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Positivity for polyharmonic problems on domains close to a disk

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Abstract We study the problem of positivity preserving of the Green operator for the polyharmonic operator $(-\Delta)^m$ under homogeneous Dirichlet boundary conditions on domains Ω of $\mathbb{R}^2$. Here we will treat only Ω, which are *ε-close* to a disk B in *$C^{m,\gamma}$-sense*, meaning, there exists a $C^{m,\gamma}$-mapping $g : \overline{B} \to \overline{\Omega}$ such that $g(\overline{B}) = \overline{\Omega}$ and $\|g - Id\|_{C^{m,\gamma}(\overline{B})} \leq \varepsilon$. We show that *$\varepsilon$-closeness* in *$C^{m,\gamma}$-sense* is enough in order to ensure positivity preserving. For the clamped plate equation (i.e. $m = 2$), this means that it is a Hölder norm of the curvature of $\partial\Omega$, which governs the positivity behavior. This improves the previous work by Grunau and Sweers, where closeness to the disk in *C^{2m}-sense* was required (in *C^4-sense* for the clamped plate).

Keywords Polyharmonic operator · Positivity · Maximum principle

Mathematics Subject Classification (2000) 35J30, 35B50

1 Introduction

When studying second order elliptic differential equations, maximum and comparison principles are very important tools. So it is an obvious question whether these instruments are available for higher order elliptic differential equations, too. It is known that neither maximum nor comparison principles hold true in general domains. However, it is useful to have as precise knowledge as possible in how far these principles extend to higher order elliptic equations.

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If we have a sufficiently smooth domain Ω, ν exterior normal and non-negative force $f \geq 0$, what can we say about the sign of the solution of the problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = \dfrac{\partial}{\partial \nu} u = 0 & \text{on } \partial\Omega \end{cases} ? \tag{1}$$

In this paper, we will restrict to two-dimensional domains. This means that problem (1) models the clamped plate equation (an elastic horizontally clamped plate subject to vertical force f).

In the first years of the 20th century, Boggio [2] and Hadamard [10] conjectured that in sufficiently regular domains Ω, $f \geq 0$ should give $u \geq 0$ in problem (1). Since 1905 [3], we know that $u \geq 0$ in the case of Ω equal to a ball. But in 1909, Hadamard [11] showed that in an annulus with small inner radius we lose the nonnegativity of u.

Later it has been proved (see [4, 5, 12]) that even in convex domains one can obtain sign-changing solutions for suitable $f \geq 0$. In the work, Grunau and Sweers [7] found that for domains sufficiently close to a disk in a suitable sense, the operator $(-\Delta)^m$ satisfies the positivity preserving property.

In this paper, we will relax the required notion of closeness: for the clamped plate Grunau and Sweers [7] worked with domains C^4-*close* to a disk, while here we will work with domains with curvature near to a constant in $C^{0,\alpha}$. The curvature itself is a quantity by far more natural to govern the positivity preserving property in the clamped plate equation than its second derivatives. Moreover, we will prove related results for polyharmonic operators.

2 Perturbation of the domain

Like in [6, in Sect. 6.2], we will use the next definition:

Definition 1 A bounded domain $\Omega \subset \mathbb{R}^n$ and its boundary are of class $C^{m,\gamma}$, $0 \leq \gamma \leq 1$, if at each point $x_0 \in \partial\Omega$ there is a ball $B = B(x_0)$ and a one-to-one mapping ψ of B onto $D \subset \mathbb{R}^n$ such that:

(i) $\psi(B \cap \Omega) \subset \mathbb{R}^n_+$;
(ii) $\psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$;
(iii) $\psi \in C^{m,\gamma}(B)$, $\psi^{-1} \in C^{m,\gamma}(D)$.

Definition 2 Let $\varepsilon \geq 0$. We call Ω ε-*close* in $C^{m,\gamma}$-*sense* to $\Omega^\star$, if there exists a $C^{m,\gamma}$ mapping $g : \overline{\Omega^\star} \to \overline{\Omega}$ such that $g(\overline{\Omega^\star}) = \overline{\Omega}$ and

$$\|g - Id\|_{C^{m,\gamma}(\overline{\Omega^\star})} \leq \varepsilon.$$

We can now formulate our main result.

Theorem 1 *Let* $m \geq 2$, $0 < \gamma < 1$. *Then there is some* $\varepsilon_0 = \varepsilon_0(m, \gamma) > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$, *the following holds:*

If the domain $\Omega \subset \mathbb{R}^2$ is $C^{m,\gamma}$-smooth and ε-close to the disk in $C^{m,\gamma}$-sense and $f \in C^{0,\gamma}(\overline{\Omega})$, $0 \not\equiv f \geq 0$, then the uniquely determined solution $u \in C^{m,\gamma}(\overline{\Omega})$ of

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ \dfrac{\partial^j}{\partial \nu^j} u = 0 & \text{on } \partial\Omega, \quad 0 \leq j \leq m-1 \end{cases} \tag{2}$$

is strictly positive.

For existence and uniqueness we refer to [1, in Sect. 8], cf. also [6]. In our paper, we focus on positivity. Then assume for a while that $C^{k,\gamma}$-*closeness* of Ω to B already implies that there is a biholomorphic mapping $h : \overline{B} \to \overline{\Omega}$ with $\|h - Id\|_{C^{k,\alpha}(\overline{B})} \leq \mathcal{O}(\varepsilon)$ (see Theorem 2). Let us see how the polyharmonic equation transforms when being pulled back to the disk by means of h.

Let $h : B \to \Omega$, $(\xi, \eta) \mapsto (x, y)$ be the holomorphic function, which maps the disk B onto Ω. Note that $h \in C^{m,\gamma}$. We compose both parts of the first equation of (2) with h:

$$((-\Delta)^m u) \circ h = f \circ h \quad \text{on} \quad B, \tag{3}$$

where Δ is the Laplacian with respect to (x, y). Let us denote Δ^* the Laplacian with respect to (ξ, η), $v := (u \circ h)$ and h' the complex derivative of h.

$$\Delta^*(u \circ h) = \Delta^* v = [(\Delta u) \circ h] \cdot |h'|^2; \tag{4}$$

$$f \circ h = \left(-\frac{1}{|h'|^2} \Delta^* \right)^m (u \circ h) = \left(-\frac{1}{|h'|^2} \Delta^* \right)^m v. \tag{5}$$

To sum up: domain transformations change into lower-order perturbations by means of the holomorphic mapping h. In this situation, a result by Grunau and Sweers [9] applies. In order to understand in which norm we need $h - Id$ small enough for this purpose, we look at the precise form of:

$$\left(-\frac{1}{|h'|^2} \Delta \right)^m.$$

Lemma 1 *Let h be a holomorphic function and let v be a function in $C^{2m}(B)$, then the operator*

$$\left(-\frac{1}{|h'|^2} \Delta \right)^m v \tag{6}$$

contains no derivatives of h of order larger than m. Here h' denotes the complex derivative of h.

Proof We identify $\mathbb{C} \simeq \mathbb{R}^2$ and denote $z = x + iy$ for $(x, y) \in \mathbb{R}^2$. We shall exploit

$$\Delta(\bullet) = 4\partial_z \partial_{\overline{z}}(\bullet), \tag{7}$$

where $\partial_z = \frac{1}{2}(\partial_x + i\partial_y)$.

We proceed by induction: obviously, the claim holds for $m = 1$. In order to show the underlying idea how to exploit h being holomorphic, we first treat the case $m = 2$, however:

$$\begin{aligned}\left(-\frac{1}{|h'|^2}\Delta\right)^2 v &= -4\frac{1}{|h'|^2}\partial_z\partial_{\overline{z}}\left[\left(-\frac{1}{|h'|^2}\Delta\right)v\right]\\ &= \frac{4}{|h'|^2}\left[\left(\partial_z\partial_{\overline{z}}\Delta v\right)\frac{1}{|h'|^2}+\left(\partial_z\Delta v\right)\left(\partial_{\overline{z}}\frac{1}{\overline{h'}}\right)\frac{1}{h'}\right.\\ &\quad\left.+\left(\partial_{\overline{z}}\Delta v\right)\left(\partial_z\frac{1}{h'}\right)\frac{1}{\overline{h'}}+\Delta v\left(\partial_z\frac{1}{h'}\right)\left(\partial_{\overline{z}}\frac{1}{\overline{h'}}\right)\right]\\ &= \frac{1}{|h'|^2}\left[\frac{1}{|h'|^2}\Delta^2 v-4\frac{\overline{h''}}{h'\overline{h'}^2}\partial_z\Delta v-4\frac{h''}{h'^2\overline{h'}}\partial_{\overline{z}}\Delta v+4\frac{|h''|^2}{|h'|^4}\Delta v\right]. \end{aligned} \tag{8}$$

We observe that the lemma is true for $m = 2$. Now, we suppose that our hypothesis is true for some $m \geq 2$.

$$\begin{aligned}\left(-\frac{1}{|h'|^2}\Delta\right)^m v = (-1)^m \sum_{\beta_1,\beta_2=1}^{m} \overset{(m)}{F}_{\beta_1\beta_2}\Big(&h^{(m+1-\beta_1)},\ldots,h',\overline{h^{(m+1-\beta_2)}},\\ &\ldots,\overline{h'}\Big)\partial_z^{\beta_1}\partial_{\overline{z}}^{\beta_2}v,\end{aligned} \tag{9}$$

where $\overset{(m)}{F}_{k,j}=\overset{(m)}{F}_{k,j}\ (x_1,\ldots,x_{m+1-k},y_1,\ldots,y_{m+1-j})$ is a smooth function of both h and $\overline{h}$ derivatives except for $h' = 0$ and with $k, j \leq m$. Then we show that (9) is also true replacing m by $m + 1$:

$$\begin{aligned}\left(-\frac{1}{|h'|^2}\Delta\right)^{m+1} v &= (-1)^{m+1}\left(\frac{1}{|h'|^2}\Delta\right)\left[\left(\frac{1}{|h'|^2}\Delta\right)^m v\right]\\ &= 4\frac{(-1)^{m+1}}{|h'|^2}\partial_z\partial_{\overline{z}}\left[\sum_{\beta_1,\beta_2=1}^{m}\overset{(m)}{F}_{\beta_1\beta_2}\Big(h^{(m-\beta_1+1)},\ldots,h',\overline{h^{(m-\beta_2+1)}},\right.\\ &\quad\left.\ldots,\overline{h'}\Big)\partial_z^{\beta_1}\partial_{\overline{z}}^{\beta_2}v\right] = 4\frac{(-1)^{m+1}}{|h'|^2}\partial_z\left\{\sum_{\beta_1,\beta_2=1}^{m}\left[\frac{\partial\overset{(m)}{F}_{\beta_1\beta_2}}{\partial y_1}\overline{h^{(m-\beta_2+2)}}\right.\right.\\ &\quad\left.\left.+\cdots+\frac{\partial\overset{(m)}{F}_{\beta_1\beta_2}}{\partial y_{m+1-\beta_2}}\overline{h''}\right]\partial_z^{\beta_1}\partial_{\overline{z}}^{\beta_2}v+\left[\sum_{\beta_1,\beta_2=1}^{m}\overset{(m)}{F}_{\beta_1\beta_2}\ \partial_z^{\beta_1}\partial_{\overline{z}}^{\beta_2}\partial_{\overline{z}}v\right]\right\}\\ &= (-1)^{m+1}\frac{4}{|h'|^2}\partial_z\left\{\sum_{\beta_1,\beta_2=1}^{m}\left[\sum_{k=1}^{m+1-\beta_2}\frac{\partial\overset{(m)}{F}_{\beta_1\beta_2}}{\partial y_k}\overline{h^{(m+3-\beta_2-k)}}\right]\partial_z^{\beta_1}\partial_{\overline{z}}^{\beta_2}v\right.\end{aligned}$$

$$+\left[\sum_{\beta_1,\beta_2=1}^{m} \overset{(m)}{F}_{\beta_1\beta_2}\left(h^{(m+1-\beta_1)},\ldots,h',\overline{h^{(m+1-\beta_2)}},\ldots,\overline{h'}\right)\partial_z^{\beta_1}\partial_{\overline{z}}^{\beta_2}\partial_{\overline{z}}v\right]\Bigg]\Bigg\}$$

$$=(-1)^{m+1}\frac{4}{|h'|^2}\left\{\sum_{\beta_1,\beta_2=1}^{m}[A+B+C+D]\right\},$$

where

$$A=\left(\sum_{j=1}^{m+1-\beta_1}\sum_{k=1}^{m+1-\beta_2}\frac{\partial^2 \overset{(m)}{F}_{\beta_1\beta_2}}{\partial x_j\partial y_k}h^{(m+3-\beta_1-j)}\overline{h^{(m+3-\beta_2-k)}}\right)\partial_z^{\beta_1}\partial_{\overline{z}}^{\beta_2}v,$$

$$B=\left(\sum_{k=1}^{m+1-\beta_2}\frac{\partial \overset{(m)}{F}_{\beta_1\beta_2}}{\partial y_k}\overline{h^{(m+3-\beta_2-k)}}\right)\partial_z^{\beta_1+1}\partial_{\overline{z}}^{\beta_2}v,$$

$$C=\left(\sum_{j=1}^{m+1-\beta_1}\frac{\partial \overset{(m)}{F}_{\beta_1\beta_2}}{\partial x_j}h^{(m+3-\beta_1-j)}\right)\partial_z^{\beta_1}\partial_{\overline{z}}^{\beta_2+1}v,$$

$$D=\left(\overset{(m)}{F}_{\beta_1\beta_2}\right)\partial_z^{\beta_1+1}\partial_{\overline{z}}^{\beta_2+1}v.$$

Every $\overset{(m)}{F}_{\beta_1\beta_2}$ and its derivatives contain derivatives of h (respectively $\overline{h}$) of order at most $m-\beta_1+1$ (resp. $m-\beta_2+1$). The derivatives of $\overset{(m)}{F}_{\beta_1\beta_2}$ are multiplied by $\overline{h^{(m+3-\beta_2-k)}}$ and/or $h^{(m+3-\beta_1-j)}$, so the highest derivatives of h and $\overline{h}$ have order $m+1$. □

So far, by referring to Grunau and Sweers [9, Theorem 5.1], we have shown that in order to prove Theorem 1 it is enough to show $\|h-Id\|_{C^{m,\gamma}(\overline{B})}=\mathcal{O}(\varepsilon)$ for the biholomorphic map h.

Theorem 2 *Let δ_0 be given. Then there is some $\varepsilon_0=\varepsilon_0(\delta_0,m)>0$ such that for $\varepsilon\in[0,\varepsilon_0)$ we have the following:*

If the $C^{m,\gamma}$ domain Ω is ε-close in $C^{m,\gamma}$-sense to B, then there is a biholomorphic mapping $h:\overline{B}\to\overline{\Omega}$, $h\in C^{m,\gamma}(\overline{B})$, $h^{-1}\in C^{m,\gamma}(\overline{\Omega})$ with $||h-Id||_{C^{m,\gamma}(\overline{B})}\le\delta_0$.

Comparing this result with the analogous one by Grunau and Sweers [7], we gain one order of derivative in the estimate for $h-Id$. In order to prove this theorem, with help of Schauder theory, we show the following lemma.

Lemma 2 *Let Ω be a domain ε-close to a disk B in $C^{m,\gamma}$-sense. Let g be a map satisfying $g:\overline{B}\to\overline{\Omega}$, with $||g-Id||_{C^{m,\gamma}(\overline{B})}<\varepsilon$, and let $\varphi(x)=\log|x|$ on the boundary of Ω.*

Then there exists a function $\hat{\varphi}\in C^{m,\gamma}(\overline{B})$ such that $\hat{\varphi}=\varphi\circ g$ on ∂B and $||\hat{\varphi}||_{C^{m,\gamma}(\overline{B})}\le\mathcal{O}(\varepsilon)$.

Let $\psi(t) = \varphi(g(\cos t, \sin t))$,

$$\varphi_1(x) = \log\left(\left|g\left(\frac{x}{|x|}\right)\right|\right)$$

and

$$\varphi_2 \in C_0^\infty(\mathbb{R}^2), \quad \varphi_2(x,y) = \begin{cases} 1 & \frac{1}{2} \le |x| \le 2, \\ 0 & |x| < \frac{1}{4}, \quad |x| > 4, \\ 0 < \varphi_2 < 1 & \text{otherwise.} \end{cases}$$

We define the function $\hat{\varphi}(x) := \varphi_2(x) \cdot \varphi_1(x)$. It satisfies $\hat{\varphi} = \varphi \circ g$ on ∂B. Obviously, $\hat{\varphi}$ is in $C^{m,\gamma}$.

$\hat{\varphi} = 0$ in $B_{\frac{1}{4}}$ and in $\overline{B} \setminus B_{\frac{1}{4}}$

$$\|\hat{\varphi}\|_{C^{m,\gamma}\left(\overline{B}\setminus B_{\frac{1}{4}}\right)} \le C_1 \left(\sum_{j=0}^{m} \left(\sum_{k=0}^{j} \left[\frac{\partial^j \hat{\varphi}}{\partial r^k \partial \theta^{j-k}} \right]_{0;\,\Omega} \right) + \left[\frac{\partial^m \hat{\varphi}}{\partial r^j \partial \theta^{m-j}} \right]_{\gamma;\,\Omega} \right),$$

where (r,θ) are the polar coordinates and C_1 is a constant. Because of $\hat{\varphi} = \varphi_2 \cdot \varphi_1$, the regularity of φ_2 and all the derivatives of φ_1 with respect to r being zero, we obtain

$$\|\hat{\varphi}\|_{C^{m,\gamma}\left(\overline{B}\setminus B_{\frac{1}{4}}\right)} \le C_2 \left(\sum_{j=0}^{m} \left\| \frac{\partial^j \varphi_1}{\partial \theta^j} \right\|_{C^0(\Omega)} + \left\| \frac{\partial^m \varphi_1}{\partial \theta^m} \right\|_{C^{0,\gamma}(\Omega)} \right);$$

$$\begin{aligned} \varphi_1(x) &= \frac{1}{2} \log\left(g_1^2\left(\frac{x}{|x|}\right) + g_2^2\left(\frac{x}{|x|}\right) \right) \\ &= \frac{1}{2} \log(g_1^2(\cos(\theta), \sin(\theta)) + g_2^2(\cos(\theta), \sin(\theta))) = \psi(\theta), \end{aligned}$$

where $x = (\cos(\theta), \sin(\theta))$. Let $\tilde{g} = g(\cos(\theta), \sin(\theta))$; then

$$\begin{aligned} \left(\frac{d}{d\theta}\right)^j \psi &= \left(\frac{d}{d\theta}\right)^j (\varphi \circ \tilde{g}) \\ &= \sum_{|\alpha|=1}^{j} ((D^\alpha \varphi) \circ \tilde{g}) \left(\sum_{\substack{p_1+\cdots+p_{|\alpha|}=j \\ 1 \le p_l}} d_{j,\alpha,\vec{p}} \prod_{l=1}^{|\alpha|} \left(\frac{d}{d\theta}\right)^{p_l} \tilde{g}^{(\beta_l)} \right), \end{aligned}$$

with some suitable coefficients $d_{j,\alpha,\vec{p}}$, $\beta_l = 1$ for $l = 1, \ldots, \alpha_1$ and $\beta_l = 2$ for $l = \alpha_1 + 1, \ldots, |\alpha|$.

Let $\tilde{g}_0(\theta) := Id \circ (\cos(\theta), \sin(\theta))$ and $f(\theta) := (\cos(\theta), \sin(\theta))$. We observe first that

$$\begin{aligned} &\|\tilde{g} - \tilde{g}_0\|_{C^{1,0}([0,2\pi])} \\ &\quad = \sup_{\theta \in [0,2\pi]} |\tilde{g}(\theta) - \tilde{g}_0(\theta)| + \sup_{\theta \in [0,2\pi]} |\tilde{g}'(\theta) - \tilde{g}_0'(\theta)| \end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{\theta\in[0,2\pi]} |g(f(\theta)) - Id(f(\theta))| \\
&\quad + \sup_{i=1,2} \sup_{\theta\in[0,2\pi]} |\partial_{x_i}(g(f(\theta)))\, f_i(\theta) - \partial_{x_i}(Id(f(\theta)))\, f_i(\theta))| \\
&\leq \sup_{x\in\overline{B}} |g(x) - Id(x)| + \sup_{i=1,2} \sup_{\substack{x\in\overline{B}\\ \theta\in[0,2\pi]}} |\partial_{x_i}((g(x) - Id(x)))\, f_i(\theta)| \\
&\leq \sup_{x\in\overline{B}} |g(x) - Id(x)| + \sup_{i=1,2} \sup_{x\in\overline{B}} |\partial_{x_i}(g(x) - Id(x))| \\
&= \|g - Id\|_{C^{1,0}(\overline{B})} \leq \mathcal{O}(\varepsilon).
\end{aligned}
$$

And further:

$$
\begin{aligned}
\left(\frac{d}{d\theta}\right)^j \psi = \sum_{|\alpha|=1}^{j} (((D^\alpha\varphi)\circ\tilde{g} - (D^\alpha\varphi)\circ\tilde{g}_0) + (D^\alpha\varphi)\circ\tilde{g}_0) \sum_{\substack{p_1+\cdots+p_{|\alpha|}=j\\ 1\leq p_l}} \\
\times d_{j,\alpha,\mathbf{p}} \prod_{l=1}^{|\alpha|} \left(\left(\left(\frac{d}{d\theta}\right)^{p_l} \tilde{g}^{(\beta_l)} - \left(\frac{d}{d\theta}\right)^{p_l} \tilde{g}_0^{(\beta_l)}\right) + \left(\frac{d}{d\theta}\right)^{p_l} \tilde{g}_0^{(\beta_l)}\right).
\end{aligned}
$$

But when we calculate $\varphi(\tilde{g}_0(\theta)) = \log|(\cos(\theta), \sin(\theta))| = 0$, all terms containing only $\tilde{g}_0$ vanish. It remains to study

$$
\left(\frac{d}{d\theta}\right)^{p_l} \tilde{g}^{(\beta_l)} - \left(\frac{d}{d\theta}\right)^{p_l} \tilde{g}_0^{(\beta_l)} \tag{10}
$$

and

$$
(D^\alpha\varphi)\circ\tilde{g} - (D^\alpha\varphi)\circ\tilde{g}_0. \tag{11}
$$

We have to estimate (11) in the norm $\|\cdot\|_{C^{0,\gamma}([0,2\pi])}$, but it is much easier to estimate the norm $\|\cdot\|_{C^{1,0}([0,2\pi])}$. We can do that because the function φ is sufficiently regular. So we have to study

$$
\begin{aligned}
&\|(D^\alpha\varphi)\circ\tilde{g} - (D^\alpha\varphi)\circ\tilde{g}_0\|_{C^{1,0}([0,2\pi])} \\
&\quad = \sup_{\theta\in[0,2\pi]} |[(D^\alpha\varphi)\circ(\tilde{g}(\theta)) - (D^\alpha\varphi)\circ(\tilde{g}_0(\theta))]| \\
&\quad\quad + \sup_{\theta\in[0,2\pi]} |[(D^\alpha\varphi)\circ(\tilde{g}(\theta)) - (D^\alpha\varphi)\circ(\tilde{g}_0(\theta))]'|.
\end{aligned} \tag{12}
$$

We remember that $\varphi(x) = \log(|x|)$, so

$$
\sup_{|\tilde{\alpha}|=1} \max_{\substack{x\in B_{1+\varepsilon}(0)\\ x\notin B_{1-\varepsilon}(0)}} |D^{\alpha+\tilde{\alpha}}\varphi(x)| < C_1, \quad \sup_{|\overline{\alpha}|=2} \max_{\substack{x\in B_{1+\varepsilon}(0)\\ x\notin B_{1-\varepsilon}(0)}} |D^{\alpha+\overline{\alpha}}\varphi(x)| < C_2,
$$

$$
\begin{aligned}
&\sup_{\theta\in[0,2\pi]} |[(D^\alpha\varphi)\circ(\tilde{g}(\theta)) - (D^\alpha\varphi)\circ(\tilde{g}_0(\theta))]| \\
&\quad \leq \sup_{|\tilde{\alpha}|=1} \max_{\substack{x\in B_{1+\varepsilon}(0)\\ x\notin B_{1-\varepsilon}(0)}} |D^{\alpha+\tilde{\alpha}}\varphi| \sup_{\theta\in[0,2\pi]} |\tilde{g}(\theta) - \tilde{g}_0(\theta)| \\
&\quad \leq C_1 \sup_{\theta\in[0,2\pi]} |\tilde{g}(\theta) - \tilde{g}_0(\theta)| \leq \mathcal{O}(\varepsilon);
\end{aligned} \tag{13}
$$

$$\sup_{\theta\in[0,2\pi]} |[(D^\alpha\varphi)\circ(\tilde g(\theta)) - (D^\alpha\varphi)\circ(\tilde g_0(\theta))]'|$$
$$= \sup_{\theta\in[0,2\pi]} \left|\sum_{i=1}^{2}((\partial_{x_i}D^\alpha\varphi)\circ\tilde g(\theta)\,\tilde g_i'(\theta) - (\partial_{x_i}D^\alpha\varphi)\circ\tilde g_0(\theta)\,\tilde g_{0i}'(\theta))\right|. \quad (14)$$

We subtract and add $\sum_{i=1}^{2}[((\partial_{x_i}D^\alpha\varphi)\circ\tilde g(\theta))\,\tilde g_{0i}'(\theta)]$ to (14) and recall that $|\tilde g_0'(\theta)| = |(-\sin(\theta), \cos(\theta))| = 1$:

$$\begin{aligned}(14) = &\sup_{\theta\in[0,2\pi]} \Bigg|\sum_{i=1}^{2}\{(\partial_{x_i}D^\alpha\varphi)\circ\tilde g(\theta)\,[\tilde g(\theta) - \tilde g_0(\theta)]_i' \\ &- [(\partial_{x_i}D^\alpha\varphi)\circ\tilde g_0(\theta) - (\partial_{x_i}D^\alpha\varphi)\circ\tilde g(\theta)]\,\tilde g_{0i}'(\theta)\}\Bigg| \\ &\le \sup_{|\tilde\alpha|=1}\max_{\substack{x\in B_{1+\varepsilon}(0)\\ x\notin B_{1-\varepsilon}(0)}} |D^{\alpha+\tilde\alpha}\varphi| \sup_{\theta\in[0,2\pi]} |[\tilde g(\theta) - \tilde g_0(\theta)]'| \\ &\quad + \sup_{|\overline\alpha|=2}\max_{\substack{x\in B_{1+\varepsilon}(0)\\ x\notin B_{1-\varepsilon}(0)}} |D^{\alpha+\overline\alpha}\varphi| \sup_{\theta\in[0,2\pi]} |[\tilde g(\theta) - \tilde g_0(\theta)]| \\ &\le C_3 \sup_{\theta\in[0,2\pi]} |[\tilde g(\theta) - \tilde g_0(\theta)]'| + C_4 \sup_{\theta\in[0,2\pi]} |[\tilde g(\theta) - \tilde g_0(\theta)]| \\ &\le C_3\mathcal{O}(\varepsilon) + C_4\mathcal{O}(\varepsilon) \le \mathcal{O}(\varepsilon). \end{aligned} \quad (15)$$

So, combining (13) and (15), we have

$$\|(D^\alpha\varphi)\circ\tilde g - (D^\alpha\varphi)\circ\tilde g_0\|_{C^{1,0}([0,2\pi])} \le \mathcal{O}(\varepsilon).$$

Now we evaluate (10) with respect to the norm $\|\cdot\|_{C^{0,\gamma}([0,2\pi])}$:

$$\begin{aligned}&\left\|\left(\frac{d}{d\theta}\right)^{p_l}\left(\tilde g^{(\beta_l)} - \tilde g_0^{(\beta_l)}\right)\right\|_{C^{0,\gamma}([0,2\pi])} \\ &\quad\le \max_{\substack{x\in\partial B\\ |\alpha|=m}} |D^\alpha\,(g(x) - Id(x))\,| \\ &\qquad + \max_{\substack{x,x'\in\partial B\\ |\alpha|=m}} \frac{|D^\alpha(g(x) - Id(x)) - D^\alpha(g(x') - Id(x'))|}{|x - x'|^\gamma}. \end{aligned} \quad (16)$$

But $\max_{\substack{x\in\partial B\\ |\alpha|=m}} \bullet \le \max_{\substack{x\in\overline B\\ |\alpha|=m}} \bullet$ and $\max_{\substack{x,x'\in\partial B\\ |\alpha|=m}} \bullet \le \max_{\substack{x,x'\in\overline B\\ |\alpha|=m}} \bullet$, too, so

$$(16) \le \|g - Id\|_{C^{m,\gamma}(\overline B)} \le \mathcal{O}(\varepsilon). \quad (17)$$

□

Now we proceed with the proof of Theorem 2. With help of [6, Corollary 6.7, paragraph 6.4] we can find a function $r(x)$ such that

$$\begin{cases}\Delta r = 0 & x\in B,\\ r(x) = \hat\varphi(x) & x\in\partial B\end{cases}$$

and such that $\|r(x)\|_{C^{m,\gamma}(\overline{B})} \leq \mathcal{O}(\varepsilon)$. So we can build the biholomorphic function h as in [7]: Set

$$\omega(x) := 2\pi G(x,0), \quad G(x,0) := -\frac{1}{2\pi}(\log|x| - r(x)).$$

Here $G(x,0)$ is the Green function for $-\Delta$ in B under homogeneous Dirichlet condition. Then let

$$\omega^*(x) := \int_{\frac{1}{2}}^{x} \left(-\frac{\partial}{\partial \xi_2}\omega(\xi)\,\mathrm{d}\xi_1 + \frac{\partial}{\partial \xi_1}\omega(\xi)\,\mathrm{d}\xi_2 \right),$$

where the integral is understood on any curve in $\Omega \setminus \{0\}$ connecting $\frac{1}{2}$ to x, well defined up to multiples of 2π. Let

$$h^{-1}(x) := \mathrm{e}^{-\omega(x)-\mathrm{i}\omega^*(x)}$$

be the function that satisfies Theorem 2: $h^{-1}(0) = 0$, $h^{-1}(\frac{1}{2}) \in \mathbb{R}_+$ and if $x \in \partial\Omega$, then $|h^{-1}(x)| = |e^{-i\omega^*(x)}| = 1$, it means that $h^{-1}(\partial\Omega) = \partial B$, for $x \in \Omega \setminus \{0\}$, $\omega(x) > 0$, $|h^{-1}(x)| < 1$ and then $h^{-1}(\Omega) \subset B$. □

Only Theorem 1 remains to be proved: With help of Theorem 2 we find a biholomorphic function $h : \overline{B} \to \overline{\Omega}$ with $\|h - Id\|_{C^{m,\gamma}(B)} \leq \mathcal{O}(\varepsilon)$. To study the solution u of (2) on $\overline{B}$, means to solve Eq. 5 with Dirichlet boundary conditions in B. According to Lemma 1, the operator in (5), multiplied with $|h'|^{2m}$ has principal part $(-\Delta)^m$; the lower-order coefficients depend on derivatives of h of order at least 2 and at most m. By Lemma 2 these are small in $C^{0,\alpha}$-*sense* and Theorem 5.1 of [9] yields positivity of v and hence of u in Ω, provided Ω close enough in $C^{m,\gamma}$-*sense* to the disk.

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Appendix A

Usually, it is not obvious how to extend $C^{m,\gamma}$ curves $\alpha : \partial B \to \partial\Omega$ to $C^{m,\gamma}$ maps $\overline{B} \to \overline{\Omega}$, since in this procedure the normal is involved. But when the curve is close to ∂B, then the situation is much easier. For the reader's convenience we prove the following lemma.

Lemma 3 *Let $\partial\Omega$ be given by the curve: $\alpha : \mathbb{R} \to \mathbb{R}^2$, α 2π-periodic, $\alpha \in C^{m,\gamma}$, with $\|\alpha(t) - (\cos(t), \sin(t))\|_{C^{m,\gamma}([0,2\pi])} \leq \varepsilon$. Then Ω is $\tilde{\varepsilon}$-close to $B_1(0)$ in $C^{m,\gamma}$-sense, where $\tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon, m) = \mathcal{O}(|\varepsilon|)$.*

Proof In order to find a function $\tilde{g} : B_1(0) \to \Omega$, such that $\|Id - \tilde{g}\|_{C^{m,\gamma}(B_1(0))} \leq \mathcal{O}(\varepsilon)$, we set

$$g : (x_1, x_2) = (\rho\cos(\varphi), \rho\sin(\varphi)) \mapsto (\rho\alpha_1(\varphi), \rho\alpha_2(\varphi)).$$

Its differential is

$$\begin{aligned} Dg &= \begin{pmatrix} \alpha_1(\varphi) & \rho\alpha_1'(\varphi) \\ \alpha_2(\varphi) & \rho\alpha_2'(\varphi) \end{pmatrix} \frac{1}{\rho} \begin{pmatrix} \rho\cos(\varphi) & \rho\sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \\ &= \frac{1}{\rho} \begin{pmatrix} \alpha_1(\varphi)\rho\cos(\varphi) - \rho\alpha_1'(\varphi)\sin(\varphi) & \alpha_1(\varphi)\rho\sin(\varphi) + \rho\alpha_1'(\varphi)\cos(\varphi) \\ \alpha_2(\varphi)\rho\cos(\varphi) - \rho\alpha_2'(\varphi)\sin(\varphi) & \alpha_2(\varphi)\rho\sin(\varphi) + \rho\alpha_2'(\varphi)\cos(\varphi) \end{pmatrix} \\ &= \begin{pmatrix} \alpha_1(\varphi)\cos(\varphi) - \alpha_1'(\varphi)\sin(\varphi) & \alpha_1(\varphi)\sin(\varphi) + \alpha_1'(\varphi)\cos(\varphi) \\ \alpha_2(\varphi)\cos(\varphi) - \alpha_2'(\varphi)\sin(\varphi) & \alpha_2(\varphi)\sin(\varphi) + \alpha_2'(\varphi)\cos(\varphi) \end{pmatrix}. \end{aligned} \tag{18}$$

So we obtain $\frac{\partial g_1}{\partial x_1} = \alpha_1(\varphi)\cos(\varphi) - \alpha_1'(\varphi)\sin(\varphi) = (\cos(\varphi) + \mathcal{O}(\varepsilon))\cos(\varphi) + (\sin(\varphi) + \mathcal{O}(\varepsilon))\sin(\varphi) = 1 + \mathcal{O}(\varepsilon)$. This means

$$(18) = \begin{pmatrix} 1+\mathcal{O}(\varepsilon) & \mathcal{O}(\varepsilon) \\ \mathcal{O}(\varepsilon) & 1+\mathcal{O}(\varepsilon) \end{pmatrix}.$$

When we calculate the second derivatives, we obtain

$$\begin{aligned} D\frac{\partial g}{\partial x_1} &= \begin{pmatrix} 0 & \alpha_1'(\varphi)\cos(\varphi) - \alpha_1(\varphi)\sin(\varphi) - \alpha_1''(\varphi)\sin(\varphi) - \alpha_1'(\varphi)\cos(\varphi) \\ 0 & \alpha_2'(\varphi)\cos(\varphi) - \alpha_2(\varphi)\sin(\varphi) - \alpha_2''(\varphi)\sin(\varphi) - \alpha_2'(\varphi)\cos(\varphi) \end{pmatrix} \\ &\quad \times \frac{1}{\rho}\begin{pmatrix} \rho\cos(\varphi) & \rho\sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \\ &= \frac{1}{\rho}\begin{pmatrix} \sin^2(\varphi)(\alpha_1(\varphi)+\alpha_1''(\varphi)) & -\sin(\varphi)\cos(\varphi)(\alpha_1(\varphi)+\alpha_1''(\varphi)) \\ \sin^2(\varphi)(\alpha_2(\varphi)+\alpha_2''(\varphi)) & -\sin(\varphi)\cos(\varphi)(\alpha_2(\varphi)+\alpha_2''(\varphi)) \end{pmatrix}. \end{aligned}$$

So there is a discontinuity for the second derivatives in the point (0, 0). To solve this problem of regularity in a neighborhood of the origin, we define a cut-off function $\psi \in C^\infty$ with

$$\begin{cases} \psi(x) = 0 & |x| < \frac{1}{4}, \\ 0 \le \psi(x) \le 1 & \frac{1}{4} \le |x| \le \frac{1}{2}, \\ \psi(x) = 1 & |x| > \frac{1}{2}. \end{cases}$$

We introduce the function

$$\tilde{g}(x) := \begin{cases} g(x) & |x| > \frac{1}{2}, \\ g(x)\,\psi(x) + x\,(1-\psi(x)) & \frac{1}{4} \le |x| \le \frac{1}{2}, \\ x & |x| < \frac{1}{4}. \end{cases}$$

We shall now prove that $\tilde{g}$ has the desired properties.

On $\Omega \setminus B_{\frac{1}{2}}(0)$

$$\frac{\partial^{|j|}(\tilde{g} - Id)_i}{\partial x_1^{j_1}\partial x_2^{j_2}} = \frac{\partial^{|j|}\left\{\rho\left[\alpha - \frac{Id}{\rho}\right]\right\}_i}{\partial x_1^{j_1}\partial x_2^{j_2}} = \frac{1}{\rho^{|j|-1}}\,\mu_{\mathbf{j}}(\varphi), \tag{19}$$

where

$$\begin{aligned} \mu_{\mathbf{j}}(\varphi) &= \sum_{h=0}^{|j|} \nu_{h,i,\mathbf{j}}(\varphi)\,\sigma_{h,i,\mathbf{j}}(\varphi), \\ \nu_{h,i,\mathbf{j}}(\varphi) &= \left(\frac{\partial}{\partial\varphi}\right)^h (\alpha_1(\varphi)-\cos(\varphi), \alpha_2(\varphi)-\sin(\varphi))_i\,, \\ \sigma_{h,i,\mathbf{j}}(\varphi) &= \sum_{\substack{k_1+k_2=|j| \\ k_1,k_2\ge 0}} c_{h,i,\mathbf{j},\mathbf{k}}(\cos(\varphi))^{k_1}(\sin(\varphi))^{k_2}, \end{aligned}$$

with some suitable coefficients $c_{h,i,\mathbf{j},\mathbf{k}}$. For a proof see Appendix B. $\|\alpha(\varphi) - (\cos(\varphi), \sin(\varphi))\|_{C^{m,\gamma}([0,2\pi])} \le \mathcal{O}(\varepsilon)$, so for every $\mathbf{j}$, such that $|j| \le m$ we have

$$\left(\frac{\partial}{\partial\varphi}\right)^h (\alpha_1(\varphi)-\cos(\varphi), \alpha_2(\varphi)-\sin(\varphi)) \le \mathcal{O}(\varepsilon) \text{ and then } (19) \le \mathcal{O}(\varepsilon).$$

We consider only the Hölder seminorm of the highest-order derivative because lower-order derivatives are more regular. If we set $x = \rho_1(\cos(\varphi_1), \sin(\varphi_1))$, $y = \rho_2(\cos(\varphi_2), \sin(\varphi_2))$, then

$$[\tilde{g} - Id]_{m,\gamma,\Omega\setminus B_{\frac{1}{2}}(0)} = \sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\frac{\partial^{|j|}(\tilde{g}-Id)_i}{\partial x_1^{j_1}\partial x_2^{j_2}}(x) - \frac{\partial^{|j|}(\tilde{g}-Id)_i}{\partial x_1^{j_1}\partial x_2^{j_2}}(y)\right|}{|x-y|^\gamma}$$

$$= \sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\frac{1}{\rho_1^{m-1}}\,\mu_{\mathbf{j}}(\varphi_1) - \frac{1}{\rho_2^{m-1}}\,\mu_{\mathbf{j}}(\varphi_2)\right|}{|x-y|^\gamma}. \tag{20}$$

Applying the notation of (19), we add and subtract to the numerator the quantity $\frac{1}{\rho_1^{m-1}}\,\mu_{\mathbf{j}}(\varphi_2)$:

$$\begin{aligned}
(20) \le{}& \sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\frac{1}{\rho_1^{m-1}}\,(\mu_{\mathbf{j}}(\varphi_1) - \mu_{\mathbf{j}}(\varphi_2))\right|}{|x-y|^\gamma} \\
&+ \sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\left(\frac{1}{\rho_1^{m-1}} - \frac{1}{\rho_2^{m-1}}\right)\mu_{\mathbf{j}}(\varphi_2)\right|}{|x-y|^\gamma} \\
\le{}& \sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\frac{1}{\rho_1^{m-1}}\,\sum_{h=0}^{|j|}[\nu_{h,i,\mathbf{j}}(\varphi_1)\,(\sigma_{h,i,\mathbf{j}}(\varphi_1) - \sigma_{h,i,\mathbf{j}}(\varphi_2))]\right|}{|x-y|^\gamma} \\
&+ \sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\frac{1}{\rho_1^{m-1}}\,\sum_{h=0}^{|j|}[(\nu_{h,i,\mathbf{j}}(\varphi_1) - \nu_{h,i,\mathbf{j}}(\varphi_2))\,\sigma_{h,i,\mathbf{j}}(\varphi_2)]\right|}{|x-y|^\gamma} \\
&+ \sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\left(\frac{1}{\rho_1^{m-1}} - \frac{1}{\rho_2^{m-1}}\right)\mu_{\mathbf{j}}(\varphi_2)\right|}{|x-y|^\gamma}
\end{aligned} \tag{21}$$

$$\le \sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{1}{\rho_1^{m-1}} \sum_{h=0}^{|j|}\left[|\nu_{h,i,\mathbf{j}}(\varphi_1)|\,\frac{|(\sigma_{h,i,\mathbf{j}}(\varphi_1) - \sigma_{h,i,\mathbf{j}}(\varphi_2))|}{|x-y|^\gamma}\right] \tag{22}$$

$$+ \sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{1}{\rho_1^{m-1}} \sum_{h=0}^{|j|}\left[\frac{|\nu_{h,i,\mathbf{j}}(\varphi_1) - \nu_{h,i,\mathbf{j}}(\varphi_2)|}{|x-y|^\gamma}\,|\sigma_{h,i,\mathbf{j}}(\varphi_2)|\right] \tag{23}$$

$$+ \sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\frac{1}{\rho_1^{m-1}} - \frac{1}{\rho_2^{m-1}}\right|}{|x-y|^\gamma}\,|\mu_{\mathbf{j}}(\varphi_2)|. \tag{24}$$

Studying piece to piece (22), (23), and (24), we can easily demonstrate that they are sufficiently small:

$$(22) \le \underbrace{\sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{1}{\rho_1^{m-1}}}_{\le 2^{m-1}} \sum_{h=0}^{|j|}\Bigg[\underbrace{\sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} |\nu_{h,i,\mathbf{j}}(\varphi_1)|}_{\le \mathcal{O}(\varepsilon)}$$

$$\times \underbrace{\sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\big|\overbrace{\sigma_{h,i,\mathbf{j}}(\varphi_1)-\sigma_{h,i,\mathbf{j}}(\varphi_2)}^{\in C^\infty}\big|}{|x-y|^\gamma}}_{\le C_5}\Bigg] \le \mathcal{O}(\varepsilon),$$

$$(23) \le \underbrace{\sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{1}{\rho_1^{m-1}}}_{\le 2^{m-1}} \sum_{h=0}^{|j|}\Bigg[\underbrace{\sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\big|\overbrace{\nu_{h,i,\mathbf{j}}(\varphi_1)-\nu_{h,i,\mathbf{j}}(\varphi_2)}^{\in C^{0,\gamma}}\big|}{|x-y|^\gamma}}_{\le \mathcal{O}(\varepsilon)}$$

$$\times \underbrace{\sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} \big|\overbrace{\sigma_{h,i,\mathbf{j}}(\varphi_2)}^{\in C^\infty}\big|}_{\le C_6}\Bigg] \le \mathcal{O}(\varepsilon),$$

$$(24) \le \underbrace{\sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\Big|\overbrace{\frac{1}{\rho_1^{m-1}}-\frac{1}{\rho_2^{m-1}}}^{\in C^\infty}\Big|}{|x-y|^\gamma}}_{C_7} \underbrace{\sup_{\substack{|j|=m \\ x,y\in\Omega\setminus B_{\frac{1}{2}}(0)}} \big|\overbrace{\mu_{\mathbf{j}}(\varphi_2)}^{\in C^{0,\gamma}}\big|}_{\le \mathcal{O}(\varepsilon)} \le \mathcal{O}(\varepsilon).$$

It means, (20) $\le \mathcal{O}(\varepsilon)$. On $A := \{x : \frac{1}{4} \le |x| \le \frac{1}{2}\}$ we have the same estimate, multiplied with another constant:

$$\|\tilde{g}-Id\|_{C^{m,\gamma}(A)} = \|g\,\psi + Id\,(1-\psi) - Id\|_{C^{m,\gamma}(A)} = \|(g-Id)\,\psi\|_{C^{m,\gamma}(A)} \le \mathcal{O}(\varepsilon).$$

$\tilde{g}$ on $B_{\frac{1}{4}}(0)$ is the Id, then

$$\|\tilde{g}-Id\|_{C^{m,\gamma}(B_{\frac{1}{4}}(0))} = 0.$$

Appendix B

In order to prove (19), take $(x_1, x_2) = (\rho\cos(\varphi), \rho\sin(\varphi))$ and consider the function $g(\rho,\varphi) = (\rho\alpha_1(\varphi), \rho\alpha_2(\varphi))$.

$$\frac{\partial(\rho,\varphi)}{\partial(x_1,x_2)} = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\frac{1}{\rho}\sin(\varphi) & \frac{1}{\rho}\cos(\varphi) \end{pmatrix}.$$

$$\begin{pmatrix} (g_i)_{x_1} \\ (g_i)_{x_2} \end{pmatrix} = \alpha_i(\varphi)\begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix} + \rho\alpha_i'(\varphi)\begin{pmatrix} -\frac{\sin(\varphi)}{\rho} \\ \frac{\cos(\varphi)}{\rho} \end{pmatrix}.$$

$$\begin{pmatrix} (g_i)_{x_1,x_1} \\ (g_i)_{x_1,x_2} \\ (g_i)_{x_2,x_2} \end{pmatrix} = \alpha_i \begin{pmatrix} -\sin(\varphi)\frac{-\sin(\varphi)}{\rho} \\ \cos(\varphi)\frac{-\sin(\varphi)}{\rho} \\ \cos(\varphi)\frac{\cos(\varphi)}{\rho} \end{pmatrix} + \alpha_i' \begin{pmatrix} 2\cos(\varphi)\frac{-\sin(\varphi)}{\rho} \\ 2\sin(\varphi)\frac{-\sin(\varphi)}{\rho} \\ 2\sin(\varphi)\frac{\cos(\varphi)}{\rho} \end{pmatrix}$$

$$+\alpha_i'' \begin{pmatrix} -\sin(\varphi)\frac{-\sin(\varphi)}{\rho} \\ \cos(\varphi)\frac{-\cos(\varphi)}{\rho} \\ \cos(\varphi)\frac{\cos(\varphi)}{\rho} \end{pmatrix}$$

$$= \frac{1}{\rho}\left[\alpha_i \begin{pmatrix} \sin^2(\varphi) \\ -\cos(\varphi)\sin(\varphi) \\ \cos^2(\varphi) \end{pmatrix} + \alpha_i' \begin{pmatrix} -2\sin(\varphi)\cos(\varphi) \\ -2\sin^2(\varphi) \\ 2\cos(\varphi)\sin(\varphi) \end{pmatrix} \right.$$

$$\left. + \alpha_2'' \begin{pmatrix} \sin^2(\varphi) \\ -\cos^2(\varphi) \\ \cos^2(\varphi) \end{pmatrix}\right].$$

Then, by induction, assume that (19) is true for some j with $|j| \geq 2$:

$$\frac{\partial^{|j|}(\tilde{g} - Id)_i}{\partial x_1^{j_1}\partial x_2^{j_2}} = \frac{1}{\rho^{|j|-1}}\mu_{\mathbf{j}}(\varphi),$$

where

$$\mu_{\mathbf{j}}(\varphi) = \sum_{h=0}^{|j|}\left(\nu_{h,i,\mathbf{j}}(\varphi)\sigma_{h,i,\mathbf{j}}(\varphi)\right),$$

$$\nu_{h,i,\mathbf{j}}(\varphi) = (\alpha_1(\varphi) - \cos(\varphi), \alpha_2(\varphi) - \sin(\varphi))_i^{(h)},$$

$$\sigma_{h,i,\mathbf{j}}(\varphi) = \left(\sum_{\substack{k_1+k_2=|j| \\ k_1,k_2\geq 0}} c_{h,i,\mathbf{j},\mathbf{k}}(\cos(\varphi))^{k_1}(\sin(\varphi))^{k_2}\right).$$

We differentiate one more time with respect to x_1 (we obtain analogue results with respect to x_2):

$$\frac{\partial^{|j|+1}(\tilde{g} - Id)_i}{\partial x_1^{j_1+1}\partial x_2^{j_2}} = \frac{1}{\rho^{|j|}}\mu_{\mathbf{j}}(\varphi)\cos(\varphi)$$

$$+\frac{1}{\rho^{|j|-1}}\sum_{h=0}^{|j|}\left[\nu_{h,i,\mathbf{j}}'(\varphi)\,\sigma_{h,i,\mathbf{j}}(\varphi)\left(-\frac{1}{\rho}\sin(\varphi)\right)\right.$$

$$+\nu_{h,i,\mathbf{j}}(\varphi)\left(\sum_{\substack{k_1+k_2=|j| \\ k_1,k_2\geq 0}} c_{h,i,\mathbf{j},\mathbf{k}}(k_1(\cos(\varphi))^{k_1-1}(\sin(\varphi))^{k_2}\right.$$

$$\left.\left.+\ k_2(\cos(\varphi))^{k_1}(\sin(\varphi))^{k_2-1})\left(-\frac{1}{\rho}\sin(\varphi)\right)\right)\right]$$

$$= \frac{1}{\rho^{|j|}}\sum_{h=0}^{|j|+1}\left[(\alpha_1(\varphi) - \cos(\varphi), \alpha_2(\varphi) - \sin(\varphi))_i^{(h)}\right.$$

$$\left.\times\left(\sum_{\substack{k_1+k_2=|j|+1 \\ k_1,k_2\geq 0}} c_{h,i,\mathbf{j}+(1,0),\mathbf{k}}(\cos(\varphi))^{k_1}(\sin(\varphi))^{k_2}\right)\right].$$

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