Annali di Matematica (2007) 186(2):359–380
DOI 10.1007/s10231-006-0010-5

**Wei Wang**

# The Yamabe problem on quaternionic contact manifolds

Received: 10 April 2004 / Revised: 24 August 2005 / Published online: 22 April 2006


**Abstract** By constructing normal coordinates on a quaternionic contact manifold $M$, we can osculate the quaternionic contact structure at each point by the standard quaternionic contact structure on the quaternionic Heisenberg group. By using this property, we can do harmonic analysis on general quaternionic contact manifolds, and solve the quaternionic contact Yamabe problem on $M$ if its Yamabe invariant satisfies $\lambda(M) < \lambda(\mathbb{H}^n)$.

**Keywords** Quaternionic contact structure · The quaternionic Heisenberg group · Quaternionic contact Yamabe problem · Normal coordinates

**Mathematics Subject Classification (2000)** 53C17, 53D10, 35J70

## 1 Introduction

Quaternionic contact manifolds are quaternionic analogues of integrable CR manifolds defined by Biquard [3] and naturally appear as the boundaries at infinity of quaternionic-Kähler asymptotic symmetric manifolds. We will consider the quaternionic contact Yamabe problem in this paper.

Let $\theta = (\theta_1, \theta_2, \theta_3)$ be a contact 1-form on a smooth manifold $M$ valued on $\mathbf{R}^3$, $V = \ker\theta$, and $\dim V_\xi = 4n$ for each $\xi \in M$, where $\dim M = 4n + 3$. A $\mathrm{Sp}_n\mathrm{Sp}_1$ *Carnot–Carathéodory metric* $g$ compatible with $\mathrm{d}\theta$ is given by a metric on $V$ if there exist three complex structures $I_1, I_2, I_3$ on $V$ such that

$$\mathrm{d}\theta_\beta(X, Y) = g(I_\beta X, Y), \quad \beta = 1, 2, 3, \tag{1.1}$$

for each $X, Y \in V$, where $I_1, I_2, I_3$ satisfy *the commutating relation of quaternions*

$$I_1^2 = I_2^2 = I_3^2 = -\mathrm{id}_V, \quad I_1 I_2 I_3 = -\mathrm{id}_V. \tag{1.2}$$

W. Wang (✉)
Department of Mathematics, Zhejiang University, Xixi campus, Zhejiang 310028, P.R. China
E-mail: wangf@mail.hz.zj.cn

The $\mathbf{R}^3$-valued 1-form $\theta$ is given only up to the action of $\mathrm{SO}_3$ on $\mathbf{R}^3$ and to a conformal factor, thus we get a $\mathrm{Sp}_n\mathrm{Sp}_1$-*structure* on $V$.

A *quaternionic contact structure* on $M$ is the data of a codimension 3 distribution $V$, equipped with a $\mathrm{Sp}_n\mathrm{Sp}_1$-structure, such that the $\mathrm{Sp}_n\mathrm{Sp}_1$-structure and contact 1-form valued on $\mathbf{R}^3$ satisfy the commutating relation of quaternions (1.2). *The quaternionic contact manifold* is a smooth manifold equipped with a quaternionic contact structure.

Biquard [3] defined a connection on a quaternionic contact manifold, which is the analogue of Webster's connection in CR geometry [17].

**Theorem 1** *([3], p. 8) Suppose codimension* 3 *distribution $V$ on $M$ has a quaternionic contact structure,* $\dim M = 4n+3$ *with* $n \geq 2$*, and $g_V$ is a compatible Carnot–Carathéodory metric. Then there exists a unique complement $W$ to $V$ and a unique connection $\nabla$ on $M$ such that*

*(1) $\nabla$ preserves $V$ and the* $\mathrm{Sp}_n\mathrm{Sp}_1$*-structure on $V$.*
*(2) For $X, Y \in V$, the torsion $T_{X,Y} = \nabla_X Y - \nabla_Y X - [X, Y]$ satisfies $T_{X,Y} = [X, Y]_W$.*
*(3) For $R \in W$, the endomorphism of $V$ defined by $X \longrightarrow (T_{R,X})_V$ is in the subspace $(sp_m \oplus sp_1)^\perp \subset \mathrm{End}V$.*
*(4) $\nabla$ preserves $W$ and $\nabla|_W$ coincides with the connection induced on a $\mathbf{R}^3$ subbundle of* $\mathrm{End}V$*, which is formed by complex structures on $V$.*

The first two conditions determine a unique $W$.

Let $R$ be the curvature of this connection, $R_{X,Y} = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$. Define Ricci tensor

$$\mathrm{Ric}_{X,Y} = \sum_{l=1}^{4n} \langle R_{e_l,X}Y, e_l\rangle, \tag{1.3}$$

where $\{e_l\}$ is an orthonormal basis of $V$ under matric $g_V$. The scalar curvature of this connection is $s_\theta = \mathrm{Tr}^V\,\mathrm{Ric}$.

Let us consider a conformal transformation $\theta' = e^{2f}\theta$, $f \in C^\infty(M)$. Then, the metric $g_V$ is changed to $g'_V = e^{2f}g_V$, which is the Carnot–Carathéodory metric compatible with $\mathrm{d}\theta'$. Biquard also gave the transformation formula for scalar curvatures under conformal transformations.

**Proposition 2** *([3], p. 74) Under conformal transformation $\theta' = f^2\theta$ for $f \in C^\infty(M)$ with $f > 0$, the scalar curvature satisfies the following formula*

$$s_{\theta'} = f^{-2}(s_\theta - 8(n+2)\mathrm{Tr}^V\nabla\omega - 16(n+1)(n+2)|\omega|^2), \tag{1.4}$$

*where $\omega = f^{-1}\mathrm{d}f$.*

The following corollary is another form of the transformation formula for scalar curvatures.

**Corollary 3** *Under conformal transformation $\theta' = v^{(4/(Q-2))}\theta$ for $v \in C^\infty(M)$ with $v > 0$, the scalar curvature satisfies the following formula:*

$$b_n\Delta_\theta v + s_\theta v = s_{\theta'}v^{\frac{Q+2}{Q-2}}, \tag{1.5}$$

*where $b_n = 4\{(Q+2)/(Q-2)\}$, $Q = 4n+6$ is the homogeneous dimension of $M$, and the SubLaplacian $\Delta_\theta$ is defined by (2.28).*

This corollary will be checked at the end of Sect. 2. Given a $\mathbf{R}^3$-valued contact form $\theta$ and a positive function $u \in C^\infty(M)$, a necessary and sufficient condition for $\theta' = u^{(4/(Q-2))}\theta$ to have constant scalar curvature, $s_{\theta'} \equiv \lambda$, is that $u$ satisfies the differential equation

$$b_n \Delta_\theta u + s_\theta u = \lambda u^{\frac{Q+2}{Q-2}}. \tag{1.6}$$

To find such $u$ is *the quaternionic contact Yamabe problem*.

As in the Riemannian and CR case, (1.6) is the Euler–Lagrangian equation for the constrained variational problem

$$\lambda(M) = \inf_\theta \{A_\theta(u); B_\theta(u) = 1\}, \tag{1.7}$$

where

$$A_\theta(u) = \int_M (b_n |\mathrm{d}_b u|_\theta^2 + s_\theta u^2)\psi_\theta, \quad B_\theta(u) = \int_M |u|^p \psi_\theta, \tag{1.8}$$

and $p = 2Q/(Q-2)$, $\psi_\theta$ is the volume element defined by (2.27). We prove the following result for the quaternionic contact Yamabe problem.

**Theorem 1.1** *Suppose $M$ is a compact quaternionic contact manifold with a contact form $\theta$ valued on $\mathbf{R}^3$, $V = \ker\theta$, and there is a Carnot–Carathéodory metric $g_\theta$ compatible with $\theta$ in sense of (1.1), where* $\dim M = 4n + 3$ *with $n \geq 2$. Then,*

*(1) $\lambda(M) \leq \lambda(\mathbb{H}^n)$, where $\mathbb{H}^n$ is the quaternionic Heisenberg group with standard contact form $\theta_{\mathbb{H}}$ defined by (2.19) and (2.20).*
*(2) If $\lambda(M) < \lambda(\mathbb{H}^n)$, then the infimum of (1.7) is achieved by a positive $C^\infty$ solution $u$ of (1.6), i.e. contact form $\theta' = u^{4/(Q-2)}\theta$ has constant scalar curvature $s_{\theta'} = \lambda(M)$.*

For the existence and uniqueness of extremal achieving the infimum $\lambda(\mathbb{H}^n)$, it is natural to conjecture that contact forms $\theta_\phi = \phi^*\theta_{\mathbb{H}}$ for $\phi \in \mathrm{Sp}(n+1, 1)\mathrm{Sp}(1)$ are only ones on $\mathbb{H}^n$ achieving the infimum $\lambda(\mathbb{H}^n)$ and hence having constant scalar curvature. The CR Yamabe problem is completely solved in [8, 9, 11–13]. The general results about the quaternionic contact Yamabe problem are working in progress. In Sect. 2, we state some basic facts about quaternions, the quaternionic Heisenberg group and the SubLaplacian. In Sect. 3, we prove that the quaternionic contact structure at each point of a quaternionic contact manifold can be osculated by that of the quaternionic Heisenberg group and construct normal coordinates locally. Results on the quaternionic Heisenberg group are used to prove the regularity of the SubLaplacians on general quaternionic contact manifolds in Sect. 4. The main theorem is proved in the last section.

## 2 Some basic facts

The element of the algebra $\mathbf{H}$ of quaternion numbers is $x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$ with $x_l \in \mathbf{R}$, $l = 1, \ldots, 4$, and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ijk} = -1. \tag{2.1}$$

The conjugate $\overline{q}$ of $q = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$ is $x_1 - x_2\mathbf{i} - x_3\mathbf{j} - x_4\mathbf{k}$. The imaginary part $\Im(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}) = x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$. The quaternionic unitary group $\mathrm{Sp}_n$ is the group of endmorphisms preserving the quaternionic quadratic form

$$\langle P, Q\rangle = \sum_{l=1}^{n} p_l \overline{q}_l, \tag{2.2}$$

where $P = (p_1, \ldots, p_n)$, $Q = (q_1, \ldots, q_n) \in \mathbf{H}^n$. Thus, Sp(1) is the set of all unit quaternions. By identifying $\mathbf{H}^n$ with $\mathbf{C}^{2n}$, $\mathrm{Sp}_n$ is the subgroup of $\mathrm{U}(2n)$ of elements commutative with the complex structure $J : \mathbf{C}^{2n} \longrightarrow \mathbf{C}^{2n}$, $J(e_i) = e_{n+i}$, $J(e_{n+i}) = e_i$, $i = 1, \ldots, n$, where $e_1, \ldots, e_{2n}$ is an orthonormal basis of $\mathbf{C}^{2n}$.

The standard model of the quaternionic contact manifold is *the quaternionic Heisenberg group* $\mathbb{H}^{\mathbf{n}} = \mathbf{H}^n \oplus \Im\mathbf{H}$. Its multiplication is given by

$$(x, t) \cdot (y, s) = (x + y, t + s + \Im(x\overline{y})), \tag{2.3}$$

where $x, y \in \mathbf{H}^n$, $t, s \in \Im\mathbf{H}$. The neutral element is $(0, 0)$ and the inverse of $(x, t)$ is $(-x, -t)$.

There are six groups of automorphisms of $\mathbb{H}^{\mathbf{n}}$ [18].

(1) The one parameter group $A$ of *dilations*:

$$D_\delta : (x, t) \longrightarrow (\delta x, \delta^2 t), \quad \delta > 0; \tag{2.4}$$

(2) The group $N$ of *translations*:

$$\tau_P : Q \longrightarrow P \cdot Q, \quad \text{for } P, Q \in \mathbb{H}^n; \tag{2.5}$$

(3) The group $M$ of *rotations*, $M$ is isomorphic to $\mathrm{Sp}_n$, the quaternionic unitary group:

$$(x, t) \longrightarrow (Ux, t), \quad \text{for } U \in \mathrm{Sp}_n; \tag{2.6}$$

(4) The *inversion* $R$, which is defined on $\mathbb{H}^n \setminus \{0\}$ by

$$R : (x, t) \longrightarrow \left(\frac{-x}{|x|^2 - t}, \frac{-t}{|x|^4 + |t|^2}\right), \tag{2.7}$$

where $x \in \mathbf{H}^{n-1}$, $t = t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k}$;

(5) The *conjugation* $\Sigma$:

$$\Sigma : (x, t) \longrightarrow (\overline{x}, \overline{t}); \tag{2.8}$$

(6) Sp(1) acts on $\mathbb{H}^{\mathbf{n}}$ as follows:

$$\sigma(x, t) = (\sigma x, \sigma t \sigma^{-1}), \tag{2.9}$$

where the action on the first factor is left multiplication by unit quaternions, and the action on the second factor $t$ is given by orthogonal matrix conjugation under the identification $\Im\mathbf{H}$ with O(3).

The group generated by $A, M, N, R, \Sigma$ and $\mathrm{Sp}(1)$ is isomorphic to the rank one Lie group $\mathrm{Sp}(n,1)\mathrm{Sp}(1)$. The stereographic projection $F$ from $\mathbf{S}^{4n+3}$ to hypersurface

$$\Re q'_{n+1} - \sum_{l=1}^{n} |q'_l|^2 = 0 \tag{2.10}$$

is given by

$$q'_l = \frac{q_l}{1+q_{n+1}}, \qquad q'_{n+1} = \frac{1-q_{n+1}}{1+q_{n+1}} \tag{2.11}$$

$l = 1, \ldots, n$. The standard contact form on the hypersurface (2.10) is given by

$$\theta_0 = \Im\, \mathrm{d}q'_{n+1} - \sum_{l=1}^{n} q'_l\, \mathrm{d}\overline{q}'_l - \overline{q}'_l\, \mathrm{d}q'_l. \tag{2.12}$$

The multiplication of the quaternionic Heisenberg group can be written in real variables as follows.

$$(x,t)\cdot(y,s) = \left( x+y, t_\beta + s_\beta + \sum_{l=0}^{n-1} \sum_{j,k=1}^{4} b^\beta_{kj} x_{4l+k} y_{4l+j} \right), \tag{2.13}$$

where $\beta = 1, 2, 3$, $x = (x_1, x_2, \ldots, x_{4n}) \in \mathbf{R}^{4n}$, $t = (t_1, t_2, t_3) \in \mathbf{R}^3$, $y$ and $s$ are defined similarly, and antisymmetric matrices

$$b^1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad b^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$
$$b^3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.14}$$

The multiplications of the quaternionic Heisenberg group in (2.3) and (2.13) are the same by direct calculation

$$\begin{aligned} &\Im\{(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k})(y_1 - y_2\mathbf{i} - y_3\mathbf{j} - y_4\mathbf{k})\} \\ &\quad = (-x_1y_2 + x_2y_1 - x_3y_4 + x_4y_3)\mathbf{i} + (-x_1y_3 + x_3y_1 + x_2y_4 - x_4y_2)\mathbf{j} \\ &\qquad + (-x_1y_4 + x_4y_1 - x_2y_3 + x_3y_2)\mathbf{k}. \end{aligned} \tag{2.15}$$

$b^1, b^2, b^3$ are the matrices of the quadratic forms of coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in (2.15), respectively. It is easy to see that matrices $b^1, b^2, b^3$ satisfy the commutating relation (1.2) of quaternions

$$(b^1)^2 = (b^2)^2 = (b^3)^2 = -\mathrm{id}, \quad b^1b^2b^3 = -\mathrm{id}. \tag{2.16}$$

The following vector fields are left invariant on the quaternionic Heisenberg group by the multiplication laws of the quaternionic Heisenberg group in (2.13):

$$Y_{4l+j}=\frac{\partial}{\partial x_{4l+j}}+\sum_{\beta=1}^{3}\sum_{k=1}^{4}b_{kj}^{\beta}x_{4l+k}\frac{\partial}{\partial t_{\beta}}, \tag{2.17}$$

and

$$[Y_{4l'+k},Y_{4l+j}]=2\delta_{ll'}\sum_{\beta=1}^{3}b_{kj}^{\beta}\frac{\partial}{\partial t_{\beta}}, \tag{2.18}$$

for $l,l'=0,\ldots,n-1,\ j,k=1,\ldots,4$. Then, the horizontal subspace $V_{\mathbb{H}}=\mathrm{span}_{\mathbf{R}}\{Y_1,\ldots,Y_{4n}\}$ generate the corresponding Lie algebra of the quaternionic Heisenberg group. Let

$$\theta_{\mathbb{H}}=(\theta_{\mathbb{H},1},\theta_{\mathbb{H},2},\theta_{\mathbb{H},3}) \tag{2.19}$$

with

$$\theta_{\mathbb{H},\beta}=\mathrm{d}t_{\beta}-\sum_{l=0}^{n-1}\sum_{j,k=1}^{4}b_{kj}^{\beta}x_{4l+k}\mathrm{d}x_{4l+j}, \tag{2.20}$$

$\beta=1,2,3$. It is obvious that $\theta_{\mathbb{H},\beta}(Y_\mu)=0$ for any $\mu=1,\ldots,4n$ and

$$\mathrm{d}\theta_{\mathbb{H},\beta}=-\sum_{l=0}^{n-1}\sum_{j,k=1}^{4}b_{kj}^{\beta}\mathrm{d}x_{4l+k}\wedge\mathrm{d}x_{4l+j}. \tag{2.21}$$

Define a metric $g_{\mathbb{H}}$ on $V_{\mathbb{H}}$ by

$$g_{\mathbb{H}}(Y_\mu,Y_\nu)=2\delta_{\mu\nu} \tag{2.22}$$

for $\mu,\nu=1,\ldots,4n$, and transformations $I_\beta$ on $V_{\mathbb{H}}$ by

$$I_\beta Y_{4l+k}=\sum_{j=1}^{4}b_{jk}^{\beta}Y_{4l+j}, \tag{2.23}$$

for $l=0,\ldots,n-1$, $k=1,2,3,4$, $\beta=1,2,3$. It is easy to see that $\{I_\beta\}$ satisfies the commutating relation (1.2) of quaternions since $b^\beta$ satisfies the commutating relation (2.16) of quaternions, and

$$\mathrm{d}\theta_{\mathbb{H},\beta}(Y_{4l+k},Y_{4l'+j})=-2b_{kj}^{\beta}\delta_{ll'}=g_{\mathbb{H}}(I_\beta Y_{4l+k},Y_{4l'+j}), \tag{2.24}$$

by (2.21). Thus, $g_{\mathbb{H}}$ is the Carnot–Carathéodory metric compatible with $\mathrm{d}\theta_{\mathbb{H}}$ on $V_{\mathbb{H}}$ in sense of (1.1).

The standard $\mathbf{R}^3$-valued contact form on $\mathbf{S}^{4n+3}$ is defined by

$$\theta_S=\sum_{l=1}^{n+1}(\overline{q}_l\mathrm{d}q_l-q_l\mathrm{d}\overline{q}_l), \tag{2.25}$$

where $q_l=x_{4l-3}+x_{4l-2}\mathbf{i}+x_{4l-1}\mathbf{j}+x_{4l}\mathbf{k}$, $l=1,\ldots,n+1$.

Now consider a general quaternionic contact manifold $M$ with a contact 1-form $\theta$ valued on $\mathbf{R}^3$. We choose a local basis $\{X_\mu\}_{\mu=1}^{4n}$ of $V = \ker\theta$ with norm $\sqrt{2}$ under the Carnot–Carathéodory metric $g_\theta$ compatible with $\mathrm{d}\theta$. This is because each element of the standard basis (2.17) of the horizontal subspace $V_{\mathbb{H}}$ of the quaternionic Heisenberg group has norm $\sqrt{2}$ under the standard metric $g_{\mathbb{H}}$ in (2.22). The metric $g_\theta$ on $V$ induces a dual metric on $V^*$. We denote it by $\langle\cdot,\cdot\rangle_\theta$. Define a norm $|\omega|_\theta^2 = \langle\omega,\omega\rangle_\theta$ for $\omega \in V^*$. It defines an $L^2$ inner product on $V^*$ by

$$\langle\omega,\omega'\rangle = \int_M \langle\omega,\omega'\rangle_\theta \psi_\theta, \tag{2.26}$$

where the volume element $\psi_\theta$ associated with $\theta$ is locally

$$\psi_\theta = 2^{2n}\theta_1\wedge\theta_2\wedge\theta_3\wedge\theta^1\wedge\cdots\theta^{4n}, \tag{2.27}$$

where $\{\theta^1,\ldots,\theta^{4n}\}$ is a local dual basis of $\{X_\mu\}_{\mu=1}^{4n}$ satisfying $\theta^\mu|_W = 0$ for each $\mu$.

Denote $\mathrm{d}_b = \pi\circ\mathrm{d}$, where $\pi$ is the projection from $T^*M$ to $V^*$. We define *the SubLaplacian* $\Delta_\theta$ associated with the contact form $\theta$ by

$$\int_M \Delta_\theta u\cdot v\cdot\psi_\theta = \int_M \langle\mathrm{d}_b u,\mathrm{d}_b v\rangle_\theta\psi_\theta. \tag{2.28}$$

Define covariant differentiation as follows. Since $\nabla$ preserves $V$, there exist 1-forms $\omega_\mu^{\ \nu}$ such that

$$\nabla_Y X_\mu = \omega_\mu^{\ \nu}(Y)X_\nu, \tag{2.29}$$

for $Y\in V$. Let $\omega_\mu^{\ \nu'}(X_\nu) = \Gamma_{\mu\nu}^{\nu'}$. Then, $\nabla_{X_\nu}X_\mu = \Gamma_{\mu\nu}^{\nu'}X_{\nu'}$. For 1-form $\omega\in\Omega^1(M)$,

$$(\nabla_X\omega)(Y) = \nabla_X(\omega(Y)) - \omega(\nabla_X Y). \tag{2.30}$$

The SubLaplacian $\Delta_\theta$ has the following expression.

**Proposition 2.1** *Let $\{X_\mu\}_{\mu=1}^{4n}$ be a local basis of $V$ such that $g_\theta(X_\mu,X_\nu) = 2\delta_{\mu\nu}$. Then, for $u\in C^\infty(M)$,*

$$\Delta_\theta u = -\mathrm{Tr}^V\nabla(\mathrm{d}u) = \frac{1}{2}\sum_{\mu=1}^{4n}\left(-X_\mu X_\mu u + \sum_{\nu=1}^{4n}\Gamma_{\nu\nu}^{\mu}X_\mu u\right). \tag{2.31}$$

*where the covariant derivatives of 1-forms are defined by (2.30).*

*Proof* It is sufficient to check (2.31) for smooth $u$ with sufficiently small support. Let $\{\theta^\mu\}_{\mu=1}^{4n}$ be a dual basis of $\{X_\mu\}_{\mu=1}^{4n}$ for $V^*$, and $\theta^\mu|_W = 0$ for each $\mu$. Let $X_\mu^*$ be the adjoint operator of densely defined operator $X_\mu$ on the Hilbert space $L^2(M)$, $\mu = 1,\ldots,4n$. We claim that

$$X_\mu^* = -X_\mu + \sum_{\nu=1}^{4n}\Gamma_{\nu\nu}^{\mu}. \tag{2.32}$$

Note that $\mathrm{d}_b u = \sum_\mu X_\mu u \cdot \theta^\mu$ and $\langle \theta^\mu, \theta^\mu \rangle_\theta = 1/2$. By the definition of the SubLaplacian in (2.28), we have that

$$\Delta_\theta = \frac{1}{2}\sum_{\mu=1}^{4n} X_\mu^* X_\mu = \frac{1}{2}\sum_{\mu=1}^{4n}\left(-X_\mu X_\mu u + \sum_{\nu=1}^{4n}\Gamma_{\nu\nu}^{\ \mu} X_\mu u\right). \tag{2.33}$$

By (2.30), we have that

$$\nabla_{X_\nu}\theta^\mu = -\Gamma_{\nu'\nu}^{\ \mu}\theta^{\nu'}. \tag{2.34}$$

Now by the definition of trace,

$$\begin{aligned}\mathrm{Tr}^V \nabla(\mathrm{d}u) &= \mathrm{Tr}^V \nabla(\mathrm{d}_b u)\\ &= \frac{1}{2}\sum_{\alpha=1}^{4n}\sum_{\mu=1}^{4n}(\nabla_{X_\alpha}(X_\mu u\cdot\theta^\mu))(X_\alpha) = \frac{1}{2}\sum_{\mu=1}^{4n}\left(X_\mu X_\mu u - \sum_{\alpha=1}^{4n}\Gamma_{\alpha\alpha}^{\ \mu}X_\mu u\right),\end{aligned} \tag{2.35}$$

since $X_\alpha$ has norm $\sqrt{2}$. (2.31) is proved.

It remains to prove the claim (2.32). Note that $Xu = i_{X_\mu}\mathrm{d}u$ by the formula of Lie derivative, where $i_X$ is the interior operator. By Stokes' formula,

$$\begin{aligned}\int_M X_\mu u\cdot v\psi_\theta &= \int_M v\,\mathrm{d}u\wedge i_{X_\mu}\psi_\theta = -\int_M u\,\mathrm{d}v\wedge i_{X_\mu}\psi_\theta - \int_M uv\,\mathrm{d}(i_{X_\mu}\psi_\theta)\\ &= -\int_M uX_\mu v\psi_\theta - \int_M uv\,\mathrm{d}(i_{X_\mu}\psi_\theta).\end{aligned} \tag{2.36}$$

Here we have used identities $\int_M i_{X_\mu}(v\,\mathrm{d}u\wedge\psi_\theta) = \int_M i_{X_\mu}(u\,\mathrm{d}v\wedge\psi_\theta) = 0$. By the standard exterior differentiation formula,

$$\mathrm{d}\phi(X,Y) = (\nabla_X\phi)(Y) - (\nabla_Y\phi)(X) + \phi(T_{X,Y}) \tag{2.37}$$

for 1-form $\phi\in\Omega^1(M)$, where $T_{X,Y}$ is the torsion, and $T_{X,Y} = [X,Y]_W$ by Theorem 1, we find that $\mathrm{d}\theta^\mu(X_\nu, X_{\nu'}) = -\Gamma_{\nu'\nu}^{\ \mu} + \Gamma_{\nu\nu'}^{\ \mu}$, and hence,

$$\mathrm{d}\theta^\mu = \frac{1}{2}\big(-\Gamma_{\nu'\nu}^{\ \mu} + \Gamma_{\nu\nu'}^{\ \mu}\big)\theta^\nu\wedge\theta^{\nu'}, \quad \mod \quad \theta_1,\theta_2,\theta_3. \tag{2.38}$$

Note that

$$\begin{aligned}\mathrm{d}(i_{X_\mu}\psi_\theta) =& (-1)^\mu 2^{2n}(\mathrm{d}\theta_1\wedge\theta_2\wedge\theta_3 - \theta_1\wedge\mathrm{d}\theta_2\wedge\theta_3 + \theta_1\wedge\theta_2\wedge\mathrm{d}\theta_3)\wedge\\ &\wedge\theta^1\wedge\cdots\hat{\theta^\mu}\cdots\wedge\theta^{4n} + \sum_{\beta\neq\mu}\mathrm{d}\theta^\beta\wedge i_{X_\beta}i_{X_\mu}\psi_\theta.\end{aligned} \tag{2.39}$$

The first term on the right-hand side of (2.39) is zero since it annihilates the Reeb vector $T_j$ by $i_{T_j}\mathrm{d}\theta_j = 0$ (Proposition II.1.7 in [3]), where $T_j\in W$ satisfies $\theta_k(T_j) = \delta_{jk}$. Inserting

$$\mathrm{d}\theta^\beta = \frac{1}{2}\big(-\Gamma_{\beta\mu}^{\ \beta} + \Gamma_{\mu\beta}^{\ \beta}\big)\theta^\mu\wedge\theta^\beta + \frac{1}{2}\big(-\Gamma_{\mu\beta}^{\ \beta} + \Gamma_{\beta\mu}^{\ \beta}\big)\theta^\beta\wedge\theta^\mu + \cdots \tag{2.40}$$

by (2.38) in the second sum in (2.39) and using antisymmetry $\Gamma_{\beta\gamma}^{\alpha} = -\Gamma_{\alpha\gamma}^{\beta}$, which follows from the fact that the connection $\nabla$ preserves the metric $g_\theta$ on $V$, we find that

$$\mathrm{d}(i_{X_\mu}\psi_\theta) = \sum_{\beta\neq\mu}(-\Gamma_{\beta\mu}^{\beta} + \Gamma_{\mu\beta}^{\beta})\psi_\theta = -\sum_{\beta\neq\mu}\Gamma_{\beta\beta}^{\mu}\psi_\theta = -\sum_{\beta=1}^{4n}\Gamma_{\beta\beta}^{\mu}\psi_\theta, \quad (2.41)$$

since $\Gamma_{\beta\mu}^{\beta} \equiv 0$ by antisymmetry. The claim (2.32) follows from (2.36) and (2.41). The proposition is proved. □

*Proof of Corollary* By substituting $f = v^{2/(Q-2)}$ in (1.4) and using the above (2.31) for $u = 2/(Q-2)\log v$, we can easily prove (1.5). □

## 3 Normal coordinates

The purpose of this section is to define normal coordinates to approximate the quaternionic contact structure at each point of a quaternionic contact manifold $M$ by the standard quaternionic contact structure on the quaternionic Heisenberg group.

Let $\xi$ be an arbitrary point of $M$ and $U$ be a sufficiently small neighborhood of $\xi$. For any two local sections $e, e' \in C^\infty(U, V)$, we have

$$\begin{aligned} g_\theta(I_\alpha e, I_\alpha e') &= \mathrm{d}\theta_\alpha(e, I_\alpha e') = -\mathrm{d}\theta_\alpha(I_\alpha e', e) \\ &= -g_\theta(I_\alpha(I_\alpha e'), e) = g_\theta(e', e) = g_\theta(e, e'), \end{aligned} \quad (3.1)$$

by (1.1). Namely $I_\alpha$ ($\alpha = 1, 2, 3$) are isometries on $V$ under the metric $g_\theta$. Now choose a section $e_1 \in C^\infty(U, V)$ such that $g(e_1, e_1) = 1$. From now on, we take $I_0 = \mathrm{id}_V$. Note that

$$g_\theta\big(I_\alpha e_1, I_\beta e_1\big) = g_\theta(I_\beta I_\alpha e_1, I_\beta^2 e_1) = -g_\theta(I_\gamma e_1, e_1) = -\mathrm{d}\theta_\gamma(e_1, e_1) = 0 \quad (3.2)$$

for some $\gamma \in \{1, 2, 3\}$ if $\alpha \neq \beta$. Hence, $e_1, I_1e_1, I_2e_1, I_3e_1$ are mutually orthogonal. Now choose a section $e_2 \in V$ orthogonal to $\mathrm{span}_{\mathbf{R}}\{I_k e_1;\ k = 0, \ldots, 3\}$. As above, $e_2, I_1e_2, I_2e_2, I_3e_2$ are mutually orthogonal and $\mathrm{span}_{\mathbf{R}}\{I_k e_1;\ k = 0, \ldots, 3\} \perp \mathrm{span}_{\mathbf{R}}\{I_k e_2;\ k = 0, \ldots, 3\}$. Repeating this procedure, we find at last vectors $e_1, \ldots, e_n$ such that $\{I_k e_j;\ j = 1, \ldots, n,\ k = 0, \ldots, 3\}$ forms a local orthonormal basis of $V$ under the metric $g_\theta$. Let

$$X_{4l+\alpha} = \sqrt{2}I_{\alpha-1}e_{l+1}, \quad (3.3)$$

for $l = 0, \ldots, n-1, \alpha = 1, \ldots, 4$. Then,

$$\mathrm{d}\theta_\beta(X_{4l+k}, X_{4l'+j}) = 2g_\theta(I_\beta I_{k-1}e_{l+1}, I_{j-1}e_{l'+1}), \quad (3.4)$$

which is zero except for $I_\beta I_{k-1} = \pm I_{j-1}$ and $l = l'$. For example, the right side of (3.4) for $\beta = 1$ vanishes except for $\{k, j\} = \{1, 2\}$ or $\{3, 4\}$. It equals to $-2\delta_{ll'}b_{kj}^1$ by direct calculation, where $b^1$ is defined by (2.14). Similarly, we can check that

$$\mathrm{d}\theta_\beta(X_{4l+k}, X_{4l'+j}) = -2\delta_{ll'}b_{kj}^\beta \quad (3.5)$$

for $\beta = 1, 2, 3$. Let $\{\theta^1, \dots, \theta^{4n}\}$ be a dual basis of $\{X_1, \dots, X_{4n}\}$. (3.5) is equivalent to

$$\mathrm{d}\theta_\beta = -\sum_{l=0}^{n-1}\sum_{j,k=1}^{4} b_{kj}^{\beta}\theta^{4l+k}\wedge\theta^{4l+j}, \quad \mathrm{mod} \quad \theta_1, \theta_2, \theta_3. \tag{3.6}$$

The dual of (3.5) is

$$[X_{4l+k}, X_{4l'+j}] = 2\delta_{ll'}\sum_{\beta=1}^{3} b_{kj}^{\beta} T_\beta, \quad \mathrm{mod} \quad V, \tag{3.7}$$

where $T_\beta$ is the Reeb vectors satisfying $\theta_\alpha(T_\beta) = \delta_{\alpha\beta}, \alpha, \beta = 1, 2, 3$, by using the formula of exterior derivative: for 1-form $\omega$,

$$\mathrm{d}\omega(X, Y) = X(\omega(Y)) - Y(X(\omega)) - \omega([X, Y]). \tag{3.8}$$

Define $X_{4n+\beta} = T_\beta$ for $\beta = 1, 2, 3$. For each $\xi \in M$, as in the CR case in [7], we define *the exponential map* $E_\xi$ at $\xi$ based on the local basis $\{X_1, \dots, X_{4n+3}\}$. For $u = (u_1, \dots, u_{4n+3}) \in \mathbf{R}^{4n+3}$, we define $E_\xi(u) \in M$ to be the endpoint of integral curve $\eta(s), 0 \le s \le 1$, of the vector field $\sum_{j=1}^{4n+3} u_j X_j$ with $\eta(0) = \xi$. Then $E_\xi$ is a smooth mapping of a star shaped neighborhood $U_\xi$ of $0 \in \mathbf{R}^{4n+3}$ into $M$. It is clear that $E_{\xi*}(\frac{\partial}{\partial u_j})|_0 = X_j|_\xi$. So $E_\xi$ is a diffeomorphism of a smaller neighborhood of $U_\xi$ of 0 (denoted also by $U_\xi$) onto a neighborhood $V_\xi$ of $M$.

Let $\Omega = \{(\xi, \eta) \in M \times M; \eta \in V_\xi\}$. $\Omega$ is a neighborhood of the diagonal in $M \times M$. Denote $\Theta_\xi$ the coordinate mapping $E_\xi^{-1} : V_\xi \longrightarrow \mathbf{R}^{4n+3}$, and $\Theta(\xi, \eta) = \Theta_\xi(\eta)$. We also write $(x(\cdot; \xi), t(\cdot; \xi))$ or $u(\eta; \xi)$ for the coordinates of $\Theta_\xi(\eta)$. Define a norm on $\mathbf{R}^{4n+3}$ by

$$\|u\| = \left(\left(\sum_{j=1}^{4n} u_j^2\right)^2 + \sum_{\beta=1}^{3} u_{4n+\beta}^2\right)^{\frac{1}{4}}. \tag{3.9}$$

Set $\rho(\xi, \eta) = \|\Theta(\xi, \eta)\|$. A function $f$ on $V_\xi$ is said to be $O^k$ if $f(\eta) = O(\rho(\xi, \eta)^k)$ as $\eta \longrightarrow \xi$.

**Proposition 3.1** *In the coordinates $x(\cdot; \xi), t(\cdot; \xi)$,*

$$X_{4l+j} = \frac{\partial}{\partial x_{4l+j}} + \sum_{\beta=1}^{3}\sum_{k=1}^{4} b_{kj}^{\beta} x_{4l+k}\frac{\partial}{\partial t_\beta} + \sum_{\mu=1}^{4n} O^1\frac{\partial}{\partial x_\mu} + \sum_{\beta=1}^{3} O^2\frac{\partial}{\partial t_\beta}, \tag{3.10}$$

*for $l = 0, \dots, n-1$ and $j = 1, \dots, 4$.*

*Proof* In the proposition, we identify $\Theta_{\xi*}X_\mu$ with $X_\mu$. Write

$$X_\mu = \sum_{\nu=1}^{4n+3} F_{\mu\nu}(u)\frac{\partial}{\partial u_\nu}, \quad \mu = 1, \dots, 4n+3. \tag{3.11}$$

The following sublemma is Lemma 14.2 and Lemma 14.5 in [7], pp. 472–474 (see also Chevalley's book [6], pp. 155–156), which hold for general vector fields $\{Z_\mu\}$ satisfying the condition that vector fields $\{Z_\mu\}$ span the tangential space at each point.

**Sublemma** *Let* $(A_{\mu\nu})$ *be the inverse transport matrix of* $(F_{\mu\nu})$. *Then,*

(1) *If* $u \in U_\xi$, $|s| < 1$, *we have*

$$\sum_{\nu=1}^{4n+3} A_{\mu\nu}(su)u_\nu = u_\mu. \tag{3.12}$$

(2) *Define real functions* $c_{\mu\nu\nu'}$ on $U_\xi$ *by*

$$[X_\mu, X_\nu] = \sum_{\nu'=1}^{4n+3} c_{\mu\nu\nu'} X_{\nu'}, \tag{3.13}$$

*and matrices* $D(s,u) = (sA_{\mu\nu}(su))$ *and* $\Gamma(s,u)$ *with entries*

$$\Gamma_{\nu\nu'}(s,u) = \sum_{\mu=1}^{4n+3} c_{\mu\nu'\nu}(su)u_\mu \tag{3.14}$$

on $u \in U_\xi$, $|s| < 1$. *We have*

$$\frac{\partial D}{\partial s} = I - \Gamma D. \tag{3.15}$$

Now by Taylor's theorem, we have the expansion

$$F(su) = F(0) + sF^{(1)}(u) + s^2F^{(2)}(u) + \dots, \tag{3.16}$$

where $F(0) = I$ by $E_{\xi*}(\frac{\partial}{\partial u_\mu})|_0 = X_\mu|_\xi$ and $F^{(1)}, F^{(2)}, \dots$ are certain matrices. Let

$$\begin{aligned} A(su) &= I + sA^{(1)}(u) + s^2A^{(2)}(u) + \cdots, \\ D(s,u) &= sI + s^2A^{(1)}(u) + \cdots. \end{aligned} \tag{3.17}$$

Since $FA^t = I$, we have $F^{(1)} = -A^{(1)t}$. Let $\Gamma(s,u) = \Gamma^{(0)}(u) + s\Gamma^{(1)}(u) + \cdots$. Eq. (3.15) implies that $(1/2)\Gamma^{(0)t} = F^{(1)}$. Since $\Gamma^{(0)}_{\nu\nu'} = \sum_\mu c_{\mu\nu'\nu}(0)u_\mu$ are real and $u_{4n+\beta} = t_\beta = O^2$ for $\beta = 1, 2, 3$, we find that

$$\begin{aligned} F^{(1)}_{\mu(4n+j)} &= \frac{1}{2}\Gamma^{(0)t}_{\mu(4n+j)} = \frac{1}{2}\sum_{\nu=1}^{4n+3} c_{\nu\mu(4n+j)}(0)u_\nu \\ &= \frac{1}{2}\sum_{\nu=1}^{4n} c_{\nu\mu(4n+j)}(0)u_\nu + O^2. \end{aligned} \tag{3.18}$$

It remains to determine $c_{\nu\mu(4n+j)}(0)$ for $\mu, \nu = 1, \ldots, 4n$ and $j = 1, 2, 3$. Note that

$$
\begin{aligned}
[X_{4l+j}, X_{4l'+j'}] &= \sum_{m=1}^{4n+3} c_{(4l+j)(4l'+j')m} X_m \\
&= 2\delta_{ll'} \sum_{\beta=1}^{3} b_{jj'}^{\beta} T_\beta \quad \text{mod} \quad V,
\end{aligned} \tag{3.19}
$$

by the definition of coefficients $c$ in (3.13) and (3.7). It follows from (3.19) at the origin that

$$
c_{(4l+j)(4l'+j')(4n+\beta)}(0) = 2\delta_{ll'} b_{jj'}^{\beta}, \tag{3.20}
$$

for $l, l' = 0, \ldots, n-1,\ j,\ \ j' = 1, \ldots, 4$ and $\beta = 1, 2, 3$. Now in coordinates chart $U_\xi$,

$$
\begin{aligned}
X_{4l+j} &= \sum_{\nu=1}^{4n+3} F_{(4l+j)\nu}(u) \frac{\partial}{\partial u_\nu} \\
&= \frac{\partial}{\partial u_{4l+j}} + \sum_{\beta=1}^{3} F_{(4l+j)(4n+\beta)}(u) \frac{\partial}{\partial u_{4n+\beta}} + \sum_{\mu=1}^{4n} O^1 \frac{\partial}{\partial u_\mu} \\
&= \frac{\partial}{\partial u_{4l+j}} + \frac{1}{2} \sum_{\beta=1}^{3} \sum_{m=1}^{4n+3} c_{m(4l+j)(4n+\beta)}(0) u_m \frac{\partial}{\partial u_{4n+\beta}} + \mathcal{R} \\
&= \frac{\partial}{\partial u_{4l+j}} + \sum_{\beta=1}^{3} \sum_{j'=1}^{4} b_{j'j}^{\beta} u_{4l+j'} \frac{\partial}{\partial u_{4n+\beta}} + \mathcal{R}
\end{aligned} \tag{3.21}
$$

by using (3.18) and (3.20), where the remainders $\mathcal{R} = \sum_{\mu=1}^{4n} O^1 \frac{\partial}{\partial u_\mu} + \sum_{\beta=1}^{3} O^2 \frac{\partial}{\partial u_{4n+\beta}}$. The Lemma is proved. □

**Theorem 1.2** *(1)* $\Theta_\xi(\eta) = -\Theta_\eta(\xi) \in \mathbf{R}^{4n+3}$*;*
*(2)* $\Theta : \Omega \longrightarrow \mathbf{R}^{4n+3}$ *is* $C^\infty$*;*
*(3)* $\Theta_\xi^*(\mathrm{d}u_1 \cdots \mathrm{d}u_{4n+3})|_\xi$ *is the volume element on* $V$ *at* $\xi$*;*
*(4) Suppose* $(\xi, \eta), (\xi, \zeta), (\zeta, \eta) \in \Omega$ *and* $\rho(\xi, \eta) \le \varepsilon_0, \rho(\zeta, \eta) \le \varepsilon_0$ *for some sufficiently small constant* $\varepsilon_0 > 0$*. Then, there exists a constant* $C > 0$ *such that*

$$
\begin{aligned}
&\|\Theta(\xi, \eta) - \Theta(\zeta, \eta)\| \le C(\rho(\xi, \zeta) + \rho(\xi, \zeta)^{\frac{1}{2}} \rho(\xi, \eta)^{\frac{1}{2}}), \\
&\rho(\zeta, \eta) \le C(\rho(\xi, \zeta) + \rho(\xi, \eta)).
\end{aligned} \tag{3.22}
$$

*Namely,* $\rho(\cdot, \cdot)$ *is a local pseudodistance on* $M$*.*

*Proof* (1) follows from the definition of exponential map $E_\xi$; (2) follows from the well-known theorem in O.D.E. on smooth dependence of solutions on parameters; (3) follows from $\Theta_{\xi*}$ mapping $X_\mu|_\xi$ to $\frac{\partial}{\partial u_\mu}$, $\mu = 1, \ldots, 4n+3$. For (4), we regard $\zeta$ as a function of $\xi \in M$ and $u \in U_\xi$ by equation $\zeta = E_\xi(u)$. We write $\Theta(\zeta, \eta) = f(\eta, \xi, u) \in \mathbf{R}^{4n+3}$ with $f(\eta, \xi, 0) = \Theta(\xi, \eta)$. We expand $f$ in Taylor series about 0,

$$\Theta_\mu(\zeta, \eta) = \Theta_\mu(\xi, \eta) + \sum_{\nu=1}^{4n+3} a_{\mu\nu}(\eta, \xi) u_\nu + O(|u|^2), \tag{3.23}$$

where $u_\mu = \Theta_\mu(\xi, \zeta)$, $a_{\mu\nu}(\eta, \xi) = \frac{\partial f_\mu(\eta,\xi,0)}{\partial u_\nu}$. Using (3.23) for $\eta = \xi$, we have $u_\mu = \Theta_\mu(\xi, \zeta) = -\Theta_\mu(\zeta, \xi) = -\sum_{\nu=1}^{4n+3} a_{\mu\nu}(\xi, \xi)\, u_\nu + O(|u|^2)$. Hence, $a_{\mu\nu}(\xi, \xi) = -\delta_{\mu\nu}$ and $|a_{\mu\nu}(\eta, \xi)| = O(\rho(\eta, \xi))$ for $\mu \neq \nu$. It follows that

$$\begin{aligned} &|\Theta_{4n+\beta}(\xi, \eta) - \Theta_{4n+\beta}(\zeta, \eta)| \leq C(\rho(\xi, \zeta)^2 + \rho(\eta, \xi)\rho(\xi, \zeta)), \quad \beta = 1, 2, 3 \\ &|\Theta_\mu(\xi, \eta) - \Theta_\mu(\zeta, \eta)| \leq C\rho(\xi, \zeta), \quad \mu = 1, \ldots, 4n. \end{aligned} \tag{3.24}$$

Now the first inequality of (3.22) follows from (3.24). The second inequality of (4) follows from the first one easily as in p. 476 of [7]. We omit details.

For the standard contact forms $\theta_{\mathbb{H}}$, the associated metric is defined by (2.22). Its curvature is identically zero. For $u \in C^1(\mathbb{H}^n)$, $\mathrm{d}u = \sum_{\mu=1}^{4n} Y_\mu u \cdot \theta^\mu + \sum_{\beta=1}^{3} \frac{\partial u}{\partial t_\beta} \cdot \theta_{\mathbb{H},\beta}$ and $\mathrm{d}_b u = \sum_{\mu=1}^{4n} Y_\mu u \cdot \theta^\mu$, where $\theta^\mu = \mathrm{d}x_\mu$. Note that $\langle Y_\mu, Y_\nu \rangle = 2\delta_{\mu\nu}$ and $\langle \theta^\mu, \theta^\nu \rangle_{\theta_{\mathbb{H}}} = \frac{1}{2}\delta_{\mu\nu}$, $\mu, \nu = 1, \ldots, 4n$. Hence,

$$\langle \mathrm{d}_b u, \mathrm{d}_b v \rangle_{\theta_{\mathbb{H}}} = \frac{1}{2} \sum_{\mu=1}^{4n} Y_\mu u \cdot Y_\mu v \tag{3.25}$$

if $u$ and $v$ are real valued. The SubLaplacian is

$$\Delta_{\mathbb{H}} = -\frac{1}{2} \sum_{\mu=1}^{4n} Y_\mu Y_\mu. \tag{3.26}$$

□

**Corollary 3.3** *Let $\{X_1, \ldots, X_{4n}\}$ be a local basis of $V$ such that $g_\theta(X_\mu, X_\nu) = 2\delta_{\mu\nu}$. Then, in the normal coordinates defined by Theorem 1.2, we have*

*(1)* $(\Theta_\xi^{-1})^* \theta_\beta = \theta_{\mathbb{H},\beta} + O^1 \mathrm{d}t + O^2 \mathrm{d}x$, $\beta = 1, 2, 3$;
*(2)* $(\Theta_\xi^{-1})^* \psi_\theta = (1 + O^1)\psi_{\mathbb{H}}$;
*(3)* $(\Theta_\xi)_* \Delta_\theta = \Delta_{\mathbb{H}} + \mathcal{E}(\partial_x) + O^1 \mathcal{E}(\partial_t; \partial_x^2) + O^2 \mathcal{E}(\partial_x \cdot \partial_t) + O^3 \mathcal{E}(\partial_t^2)$.
*where $\psi_{\mathbb{H}} = \psi_{\theta_{\mathbb{H}}}$ and $O^k\mathcal{E}$ indicates an operator involving combinations of the indicated derivatives with coefficients in $O^k$.*

*Proof* (1) follows from the expansion of $X_\mu$ in Proposition 3.1; (2) follows from (1). Since $\Gamma_{\mu\nu}^{\nu'}$ bounded, $\Gamma_{\mu\nu}^m X_m = \mathcal{E}(\partial_x) + O^1\mathcal{E}(\partial_t)$ by expansion (3.10). Part (3) follows from the expansion of $X_\mu$ in Proposition 3.1 and the expression of the SubLaplacian in (2.31). □

*Remark 3.4* Folland and Stein [7] used normal coordinates to osculate strictly pseudoconvex CR structure by the Heisenberg group (see also [14]). For CR manifolds of high codimension, we constructed normal coordinates and osculated the CR structure at each point by a nilpotent Lie group of step two in [15].

## 4 The regularity of $\Delta_\theta$

The Green functions of the SubLaplacians on general step-two nilpotent Lie groups are constructed in [2]. We summarize the results in Theorems 1–3 and their proofs in [2] in the following theorem.

**Theorem 4.1** *Let* $a^\alpha = (a^\alpha_{jk})$ *be an antisymmetric* $N \times N$ *matrix, and the eigenvalues of* $\Omega(\tau) = -i\sum_{\alpha=1}^{q} a^\alpha \tau_\alpha$ *be* $\lambda_1(\tau), \ldots, \lambda_N(\tau)$*, and*

$$\Delta = \frac{1}{2}\sum_{j=1}^{N} L_j^2, \tag{4.1}$$

*where*

$$L_j = \frac{\partial}{\partial x_j} + \sum_{\alpha=1}^{r}\sum_{k=1}^{N} a^\alpha_{jk} x_k \frac{\partial}{\partial t_\alpha}, \tag{4.2}$$

$j = 1, \ldots, N$, *on* $\mathbf{R}^{N+r}$*. Suppose vector fields satisfy the condition* $\mathcal{C}$*, i.e.* $[L_j, L_k]$ *and* $L_j$, $j, k = 1, \ldots, N$*, span the tangential space of* $\mathbf{R}^{N+r}$*. Let*

$$f(x,t,\theta) = \frac{1}{2}\langle \Omega(\tau)\coth\Omega(\tau)x, x\rangle - i\sum_{\alpha=1}^{r} t_\alpha\tau_\alpha, \quad t \in \mathbf{R}^r,\ x \in \mathbf{R}^N,$$
$$V(\theta) = \left(\frac{\det\Omega(\tau)}{\det\sinh\Omega(\tau)}\right)^{\frac{1}{2}} = \prod_{k=1}^{N}\left(\frac{\lambda_k(\tau)}{\sinh\lambda_k(\tau)}\right)^{\frac{1}{2}}. \tag{4.3}$$

*Then the following integral converges provided* $|z| + |t| \neq 0$ *and is the Green function of* $\Delta$,

$$G(x,t) = -\frac{\Gamma(s)}{(2\pi)^{s+1}}\int_{\mathbf{R}^r}\frac{V(\tau + i\varepsilon\hat{t})}{f(x,t,\tau+i\varepsilon\hat{t})^s}\,\mathrm{d}\tau, \tag{4.4}$$

*where* $s = \frac{N}{2} + r - 1$, $\hat{t}$ *is the unit vector of* $t$, $\varepsilon$ *is a small positive number. If* $|z| \neq 0$,

$$G(x,t) = -\frac{\Gamma(s)}{(2\pi)^{s+1}}\int_{\mathbf{R}^r}\frac{V(\tau)}{f(x,t,\tau)^s}\,\mathrm{d}\tau, \tag{4.5}$$

*(4.4) is the integral changing the contour of (4.5). Equation (4.4) is independent on* $\varepsilon$ *for* $|z| \neq 0$.

*Furthermore,* $G$ *is* $C^\infty$ *on* $\mathbf{R}^{N+r} \setminus \{0\}$ *and is homogeneous of degree* $-N-2r+2$*, i.e.* $G(\delta x, \delta^2 t) = \delta^{-N-2r+2}G(x,t)$*, and*

$$|G(x,t)| \leq C\|(x,t)\|^{-N-2r+2} \tag{4.6}$$

*for some constant* $C > 0$*, where the norm* $\|(x,t)\| = (|x|^4 + |t|^2)^{1/4}$.

In the case of the quaternionic Heisenberg group, $r = 3$, $N = 4n$ and $\Delta = -\Delta_{\mathbb{H}}$. It is easy to see that the condition $\mathcal{C}$ is satisfied for left invariant vector fields (2.17) by relations (2.18).

**Proposition 4.2** (Sobolev-type inequality for the critical exponent on $\mathbb{H}^n$) *There exists a constant $C_n > 0$ such that*

$$\left(\int_{\mathbb{H}^n} |u|^p \psi_{\mathbb{H}}\right)^{\frac{2}{p}} \leq C_n \int_{\mathbb{H}^n} \sum_{\mu=1}^{4n} |Y_\mu u|^2 \psi_{\mathbb{H}} \tag{4.7}$$

*for each $u \in C_0^\infty(\mathbb{H}^n)$, where $p = 2Q/(Q-2)$.*

This proposition is a special case of Sobolev inequalities on equiregular Carnot–Carathéodory spaces (including quaternionic contact manifolds and quaternionic Heisenberg group) proved by Gromov in Sect. 2.4 in [10].

**Corollary 4.3** $0 < \lambda(\mathbb{H}^n) < \infty$.

Now let $U$ be a relatively compact open set of a normal coordinates neighborhood $\Omega_\xi \subset M$. $\{X_1, \ldots, X_{4n}\}$ be an orthonormal basis of $V$ under $g_\theta$. Define

$$\|u\|_{S_k^m(U)} = \sum_{k' \leq k} \|X_{\mu_1} \cdots X_{\mu_{k'}} u\|_{L^m(U)} \tag{4.8}$$

for $u \in C^\infty(M)$, where the summation takes over all multiindices $(\mu_1, \ldots, \mu_{k'})$ with $k' \leq k$. The *Folland–Stein space* $S_k^m(U)$ is the completion of $C^\infty(M)$ with respect to this norm. Define

$$\|u\|_{\Gamma_\beta(U)} = \sup_{x \in U} |u(x)| + \sup_{x,y \in U} \frac{|u(x) - u(y)|}{\rho(x,y)^\beta}, \quad \text{for } 0 < \beta < 1, \tag{4.9}$$

and $\Gamma_\beta(U)$ is the completion of $C^\infty(M)$ with respect to this norm. For $k < \beta < k+1$,

$$\Gamma_\beta(U) = \{u \in C^0(U); X^L u \in \Gamma_{\beta-k} \text{ for any multiindices } L \text{ with } |L| \leq k\} \tag{4.10}$$

with norm

$$\|u\|_{\Gamma_\beta(U)} = \sup_{|L| \leq k} \|X^L u\|_{\Gamma_{\beta-k}(U)}, \tag{4.11}$$

where $X^L = X_{\mu_1} \cdots X_{\mu_{k'}}$ for $L = (\mu_1, \ldots, \mu_{k'})$, $|L| = k'$.

Now choose an open covering $\{U_1, \ldots, U_s\}$ of $M$ such that each $U_j$ is a relatively compact open set of a normal coordinates neighborhood $\Omega_\xi \subset M$ for some $\xi \in M$. Let $\{\phi_j\}_{j=1}^s$ be a unit partition of $M$ such that $\operatorname{supp} \phi_j \subset U_j$. Define

$$S_k^m(M) = \left\{u; \phi_j u \in S_k^m(U_j) \text{ for all } j\right\}. \tag{4.12}$$

**Proposition 4.4** *For each positive non-integer $\beta$, $0 < r < \infty$ and integer $k \geq 1$, there exists a constant $C > 0$ such that for $u \in C_0^\infty(U)$,*

*(1)* $\|u\|_{\Gamma_\beta(U)} \le C\|u\|_{S_k^m(U)}$ *with* $\beta = k - \frac{Q}{m}$*;*
*(2)* $\|u\|_{\Lambda_{\frac{\beta}{2}}(U)} \le C\|u\|_{\Gamma_\beta(U)}$*;*
*(3)* $\|u\|_{S_2^m(U)} \le C(\|\Delta_\theta u\|_{L^m(U)} + \|u\|_{L^m(U)})$*;*
*(4)* $\|u\|_{\Gamma_{\beta+2}(U)} \le C(\|\Delta_\theta u\|_{\Gamma_\beta(U)} + \|u\|_{\Gamma_\beta(U)})$.

For strictly pseudoconvex CR manifolds, such estimates for $\Box_b$ were proved by Folland and Stein in Theorem 21.1, 20.1, 16.6 and 15.20 in [7]. Their arguments also apply to $\Delta_b$ (Proposition 5.7 in [11]). If we use the Green function of $\Delta_{\mathbb{H}}$ provided by Theorem 4.1 and normal coordinates provided by Theorem 1.2, their arguments apply verbatim to $\Delta_\theta$.

Harnack inequality and Poincaré-type inequality for general Hörmander system of vector fields are well known now. (See [4, 5], for example).

**Proposition 4.5** (Harnack inequality) *Suppose* $f \in L^\infty(U)$, $u \in L^p(U)$, $u \ge 0$ *and* $(\Delta_\theta + f)u = 0$ *in sense of distribution on* $U$*. Then, for any* $K \subset\subset U$,

$$\max_{x\in K} u(x) \le C \min_{x\in K} u(x), \tag{4.13}$$

*the constant* $C$ *depends only on* $K$, $\|u\|_{L^p(U)}$, $\|f\|_{L^\infty(U)}$ *and the frame constants.*

**Proposition 4.6** (Poincaré-type inequality) *Let* $B_a \subset U$ *be a ball of radius* $a$ *with respect to quasidistance* $\rho$*. Then, for each* $f$ *with* $\sum_{\mu=1}^{4n} |X_\mu f|^q \in L^1(U)$*, we have*

$$\int_{B_a} |f - f_{B_a}|^q \psi_\theta \le Ca^q \int_M \sum_{\mu=1}^{4n} |X_\mu f|^q \psi_\theta, \tag{4.14}$$

*where* $f_{B_a} = (\int_{B_a} f\psi_\theta)/(\int_{B_a} \psi_\theta)$*, the average of* $f$.

By estimates in Proposition 4.4 and Harnack inequality, the following regularity results involving critical exponent can be proved in the same way as Proposition 5.10 and 5.15 in [11].

**Proposition 4.7** *Suppose* $f \in L^{Q/2}(U)$, $u \in L^p(U)$, $u \ge 0$ *and* $(\Delta_\theta + f)u = 0$ *in sense of distribution on* $U$*. Then, for any* $\eta \in C_0^\infty(U)$, $\eta u \in L^s(U)$ *for each* $0 < s < \infty$*. If* $f \in L^s(U)$ *with* $s > Q/2$*, then* $u \in \Gamma_\beta(K)$ *for some* $\beta > 0$ *and any* $K \subset\subset U$.

**Theorem 4.8** *Suppose* $f, g \in C^\infty(U)$, $u \ge 0$ *on* $U$, $u \in L^r(U)$, $r > p$ *and* $\Delta_\theta u + gu = fu^{q-1}$ *in sense of distribution on* $U$ *for some* $2 \le q \le p = 2Q/(Q-2)$*. Then,* $u \in C^\infty(U)$ *and* $u > 0$*. If* $K \subset\subset U$*, then* $\|u\|_{C^k(K)}$ *depends only on* $K$, $\|u\|_{L^r(K)}$, $\|f\|_{C^k(K)}$, $\|g\|_{C^k(K)}$*, the frame constants, but not on* $q$.

The following interpolation inequality for the space $S_1^m$ is just the consequence of the above Poincaré-type inequality.

**Proposition 4.9** (Interpolation inequality for the space $S_1^m$) *If* $u \in L^1(U)$ *and* $\sum_{\mu=1}^{4n} |X_\mu u|^m \in L^1(U)$ *with* $1 < m < \infty$*, then* $u \in S_1^m(U)$ *and*

$$\|u\|_{S_1^m(U)} \le C\left(\left\|\sum_{\mu=1}^{4n} |X_\mu u|^m\right\|_{L^1(U)} + \|u\|_{L^1(U)}\right). \tag{4.15}$$

**Proposition 4.10** *If* $1 < s < p = 2Q/(Q-2)$, *then imbedding* $S_1^2(M) \subset\subset L^s(M)$ *is compact.*

*Proof* By arguments in [7], we have continuous inclusion $S_1^2(M) \subset W_{1/2}^2(M)$, where $W_{1/2}^2(M)$ is the usual Sobolev space. Now the usual Sobolev imbedding theorem guarantees the inclusion $W_{1/2}^2(M) \subset L^1(M)$ to be compact. Suppose $\{u_l\}$ is a bounded sequence of $S_1^2(M)$. Then, there exists a subsequence, which is still denoted by $u_l$, converging to $u$ in $L^1(M)$. Note that

$$\|u_l - u_{l'}\|_{L^s(M)} \le \|u_l - u_{l'}\|_{L^1(M)}^a \|u_l - u_{l'}\|_{L^p(M)}^{1-a}, \quad a = \frac{(1/s) - (1/p)}{1 - (1/p)} \tag{4.16}$$

by Hölder's inequality (cf. [1], p. 89), where $\|u_l - u_{l'}\|_{L^p(M)}$ is bounded by the Sobolev-type inequality (4.7) for critical exponent $p$, which also holds on manifold $M$. We see that $\{u_l\}$ is also a Cauchy sequence in $L^s$. The compactness follows. □

## 5 The existence of extremal

The extremal problem (1.7) in $\mathbb{H}^n$ is

$$\lambda(\mathbb{H}^n) = \inf\left\{\frac{1}{2}\int_{\mathbb{H}^n} b_n \sum_{\mu=1}^{4n} |Y_\mu u|^2 \psi_{\mathbb{H}};\ \int_{\mathbb{H}^n} |u|^p \psi_{\mathbb{H}} = 1\right\}, \tag{5.1}$$

where $\psi_{\mathbb{H}} = \psi_\theta$ defined by (2.27) with $\theta = \theta_{\mathbb{H}}$.

**Lemma 5.1** $\lambda(M) \le \lambda(\mathbb{H}^n)$ *for any compact quaternionic contact manifold* $M$.

*Proof* The class of test functions defining $\lambda(\mathbb{H}^n)$ can be restricted to $C^\infty$ function with compact support. The proof is similar to that in [11, 16]. For each $\varepsilon > 0$, choose $u \in C_0^\infty(\mathbb{H}^n)$ such that $B_{\theta_{\mathbb{H}}}(u) = 1$ and $A_{\theta_{\mathbb{H}}}(u) < \lambda(\mathbb{H}^n) + \varepsilon$. Let $u_\delta(\zeta) = \delta^{-(Q-2)/2} u(D_{\delta^{-1}}\zeta)$ for $\zeta \in \mathbb{H}^n$. For fixed $\xi \in M$, set $v_\delta(\eta) = u_\delta(\Theta_\xi(\eta))$. For $\delta$ sufficiently small, supp $u_\delta$ is contained in $\Theta_\xi(\Omega_\xi)$. Hence, $v_\delta$ has compact support in $\Omega_\xi$. It can be extended to a $C^\infty$ function on $M$. It is easy to check that $B_{\theta_{\mathbb{H}}}(u_\delta) = B_{\theta_{\mathbb{H}}}(u) = 1$ and $A_{\theta_{\mathbb{H}}}(u_\delta) = A_{\theta_{\mathbb{H}}}(u) < \lambda(\mathbb{H}^n) + \varepsilon$ by rescaling. Also, $\int_{\mathbb{H}^n} |u_\delta|^2 \psi_{\mathbb{H}} = \delta^2 \int_{\mathbb{H}^n} |u|^2 \psi_{\mathbb{H}} \longrightarrow 0$ as $\delta \longrightarrow 0$. By Corollary 3.3, we have

$$\begin{aligned}
((D_{\delta^{-1}}\Theta_\xi)^{-1})^*\theta_\beta &= \delta^2\theta_{\mathbb{H},\beta} + \delta^3(O^1 \mathrm{d}t + O^2 \mathrm{d}x) \\
((D_{\delta^{-1}}\Theta_\xi)^{-1})^*\psi_\theta &= \delta^Q(1 + \delta O^1)\psi_{\mathbb{H}} \\
((D_{\delta^{-1}}\Theta_\xi)_* X_\mu &= \delta^{-1}(Y_\mu + \delta O^1\mathcal{E}(\partial_x) + \delta^1 O^2 \mathcal{E}(\partial_t)), \quad \mu = 1, \ldots, 4n, \\
(D_{\delta^{-1}}\Theta_\xi)_*\Delta_\theta &= \delta^{-2}\big(\Delta_{\mathbb{H}} + \delta\mathcal{E}(\partial_x) + \delta O^1\mathcal{E}\big(\partial_t; \partial_x^2\big) + \delta O^2\mathcal{E}(\partial_x\partial_t) \\
&\quad + \delta O^3\mathcal{E}\big(\partial_t^2\big)\big),
\end{aligned} \tag{5.2}$$

where $(Y_1, \dots, Y_{4n})$ are left invariant vector fields of the quaternionic Heisenberg group in (2.17). Therefore,

$$B_\theta(v_\delta) = \int_M |u_\delta(\Theta_\xi(\eta))|^p \psi_\theta = \int_{\mathbb{H}^n} \delta^{-Q} |u|^p ((D_{\delta^{-1}} \cdot \Theta_\xi)^{-1})^* \psi_\theta$$
$$= \int_{\mathbb{H}^n} |u|^p (1 + \delta O^1) \psi_{\mathbb{H}} \longrightarrow B_{\theta_{\mathbb{H}}}(u) = 1. \quad (5.3)$$

For any fixed $a > 0$, $D_{\delta^{-1}} \Theta_\xi(\Omega_\xi) \supset B_a$ for sufficiently small $\delta$. Similarly, $A_\theta(v_\delta) \longrightarrow A_{\theta_{\mathbb{H}}}(u) < \lambda(\mathbb{H}^n) + \varepsilon$. Since $\varepsilon$ is arbitrary, the lemma follows.

For each $2 \le q < p = 2Q/(Q-2)$, consider the following variational problem,

$$\lambda_q(M) = \inf\{A_\theta(u); u \in S_1^2(M), B_{\theta,q}(u) = 1\}, \quad (5.4)$$

where

$$B_{\theta,q}(u) = \int_M |u|^q \psi_\theta. \quad (5.5)$$

□

**Theorem 5.2** *For each $2 \le q < p = 2Q/(Q-2)$, there exists a positive $C^\infty$ solution $u_q$ to the equation*

$$b_n \Delta_\theta u_q + s_\theta u_q = \lambda_q(M) u_q^{q-1}, \quad (5.6)$$

*satisfying $A_\theta(u_q) = \lambda_q(M)$ and $B_{\theta,q}(u_q) = 1$.*

The proof of this theorem is exactly the same as that of Theorem 6.2 in [11] by using the compactness of Sobolev embedding in Proposition 4.10 and regularity results in Proposition 4.7 and Theorem 4.8. To prove Theorem 1.1, it is enough to prove the following theorem.

**Theorem 5.3** *If $\lambda(M) < \lambda(\mathbb{H}^n)$, then there exists a sequence $q_l$ tending to $p$ from below such that $u_{q_l}$ converges in $C^m(M)$ for any $m$ to a function $u \in C^\infty(M)$ such that $u > 0$, and*

$$b_n \Delta_\theta u + s_\theta u = \lambda(M) u^{p-1}. \quad (5.7)$$

*with $A_\theta(u) = \lambda(M)$ and $B_{\theta,p}(u) = 1$.*

The following behavior of $\lambda_q$ can be proved as Lemma 6.4 in [11].

**Proposition 5.2** *(1) If $\lambda_q(M) < 0$ for some $q$, then $\lambda_q(M) < 0$ for all $q \ge 2$. $\lambda_q$ is a nondecreasing function of $q$.*

*(2) If $\lambda_q(M) \ge 0$ for some $q \ge 2$, then $\lambda_q$ is nonincreasing of $q$ and is continuous from left.*

*Proof of Theorem 5.3. Case 1.* $\lambda(M) < 0$. □

Let $u_q$ be a $C^\infty$ solution of Eq. (5.6) given by Theorem 5.2. For each $2 \le q < p = 2Q/(Q-2)$, $\phi \in S_1^2(M)$, we have

$$\int_M (\langle \mathrm{d}_b u_q, \mathrm{d}_b \phi \rangle_\theta + s_\theta u_q \phi) \psi_\theta = \int_M \lambda_q u_q^{q-1} \phi \psi_\theta. \quad (5.8)$$

Let $\phi = u_q^{q-1}$. Since $\lambda_q(M) < 0$,

$$\int_M (q-1)u_q^{q-2}|\mathrm{d}_b u_q|_\theta^2 \psi_\theta \leq \int_M |s_\theta u_q^q|\psi_\theta. \tag{5.9}$$

It follows that

$$\int_M |\mathrm{d}_b w_q|^2 \psi_\theta \leq C \int_M w_q^2 \psi_\theta = C \int_M u_q^q \psi_\theta = C \tag{5.10}$$

for $w_q = u_q^{q/2}$ and some constant $C > 0$. By Sobolev-type inequality (4.7), we have $\int_M w_q^p \leq C'$ for some positive constant $C'$. Now let $q_0 > 2$ and $r = q_0/2p > p$. Then, for $q \geq q_0$, $\|u_q\|_{L^r(M)}$ is uniformly bounded as $q \longrightarrow p$. It follows from regularity result in Theorem 4.8 that $\|u_q\|_{C^k(M)}$ is uniformly bounded, and there exists a subsequence $u_{q_j}$ converging in $C^k(M)$ to $u$ for any $k$. The limit $u$ satisfies $b_n \Delta_\theta u + s_\theta u = \lambda u^{p-1}$ with $\lambda = A_\theta(u) = \lim \lambda_{q_j}$ and $B_{\theta,p}(u) = 1$. We also have $\lambda \leq \lambda(M)$. Consequently, $\lambda = \lambda(M)$ by the definition of $\lambda(M)$.

*Case 2.* $\lambda(M) \geq 0$.

*Case 2i. For some sequence $q_j \longrightarrow p$, $\sup_M |\mathrm{d}_b u_{q_j}|_\theta$ is uniformly bounded.* By interpolation inequality (4.15), $u_{q_j}$ are uniformly bounded in $S_1^m(M)$ for any $m$. Consequently, $u_{q_j}$ are in $L^m(M)$ for any $m$. The result follows as in the case 1.

*Case 2ii.* $\sup_M |\mathrm{d}_b u_q|_\theta \longrightarrow \infty$ *as* $q \longrightarrow p$.

Choose $\xi_q \in M$ such that $|\mathrm{d}_b u(\xi_q)|_\theta = \sup_M |\mathrm{d}_b u_q|_\theta$. Let $\Theta_{\xi_q}$ be the normal coordinates defined in Theorem 1.2. Without loss of generality, we can assume that there is a fixed neighborhood $U$ of the origin of $\mathbb{H}^n$ contained in the image of $\Theta_{\xi_q}$ for all $q$. We will identify $U$ with a neighborhood of $\xi_q$ by $(x,t) = \Theta_{\xi_q}(\xi)$, $\xi \in U$. Define $(\tilde{x}, \tilde{t}) = D_{\delta_q^{-1}}(x,t)$,

$$\tilde{\theta}_{\mathbb{H},\beta} = \mathrm{d}\tilde{t}_\beta - \sum_{l=0}^{n-1} \sum_{j,k=1}^{4} b_{kj}^\beta \tilde{x}_{4l+k} \mathrm{d}\tilde{x}_{4l+j} = \delta_q^{-2}(D_{\delta_q}^* \theta_{\mathbb{H}^n,\beta}), \tag{5.11}$$

and on the set $D_{\delta_q^{-1}} U$ with coordinates $(\tilde{x}, \tilde{t})$, define

$$h_q(\tilde{x}, \tilde{t}) = \delta_q^{\frac{2}{q-2}} u_q\big(\delta_q \tilde{x}, \delta_q^2 \tilde{t}\big) \tag{5.12}$$

where $\delta_q > 0$ is so chosen that $|\mathrm{d}_b h_q(0)|_{\tilde{\theta}_{\mathbb{H}^n}} = 1$. Since

$$|\mathrm{d}_b h_q(0)|_{\tilde{\theta}_{\mathbb{H}^n}} = \delta_q^{1+\frac{2}{q-2}} |\mathrm{d}_b u_q(\xi_q)|_\theta, \tag{5.13}$$

we see that $\delta_q \longrightarrow 0$ as $q \longrightarrow p$ and $D_{\delta_q{}^{-1}} U \longrightarrow \mathbb{H}^n$.

Denote

$$\theta_{(q)} = \delta_q^{-2}(D_{\delta_q}^* \theta), \quad \tilde{\Delta}_q = \Delta_{\theta_{(q)}} \tag{5.14}$$

in coordinates $(\tilde{x}, \tilde{t})$ on the region $D_{\delta_q{}^{-1}} U$. Note that

$$\big(\Delta_{\delta_q^{-2}(D_{\delta_q}^* \theta)} u_q\big)(\eta) = \delta_q^2 \big(\Delta_{D_{\delta_q}^* \theta} u_q\big)(\eta) = \delta_q^2 \big(\Delta_\theta u_q\big)(D_{\delta_q} \eta), \tag{5.15}$$

by definition of the SubLaplacians. The equation for $h_q$ is

$$b_n \tilde{\Delta}_q h_q + s_q \delta_q^2 h_q = \lambda_q(M) h_q^{q-1}, \tag{5.16}$$

where $s_q$ is the scalar curvature of $\theta$ in the coordinates $(x, t)$. Also, $|s_q| \leq \|s_\theta\|_{L^\infty(M)}$.

If necessary by passing to a subsequence, we can assume $\xi_q \longrightarrow \xi \in M$. Denote by $(X_1^q, \ldots, X_{4n}^q)$ the frame (i.e. local orthonormal basis with norm $\sqrt{2}$) used to define $\Theta_{\xi_q}$, we may assume it converging to $(X_1, \ldots, X_{4n})$. Let

$$Y_\mu^q = \delta_q D_{\delta_q^{-1} *} X_\mu^q, \quad \mu = 1, \ldots, 4n. \tag{5.17}$$

Then $(Y_1^q, \ldots, Y_{4n}^q)$ is a frame for $\theta_{(q)}$. By using (5.2), we see that $(Y_1^q, \ldots, Y_{4n}^q)$ converge in $C^k(B_a)$ to $(Y_1, \ldots, Y_{4n})$ for each $k, a > 0$ as $\delta_q \longrightarrow 0$, where $(Y_1, \ldots, Y_{4n})$ are left invariant vector fields of the quaternionic Heisenberg group in (2.17). Similarly, $\theta_{(q)}$ and $\tilde{\Delta}_q$ converge uniformly in $C^k(B_a)$ to $\theta_{\mathbb{H}}$ and $\Delta_{\mathbb{H}}$ for each $k, a > 0$ by using (5.2), respectively.

Note that $|dh_q|_{\theta_{(q)}}$ is bounded in $B_{2a}$ since it attains its maximum at the origin, and

$$\int_{|(\tilde{x},\tilde{t})|<a} |h_q(\tilde{x}, \tilde{t})|^q \psi_{\theta_{(q)}} = \delta_q^{\frac{2q}{q-2} - Q} \int_{|(x,t)|<\delta_q a} |u_q(x, t)|^q \psi_\theta. \tag{5.18}$$

When $q < p$, we have that $2q/(q-2) - Q > 0$ and the right-hand side of (5.18) is bounded. Moreover, $\psi_{\theta_{(q)}} = (1 + \delta O^1)\psi_{\mathbb{H}}$ on $B_a$ by (5.2). We find that $h_q \in L^q(B_a, \psi_{\mathbb{H}})$ with uniform bound. Consequently, $h_q \in L^1(B_a, \psi_{\mathbb{H}})$ with a uniform bound. This fact together with $|d_b h_q|_{\theta_{(q)}}$ uniformly bounded by 1 gives $h_q \in S_1^m(B_a, \psi_{\mathbb{H}})$ for each $m < \infty$ by interpolation inequality (4.15). Consequently, $\eta h_q \in L^m(M)$ for each $m \geq 1$, and by Theorem 4.8, $\eta h_q$ is uniformly bounded in $C^k(B_a)$ for each $k$.

Now take a subsequence $q_j \longrightarrow p$ such that $h_{q_j}$ converges in $C^2(B_a)$. Thus, it defines a function $u$ on $\mathbb{H}^n$ by first choosing a subsequence $h_{q_j}$ convergent in $C^2(B_1)$; then choosing a subsequence of $h_{q_j}$ convergent in $C^2(B_2)$, etc. Note $u \geq 0$, $u \in C^2(\mathbb{H}^n)$ and $u$ is not zero since $|du(0)|_{\theta_{\mathbb{H}}} = 1$. Since $\theta_{(q_j)} \longrightarrow \theta_{\mathbb{H}}$ and $\lambda_{q_j}(M) \longrightarrow \lambda(M)$ by the continuity of $\lambda_q(M)$ from left in $q$ in Proposition 5.2, by letting $q_j \longrightarrow p$ in (5.16), we have that for $\phi \in C_0^\infty(\mathbb{H}^n)$,

$$\int_{\mathbb{H}^n} (b_n \langle d_b u, d_b \phi \rangle_{\theta_{\mathbb{H}}} - \lambda(M) u^{p-1} \phi) \psi_{\mathbb{H}} = 0. \tag{5.19}$$

Since $\psi_{\theta_{(q_j)}} \longrightarrow \psi_{\mathbb{H}}$ by (5.2), (5.18) implies $\int_{B_a} u^p \psi_{\mathbb{H}} \leq 1$ for each $a > 0$. Hence, $\int_{\mathbb{H}^n} u^p \psi_{\mathbb{H}} \leq 1$. On the other hand,

$$\begin{aligned} \int_{B_a} |d_b u|^2_{\theta_{\mathbb{H}^n}} \psi_{\mathbb{H}^n} &= \lim_{j \longrightarrow \infty} \int_{B_a} |d_b h_{q_j}|^2_{\theta_{(q_j)}} \psi_{\theta_{(q_j)}} \\ &= \lim_{j \longrightarrow \infty} \int_M \delta_{q_j}^{\frac{2q_j}{q_j - 2} - Q} |d_b u_{q_j}|^2_\theta \psi_\theta \\ &\leq \limsup_{j \longrightarrow \infty} \int_M |d_b u_{q_j}|^2_\theta \psi_\theta < \infty \end{aligned} \tag{5.20}$$

by $\delta_{q_j}^{2q_j/(q_j-2)-Q} \leq 1$. By taking a subsequence $\phi_l \in C_0^\infty(\mathbb{H}^n)$ to approximate $u$ in (5.19), we find that

$$b_n \int_{\mathbb{H}^n} |\mathrm{d}u|^2_{\theta_{\mathbb{H}}} \psi_{\mathbb{H}} = \lambda(M) \int_{\mathbb{H}^n} |u|^p \psi_{\mathbb{H}} \tag{5.21}$$

Now taking $\tilde{u} = \frac{u}{\|u\|_p}$, we have

$$b_n \int_{\mathbb{H}^n} |\mathrm{d}\tilde{u}|^2_{\theta_{\mathbb{H}}} \psi_{\mathbb{H}} = \lambda(M) < \lambda(\mathbb{H}^n), \quad \|\tilde{u}\|_p = 1 \tag{5.22}$$

which contradicts the definition of $\lambda(\mathbb{H}^n)$. Thus, case 2ii is impossible and the theorem is proved. $\square$

Similar to [11], we have

**Proposition 5.5** *If $\lambda(M) \leq 0$, any two contact forms conformal to $\theta$ with constant scalar curvatures are constant multiplier of each other.*

**Acknowledgements** This study was partially supported by the National Nature Science Foundation in China.

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