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Marino—Prodi perturbation type results and Morse indices of minimax critical points for a class of functionals in Banach spaces

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Abstract In this work, we study a class of Euler functionals defined in Banach spaces, associated with quasilinear elliptic problems involving p-Laplace operator ($p > 2$). First we obtain perturbation results in the spirit of the remarkable paper by Marino and Prodi (Boll. U.M.I. (4) 11(Suppl. fasc. 3): 1–32, 1975), using the new definition of *nondegeneracy* given in (Ann. Inst. H. Poincaré: Analyse Non Linéaire. 2:271–292, 2003). We also extend Morse index estimates for minimax critical points, introduced by Lazer and Solimini (Nonlinear Anal. T.M.A. 12:761–775, 1988) in the Hilbert case, to our Banach setting.

Keywords p-Laplace operator · Perturbation results · Morse theory

Mathematics Subject Classification (1991) 58E05, 35B20, 35J60, 35J70

1 Introduction

In recent years, there has been a growing interest towards variational differential problems, involving p-Laplacian ($p > 2$), which arise naturally in various physical contexts, for instance, in the study of non-Newtonian fluids and in the study of elasticity problems (cf. [21, 25]).

It seems to be a very interesting problem to understand if global Morse relations can be used to obtain multiplicity results of solutions for such quasilinear equations.

It is well known that Morse theory in Hilbert spaces has been largely used to obtain multiplicity results of solutions for semilinear elliptic equations, having a variational structure. It is standard to find such solutions as critical points of an

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Euler functional f, defined on a Hilbert space H. Under a compactness assumption on the sublevels of the Euler functional, the so-called (P.S.) condition, global Morse relations can be applied to find a lower bound on the number of critical points of f. A basic requirement, for obtaining multiplicity of different solutions, is the *nondegeneracy* condition for the critical points of f, and this is a reasonable assumption, as a remarkable perturbation result in [26], due to Marino and Prodi, holds. This perturbation theorem guarantees that any C^2 functional f on H, having an isolated critical point u, such that the second derivative $f''(u)$ of f in u is a Fredholm operator, can be approximated locally by a C^2 functional g, having a finite number of *nondegenerate* critical points, i.e., critical points in which g'' is an isomorphism from H to its dual space. Furthermore, in [26] similar perturbation results are proved near compact sets of critical points of the functional.

We mention that perturbation results for G-invariant functionals with isolated critical G-orbits are successively proved by Viterbo in [37].

We remark that all these perturbation results by Marino and Prodi rely on an infinite dimensional version of Sard's Theorem, due to Smale [31], and thus they work under the crucial assumption that the second derivative of the Euler functional $f''(u)$ in the critical point u is a Fredholm operator.

Finally, for the approximating functional g, a classical result in Morse theory for Hilbert spaces, based on Morse Lemma, can be applied to evaluate the local behavior of the Euler functional near nondegenerate critical points, via differential notions (see Theorem 4.1 in [13, Chapter 1]).

For quasilinear equations involving p-Laplacian, it happens that the associated Euler functionals are in general defined in a Banach space, which cannot be equipped with an equivalent Hilbert norm. It is known that global Morse relations hold in a Banach setting if the (P.S.) condition holds (c.f. [13]). However, it happens that some conceptual difficulties arise in order to develop local Morse theory, and in particular to evaluate the critical groups of the Euler functional in the critical points via Hessian type notions.

Now we list some difficulties that arise in the passage to a Banach (not Hilbert) variational framework. Let X be a Banach space and f be a C^2 functional. A first problem is that the classical definition of nondegenerate critical point introduced in a Hilbert space, i.e., $f''(u)$ is an isomorphism from X to the dual space X^*, does not seem very reasonable, since there are several examples of Banach spaces which are not isomorphic to their dual spaces. Furthermore, the existence of a nondegenerate critical point $u \in X$ of f, having finite Morse index, implies the existence of an equivalent Hilbert structure (see [27] for the proof).

We also notice that if $f''(u)$ is a Fredholm operator, then X is isomorphic to its dual space X^*. So, in a Banach (not Hilbert) setting, the classical Morse lemma does not hold and also generalized versions of Morse lemma (see [13]), due to Gromoll and Meyer, fail.

Therefore, in a Banach space, we do not have the main ingredients to apply Smale's version of Sard's Theorem and so to obtain Marino–Prodi perturbation theorems in [26]. As developed in [27], the perturbation result by Marino and Prodi can be extended to a Banach space X, if there exists on X an equivalent Hilbert norm.

In this work, we are interested to obtain perturbation results, in the spirit of Marino–Prodi theorems in [26], for some functionals defined in Banach space, which cannot be equipped with an equivalent Hilbert norm.

Precisely, let A be an open subset of $W_0^{1,p}(\Omega)$ and let us consider the C^2 functional $F_\lambda : A \to \mathbb{R}$ defined by setting

$$F_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{\lambda}{2}\int_\Omega |\nabla u|^2\,dx + \int_\Omega G(u)\,dx \tag{1}$$

where $2 < p < \infty$, $\lambda \geq 0$, and Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$, with sufficiently regular boundary $\partial\Omega$. Here $G(t) = \int_0^t g(s)\,ds$ and $g \in C^1(\mathbb{R},\mathbb{R})$ satisfies the following assumption:

(g): $|g'(t)| \leq c_1|t|^q + c_2$ *with* c_1, c_2 *positive constants and* $0 \leq q < p^* - 2$, $p^* = Np/(N-p)$ *if* $N > p$, *while* q *is any positive number, if* $N = p$.

Otherwise, if $N < p$, no restrictive assumption on the growth of g is required.

We notice that for $\lambda = 0$ the functional F_0 involves only the p-Laplacian.

At this point, fixed $\lambda \geq 0$, $p > 2$, and g, we introduce the following class of C^2 functionals

$$\mathcal{F}_\lambda(A) = \{f : A \to \mathbb{R} : f(u) = F_\lambda(u) + h(P_{\tilde{V}}u)\} \tag{2}$$

where $h : \tilde{V} \to \mathbb{R}$ is a C^2 function, $\tilde{V}$ is a finite dimensional subspace of $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and $P_{\tilde{V}} : W_0^{1,p}(\Omega) \to \tilde{V}$ is a continuous and linear projection on $\tilde{V}$.

In what follows, we shall use the following notation

$$\mathcal{F}(A) = \bigcup_{\lambda \geq 0} \mathcal{F}_\lambda(A).$$

We emphasize that any functional f belonging to $\mathcal{F}(A)$ is defined on an open set of a Banach space and the second derivative $f''(u)$, in each critical point, is not a Fredholm operator. Moreover we remark that using the classical definition given for Hilbert spaces, any critical point u of f is degenerate, as, being $p > 2$, $W_0^{1,p}(\Omega)$ is not isomorphic to the dual space $W^{-1,p'}(\Omega)$, with $1/p + 1/p' = 1$ (cf. [20]).

In the recent papers [15, 16] we have introduced a new weak notion of nondegeneracy in the setting of functionals (1).

Definition 1.1 Let A be an open set in $W_0^{1,p}(\Omega)$ and $f \in C^2(A)$. A critical point $u \in A$ of f is said to be nondegenerate if $f''(u) : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is injective.

Definition 1.2 Let A be an open set in $W_0^{1,p}(\Omega)$. We say that a functional $f : A \to \mathbb{R}$ belongs to the class $\mathcal{M}(A)$ if $f \in C^2(A)$, the critical points of f are nondegenerate in A (in the sense of Definition 1.1) and, for any critical point $u \in A$ of f, there exists an open neighborhood U of u such that $f_{|U} \in \mathcal{F}_\lambda(U)$, with $\lambda > 0$.

When no confusion can arise, we shall denote by f the functionals in $\mathcal{F}(A)$ and by $\tilde{g}$ the functionals in $\mathcal{M}(A)$.

In order to recall the main result in [16, 17], we need to introduce some useful definitions (cf. [13]).

Definition 1.3 Let X be a Banach space and f be a C^2 functional on X. Let $\mathbb{K}$ be a field. Let u be a critical point of f, $c = f(u)$, and U be a neighborhood of u. We call

$$C_q(f, u) = H^q(f^c \cap U, (f^c \setminus \{u\}) \cap U)$$

the q-th critical group of f at u, $q = 0, 1, 2, \dots$, where $f^c = \{v \in X | f(v) \leq c\}$ and $H^q(A, B)$ stands for the q-th Alexander–Spanier cohomology group of the pair (A, B) with coefficients in $\mathbb{K}$ (cf. [13]).

Definition 1.4 Let X be a Banach space and f be a C^2 functional on X. If u is a critical point of f, the Morse index of f in u is the supremum of the dimensions of the subspaces of X on which $f''(u)$ is negative definite. It is denoted by $m(f, u)$. Moreover, the large Morse index of f in u is the sum of $m(f, u)$ and the dimension of the kernel of $f''(u)$. It is denoted by $m^*(f, u)$.

In [16, 17], we have computed the critical groups of the functionals belonging to the class $\mathcal{M}(A)$, in each critical point u, via Morse index.

Theorem 1.5 *Let A be an open set in $W_0^{1,p}(\Omega)$. Let $\tilde{g} \in \mathcal{M}(A)$ and $u \in A$ be a critical point of $\tilde{g}$. Then $m(\tilde{g}, u)$ is finite and*

$$C_q(\tilde{g}, u) \cong \mathbb{K}, \quad \text{if } q = m(\tilde{g}, u),$$
$$C_q(\tilde{g}, u) = \{0\}, \quad \text{if } q \neq m(\tilde{g}, u).$$

Injectivity of $\tilde{g}''(u)$ is sufficient to obtain a suitable finite dimensional reduction, which allows us to identify the critical groups of $\tilde{g}$ in u with the critical groups of a suitable function defined on a finite dimensional subspace of $W_0^{1,p}(\Omega)$. So for the class of functionals in $\mathcal{M}(A)$, we are able to develop a local Morse theory and to overcome all the mentioned difficulties in a Banach setting.

We underline that the notion of nondegenerate critical point, in Definition 1.1, is weaker than the usual nondegeneracy condition in a Hilbert setting. Furthermore, we shall prove in Corollary 2.4 that a nondegenerate critical point, in our new sense, is isolated.

The aim of this work is to prove that any functional (1) is near, in C^2 norm, to a functional belonging to the class $\mathcal{M}(A)$, in the spirit of Marino–Prodi perturbation results.

We also mention that, in the literature, Uhlenbeck et al. [11, 35, 36] have introduced the definition of *weakly nondegenerate* critical point for C^2 functionals defined on a Banach space X. Such a definition requires that u is *a priori* isolated and involves the existence of an hyperbolic operator $L : X \to X$ which commutes with the second derivative of the Euler functional f in the critical point u and satisfies other additional conditions relating the first derivative of f in a neighborhood of the critical point. In fact it does not seem very easy to prove the existence of such operator and to obtain Marino–Prodi perturbation type results in terms of functions having *weakly nondegenerate* critical points. Moreover, as Uhlenbeck wrote in [36], in 1969 Smale conjectured that the only injectivity would be enough for developing Morse theory in a Banach setting. So we can say that, in the setting of class $\mathcal{M}(A)$, Smale's conjecture is true. Furthermore, it is easy to see that a *weakly nondegenerate* critical point, in the sense introduced in [11, 35, 36], is also a nondegenerate critical point, in the new sense introduced in Definition 1.1.

We begin to consider the case $f \in \mathcal{F}_\lambda(A)$ with $\lambda > 0$ with an isolated critical point. We state the following theorem, which is proved in Sect. 2.

Theorem 1.6 *Let A be an open subset of $W_0^{1,p}(\Omega)$. Suppose that $f \in \mathcal{F}_\lambda(A)$ with $\lambda > 0$ has a unique critical point $u \in A$. Then for any $\varepsilon > 0$ and $\gamma > 0$ such that $\bar{B}_\gamma(u) \subset A$, where $B_\gamma(u) = \{v \in W_0^{1,p}(\Omega) | \|v - u\| < \gamma\}$, there exists a functional $\tilde{g} \in \mathcal{M}(A)$ with the following properties:*

(1) $\|f^{(i)}(v) - \tilde{g}^{(i)}(v)\| < \varepsilon$ *for any $v \in A$, $i = 0, 1, 2$,*

$$f(v) = \tilde{g}(v) \quad \text{if } v \in A, \ \|v - u\| \geq \gamma;$$

(2) the critical points of $\tilde{g}$, if any, are in $B_\gamma(u)$ and finitely many;

(3) $\tilde{g}$ verifies the $(P.S.)$ condition on $\bar{B}_\gamma(u)$.

As said before, in our setting we cannot apply the results in [26, 27]. In this work, we shall overcome these difficulties, applying Sard's Lemma, in the finite dimensional version (see [28, 29]), to an *ad hoc* reduction function.

In Sect. 2, we establish that u is a critical point of a functional $f \in \mathcal{F}_\lambda(A)$, with $\lambda > 0$, if and only if 0 is a critical point of a suitable C^2 function φ defined on a finite dimensional space. The construction of a continuous reduction map φ is already contained in [16], but only in Sect. 2 of the present work we show that φ is C^2. The regularity of the map φ is the crucial ingredient to apply Sard's Lemma, in the finite dimensional version, and to find a C^2 approximating function $\tilde{\varphi}$ of φ, having nondegenerate critical points in the classical sense. Finally, using $\tilde{\varphi}$, we construct a new functional $\tilde{g} \in \mathcal{M}(A)$, approximating f, in a C^2 sense, near the critical point u and so we prove Theorem 1.6.

In Sects. 3 and 4, we drop the assumption that the critical point is isolated and we deal with a compact set of critical points of $f \in \mathcal{F}_\lambda(A)$, $\lambda > 0$. Precisely in Sect. 3 we carry out a suitable C^2 finite dimensional reduction for any functional $f \in \mathcal{F}_\lambda(A)$, $\lambda > 0$ near a compact set of critical points. In Sect. 4, using this reduction and Sard's Lemma, we approximate f near a compact set of critical points by a functional $\tilde{g} \in \mathcal{M}(A)$ (cf. Theorem 4.3). We emphasize that this compact set of critical points is not required to be isolated in the critical set.

Furthermore, we prove upper (lower) semicontinuity properties of the large (strict) Morse index of the functional $\tilde{g}$ approximating f.

We remark that the idea of combining the Splitting Theorem and Sard's Lemma, in the finite dimensional case, can be traced back to Chang [10] in the special case of a C^2 functional, defined on a Hilbert space, having an isolated critical point.

In Sect. 5, we focus on the case $\lambda = 0$, which is more delicate. In order to obtain perturbation results like in [26] for a functional $f_0 \in \mathcal{F}_0(A)$, involving only p-Laplacian, we need to perform two approximations. The first step is to penalize the functional f_0 with a 'small' quadratic term $\lambda \int_\Omega |\nabla u|^2 \, dx$, with $\lambda > 0$, obtaining a new functional f_λ, still belonging to $\mathcal{F}_\lambda(A)$, having a compact set of critical points.

Using the approximating results in Sect. 4 for a compact set of critical points of f_λ, $\lambda > 0$, we obtain an approximating functional of f_0 near an isolated critical point.

Theorem 1.7 *Let A be an open subset in $W_0^{1,p}(\Omega)$. Let $f \in \mathcal{F}_0(A)$ and $u \in A$ the only critical point of f. Then for any $\varepsilon > 0$ and $\gamma > 0$ such that $\bar{B}_\gamma(u) \subset A$, there exists a functional $\tilde{g} \in \mathcal{M}(A)$ with the following properties:*

(1) $\|f^{(i)}(v) - \tilde{g}^{(i)}(v)\| < \varepsilon$ for any $v \in A,\ i = 0, 1, 2,$
$f(v) = \tilde{g}(v) \quad$ if $\ v \in A, \|v - u\| \geq \gamma.$
(2) The critical points of $\tilde{g}$, if any, are in $B_\gamma(u)$ and finitely many.
(3) $\tilde{g}$ verifies $(P.S.)$ condition on $\bar{B}_\gamma(u)$.

We notice that the computations of critical groups in [16, 17] does not cover the case $\lambda = 0$. Nevertheless Theorem 1.7 keeps holding for f_0. This guarantees that near a functional involving only the p-Laplacian there is a functional containing a semilinear term, for which we are able to evaluate the critical groups.

In Theorem 5.2, we generalize Theorem 1.7 with $\lambda = 0$ for a compact set of critical points of f_0.

Finally Sect. 6 is devoted to some applications of the perturbations results, proved in the previous sections. First we extend some well-known results in a Hilbert framework, due to Lazer and Solimini [24], to our Banach framework. More precisely, we evaluate the Morse indices of minimax critical points for functionals $f \in \mathcal{F}_\lambda(A)$, $\lambda > 0$ satisfying the assumptions of the Saddle-Point Theorem by Rabinowitz [30] and we prove the following theorem (see also Theorem 6.4 in Sect. 6).

Theorem 1.8 *Let $f \in \mathcal{F}_\lambda(W_0^{1,p}(\Omega))$, with $\lambda > 0$. Assume that $W_0^{1,p}(\Omega) = Y \oplus Z$, where Y, Z are closed subspaces of $W_0^{1,p}(\Omega)$ and $1 \leq \dim Y = k$. Let $a < b$ be two real numbers such that f satisfies the Palais–Smale condition on $\{z \in W_0^{1,p}(\Omega) : a \leq f(z) \leq b\}$. Moreover suppose that there exists $R > 0$ such that*

$$\max_{y \in Y \cap \partial B_R(0)} f(y) \leq a < \inf_{z \in Z} f(z), \qquad \max_{y \in Y \cap \bar{B}_R(0)} f(y) \leq b.$$

Then there exists a critical point x_0 of f, with $x_0 \in f^{-1}([a, b])$, such that $m(f, x_0) \leq k \leq m^*(f, x_0)$.

We remark that the results in [24] work in a Hilbert setting and require the Fredholm properties of the second derivative of the Euler functional, as the authors need to exploit the perturbation result of Marino and Prodi, which is based on Sard–Smale's theorem. Nevertheless the main ingredients in [24] fail in our Banach setting, we shall overcome the lack of the Fredholm property of the second derivative, using the Marino–Prodi type perturbation results, developed in our work, and the upper (lower) semicontinuity properties of the large (strict) Morse index of the approximating functionals.

In this paper, we furnish an application of Theorem 1.8 for a quasilinear problem with an asymptotic nonlinearity as u^{p-1} at infinity. To this aim, we recall the following facts. Let $E : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by $E(u) = \int_\Omega |\nabla u|^p \, dx$, and set $M = \{u \in W_0^{1,p}(\Omega) : \int_\Omega |u|^p = 1\}$. We denote by $\sigma(-\Delta_p)$ the spectrum of the p-Laplace operator, namely the set of critical values of $E_{|M}$. It is known that the first eigenvalue $\mu_1 = \min_{u \in M} E(u)$ is positive and it has associated an eigenfunction u_1 which is positive in Ω (cf. [2]). Moreover the spectrum $\sigma(-\Delta_p)$ of $-\Delta_p$ contains at least an increasing eigenvalue sequence μ_k having a variational characterization [4, 5]. In [3] (see also [19]), it is proved that $]\mu_1, \mu_2[\cap \sigma(-\Delta_p) = \emptyset$.

Now let us define

$$\bar{\mu}_2 = \sup_Y \left\{ \inf_{u \in M \cap Y} E(u) : W_0^{1,p}(\Omega) = Y \oplus (\mathbb{R} u_1) \right\}. \tag{3}$$

By Theorem 5.1 in [19], we have that $\mu_1 < \bar{\mu}_2 \le \mu_2$.

We derive the following corollary of Theorem 1.8:

Corollary 1.9 *Let us consider the problem*

$$(P) \quad \begin{cases} -\Delta_p u - \lambda \Delta u = \mu |u|^{p-2} u + q(u) & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $p > 2$, $\lambda > 0$, $\mu_1 < \mu < \bar{\mu}_2$ ($\bar{\mu}_2$ is defined by (3)), $q : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that

$$\lim_{|s| \to +\infty} \frac{q'(s)}{|s|^{p-2}} = 0. \tag{4}$$

Then there exists a solution $\bar{u}$ of problem (P) such that $m(F, \bar{u}) \le 1 \le m^*(F, \bar{u})$, being F the Euler functional associated to problem (P).

In this work, we restrict ourself to study problem (P) in the case $\mu_1 < \mu < \bar{\mu}_2$, as we can take advantage of a saddle structure connected with the decomposition $W_0^{1,p}(\Omega) = Y \oplus (\mathbb{R}\varphi_1)$. The more complicated case $\mu \ge \bar{\mu}_2$ has been considered in a recent work [14], in which a suitable homological linking structure has been recognized (for the case $p = 2$ see [1, 10, 12, 24]).

Finally we mention that the perturbation results, here obtained, are applied in a recent paper [18] for obtaining a multiplicity result of solutions for quasilinear problem using Morse relations and domain topology, in the spirit of a remarkable paper by Benci and Cerami [7] for the semilinear case.

Notations

1. $(\cdot|\cdot)$ denotes the scalar product in $\mathbb{R}^N$.
2. $\|\cdot\|$ denotes the usual norm in $W_0^{1,p}(\Omega)$ and also the usual norm in $W^{-1,p}(\Omega)$.
3. $\|\cdot\|_{1,2}$ denotes the usual norm in $W_0^{1,2}(\Omega)$.
4. $\|\cdot\|_\infty$ denotes the usual norm in $L^\infty(\Omega)$.
5. $\langle \cdot, \cdot \rangle : W^{-1,p'}(\Omega) \times W_0^{1,p}(\Omega) \to \mathbb{R}$ denotes the duality pairing.
6. $B_r(u) = \{v \in W_0^{1,p}(\Omega) : \|v - u\| < r\}$.
7. $K = \{v \in W_0^{1,p}(\Omega) : f'(v) = 0\}$, $K_c = \{v \in W_0^{1,p}(\Omega) : f'(v) = 0, f(v) = c\}$.

8. $f^c = \{v \in W_0^{1,p}(\Omega) : f(v) \le c\}$, $f_a^b = \{v \in W_0^{1,p}(\Omega) : a \le f(v) \le b\}$.
9. Let $M \subset W_0^{1,p}(\Omega)$ and $r > 0$. $M^r = \{v \in W_0^{1,p}(\Omega) : d(v, M) < r\}$, where $d(\cdot,\cdot)$ is the usual distance function.
10. From time to time, we omit the symbol dx in integrals over Ω.

2 A C^2 finite dimensional reduction for an isolated critical point

Let X be a Banach space and f be a C^2 real function on X. Let C be a closed subset of X. A sequence (u_n) in C is a Palais–Smale sequence for f if $\|f(u_n)\| \le M$ uniformly in n, while $\|f'(u_n)\| \to 0$ as $n \to \infty$. We say that f satisfies $(P.S.)$ on C if any Palais–Smale sequence in C has a strongly convergent subsequence (cf. [32]).

Proposition 2.1 *Let A be an open subset of $W_0^{1,p}(\Omega)$ and let $f \in \mathcal{F}_\lambda(A)$ be with $\lambda \ge 0$. Then f verifies $(P.S.)$ condition on the bounded closed subsets of A. Moreover f' is a proper map on the bounded closed subsets of A.*

Proof We introduce the maps $D, H : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by

$$\langle Dz, \varphi\rangle = \int_\Omega (\lambda + |\nabla z|^{p-2})(\nabla z|\nabla\varphi)\, dx,$$

$$\langle Hz, \varphi\rangle = \int_\Omega g(z)\varphi\, dx + \langle h'(P_{\tilde V}(z), P_{\tilde V}\varphi\rangle \tag{5}$$

for each $z, \varphi \in W_0^{1,p}(\Omega)$, so that $f' = D + H$. It is known (see [6], Appendix B) that D is an invertible operator and D^{-1} is continuous. As H is compact, the assert comes. □

The aim of this section is to prove Theorem 1.6, stated in Sect. 1. We begin to consider an open subset A of $W_0^{1,p}(\Omega)$ and a functional $f_\lambda \in \mathcal{F}_\lambda(A)$ with $\lambda > 0$. For simplicity we set $\lambda = 1$, denoting f_1 by f. Furthermore, let us fix a critical point $u \in A$ of f at level $c = f(u)$. By [33, 34], $u \in C^1(\bar\Omega)$, so the function $b(x) = |\nabla u(x)|^{(p-4)/2}\nabla u(x) \in L^\infty(\Omega)$. As in [16, 17], we introduce a Hilbert space H_b, defined as the closure of $C_0^\infty(\Omega)$ under the scalar product $(v, w)_b = \int_\Omega (1 + |b|^2)(\nabla v|\nabla w)\, dx + (p-2)(b|\nabla v)(b|\nabla w)\, dx$. The space H_b is $W_0^{1,2}(\Omega)$ equipped by an equivalent Hilbert structure and thus $W_0^{1,p}(\Omega) \subset H_b$ continuously. Denoting by $\langle\cdot,\cdot\rangle_b : H_b^* \times H_b \to \mathbb{R}$ the duality pairing in H_b, $f''(u)$ can be extended to a Fredholm operator $L_b : H_b \to H_b^*$ defined by setting

$$\langle L_b v, w\rangle_b = (v, w)_b + \int_\Omega g'(u)vw\, dx + \langle h''(P_{\tilde V}(u))\, P_{\tilde V} v, P_{\tilde V} w\rangle.$$

We can consider the natural splitting $H_b = H^- \oplus H^0 \oplus H^+$, where H^-, H^0, H^+ are, respectively, the negative, null, and positive spaces, according to the spectral decomposition of L_b in $L^2(\Omega)$. We remark that H^- and H^0 have finite dimensions and

$$\langle L_b v, w\rangle = 0 \quad \forall v \in H^- \oplus H^0, \quad \forall w \in H^+. \tag{6}$$

Since $u \in C^1(\bar{\Omega})$, we can deduce from standard regularity theory that $H^- \oplus H^0 \subset C^1(\bar{\Omega})$. Consequently, denoted by $W = H^+ \cap W_0^{1,p}(\Omega)$ and $V = H^- \oplus H^0$, we get the splitting $W_0^{1,p}(\Omega) = V \oplus W$.

Arguing as in [16, Lemma 4.5] we can prove that there exist $r > 0$ and $\varrho \in]0, r[$ such that $u + (V \cap \bar{B}_\varrho(0)) + (W \cap \bar{B}_r(0)) \subset A$ and, for each v in $V \cap \bar{B}_\varrho(0)$, there exists one and only one $\bar{w} \in W \cap B_r(0) \cap L^\infty(\Omega)$ such that for any $z \in W \cap \bar{B}_r(0)$, $f(v+\bar{w}+u) \le f(v+z+u)$. Moreover $\bar{w}$ is the only element of $W \cap \bar{B}_r(0)$ such that

$$\langle f'(u + v + \bar{w}), z\rangle = 0 \qquad \forall z \in W. \tag{7}$$

So we can introduce the map $\psi : V \cap \bar{B}_\varrho(0) \to W \cap B_r(0)$ defined by $\psi(v) = \bar{w}$ and the function $\varphi : V \cap \bar{B}_\varrho(0) \to \mathbb{R}$ defined by $\varphi(v) = f(u + v + \psi(v))$. In [16, Sect. 5] we proved that ψ is a continuous map.

In the next lemma we shall show that ψ is also differentiable from V to W with respect to the weaker norm $\|\cdot\|_b$ and thus we shall infer a regularity result for the reduction map φ.

Lemma 2.2 *The map φ is C^2 and for any $v \in V \cap \bar{B}_\varrho(0)$, $z \in V$, $w \in V$ we have*

$$\langle \varphi'(v), z\rangle = \langle f'(u + v + \psi(v)), z\rangle \tag{8}$$

$$\langle \varphi''(v)z, w\rangle = \langle f''(u + v + \psi(v))(z + \psi'(v)(z)), w\rangle. \tag{9}$$

Moreover $\varphi''(v)$ is an isomorphism if and only if $f''(u + v + \psi(v))$ is injective.

Proof Firstly we show that for any fixed $v \in V \cap \bar{B}_\varrho(0)$, we have $\psi(v) \in C^1(\bar{\Omega})$ and the map $\psi : V \cap \bar{B}_\varrho(0) \to W$ is C^1 with respect to the norm $\|\cdot\|_b$.

Indeed, from Lemma 4.3 in [16] we know that $z_v = u + v + \psi(v) \in L^\infty(\Omega)$ and there exists $k > 0$ such that $\|z_v\|_\infty \le k$, uniformly with respect to v. By [33, 34], we can infer that $z_v \in C^1(\bar{\Omega})$, and thus $\psi(v) \in C^1(\bar{\Omega})$, as $V \subset C^1(\bar{\Omega})$.

Moreover, in consequence of Theorem 4.1 in [22, Chapter 4], $\|\psi(v)\|_{C^1(\bar{\Omega})}$ is bounded from above by a suitable constant k_0. Therefore, there exists a constant $R_1 > 0$ such that $\|u + v + \psi(v)\|_{C^1} \le R_1$ for any $v \in V \cap \bar{B}_\varrho(0)$. Now, fixed $R_2 > R_1$, let us consider a non increasing C^∞ function $\omega : [0, +\infty[\to \mathbb{R}$ such that $\omega(t) = 1$ if $t \in [0, R_1]$, $\omega(t) = 0$ if $t \ge R_2$. By a suitable choose of R_2, we can build a function $\vartheta : \mathbb{R}^N \to \mathbb{R}$ such that $\vartheta(\xi) = \omega(|\xi|)\frac{1}{p}|\xi|^p + \frac{1}{2}|\xi|^2$ is strictly convex in $\mathbb{R}^N$. Moreover we consider the function $\bar{G} : \mathbb{R} \to \mathbb{R}$ defined by $\bar{G}(t) = \omega(|t|)G(t)$ and we can introduce the following C^2 functional

$$\bar{f}(v) = \int_\Omega \vartheta(\nabla v)\,dx + \int_\Omega \bar{G}(v)\,dx + h(\bar{P}_{\tilde{V}} v) \qquad \forall\, v \in H_b$$

where $\bar{P}_{\tilde{V}} : H_b \to \tilde{V}$ is the continuous extension of $P_{\tilde{V}}$. We underline that for any $v \in A \cap C^1(\bar{\Omega})$ with $\|v\|_{C^1} \le R_1$, we have

$$\bar{f}(v) = f(v), \quad \bar{f}'(v)|_{W_0^{1,p}(\Omega)} = f'(v), \quad \bar{f}''(v)|_{(W_0^{1,p}(\Omega))^2} = f''(v). \tag{10}$$

Now recalling that $H_b = V \oplus H^+$ and applying the Implicit Function Theorem to the map $B : (V \cap \bar{B}_\varrho(0)) \times H^+ \to (H^+)^*$ given by $B(v, w) = \bar{f}'(u+v+w)|_{H^+}$, we can infer that $\psi : V \cap \bar{B}_\varrho(0) \mapsto W$ is C^1 with respect to $\|\cdot\|_b$. Now by (10), it follows that φ is a C^1 functional. By the chain rule, we infer (8), as $\bar{f}'(u+v+\psi(v))|_{H^+} = 0$ and $\psi'(v)(z) \in W$. This shows that φ is also a C^2 functional and, again by the chain rule, (9) derives.

In order to complete the proof, fix $v \in V \cap \bar{B}_\varrho(0)$ and suppose that $\varphi''(v)$ is an isomorphism. By contradiction, if $f''(u+v+\psi(v))$ is not injective, there exists $\bar{z} \in W_0^{1,p}(\Omega)$, $\bar{z} \neq 0$ such that

$$\langle f''(u+v+\psi(v))(z+\psi'(v)(z)), \bar{z}\rangle = 0 \quad \forall z \in V. \tag{11}$$

If we write $\bar{z}$ as $z_1 + z_2$ with $z_1 \in V$ and $z_2 \in W$, by (9) and (11), as the function $v \in V \cap \bar{B}_\varrho(0) \mapsto \langle f'(u+v+\psi(v), w\rangle \in \mathbb{R}$ for any fixed $w \in W$ is constantly equal to zero, we deduce $\langle \varphi''(v)z_1, z\rangle = 0$ for any $z \in V$, which is a contradiction. Clearly we can deduce in a similar way that if $f''(u+v+\psi(v))$ is injective, then $\varphi''(v)$ is an isomorphism. □

Corollary 2.3 *V and W are orthogonal with respect to $f''(u)$. Moreover we have*

(i) $\psi'(0) = 0$;
(ii) $\langle \varphi''(0)z, w\rangle = 0 \quad \forall z \in V, \quad w \in W.$

Proof By (6), it follows that V and W are orthogonal with respect to $f''(u)$.

In order to verify (i), let us fix $v, z \in V$. As V and W are orthogonal with respect to $f''(u)$ and $\langle \bar{f}'(u+v+\psi(v)), w\rangle = 0$ for any $w \in W$, we deduce that, for any $w \in W$, $\langle \bar{f}''(u)(\psi'(0)z), w\rangle = 0$, so $\psi'(0) = 0$, as $\psi'(0)z \in W$.

Moreover by (9), we have, for any $z \in V$, $w \in W$ $\langle \varphi''(0)z, w\rangle = \langle f''(u)z, w\rangle = 0$, □

Corollary 2.4 *Each nondegenerate critical point, according to Definition 1.1, is isolated.*

Proof Let u be a critical point for f such that $f''(u) : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is injective. By Lemma 2.2, 0 is a critical point of φ and $\varphi''(0) : V \to V^*$ is an isomorphism. Being V a finite dimensional Hilbert space, the fact that 0 is nondegenerate implies that 0 is an isolated critical point of φ. So by (8), u is an isolated critical point of f. □

We are ready to prove Theorem 1.6 given in Sect. 1.

Proof of Theorem 1.6 For simplicity, fix $\lambda = 1$ and consider $f \in \mathcal{F}_1(A)$. As u is the only critical point of f in A, by (8), 0 is the only critical point of φ in the ball $V \cap \bar{B}_\varrho(0)$. Let us fix $\varepsilon > 0$ and $\gamma > 0$ such that $B_\gamma(u) \subset A$. Fixed $\delta \in]0, \gamma[$, there exists $\nu > 0$ such that

$$\|f'(z)\| \geq \nu \quad \text{for any } z \in B_\gamma(u) \setminus B_\delta(u). \tag{12}$$

Since V is a finite dimensional space and $\varphi : V \cap \bar{B}_\varrho(0) \to \mathbb{R}$ is a C^2 function, having only one critical point in 0, we can apply well-known perturbation results based on Sard's Lemma (cf. [28, 29]). So that, in correspondence of $\bar{\varepsilon} > 0$, there exists a C^2 function $\alpha : V \cap \bar{B}_\varrho(0) \to \mathbb{R}$ such that,

(i) $\|\alpha^{(i)}(v)\| < \bar{\varepsilon}$ for any $v \in V \cap \bar{B}_\varrho(0)$, $\quad i = 0, 1, 2$
$\alpha(v) = 0$ if $\|v\| \geq \delta$;

(ii) the critical points of $\varphi + \alpha$, if any, are nondegenerate.

Moreover these critical points are in $B_\delta(0)$ and they are finitely many.
Denoted by $\beta : V \to \mathbb{R}$ the natural extension to V of the function α, namely

$$\beta(v) = \begin{cases} \alpha(v) & \text{if} \|v\| \leq \varrho \\ 0 & \text{otherwise,} \end{cases}$$

we consider the C^2 perturbed functional $g : A \to \mathbb{R}$ defined by setting

$$\tilde{g}(z) = f(z) + \xi(z) \cdot \beta\left(P_V(z - u)\right)$$

where $\xi : W_0^{1,p}(\Omega) \to \mathbb{R}$ is a C^2 function such that $\xi(z) = 0$ if $z \notin B_\gamma(u)$, $\xi(z) = 1$ if $z \in B_\delta(u)$.

As ξ and its derivatives are uniformly bounded, choosing $\bar{\varepsilon}$ sufficiently small we get

$$\|f^{(i)}(v) - \tilde{g}^{(i)}(v)\| \leq \min\{\varepsilon, \nu/2\} \quad \text{for any} \quad v \in A, \quad i = 0, 1, 2. \tag{13}$$

So it is clear that g verifies (2) by construction, and its critical points belong to $B_\delta(u)$.

Let $u_0 \in A$ be a critical point of g. By (12) and (13) we infer also that $\|u_0 - u\| \leq \delta$, so that $B_\delta(u) \subset A$ is a neighborhood of u_0 in which

$$\tilde{g}(z) = f(z) + \beta\left(P_V(z - u)\right). \tag{14}$$

We notice that, for any $z \in W_0^{1,p}(\Omega)$, $0 = \langle \tilde{g}'(u_0), z \rangle = \langle f'(u_0), z \rangle + \langle \beta'\left(P_V(u_0 - u)\right), P_V z \rangle$. So in particular $\langle f'(u_0), w \rangle = 0$, for any $w \in W$, and this means that

$$u_0 \in Y = \{u + v + \psi(v) : \ v \in V \cap \bar{B}_\varrho(0)\}. \tag{15}$$

Our aim is to prove that u_0 is nondegenerate, i.e., $\tilde{g}''(u_0)$ is injective.

Let $z_1 \in \operatorname{Ker} \tilde{g}''(u_0)$. Denoted by $v_0 = P_V(u_0 - u)$, $v_1 = P_V z_1$, $w_1 = P_W z_1$, we have that $u_0 = u + v_0 + \psi(v_0)$ and $z_1 = v_1 + w_1$. First we note that $\langle (\varphi + \alpha)'(v_0), v \rangle = \langle \tilde{g}'(u_0), v \rangle = 0$ for any $v \in V$ so that v_0 is a critical point of $\varphi + \alpha$. Moreover $\langle \tilde{g}''(u_0) z_1, v + \psi'(v_0) v \rangle = 0$, for any $v \in V$. Hence

$$\begin{aligned} 0 &= \langle f''(u_0) z_1, v + \psi'(v_0) v \rangle + \langle \beta''(v_0) P_V z_1, P_V(v + \psi'(v_0) v) \rangle \\ &= \langle f''(u_0) v_1, v + \psi'(v_0) v \rangle + \langle f''(u_0) w_1, v + \psi'(v_0) v \rangle + \langle \beta''(v_0) v_1, v \rangle \\ &= \langle \varphi''(v_0) v_1, v \rangle + \langle \beta''(v_0) v_1, v \rangle = \langle \varphi''(v_0) v_1, v \rangle \qquad \forall v \in V. \end{aligned} \tag{16}$$

Therefore, (ii) assures that $v_1 = 0$, and $0 = \langle \tilde{g}''(u_0) w_1, w \rangle = \langle f''(u_0) w_1, w \rangle$, for any $w \in W$. As Y is bounded in L^∞ and δ is small, by [16, Lemma 4.4] there exists $C > 0$ such that $0 = \langle f''(u_0) w_1, w_1 \rangle \geq C \|w_1\|_b^2$, so finally $z_1 = w_1 = 0$ and u_0 is nondegenerate, as claimed. Moreover all the critical points of g are in the compact set Y, so they have to be finitely many, otherwise Corollary 2.4 would be contradicted.

Arguing as in the proof of Proposition 2.1, one can derives (3) of Theorem 1.6.

3 A C^2 finite dimensional reduction for a compact set of critical points

In this section, we also consider $f \in \mathcal{F}_\lambda(A)$ with $\lambda > 0$. For simplicity, we can choose $\lambda = 1$. Let us denote by Z a compact set of critical points of f.

Now we need to obtain a finite dimensional reduction, for this more general case, so we generalize the arguments developed in [16, 17] (see also Sect. 2). Firstly, by standard regularity theory, we have the uniform C^1-bound of Z.

Lemma 3.1 *For any $u \in Z$, $u \in C^1(\bar{\Omega})$. Moreover there exists $k > 0$ such that for any $u \in Z$, $\|u\|_{C^1} \leq k$.*

For each $u \in Z$, arguing as in Sect. 2, we can consider the splitting $W_0^{1,p}(\Omega) = V_u \oplus W_u$, where V_u is a finite dimensional subspace of $W_0^{1,p}(\Omega)$ and there exists $\mu > 0$ such that $\langle f''(u)v, v\rangle \geq \mu \|v\|_{1,2}^2$ for any $v \in W_u$. Taking into account Lemma 3.1, by Lemma 4.4 in [16], we can easily deduce the following result.

Lemma 3.2 *For any $u \in Z$, there exist $r_u > 0$ and $C_u > 0$ such that $B_{r_u}(u) \subset A$ and for any $z \in Z \cap B_{r_u}(u)$ we have $\langle f''(z)v, v\rangle \geq C_u \|v\|_{1,2}^2$, for any $v \in W_u$.*

Furthermore, as Z is a compact set, there exist $u_1, u_2, \ldots, u_n$ points in Z, and $r_1, r_2, \ldots, r_n$ positive numbers such that $Z \subset \bigcup_{i=1}^n B_{r_i}(u_i) \subset A$ and, for any $z \in Z$ with $\|z - u_i\| < r_i$, we have $\langle f''(z)v, v\rangle \geq C_{u_i} \|v\|_{1,2}^2$, for any $v \in W_{u_i}$.

In what follows, we set

$$V = V_{u_1} + V_{u_2} + \cdots V_{u_n} \tag{17}$$

so that V is a finite dimensional subspace of $W_0^{1,p}(\Omega)$. Let us denote by W the topological supplement in $W_0^{1,p}(\Omega)$ of V. By standard computations it follows that

$$W = \bigcap_{i=1}^n W_{u_i}. \tag{18}$$

By construction, denoting by $\bar{C} = \min\{C_{u_i} : i = 1, 2 \ldots n\}$, we immediately infer the following result.

Lemma 3.3 *There are a finite codimensional subspace W of $W_0^{1,p}(\Omega)$ and a constant $\bar{C} > 0$ such that*

$$\langle f''(z)v, v\rangle \geq \bar{C} \|v\|_{1,2}^2 \quad \forall z \in Z, \quad \forall v \in W. \tag{19}$$

As $W_0^{1,p}(\Omega) = V \oplus W$, it makes sense to consider the projection $P_V : W_0^{1,p}(\Omega) \to V$. Moreover, also $L^2(\Omega)$ and $H_0^1(\Omega)$ can be projected on V and the three functions

$$P_V : W_0^{1,p}(\Omega) \to V, \quad \tilde{P}_V : H_0^1(\Omega) \to V, \quad \bar{P}_V : L^2(\Omega) \to V$$

are all continuous with respect to each natural norm. As, in particular, $\tilde{P}_V$ is an extension of P_V, we can say that

$$\exists c > 0 \quad \text{s.t.} \quad \|P_V(u)\| \leq c\|u\|_{1,2} \quad \forall u \in W_0^{1,p}(\Omega). \tag{20}$$

Remark 3.4 Reasoning as in Lemma 4.3 of [16], we infer that, if X is a subset of A which is bounded in $W_0^{1,p}(\Omega)$-norm and if $\langle f'(z), w\rangle = 0$ for any $z \in X$, $w \in W$, then $X \subset L^\infty(\Omega)$. Moreover, there exists $k > 0$, k only depending on X, such that $\|z\|_\infty \le k$, for each $z \in X$.

At this stage, taking into account Lemma 3.3, we can deduce the next result, whose proof follows the ideas contained in that of [16, Lemma 4.4].

Lemma 3.5 *Let W be defined from (18). For any $M > 0$, there exist $r > 0$, $\gamma > 0$ such that*

$$Z^r = \{z \in W_0^{1,p}(\Omega) : d(z, Z) < r\} \subset A$$

and, for any $z \in Z^r \cap L^\infty(\Omega)$, with $\|z\|_\infty < M$, we have

$$\langle f''(z)v, v\rangle \ge \gamma\|v\|_{1,2}^2 \quad \forall v \in W. \tag{21}$$

This allows to obtain a finite dimensional reduction for each $u \in Z$. Firstly we need the following lemma.

Lemma 3.6 *There exists $\delta > 0$ such that for any $u \in Z$, for any $w \in W \setminus \{0\}$ with $\|w\| < \delta$ we have*

$$f(u + w) > f(u). \tag{22}$$

Proof By assumption (g), and Lemma 3.1 there exist two constants $c > 0$, $d > 0$ such that, for any $u \in Z$, $x \in \Omega$ and $s \in \mathbb{R}$, we have

$$|g'(s)| \le c + d|s - u(x)|^{p^*-2}. \tag{23}$$

Now let us define for any $(x, s) \in \Omega \times \mathbb{R}$

$$\bar g(u, x, s) = g(s) + \frac{d}{p^* - 1}|s - u(x)|^{p^*-2}(s - u(x))$$

and $\bar G(u, x, s) = G(s) + \frac{d}{p^*(p^*-1)}|s - u(x)|^{p^*}$. By (23) it is immediate to check that

$$D_s\bar g(u, x, s) \ge -c \tag{24}$$

Now set $q(\xi) = |\xi|^p/p$ for any $\xi \in \mathbb{R}^N$. Obviously there exist $C_1 > 0$ and $C_2 > 0$ such that $|q''(\xi)| \le C_1|\xi|^{p-2}$ for any $\xi \in \mathbb{R}^N$ and $|q''(\xi_1 - \xi_2)| \le C_2(|\xi_1|^{p-2} + |\xi_2|^{p-2})$ for any $\xi_1, \xi_2 \in \mathbb{R}^N$. Let us fix $\varepsilon > 0$ such that

$$1 - C_2\varepsilon - C_2\varepsilon K^{p-2} \ge 1/2, \quad \bar C - 2C_2\varepsilon K^{p-2} \ge \bar C/2 \tag{25}$$

where $K > 0$ is defined in Lemma 3.1 and $\bar C > 0$ is introduced in (19).

Moreover, let us define the functional $t_\varepsilon : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$t_\varepsilon(z) = \frac{\varepsilon}{p}\int_\Omega |\nabla z|^p\,dx - \frac{d}{p^*(p^* - 1)}\int_\Omega |z|^{p^*}\,dx, \quad z \in W_0^{1,p}(\Omega)$$

and let us set $k(u, z) = f(z) - t_\varepsilon(z - u)$, for any $u \in Z$, $z \in A$. Firstly, we observe that there exist $\gamma' > 0$ and $\varepsilon' > 0$ such that

$$t_\varepsilon(z) \ge \varepsilon'\int_\Omega |\nabla z|^p\,dx \qquad \forall z \in A, \quad \|z\| \le \gamma'. \tag{26}$$

Now we shall prove that there exist $\sigma > 0$, $\tilde{C} > 0$ such that $B_\sigma(u) \subset A$ and for any $z \in Z \cap B_\sigma(u)$, we have

$$\langle D_{zz}k(u,z)w, w\rangle \geq \tilde{C}\|w\|_{1,2}^2 \quad \forall w \in W. \tag{27}$$

By contradiction, we assume that there exist three sequences u_n in Z, $z_n \in A$ and $v_n \in W\setminus\{0\}$, such that $\|v_n\|_1 = 1$, $\|u_n - z_n\| \to 0$ and

$$\liminf_{n\to\infty}\langle D_{zz}k(u_n, z_n)v_n, v_n\rangle \leq 0. \tag{28}$$

Since $\{v_n\}$ is bounded in H^1, there exists $v \in H^1$ such that v_n weakly converges to v in H^1 and strongly in $L^2(\Omega)$, up to subsequences. Moreover, $v \in W$. Since Z is compact, there exists $u \in Z$ such that $z_n \to u$, $u_n \to u$, strongly in $W_0^{1,p}(\Omega)$, up to subsequences.

Firstly we prove that $v \neq 0$. By contradiction, assume that $v = 0$. By (28), we infer

$$\begin{aligned}
\langle D_{zz}k(u_n, z_n)v_n, v_n\rangle &= \int_\Omega |\nabla v_n|^2\,dx + \int_\Omega (q''(\nabla z_n)\nabla v_n|\nabla v_n)\,dx \\
&\quad - \varepsilon\int_\Omega (q''(\nabla z_n - \nabla u_n)\nabla v_n|\nabla v_n)\,dx \\
&\quad + \int_\Omega \tilde{g}'(u_n, x, z_n)v_n^2\,dx + \langle h''(P_{\tilde{V}}z_n)P_{\tilde{V}}v_n, P_{\tilde{V}}v_n\rangle \\
&\geq 1 + \int_\Omega (q''(\nabla z_n)\nabla v_n|\nabla v_n)\,dx - C_2\varepsilon\int_\Omega |\nabla z_n|^{p-2}|\nabla v_n|^2\,dx \\
&\quad - C_2\varepsilon\int_\Omega |\nabla u_n|^{p-2}|\nabla v_n|^2\,dx - c\int_\Omega v_n^2\,dx + \langle h''(P_{\tilde{V}}z_n)P_{\tilde{V}}v_n, P_{\tilde{V}}v_n\rangle \\
&\geq 1 + (1 - C_2\varepsilon)\int_\Omega |\nabla z_n|^{p-2}|\nabla v_n|^2\,dx - C_2\varepsilon K^{p-2}\int_\Omega |\nabla v_n|^2\,dx \\
&\quad - c\int_\Omega v_n^2\,dx + \langle h''(P_{\tilde{V}}z_n)P_{\tilde{V}}v_n, P_{\tilde{V}}v_n\rangle \\
&\geq 1 - \varepsilon C_2 K^{p-2} - c\int_\Omega v_n^2\,dx + \langle h''(P_{\tilde{V}}z_n)P_{\tilde{V}}v_n, P_{\tilde{V}}v_n\rangle.
\end{aligned}$$

As $v_n \to 0$ in $L^2(\Omega)$, the above inequality contradicts (28) as $n \to \infty$, thus $v \neq 0$.

At this point,we notice that

$$\int_\Omega (q''(\nabla u)\nabla v|\nabla v)\,dx \leq \liminf_{n\to\infty}\int_\Omega (q''(\nabla z_n)\nabla v_n|\nabla v_n)\,dx. \tag{29}$$

Moreover, by (24) and as $v_n \to v$ in $L^2(\Omega)$, we also obtain that

$$\int_\Omega g'(u)v^2\,dx \leq \liminf_{n\to\infty}\int_\Omega \tilde{g}'(u_n, x, z_n)v_n^2. \tag{30}$$

Finally (29) and (30) imply

$$\begin{aligned}
\langle D_{zz}k(u_n,z_n)v_n,v_n\rangle &= \int_\Omega |\nabla v_n|^2\,dx + \int_\Omega (q''(\nabla z_n)\nabla v_n|\nabla v_n)\,dx \\
&- \varepsilon\int_\Omega (q''(\nabla z_n-\nabla u_n)\nabla v_n|\nabla v_n)\,dx + \int_\Omega \tilde g'(u_n,x,z_n)v_n^2\,dx \\
&\geq \int_\Omega (1-\varepsilon C_2|\nabla u_n|^{p-2})|\nabla v_n|^2\,dx + \int_\Omega (1-\varepsilon C_2)|\nabla z_n|^{p-2}|\nabla v_n|^2\,dx \\
&+ (p-2)\int_\Omega |\nabla z_n|^{p-4}|(\nabla z_n|\nabla v_n)|^2\,dx + \int_\Omega \tilde g'(u_n,x,z_n)v_n^2\,dx \\
&+ \langle h''(P_{\tilde V}z_n)P_{\tilde V}v_n, P_{\tilde V}v_n\rangle.
\end{aligned}$$

Then by (28), we deduce

$$\begin{aligned}
0 &\geq \liminf_{n\to\infty}\langle D_{zz}k(u_n,z_n)v_n,v_n\rangle \\
&\geq \liminf_{n\to\infty}\Big(\int_\Omega (1-\varepsilon C_2|\nabla u_n|^{p-2})|\nabla v_n|^2\,dx + \int_\Omega (1-\varepsilon C_2)|\nabla z_n|^{p-2}|\nabla v_n|^2\,dx \\
&+ (p-2)\int_\Omega |\nabla z_n|^{p-4}|(\nabla z_n|\nabla v_n)|^2\,dx + \int_\Omega \tilde g'(u_n,x,z_n)v_n^2\,dx \\
&+ \langle h''(P_{\tilde V}z_n)P_{\tilde V}v_n, P_{\tilde V}v_n\rangle\Big) \\
&\geq (1-\varepsilon C_2K^{p-2})\liminf_{n\to\infty}\int_\Omega |\nabla v_n|^2\,dx + (1-\varepsilon C_2)\liminf_{n\to\infty}\int_\Omega |\nabla z_n|^{p-2}|\nabla v_n|^2\,dx \\
&+ (p-2)\liminf_{n\to\infty}\int_\Omega |\nabla z_n|^{p-4}|(\nabla z_n|\nabla v_n)|^2\,dx + \liminf_{n\to\infty}\int_\Omega \tilde g'(u_n,x,z_n)v_n^2\,dx \\
&+ \liminf_{n\to\infty}\langle h''(P_{\tilde V}z_n)P_{\tilde V}v_n, P_{\tilde V}v_n\rangle \\
&\geq (1-2\varepsilon C_2K^{p-2})\int_\Omega |\nabla v|^2\,dx + \int_\Omega (h''(\nabla u)\nabla v|\nabla v)\,dx + \int_\Omega g'(u)v^2\,dx \\
&+ \langle h''(P_{\tilde V}u)P_{\tilde V}v, P_{\tilde V}v\rangle \\
&= \langle f''(u)v,v\rangle - 2\varepsilon C_2K^{p-2}\int_\Omega |\nabla v|^2 \geq \frac{\bar C}{2}\|v\|_{1,2}^2
\end{aligned}$$

which is a contradiction.

At this point, for any $u\in Z$, for any $w\in W$ with $\|w\|\leq \min\{\gamma',\sigma\}$ we have

$$f(u+w)-f(u)=t_\varepsilon(w)+k(u,u+w)-k(u,u). \tag{31}$$

Moreover, for any $w\in W$ with $\|w\|\leq\min\{\gamma',\sigma\}$, there exists $z\in W_0^{1,p}(\Omega)$ with $\|z-u\|\leq\min\{\gamma',\sigma\}$ such that $k(u,u+w)-k(u,u)=\frac{1}{2}\langle D_{zz}k(u,z)w,w\rangle$.

At this point, by (27), we infer

$$k(u,u+w)-k(u,u)\geq \tilde C\|w\|_{1,2}^2 \tag{32}$$

$\tilde C$ being a suitable positive constant. Finally by (26), (31), and (32) we infer (22). □

Using arguments strictly related to Lemma 4.6 in [16], we derive the following lemma.

Lemma 3.7 *Let V, W be defined by (17) and (18), and $\delta > 0$ the constant introduced in the previous lemma. There exist $r \in]0, \delta[$ and $\varrho \in]0, r[$ such that for any $u \in Z$,*

$$u + \left(V \cap \bar{B}_\varrho(0)\right) + \left(W \cap B_r(0)\right) \subset A$$

and for any $u \in Z$, $v \in V \cap \bar{B}_\varrho(0)$ there exists one and only one $\bar{w} = \bar{w}(u, v) \in W \cap B_r(0) \cap L^\infty(\Omega)$ such that for any $z \in W \cap \bar{B}_r(0)$ we have

$$f(u + v + \bar{w}) \leq f(u + v + z).$$

Moreover $\bar{w}$ is the only element of $W \cap \bar{B}_r(0)$ such that

$$\langle f'(u + v + \bar{w}), z\rangle = 0 \qquad \forall z \in W. \tag{33}$$

Proof We notice that if $z \in A$ with $d(z, Z) < \delta$, where δ is the positive constant in Lemma 3.6, is a solution of $\langle f'(z), w\rangle = 0$ for any $w \in W$, then, by Remark 3.4, $z \in L^\infty(\Omega)$ and $\|z\|_\infty \leq M$, where $M > 0$ only depends from Z^δ. Now by Lemma 3.5, in correspondence of $2M$, there exists $r_0 \in]0, \delta[$ and $\gamma > 0$ such that (21) holds.

Let $r \in]0, \frac{r_0}{3}[$. Firstly, f is sequentially lower semicontinuous with respect to the weak topology of $W_0^{1,p}(\Omega)$. Therefore, for any $u \in Z$, $v \in V \cap B_r(0)$, there exists a minimum point $\bar{w} = \bar{w}(u, v) \in W \cap \bar{B}_r(0)$ of the function $w \in W \cap \bar{B}_r(0) \mapsto f(u + v + w)$.

We shall prove the existence of $\varrho \in]0, r[$ such that, for any $u \in Z$ and $v \in V \cap \bar{B}_\varrho(0)$,

$$\inf\{f(u + v + w) : w \in W, \|w\| = r\} > f(u + v). \tag{34}$$

Arguing by contradiction, we assume that there exist three sequences $\{u_n\}$ in Z, $\{w_n\}$ in $W \cap \partial B_r(0)$ and $\{v_n\}$ in V, with $\|v_n\| \to 0$ as $n \to \infty$, such that

$$f(u_n + v_n + w_n) \leq f(u_n + v_n). \tag{35}$$

Since Z is compact, and $\{w_n\}$ is bounded, there exist $\tilde{u} \in Z$, $\tilde{w} \in W$ such that $u_n \to \tilde{u}$ and $w_n \rightharpoonup \tilde{w}$ in $W_0^{1,p}(\Omega)$, up to subsequences.

By Lemma 3.6, 0 is the unique minimum point of the function $w \in W \cap \bar{B}_r(0) \mapsto f(\tilde{u} + w)$, therefore we have $f(\tilde{u}) \leq f(\tilde{u} + \tilde{w})$. Hence, by (35), it follows that

$$f(\tilde{u} + \tilde{w}) \leq \liminf_{n\to\infty} f(u_n + v_n + w_n) \leq \limsup_{n\to\infty} f(u_n + v_n) = f(\tilde{u}) \leq f(\tilde{u} + \tilde{w})$$

so that

$$f(\tilde{u}) = f(\tilde{u} + \tilde{w}). \tag{36}$$

Moreover, we deduce that

$$\lim_{n\to+\infty}\int_\Omega |\nabla u_n + \nabla v_n + \nabla w_n|^p\,dx = \int_\Omega |\nabla u_n + \nabla \tilde{w}|^p\,dx,$$

consequently $w_n \to \tilde{w}$ strongly in $W_0^{1,p}(\Omega)$ and $\|\tilde{w}\| = r$ which contradicts (36).

Hence, there exists $\varrho \in]0, r[$ such that, for any $u \in Z$, $v \in V \cap \bar{B}_\varrho(0)$, (34) holds.

Therefore, for any $u \in Z$, $v \in V \cap \bar{B}_\varrho(0)$, the minimum point $\bar{w}$ belongs to $W \cap B_r(0)$ and $\langle f'(u+v+\bar{w}), z\rangle = 0$, for any $z \in W$. Moreover $\bar{w} \in L^\infty(\Omega)$ and $\|u+v+\bar{w}\|_\infty \le M$.

Now we shall prove that for any fixed $(u, v) \in Z \times V \cap \bar{B}_\varrho(0)$, the minimum point $\bar{w}$ is unique. In fact, we shall prove even more, namely that $\bar{w}$ is the only element of $W \cap \bar{B}_r(0)$ such that $\langle f'(u+v+\bar{w}), z\rangle = 0$, for any $z \in W$.

By contradiction, let $u_1, u_2 \in Z$ and $v_1, v_2 \in V$ be such that $P_V u_1 + v_1 = P_V u_2 + v_2$. We suppose that there exist $w_1, w_2 \in W \cap B_r(0) \cap L^\infty(\Omega)$, such that $P_W u_1 + w_1 = P_W u_2 + w_2$.

We have $\langle f'(u_1+v_1+w_1), z\rangle = 0$ and $\langle f'(u_2+v_2+w_2), z\rangle = 0$, for any $z \in W$.

We notice that $\|v + w_1 + t(w_2 - w_1)\| = \|v + w_1(1-t) + w_2\| \le 3r$, for any $t \in [0, 1]$, hence $\|v + w_1 + t(w_2 - w_1)\| \le 3r < r_0$, for any $t \in [0, 1]$ and thus $d(u + v + w_1 + t(w_2 - w_1), Z) < r_0$ for any $t \in [0, 1]$. Furthermore, we have $\|u + v + w_1 + t(w_2 - w_1)\|_\infty \le 2M$. Therefore, by (21), we deduce

$$\begin{aligned} 0 &= \langle f'(u+v+w_1) - f'(u+v+w_2), w_1 - w_2\rangle \\ &= \int_0^1 \langle f''(u+v+w_1+t(w_2-w_1))(w_1-w_2), w_1-w_2\rangle \quad dt > 0. \end{aligned}$$

The claim is proved. □

Since Z is compact, there exist $u_1, \ldots u_k$, k points in Z such that

$$Z \subset \bigcup_{i=1}^{k} (u_i + (B_{\varrho/2}(0) \cap V) + (B_r(0) \cap W)) = B \subset A. \tag{37}$$

Moreover, for each $u_i + B_\varrho(0) \cap V + B_r(0) \cap W$, $i = 1, 2, \ldots k$, the thesis of Lemma 3.7 holds. So arguing as in [16], for any $i = 1, 2, \ldots, k$ we can introduce the maps

$$\psi_i : V \cap \bar{B}_\varrho(0) \to W \cap B_r(0) \tag{38}$$

where $\psi_i(v) = \bar{w}$ is the unique minimum point of the function $w \in W \cap \bar{B}_r(0) \mapsto f(u_i + v + w)$, and the function

$$\varphi_i : V \cap \bar{B}_\varrho(0) \to \mathbb{R} \tag{39}$$

defined by $\varphi_i(v) = f(u_i + v + \psi(v))$. Arguing as in Sect. 2, one can prove the following result.

Lemma 3.8 *For any $i = 1, 2, \dots, k$ the map φ_i is C^2, moreover, for any $v \in V \cap \bar{B}_\varrho(0)$ and $z, w \in V$, we have*

$$\langle \varphi_i'(v), z \rangle = \langle f'(u_i + v + \psi_i(v)), z \rangle \tag{40}$$

$$\langle \varphi_i''(v)z, w \rangle = \langle f''(u_i + v + \psi_i(v))(z + \psi_i'(v)(z)),\ w \rangle. \tag{41}$$

Furthermore, $\varphi_i''(v)$ is an isomorphism if and only if $f''\,(u_i + v + \psi_i(v))$ is injective.

4 An approximating functional of $f \in \mathcal{F}_\lambda(A)$, $\lambda > 0$ near a compact set of critical points

We begin to prove a result concerning lower semicontinuity properties of the Morse index for C^2 functionals.

Proposition 4.1 *Let A be a open set of a Banach space X. Let $f : A \to \mathbb{R}$ be a C^2 functional. Let u_0 a critical point of f. For any sequence $\{g_n\}$ of functionals in $C^2(A, \mathbb{R})$ with $g_n^{(i)} \to f^{(i)}$ uniformly for $i = 0, 1, 2$, as $n \to \infty$, and for any sequence $\{u_n\}$ of critical points of g_n such that $u_n \to u_0$, as $n \to \infty$, we have*

$$m(f, u_0) \le \liminf_{n\to\infty} m(g_n, u_n).$$

Proof Let X^- be a finite dimensional subspace of X such that, for some constant $\mu > 0$, $\langle f''(u_0)v, v \rangle \le -\mu\|v\|^2$, for any $v \in X^-$. Since, for $i = 0, 1, 2$, $g_n^{(i)} \to f^{(i)}$ uniformly and $u_n \to u_0$, as $n \to \infty$, we have, for $v \in X^-$

$$\begin{aligned}\langle g_n''(u_n)v, v \rangle &= \langle f''(u_0)v, v \rangle + \langle (f''(u_n) - f''(u_0))v, v \rangle \\ &\quad + \langle (g_n''(u_n) - f''(u_n))v, v \rangle \le -\mu\|v\|^2 + \|f''(u_n) \\ &\quad - f''(u_0)\|\|v\|^2 + \|g_n'' - f''\|\|v\|^2 \le -\mu/2\|v\|^2.\end{aligned}$$

Hence, $\liminf_{n\to\infty} m(g_n, u_n) \ge \dim X^-$. □

Now using the same notations of the previous section, we deduce the following results, based on Sard's Lemma (cf. [28, 29]).

Lemma 4.2 *Let V be a finite dimensional subspace of $W_0^{1,p}(\Omega)$, $\varrho > 0$ and $\varphi : V \cap \bar{B}_\varrho(0) \to \mathbb{R}$ a C^2 function.*
For any triple $(\varepsilon, \delta, \delta')$, such that $\varepsilon > 0$ and $0 < \delta' < \delta < \varrho$, there exists a C^2 function $\alpha : V \cap \bar{B}_\varrho(0) \to \mathbb{R}$ such that,

(i) $\|\alpha^{(l)}(v)\| < \varepsilon$ *for any v, $l = 0, 1, 2$,*
$\alpha(v) = 0$ *if* $\|v\| \ge \delta$;
(ii) the critical points of $\varphi + \alpha$ in $V \cap \bar{B}_{\delta'}(0)$, if any, are nondegenerate and finitely many.

Theorem 4.3 *Suppose that $f \in \mathcal{F}_\lambda(A)$ with $\lambda > 0$. Let U, U' U'' be open bounded sets in A where $\bar{U} \subset U'$ and $\bar{U}' \subset U''$. Then there exist an open neighborhood $\tilde{U}$ of $\{v \in \bar{U} : f'(v) = 0\}$, $\tilde{U} \subset U'$, and a C^2 sequence $\tilde{g}_n : A \to \mathbb{R}$ with the following properties:*

(1) $\lim_{n\to\infty} \| f^{(l)}(v) - \tilde{g}_n^{(l)}(v)\| = 0$ *for any $v \in A$, $l = 0, 1, 2$;*

$$f(v) = \tilde{g}_n(v) \quad \text{if } v \in A, \quad v \notin U'';$$

(2) The critical points of $\tilde{g}_n$ in $\bar{U}$, if any, are in $\bar{U} \cap \tilde{U}$ and finitely many. Moreover $\tilde{g}_n \in \mathcal{M}(\tilde{U})$.
(3) $\tilde{g}_n'$ is proper on $\bar{U}'$.
(4) If, for any $n \in \mathbb{N}$, $u_n \in \tilde{U} \cap \bar{U}$ is a critical point of $\tilde{g}_n$ with $u_n \to u_0 \in \tilde{U}$ as $n \to \infty$, then u_0 is a critical point of f and

$$\liminf_{n\to\infty} m(\tilde{g}_n, u_n) \geq m(f, u_0),$$

$$\limsup_{n\to\infty} m(\tilde{g}_n, u_n) \leq m^*(f, u_0).$$

Proof For simplicity, we can choose $\lambda = 1$ and consider $f \in \mathcal{F}_1(A)$. Fixed $\varepsilon > 0$, let us consider $Z = \{v \in \bar{U} : f'(v) = 0\}$. Since, by Proposition 2.1, f' is proper on $\bar{U}$, Z is a compact set of critical points of f. Consequently, we can carry out the reduction as in the previous section. Let us fix $\delta > 0$ such that $Z^\delta \subset B \subset U'$, where B is the set introduced in (37) with A replaced by U'. Z^δ is a neighborhood of Z which will play the role of $\tilde{U}$ in the statement. Moreover, let ψ_i and φ_i, $i = 1, \dots k$, be defined respectively as in (38) and (39).
We introduce a C^2 function $\xi : W_0^{1,p}(\Omega) \to \mathbb{R}$, such that $\xi(z) = 0$ if $z \notin U''$, $\xi(z) = 1$ if $z \in U'$ (see [8] for the construction). For any $i = 1, \dots k$, Lemma 4.2 can be applied to the function φ_i and the triple $(\varepsilon_i, 3\varrho/4, \varrho/2)$, where $\varepsilon_i > 0$ will be fixed later, so that a perturbation function α_i is obtained. Let us denote by $\beta_i : V \to \mathbb{R}$ the natural extension to V of the corresponding functions α_i, namely

$$\beta_i(v) = \begin{cases} \alpha_i(v) & \text{if } \|v\| \leq \varrho \\ 0 & \text{otherwise.} \end{cases}$$

Moreover let us consider the C^2 perturbed functional $g : A \to \mathbb{R}$ defined by setting

$$\tilde{g}(z) = f(z) + \xi(z) \cdot (\beta_1(P_V(z-u_1)) + \beta_2(P_V(z-u_2)) + \cdots + \beta_k(P_V(z-u_k))).$$

As ξ and its derivatives are uniformly bounded, there exists $\bar{\varepsilon} = \bar{\varepsilon}(\xi, \varepsilon) > 0$, such that, choosing each ε_i less or equal to $\bar{\varepsilon}$,

$$\|f^{(l)}(v) - \tilde{g}^{(l)}(v)\| \leq \varepsilon \quad \text{for } v \in A, \quad l = 0, 1, 2. \tag{42}$$

Now let us see as $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k$ have to be chosen. For $i = 1$, let $\varepsilon_1 = \bar{\varepsilon}$ and α_1 be a corresponding perturbation, according to Lemma 4.2, so that $\varphi_1 + \alpha_1$ has only nondegenerate critical points in $V \cap B_{\varrho/2}(0)$. Now there exists $\bar{\varepsilon}_1 > 0$ such that, if $\alpha \in C^2(V \cap B_{\varrho/2}(0))$ and $\|\alpha^{(l)}(v)\| < \bar{\varepsilon}_1$ for any $v \in V \cap B_{\varrho/2}(0)$, $l = 0, 1, 2$, then also $\varphi_1 + \alpha_1 + \alpha$ has only nondegenerate critical points in $V \cap B_{\varrho/2}(0)$.

For $i = 2$, we choose $\varepsilon_2 = \min\{\bar{\varepsilon}, \bar{\varepsilon}_1/(k-1)\}$. As before, a corresponding $\varphi_2 + \alpha_2$ has only nondegenerate critical points in $V \cap B_{\varrho/2}(0)$ and there exists $\bar{\varepsilon}_2 > 0$ such that if $\alpha \in C^2(V \cap B_{\varrho/2}(0))$ and $\|\alpha^{(l)}(v)\| < \bar{\varepsilon}_2$ for any $v \in V \cap B_{\varrho/2}(0)$, $l = 0, 1, 2$, then also $\varphi_2 + \alpha_2 + \alpha$ has only nondegenerate critical points in $V \cap B_{\varrho/2}(0)$. For $i = 3$ we choose $\varepsilon_3 = \min\{\bar{\varepsilon}, \bar{\varepsilon}_1/(k-1), \bar{\varepsilon}_2/(k-2)\}$, and so on.

Moreover as f' is proper in $\bar{U}$, then f satisfies $(P.S.)$ and the critical points of $\tilde{g}$ in $\bar{U}$ are in $\bar{U} \cap Z^\delta$, if δ is small enough.

Let $u_0 \in Z^\delta$ be a critical point of $\tilde{g}$. Then there exists $i \in \{1, 2, \ldots, k\}$ such that $u_0 \in U_i$, where $U_i = u_i + (B_{\varrho/2}(0) \cap V) + (B_r(0) \cap W)$.
Now we want to prove that u_0 is nondegenerate, i.e., $\tilde{g}''(u_0)$ is injective.

If $u_0 \in U_i$ and u_0 does not belong to U_j with $i \neq j$, then in a suitable neighborhood $U_0 \subset U_i$ of u_0 we have $\tilde{g}(z) = f(z) + \beta_i\,(P_V(z - u_i))$. Hence, for any $w \in W$, $0 = \langle \tilde{g}'(u_0), w\rangle = \langle f'(u_0), w\rangle$, so that $u_0 = u_i + v_0 + \psi_i(v_0)$, where $v_0 = P_V(u_0 - u_i)$.

Let $z_1 \in \mathrm{Ker}\tilde{g}''(u_0)$. Denoted by $v_1 = P_V z_1$, $w_1 = P_W z_1$, we have that $u_0 = u_i + v_0 + \psi_i(v_0)$ and $z_1 = v_1 + w_1$. First we note that $\langle(\varphi_i + \alpha_i)'(v_0), v\rangle = \langle \tilde{g}'(u_0), v\rangle = 0$, for any $v \in V$, so that v_0 is a critical point of $\varphi_i + \alpha_i$. So arguing as in the proof of Theorem 1.6 and using Lemma 4.2, we deduce that u_0 is a nondegenerate critical point for $\tilde{g}$, i.e., $\tilde{g}''(u_0)$ is injective.

If, instead, u_0 belongs to $U_i \cap U_j$, with $i < j$ and u_0 does not belong to U_r, with $r \neq i, r \neq j$, then in a suitable neighborhood $U_0 \subset U_i \cap U_j$ of u_0 we have $\tilde{g}(z) = f(z) + \beta_i(P_V(z - u_i)) + \beta_j(P_V(z - u_j))$ and so, for any $z \in W_0^{1,p}(\Omega)$,

$$0 = \langle \tilde{g}'(u_0), z\rangle = \langle f'(u_0), z\rangle + \langle \beta_i'\,(P_V(u_0 - u_i))\,, P_V z\rangle + \langle \beta_j'(P_V(u_0 - u_j)), P_V z\rangle.$$

Since $0 = \langle \tilde{g}'(u_0), w\rangle = \langle f'(u_0), w\rangle$, for any $w \in W$, we infer that $u_0 = u_i + v_i^0 + \psi_i(v_i^0)$, where $v_i^0 = P_V(u_0 - u_i)$, and $u_0 = u_j + v_j^0 + \psi_j(v_j^0)$, where $v_j^0 = P_V(u_0 - u_j)$. Now we note that, for any $v \in V$,

$$\langle(\varphi_i' + \alpha_i + \alpha_j^i)(v_i^0), v\rangle = \langle f'(u_0), v\rangle + \langle \beta_i'(v_i^0), v\rangle + \langle \beta_j'(v_j^0), v\rangle = \langle \tilde{g}'(u_0), v\rangle = 0,$$

so that v_i^0 is a critical point of $\varphi_i + \alpha_i + \alpha_j^i$, where we define $\alpha_j^i(v) = \alpha_j(v + P_V(u_i - u_j))$.

Let $z_1 \in \mathrm{Ker}\tilde{g}''(u_0)$. Denoted by $v_1 = P_V z_1$, $w_1 = P_W z_1$, we have that $z_1 = v_1 + w_1$.

We can deduce that

$$\begin{aligned} 0 = \left\langle \tilde{g}''(u_0) z_1, v + \psi_i'\left(v_i^0\right) v\right\rangle &= \left\langle \varphi_i''\left(v_i^0\right) v_1, v\right\rangle \\ &\quad + \left\langle \beta_i''\left(v_i^0\right) v_1, v\right\rangle + \left\langle \beta_j''\left(v_j^0\right) v_1, v\right\rangle \qquad \forall v \in V. \end{aligned}$$

Since $\|D^l \alpha_j^i(v)\| = \|D^l \alpha_j(v)\| \le \bar{\varepsilon}_i$ for any v, $l = 0, 1, 2$, we infer $\varphi_i + \alpha_i + \alpha_j^i$ has nondegenerate finitely many critical points in $B_\varrho(0) \cap (u_j - u_i + B_\varrho(0))$. Hence, $v_1 = 0$.

If u_0 belongs to more than two neighborhoods U_i, we argue in analogous way.

As δ is small, by (21) there exists $C > 0$ such that $0 = \langle f''(u_0)w_1, w_1\rangle \geq C\|w_1\|_b$, so finally $z_1 = w_1 = 0$ and u_0 is nondegenerate, as claimed. As ε is arbitrarily chosen, the proof of points (1) and (2) is complete.
As regards point (3), the same arguments of the proof of Proposition 2.1 can be repeated, adding to H defined in (5) a suitable compact term.

Now we shall prove the last statement of the theorem. Let $u_0 \in \tilde{U} \cap \bar{U}$ be a critical point of f. We can consider an approximating sequence of functionals $\{g_n\}$ satisfying the points (1)–(3) of the theorem. Let us assume that $\{u_n\} \in W_0^{1,p}(\Omega)$ is a sequence of points such that $\tilde{g}_n'(u_n) = 0$ and $u_n \to u_0$ as $n \to \infty$. By Proposition 4.1, we have that $m(f, u_0) \leq \liminf_{n\to\infty} m(g_n, u_n)$. Moreover arguing as in Sect. 2, we can consider V_{u_0} and W_{u_0} closed subspace of $W_0^{1,p}(\Omega)$ such that $W_0^{1,p}(\Omega) = V_{u_0} \oplus W_{u_0}$ with $\dim V_{u_0} = m^*(f, u_0) < +\infty$ and

$$\langle f''(u_0)w, w\rangle \geq C\|w\|_{1,2}^2 \quad \forall w \in W_{u_0}.$$

By standard regularity theory we have that $u_n \in L^\infty(\Omega)$ and there exists $M > 0$ such that $\|u_n\|_\infty \leq M$. Furthermore, by Lemma 4.4 in [16], we have $\langle f''(u_n)w, w\rangle \geq C\|w\|_{1,2}^2$, for any $w \in W_{u_0}$ and for $n \in \mathbb{N}$ large enough. By construction, we can write, for any $z \in U'$, $g_n(z) = f(z) + \gamma_n(P_V z)$, where $\gamma_n \in C^2(A, \mathbb{R})$ is a perturbation such that $\|\gamma_n''\| \to 0$, as $n \to \infty$. Hence, using (20), we have, for n sufficiently large,

$$\begin{aligned}\langle g_n''(u_n)w, w\rangle &= \langle f''(u_n)w, w\rangle + \langle \gamma_n''(P_V u_n)P_V w, P_V w\rangle \\ &\geq C\|w\|_{1,2}^2 - c\|\gamma_n''\|\|w\|_{1,2}^2 \geq C/2\|w\|_{1,2}^2 \quad \forall w \in W_{u_0}.\end{aligned}$$

Therefore, it follows that $\limsup_{n\to\infty} m(g_n, u_n) \leq \dim V_{u_0} \equiv m^*(f, u_0)$. □

If the compact critical set of f is isolated, the statement of the Theorem 4.3 becomes more precise.

Corollary 4.4 *Suppose that $f \in \mathcal{F}_\lambda(A)$ with $\lambda > 0$. Let U'' be an open bounded set in A. Assume that $Z \subset U''$ is a compact set of critical points of f such that $K \cap U'' = Z$ and let U' be a neighborhood of Z such that $\bar{U}' \subset U''$. Then there exist an open neighborhood $\tilde{U}$ of Z with $\tilde{U} \subset U'$ and a C^2 sequence $\tilde{g}_n : A \to \mathbb{R}$ with the following properties:*

(1) $\lim\limits_{n\to\infty} \|f^{(l)}(v) - \tilde{g}_n^{(l)}(v)\| = 0$ *for any* $v \in A$, $l = 0, 1, 2$;
$f(v) = \tilde{g}_n(v)$ *if* $v \in A$, $v \notin U''$.
(2) The critical points of $\tilde{g}_n|_{U''}$, if any, are in $\tilde{U}$ and finitely many. Moreover $\tilde{g}_n \in \mathcal{M}(U'')$.
(3) $\tilde{g}_n'$ is proper on $\bar{U}'$. Furthermore, $\tilde{g}_n'$ satisfies $(P.S.)$ on the closed subset of U''.
(4) If $u_n \in U''$ is a critical point of $\tilde{g}_n$ with $u_n \to u_0$ as $n \to \infty$, then $u_0 \in \tilde{U}$, it is a critical point of f and

$$\liminf_{n\to\infty} m(\tilde{g}_n, u_n) \geq m(f, u_0),$$

$$\limsup_{n\to\infty} m(\tilde{g}_n, u_n) \leq m^*(f, u_0).$$

Proof Using the same notations of the previous proof, we can fix $\nu > 0$ such that $\|f'(z)\| \geq \nu$, for any $z \in U'' \setminus \tilde{U}$. Let us consider a C^2 function $\xi : W_0^{1,p}(\Omega) \to \mathbb{R}$ as before. As ξ and its derivatives are uniformly bounded, there exists $\bar{\varepsilon} = \bar{\varepsilon}(\xi, \varepsilon, \nu) > 0$, such that if we choose each $\varepsilon_i \leq \bar{\varepsilon}$, then $\|f^{(l)}(v) - \tilde{g}^{(l)}(v)\| \leq \min\{\varepsilon, \nu/2\}$, for any $v \in A$, $\quad l = 0, 1, 2$. By construction, it is clear that the critical points of $\tilde{g}|_{U''}$ belong to $\tilde{U}$. The proof can be completed arguing as in Theorem 4.3. □

5 Perturbation results for functionals involving only the p-Laplacian

In this section, we focus on the case $\lambda = 0$ and we prove Theorem 1.7 stated in Sect. 1. Let us fix a functional $f_0 \in \mathcal{F}_\lambda(A)$ with $\lambda = 0$ and an isolated critical point $u_0 \in A$ of f_0. Without restriction, we can suppose that u_0 is the only critical point of f_0 in A.

Now we need to perform two approximations. Firstly the following lemma proves that f_0 can be approximated by a functional $f \in \mathcal{F}_\lambda(A)$, with $\lambda > 0$.

Lemma 5.1 *For any $\varepsilon > 0$ and $\gamma > 0$ sufficiently small, there exist $0 < \gamma_1 < \gamma$, a positive number $\lambda = \lambda(\varepsilon, \gamma)$ and a C^2 functional h such that*

(1) $\|f_0^{(i)}(v) - h^{(i)}(v)\| < \varepsilon/2$ *for any* $v \in A$, $i = 0, 1, 2$.
(2) The critical points of h are a compact set contained in the ball $B_{\gamma_1}(u_0)$ and $h|_{B_{\gamma_1}(u_0)} \in \mathcal{F}_\lambda(B_{\gamma_1}(u_0))$.
(3) $f_0(v) = h(v)$ *if* $v \notin B_\gamma(u_0)$.

Proof Fixed $\gamma_1 \in]0, \gamma[$, let us consider a C^2 function $\chi : A \to \mathbb{R}$ such that

$$\chi(v) = 1 \quad if\ \|v - u_0\| \leq \gamma_1, \qquad \chi(v) = 0 \quad if\ \|v - u_0\| \geq \gamma.$$

We can also require that χ is uniformly bounded together with its derivatives.

Defining h by

$$h(v) = f_0(v) + \frac{\lambda}{2}\chi(v)\int_\Omega |\nabla v|^2\, dx,$$

the lemma follows from Proposition 2.1. □

Proof of Theorem 1.7. Let us denote by h the functional given by Lemma 5.1 applied to f_0 in correspondence to the couple (ε, γ).

Let us denote by Z the critical set of h which is a compact subset of $B_{\gamma_1}(u_0)$ and by f_λ the function $h|_{B_{\gamma_1}(u_0)} \in \mathcal{F}_\lambda\left(B_{\gamma_1}(u_0)\right)$, where $\lambda > 0$.

Let $\alpha > 0$ be such that $Z^\alpha \subset B_{\gamma_1}(u_0)$. Fixing $\alpha_1 \in]0, \alpha[$, we can apply Corollary 4.4 to the penalized functional $f_\lambda \in \mathcal{F}(B_{\gamma_1}(u_0))$, where $U' = Z^{\alpha_1}$ and $U'' = Z^\alpha$.

Consequently there exists $g \in \mathcal{M}(B_{\gamma_1}(u_0))$ such that

- $\|g^{(i)}(v) - f_\lambda^{(i)}(v)\| < \varepsilon/2$ for any $v \in A$, $i = 0, 1, 2$;
- $g(v) = f_\lambda(v)$ if $v \notin Z^\alpha$;
- the critical points of g, if any, are in Z^{α_1} and are finitely many.

Denoting by $\tilde{g} : A \to \mathbb{R}$ the function defined by

$$\tilde{g}(v) = g(v) \quad if\ v \in B_{\gamma_1}(u_0), \qquad \tilde{g}(v) = h(v) \quad if\ v \notin B_{\gamma_1}(u_0),$$

the theorem is completely proved.

Finally we state the following theorem, which deals with a compact subset of the critical set of f_0.

Theorem 5.2 *Suppose that $f_0 \in \mathcal{F}_0(A)$. Let $U, U'\ U''$ be open bounded sets in A where $\bar{U} \subset U'$ and $\bar{U}' \subset U''$. Then there exist a sequence of open neighborhoods $\tilde{U}_n$ of $\{v \in \bar{U} : f_0'(v) = 0\}$ and a C^2 sequence $\tilde{g}_n : A \to \mathbb{R}$ with the following properties:*

(1) $\tilde{g}_n^{(l)} \to f_0^{(l)}$ uniformly, for any $l = 0, 1, 2$;

$$f_0(v) = \tilde{g}_n(v) \quad \text{if } v \in A, \quad v \notin U''.$$

(2) The critical points of $\tilde{g}_n$ in $\bar{U}$, if any, are in $\bar{U} \cap \tilde{U}_n$ and finitely many. Moreover $\tilde{g}_n \in \mathcal{M}(\tilde{U}_n)$.
(3) $\tilde{g}_n'$ is proper on $\bar{U}'$.

Proof The proof of statements (1)–(3) follows performing two approximations as just made in the previous proof. □

Now we restrict ourself to consider the functional introduced in (1) with $\lambda = 0$

$$F_0(u) = \frac{1}{p} \int_\Omega |\nabla u|^p + \int_\Omega G(u)\, dx. \tag{43}$$

In the next result, we shall deal with the case in which F_0 has the origin like isolated critical point. We shall prove some semicontinuity properties of the large Morse index of F_0 in 0, which are crucial in the applications to avoid the trivial solutions (cf. Proposition 6.8 and [14]).

Theorem 5.3 *Let us consider the functional $F_0 \in \mathcal{F}_0(A)$ defined in (43). Assume that $0 \in A$ is the only critical point for F_0 in A. For any $\gamma > 0$ such that $\bar{B}_\gamma(0) \subset A$, there exists a sequence $h_n \in \mathcal{M}(A)$ such that*

(1) $h_n^{(l)} \to F_0^{(l)}$ uniformly, for any $l = 0, 1, 2$;

$$F_0(v) = h_n(v) \quad \text{if } v \in A, \quad \|v\| \geq \gamma.$$

(2) The critical points of h_n, if any, are in $B_\gamma(u)$ and finitely many.
(3) h_n verifies (P. S.) condition on $\bar{B}_\gamma(u)$.
(4) If $u_n \in A$ is a critical point of h_n, $u_n \to 0$ as $n \to +\infty$, then

$$\liminf_{n\to+\infty} m(h_n, u_n) \geq m(F_0, 0), \tag{44}$$

$$\limsup_{n\to+\infty} m(h_n, u_n) \leq m^*(F_0, 0). \tag{45}$$

Proof We begin to consider the case $g'(0) \leq 0$. By Theorem 1.7 we can construct a sequence h_n satisfying (1)–(3). Let u_n be a critical point of the functional h_n such that $u_n \to 0$ as $n \to +\infty$. By Proposition 4.1 we deduce (44) and as $m^*(F_0, 0) = +\infty$ we have

$$\limsup_{n\to+\infty} m(h_n, u_n) \leq m^*(F_0, 0).$$

Conversely if $g'(0) > 0$, we have $m^*(F_0, 0) = 0$. Fix $0 < r_1 < r$ small such that $\bar{B}_r(0) \subset A$ and $\|F_0'(v)\| \geq m > 0$ if $r_1 \leq \|v\| \leq r$, $v \in A$.

Let us define the sequence of functionals

$$h_n(v) = F_0(v) + \frac{\chi(v)}{2n} \int |\nabla v|^2$$

where $\chi : W_0^{1,p}(\Omega) \to \mathbb{R}$ is a C^2 function such that $\chi = 1$ if $\|v\| \leq r_1$, $\chi(v) = 0$ if $\|v\| \geq r$ and χ is uniformly bounded together with its derivatives.

By construction 0 is critical point of h_n and $m(h_n, 0) = m^*(h_n, 0) = 0$. It follows that 0 is nondegenerate for h_n.

Now assume that u_n is a critical point of h_n and $u_n \to 0$ as $n \to +\infty$. By standard regularity theory [22], we have that $u_n \in C^1(\bar{\Omega})$ and there exists $M > 0$ such that $\|u_n\|_{C^1} \leq M$ for any $n \in \mathbb{N}$. Hence we have $u_n \to 0$ in $L^\infty(\Omega)$ as $n \to +\infty$.

As $g'(0) > 0$, we can conclude that for n large enough

$$0 = \langle h_n'(u_n), u_n\rangle \geq \int_\Omega u_n g(u_n) \geq \varepsilon \|u_n\|_{L^2}^2 \tag{46}$$

where $\varepsilon > 0$ and thus $u_n \equiv 0$ eventually. We conclude that

$$0 = \limsup_{n\to+\infty} m(h_n, u_n) \leq m^*(F_0, 0).$$

By Proposition 4.1 we have deduce that h_n satisfies (44). Finally by Proposition 2.1 and by construction we derive (1)–(3). □

6 Morse index estimate for minimax critical points and applications

This section is devoted to generalize some classical result in a Hilbert setting, due to Lazer and Solimini [24], concerning Morse index estimates of minimax critical points (cf. [30]).

For reader's convenience, we start to recall the following result which is a simple variant of Theorem 1.5 (Chapter II) in [13], where homological notions are replaced by cohomological ones. We start to introduce the following assumption.

(S) Let $f \in C^2(W_0^{1,p}(\Omega), \mathbb{R})$, $W_0^{1,p}(\Omega) = X \oplus Z$, where X and Z are closed subspaces with $1 \leq k \equiv \dim X < +\infty$. Let $D = X \cap B_r(0)$, where $B_r(0) = \{v \in W_0^{1,p}(\Omega) | \|v\| < r\}$. There exists $a \in \mathbb{R}$ such that $f(v) > a$ for any $v \in Z$ and $f(v) \leq a$ for any $v \in \partial D$.

Theorem 6.1 *Let $f : W_0^{1,p}(\Omega) \to \mathbb{R}$ be a functional, satisfying the hypothesis (S).*

Let us denote by j^* the homomorphism induced by the inclusion map j of ∂D into f^c, $c \in [a, +\infty[$. Let us define

$$\bar{c} = \sup\{c \in [a, +\infty[: j^* : H^{k-1}(f^c) \to H^{k-1}(\partial D) \quad \text{is surjective}\}. \tag{47}$$

Then $inf_Z f \leq \bar{c} \leq \sup_D f$. Suppose that f satisfies the $(P.S.)$ condition at level $\bar{c}$ and $K_{\bar{c}}$ is discrete in K, then there exists $u_0 \in K_{\bar{c}}$ such that $f(u_0) = \bar{c}$ and $C_k(f, u_0) \neq \{0\}$.

In the next theorem we shall evaluate the Morse index of nondegenerate minimax critical points in our Banach framework, in the spirit of Theorem 2.3 in [24].

Theorem 6.2 *Let $f \in \mathcal{M}(W_0^{1,p}(\Omega))$ be satisfying the hypothesis (S) and the $(P.S.)$ condition at level $\bar{c}$, defined in (47). Then there exists a critical point u_0 of f such that $m(f, u_0) = k$ and $f(u_0) = \bar{c}$.*

Proof By Theorem 6.1, there exists a critical point u_0 of f at level $\bar{c}$, introduced in (47), such that $C_k(f, u_0) \neq \{0\}$. Moreover as u_0 is nondegenerate, we can apply Theorem 1.2 in [17] (see also Theorem 1.1 in [16]) and we can infer that $C_q(f, u_0) = \{0\}$, if $q \neq m(f, u_0)$. Therefore, we deduce that $m(f, u_0) = k$. □

Remark 6.3 Let us consider the case $\lambda = 0$ and a functional $f \in \mathcal{F}_0(W_0^{1,p}(\Omega))$ satisfying the hypothesis (S). If f also satisfies the $(P.S.)$ condition at level $\bar{c}$ introduced in (47) and $K_{\bar{c}}$ is discrete, then by the paper of Lancelotti [23], we can easily infer that there exists a critical point u_0 of f at level $\bar{c}$ such that $m(f, u_0) \leq k$.

Now we deal with the case in which the functional $f \in \mathcal{F}_\lambda(W_0^{1,p}(\Omega))$ with $\lambda > 0$ has degenerate critical points at level $\bar{c}$. In the next theorem we shall extend Theorem 2.6 in [24], to our Banach setting. We point out that in Theorem 2.6 in [24], Lazer and Solimini require that the second derivative of the functional is a Fredholm map in the critical points at level $\bar{c}$, as they need to exploit the perturbation result due to Marino and Prodi, based on Sard–Smale's theorem.

In our case, the functional $f \in \mathcal{F}_\lambda(W_0^{1,p}(\Omega))$ with $\lambda > 0$ does not satisfy this assumption. We are able to overcome the lack of the Fredholm property of the second derivative, using the Marino–Prodi type perturbation result, developed in Theorem 4.3.

Theorem 6.4 *Let $f \in \mathcal{F}_\lambda(W_0^{1,p}(\Omega))$ be with $\lambda > 0$, satisfying the hypothesis (S) and the $(P.S.)$ condition at level $\bar{c}$, defined in (47). Then there exists a critical point $u_0 \in K_{\bar{c}}$ such that $m(f, u_0) \leq k \leq m^*(f, u_0)$.*

Proof Fix $\varepsilon' > 0$ and denote by $A = f_{\bar{c}-\varepsilon'}^{\bar{c}+\varepsilon'} \cap K$ where K denotes the set of the critical points of f.

As f satisfies $(P.S.)$ at level $\bar{c}$, then any $(P.S.)$ sequence of f at level c in a suitable neighborhood of $\bar{c}$ is bounded, hence, by Proposition 2.1, it converges strongly, up to subsequences. Therefore, we can choose ε' so small that A is compact. Chosen $\varepsilon \in]0, \varepsilon'[$, let us introduce the sets

$$U' = f^{-1}]\bar{c} - \varepsilon', \bar{c} + \varepsilon'[\cap A^{\varepsilon'} \qquad U = f^{-1}]\bar{c} - \varepsilon, \bar{c} + \varepsilon[\cap A^{\varepsilon},$$

so that U' and U are open and bounded and $\bar{U} \subset U'$.

For any $n \in \mathbb{N}$ there exists $\nu_n > 0$ such that, if $u \in f_{\bar{c}-\varepsilon}^{\bar{c}+\varepsilon} \setminus A^{1/n}$, then $\|f'(u)\| > \nu_n$.

Hence we can apply Theorem 4.3 and, denoted by $Z = \bar{U} \cap K$, we can construct an open bounded neighborhood $\tilde{U}$ of Z, and a sequence $\tilde{g}_n \in C^2(W_0^{1,p}(\Omega), \mathbb{R})$ such that $\|\tilde{g}_n^{(i)}(v) - f^{(i)}(v)\| < \min\{1/n, \nu_n/2\}$ for any $v \in W_0^{1,p}(\Omega)$, $i = 0, 1, 2$, the critical points of $\tilde{g}_n$ in $\bar{U}$, if any, are in $\tilde{U} \cap \bar{U}$ and they are nondegenerate.

If n is great enough,

1. $\tilde{g}_n$ has no critical points in $(\tilde{g}_n)_{\bar{c}-\varepsilon/2}^{\bar{c}+\varepsilon/2} \setminus Z^{1/n}$,
2. $Z^{1/n} \subset \tilde{U}$,

consequently each critical point of $\tilde{g}_n$ in $(\tilde{g}_n)_{\bar{c}-\varepsilon/2}^{\bar{c}+\varepsilon/2}$ is nondegenerate.

Moreover $\tilde{g}_n$, like f, verifies the hypotheses of Theorem 6.1 where $\bar{c}_n$ denotes the critical value of $\tilde{g}_n$ defined by (47). Therefore, for n sufficiently big, Theorem 6.2 can be applied to $\tilde{g}_n$. So there exists a critical point u_n of $\tilde{g}_n$ such that $m(\tilde{g}_n, u_n) = k$.

As $u_n \in Z^{1/n}$, we have that $d(u_n, Z) \to 0$ as $n \to +\infty$. Furthermore, as Z is compact, there exists a point $u_0 \in Z$, such that u_n converges to u_0, as $n \to +\infty$, up to subsequences. By Proposition 4.1 and by (4) of Theorem 4.3, we deduce that $m(f, u_0) \le k \le m^*(f, u_0)$. □

We remark that Theorem 1.8 follows from the previous Theorem 6.4.

Remark 6.5 In the case f satisfies the assumptions of the Mountain Pass Theorem, arguing as in Theorem 6.4, we can infer the existence of a minimax critical point x_0 such that $m(f, x_0) \le 1 \le m^*(f, x_0)$.

We prove Corollary 1.9, stated in Sect. 1.

Proof of Corollary 1.9. It is standard that solutions of (P) correspond to critical points of the functional

$$F(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{\lambda}{2}\int_\Omega |\nabla u|^2 - \frac{\mu}{p}\int_\Omega |u|^p - \int_\Omega Q(u)$$

where $Q(s) = \int_0^s q(t)dt$.

Moreover as $\lim_{|s|\to+\infty} \frac{q'(s)}{|s|^{p-2}} = 0$, then F has a saddle geometry. Indeed, $F(tu_1) \to -\infty$ as $t \to +\infty$.

Furthermore, as $\mu < \bar{\mu}_2$, we can fix $\varepsilon > 0$ small such that $\mu \le \bar{\mu}_2 - \varepsilon$. In correspondence of $\varepsilon/2 > 0$, there exists Y, a subspace of $W_0^{1,p}(\Omega)$ complementing $\mathbb{R}u_1$, such that for any $v \in Y$

$$\int_\Omega |\nabla v|^p > \left(\bar{\mu}_2 - \frac{\varepsilon}{2}\right)\int_\Omega |v|^p. \tag{48}$$

Hence by (48) we deduce that

$$F(v) \ge \frac{\varepsilon}{p(2\bar{\mu}_2 - \varepsilon)}\int_\Omega |\nabla v|^p - \int_\Omega Q(v). \tag{49}$$

As $\lim_{|s|\to+\infty}\frac{q'(s)}{|s|^{p-2}}=0$, by (49), we infer $F(v)\to+\infty$ as $\|v\|\to+\infty$, $v\in Y$.

As $\mu_1<\mu<\bar{\mu}_2\le\mu_2$, we can deduce by standard arguments that F satisfies the $(P.S.)$ condition. Arguing by contradiction, we assume that there exists a sequence u_n, with $\|u_n\|\to+\infty$ as $n\to+\infty$, such that $F(u_n)\to c$ and that there exists a decreasing sequence ξ_n converging to 0 such that

$$|\langle F'(u_n),v\rangle|\le\xi_n\|v\|\quad\forall n\in\mathbb{N},\quad\forall v\in W_0^{1,p}(\Omega). \tag{50}$$

Set $z_n=\frac{u_n}{\|u_n\|}$, then z_n converges weakly to some z in $W_0^{1,p}(\Omega)$ and strongly in L^q where $q\in[p,p^*[$ if $N>p$ or $q\ge p$ if $1\le N\le p$.

Now dividing (50) by $\|u_n\|^{p-1}$, and then taking $v=z_n-z$, we derive by the assumption $\lim_{|s|\to+\infty}\frac{q'(s)}{|s|^{p-2}}=0$ that

$$\lim_{n\to+\infty}\int|\nabla z_n|^{p-2}\nabla z_n\nabla(z_n-z)\,dx=0$$

which implies that $z_n\to z$ strongly in $W_0^{1,p}(\Omega)$, as $n\to+\infty$. Choosing $v=\varphi/\|u_n\|^{p-1}$ in (50) we infer that $-\Delta_p z=\mu|z|^{p-2}z$ with homogeneous Dirichlet boundary conditions on $\partial\Omega$. But this equation, being $\mu\in]\mu_1,\bar{\mu}_2[$, has zero as the only solution by definition of $\bar{\mu}_2$. Thus we conclude $z=0$, which contradicts the fact $\|z_n\|=1$. Thus $\|u_n\|$ is bounded. Standard arguments prove that u_n converges strongly in $W_0^{1,p}(\Omega)$, up to subsequences.

We can apply Theorem 6.4 and deduce the existence of a critical point $\bar{u}\in W_0^{1,p}(\Omega)$ of F such that $m(F,\bar{u})\le1\le m^*(F,\bar{u})$.

We notice that the Morse index estimates can be useful to avoid the trivial solution. By Theorem 6.4, we can directly deduce an existence result, analogous to Theorem 1.1 of [24], in a Hilbert setting, dropping the assumption on the Fredholm properties of $f''(0)$.

Theorem 6.6 *Let $f\in\mathcal{F}_\lambda(W_0^{1,p}(\Omega))$ be with $\lambda>0$, satisfying the hypothesis (S) and the $(P.S.)$ condition at level $\bar{c}$, defined in (47). If 0 is a critical point and either $\dim X<m(f,0)$ or $m^*(f,0)<\dim X$, then there exists a critical point u_0 of f with $u_0\ne0$.*

Remark 6.7 By Theorem 6.6 we can deduce that if $q(0)=0$ in Corollary 1.9 (thus 0 solves problem (P)), then the found solution $\bar{u}$ is not trivial, provided that $1\notin[m(F,0),m^*(F,0)]$.

On the contrary if we consider a functional $f_0\in\mathcal{F}_\lambda(W_0^{1,p}(\Omega))$ with $\lambda=0$, (which is not covered by Theorem 6.6), having a critical point in 0, we prove by regularity results in [22] that the large Morse index of the functionals approximating f_0 near the origin (see Theorem 5.3) is upper semicontinuous. These results are useful to avoid the trivial solution for quasilinear problems involving only p-Laplacian.

Proposition 6.8 *Let us consider the Euler functional $I:W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by*

$$I(u)=\frac{1}{p}\int_\Omega|\nabla u|^p dx+\int_\Omega G(u)\,dx \tag{51}$$

where $p > 2$, Ω is a bounded open set in $\mathbb{R}^N$ with smooth boundary and $G(s) = \int_0^s g(t)dt$, $g \in C^1(\mathbb{R}, \mathbb{R})$ which satisfies assumption (g).

Assume that I satisfies the assumption (S) and the $(P.S.)$ condition at level $\bar{c}$, defined in (47).

If $g(0) = 0$ and $g'(0) \neq 0$, then there exists a nontrivial critical point u_0 of I.

Proof We assume that 0 is the only critical point of I, in the other case the statement is trivial.

If $g'(0) < 0$, the Morse index $m(f, 0) = m^*(f, 0) = +\infty$. Moreover by Theorem 3.1 in [23], due to Lancelotti, it is known that $C_q(I, 0) = \{0\}$ for any $q \geq 0$. So by Theorem 6.1, we deduce that there exists a critical point u_0 of I such that $C_k(I, u_0) \neq \{0\}$ and this is absurd.

If $g'(0) > 0$, then we have $m(I, 0) = m^*(I, 0) = 0$. By Theorem 5.3, we can consider an approximating sequence of functionals h_n of I in C^2, such that $h_n(v) = I(v)$ if $\|v\| > r$ $(r > 0)$ and the critical points of h_n in $\bar{B}_r(0)$ are in $B_{1/n}(0)$ and nondegenerate (in the new sense). Now for $n \in \mathbb{N}$ large enough we can apply Theorem 6.2 and we can infer that there exists $u_n \neq 0$ such that $h'_n(u_n) = 0$ and $m(h_n, u_n) = k$. As g_n have no critical point outside the ball $B_{1/n}(0)$, we deduce $u_n \to 0$ as $n \to +\infty$. Thus by (45) in Theorem 5.3, we have

$$k = \limsup_{n \to +\infty} m(h_n, u_n) \leq m^*(I, 0) = 0.$$

This is absurd. □

Remark 6.9 If $g'(0) = 0$, the Morse index $m(I, 0) = 0$ and $m^*(I, 0) = +\infty$. In this case, the statement of Proposition 6.8 is not true, in general.

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