_1|)\nabla v_1 - \phi(|\nabla v_0|)\nabla v_0) \cdot \nabla(v_1 - v_0)_+ \mathrm{d}x \\ &\leq \int_\Omega (\phi(|\nabla v_1|)\nabla v_1 - \phi(|\nabla v_0|)\nabla v_0) \cdot \nabla(v_1 - v_0)_+ \mathrm{d}x \\ &\quad + \lambda K \int_\Omega (\psi(|v_1|)v_1 - \psi(|v_0|)v_0)(v_1 - v_0)_+ \mathrm{d}x \\ &\leq 0, \end{aligned}$$

where $(v_1 - v_0)_+(x) = \max\{v_1(x) - v_0(x), 0\}$ and $\Omega^+ = \{x \in \Omega;\ v_1(x) > v_0(x)\}$. From Lemma 3.2, $\nabla(v_1 - v_0)_+ = 0$ on Ω. Since $(v_1 - v_0)_+ = 0$ on $\partial\Omega$, we see

$(v_1 - v_0)_+ = 0$, that is, $v_1 \le v_0$ on Ω. Let u be any positive solution of (P_λ). Then by (6.5), $u \le M \le v_0$ in Ω. By assumption (f_5), the inequality

$$h \equiv \lambda K \psi(|u|)u + \lambda f(x, u) \le h_0$$

holds. Then we have

$$\begin{aligned}
&\int_{\Omega^+} (\phi(|\nabla u|)\nabla u - \phi(|\nabla v_1|)\nabla v_1) \cdot \nabla(u - v_1)_+ \mathrm{d}x \\
&\quad \le \int_{\Omega} (\phi(|\nabla u|)\nabla u - \phi(|\nabla v_1|)\nabla v_1) \cdot \nabla(u - v_1)_+ \mathrm{d}x \\
&\qquad + \lambda K \int_{\Omega} (\psi(|u|)u - \psi(|v_1|)v_1)(u - v_1)_+ \mathrm{d}x \\
&\quad = \int_{\Omega} (h - h_0)(u - v_1)_+ \mathrm{d}x \le 0.
\end{aligned}$$

Hence, $u \le v_1 \le v_0$ on Ω. Especially v_1 belongs to $L^\infty(\Omega)$, and v_2 is well defined. Similarly, we can show iteratively that v_n is well defined and

$$u \le v_{n+1} \le v_n \quad \text{on } \Omega \tag{6.8}$$

for any integer n. Thus, $\|h_n - \lambda K \psi(|v_{n+1}|)v_{n+1}\|_{L^\infty(\Omega)}$ is bounded above by a constant independent of n. Applying Lemma 3.3 for Eq. (6.6), we see that v_n belongs to $C^{1,\alpha}(\overline{\Omega})$ with some $\alpha > 0$ and the sequence $\{v_n\}$ is bounded in $C^{1,\alpha}(\overline{\Omega})$. Then there exists a subsequence of $\{v_n\}$, which converges to some ω_λ in $C^1(\overline{\Omega})$. Taking a limit in (6.6) and (6.8), we easily see that ω_λ solves (P_λ) and $u \le \omega_\lambda$ for any positive solution u of (P_λ). This shows that ω_λ is the maximal solution of (P_λ). □

Remark 6.1 It can be shown in the same way that if (f_5) is assumed in addition to the assumptions in Theorem 6.1, then there also exists the maximal solution for $\lambda = \Lambda$.

Remark 6.2 In the case when the integration in (6.2) is convergent, the conclusion of Theorem 6.1 holds under the assumption

(f_7') $f(x, t) + K\Psi'(t)$ is nondecreasing in $t > 0$ for each $x \in \Omega$ for some N-function Ψ,

instead of (f_7).

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